Theoretical Physics
Chalmers & Göteborg University

Final Exam in Quantum Mechanics, FKA081

Tuesday December 12, 2000, 8.45 - 13.45, MB and MC

Examiner: Henrik Johannesson, phone: 3185.
Allowed references: J. Sakurai, "Modern Quantum Mechanics", mathematical tables, one sheet of hand-written formulas.

Please put your name on *each solution sheet*, and don't forget to include your e-mail address on the cover sheet.

Structure your solutions carefully. State precisely which assumptions, theoretical results, approximations, etc. you use. The logic of your arguments must be transparent, and you should strive for optimal readability! All problems are equally weighted (10p/problem).

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1. Observables, measurements, expectation values, and all that...

Consider the operators Λ_x, Λ_y and Λ_z on a linear vector space $V^2(C)$, represented by the matrices

$$\Lambda_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Lambda_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \Lambda_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(i) If Λ_y is measured in the state

$$|\,\psi\rangle = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} ,$$

what are the possible outcomes and their respective probabilities? If then Λ_z is measured, what are the possible outcomes and respective probabilities?

(ii) Let λ_{y1} and λ_{y2} be the possible outcomes of measuring Λ_y. What is the most general, normalized state $|\,\phi\rangle$ with the property that if one measures Λ_y in this state, λ_{y1} and λ_{y2} are equally probable? Calculate the expectation value of Λ_z in this state.

(iii) Consider a Hamiltonian $\mathcal{H}$ on $V^2(C)$ which commutes with Λ_y and has eigenvalues $\pm\epsilon$. If $\mathcal{H}$ governs the time evolution of the state $|\,\phi\rangle$ that you constructed in (ii), is there a time at which the expectation value of Λ_z changes sign?

2. Transition probabilities

A one-dimensional charged-particle harmonic oscillator with fundamental frequency ω and mass m is in a time-dependent homogeneous electric field given by

$$E(t) = \frac{A}{\sqrt{\pi}\tau} e^{-(t/\tau)^2} \ ,$$

where A and τ are constants, and where the field is directed along the line of displacement of the oscillator. If, at $t = -\infty$, the oscillator is in its groundstate, find, to a first approximation, the probability that it will be in its first excited state at $t = \infty$. For what frequencies ω is your perturbative solution guaranteed to be valid?

3. Alice and Bob

(i) Alice has found an amazing collection of quantum mechanical systems, all confined to move along a fixed straight line. She believes that she can describe each system as a superposition of two states of momenta $\pm p_0$, which at some reference time $t = 0$ takes the form

$$|\phi\rangle = \frac{1}{\sqrt{2}}(e^{i\pi/4}|p_0\rangle + e^{-i\pi/4}|-p_0\rangle) \ ,$$

where $|\pm p_0\rangle$ are normalized momentum kets. Bob comes along, telling Alice that she is making life unnecessarily complicated: "Alice! You are obviously dealing with an ordinary statistical mixture: half of the systems carry momentum p_0 and the other half momentum $-p_0$. In other words, each system has momentum p_0 *or* $-p_0$. Nothing could be simpler! What is is this nonsense about quantum superpositions? Do you really believe a single system can somehow carry momentum p_0 *and* $-p_0$?" To get his point through Bob argues that his interpretation is much simpler and moreover gives the same prediction for the probability distribution for momentum measurements as does Alice's description. Is Bob right?

(ii) Could you suggest another series of measurements that would settle the question - "pure or mixed ensemble?" - once and for all?

4. Clebsch-Gordan decomposition

Two atoms, A and B with angular momentum $j_A = j_B = 1$ are combined into a bound state of *total* angular momentum $j = 0$. Use a Clebsch-Gordan decomposition of this state to derive the probabilities to find atom A with z-component of its angular momentum $j_A^z = 0, \pm\hbar$. Could you have used a short cut to solve this problem, not employing Clebsch-Gordan technology?

5. Spin Hamiltonians and time reversal

The Hamiltonian for a spin-1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2).$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in condensed matter physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?