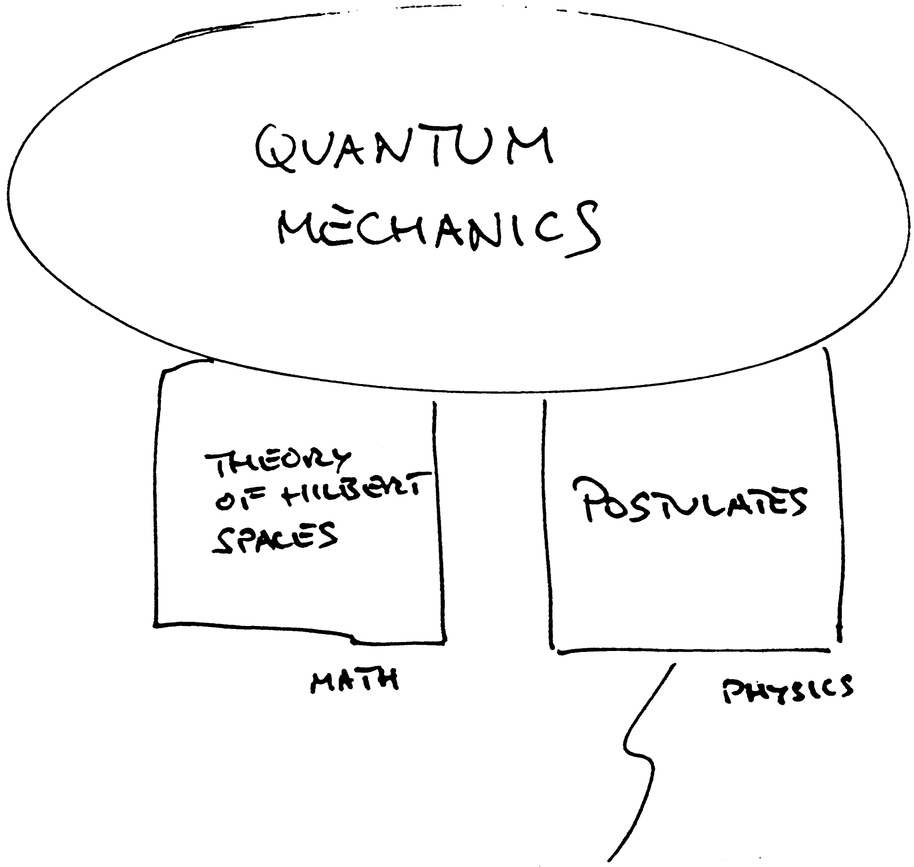


HOW TO THINK ABOUT PHYSICAL SYSTEMS AND MEASUREMENTS
$\left\{\begin{array}{l} \text{I. } \ldots\ldots \\ \text{II. } \ldots\ldots \\ \text{III. } \ldots\ldots \end{array}\right.$

TIME EVOLUTION $\left\{ \text{IV. } i\hbar|\dot{\psi}\rangle = H|\psi\rangle \right.$

The Postulates of (non-relativistic) Quantum Mechanics

I – III

$\downarrow$

Particle in a state $|\psi(t)\rangle$

$\downarrow$ ***four-step program** (← postulates I - III)*

<u>step 1</u>: construct $\Omega = \omega(x \to X, p \to P)$

<u>step 2</u>: find the eigenkets $|\omega_i\rangle$ and eigenvalues ω_i of Ω

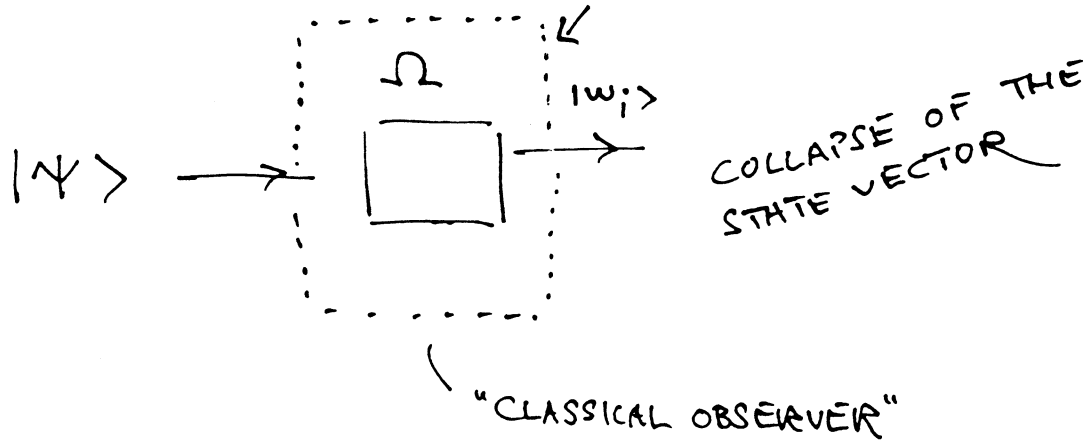
<u>step 3</u>: expand $|\psi\rangle$ in this basis :

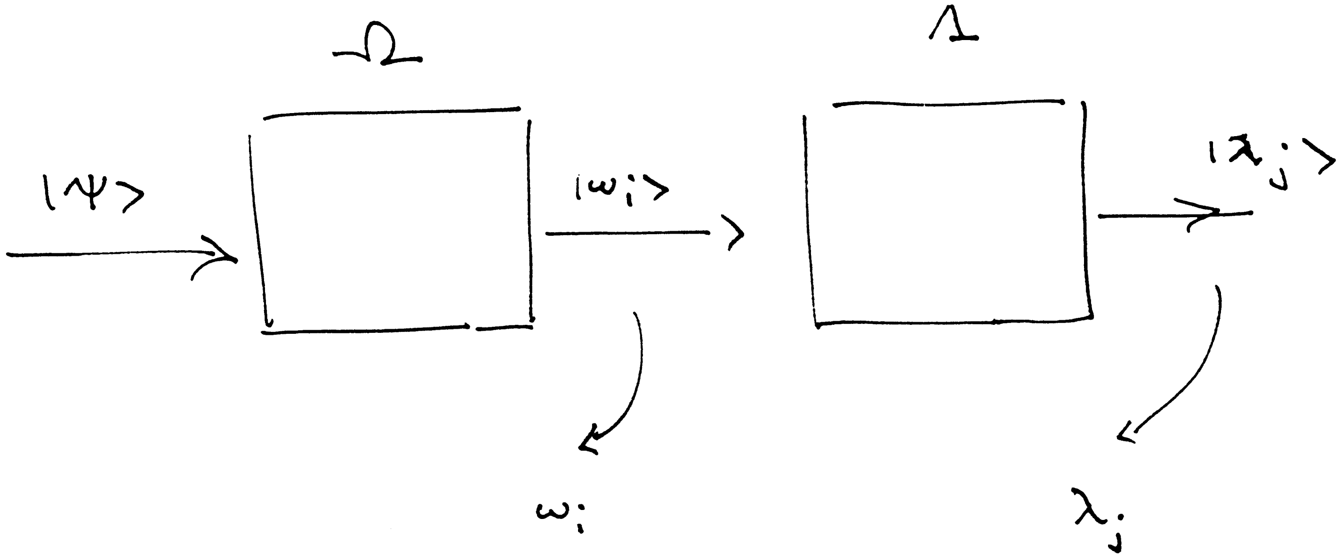
$$|\psi\rangle = \sum_{i=1}^{n} |\omega_i\rangle\langle\omega_i|\psi\rangle$$

<u>step 4</u>: the probability $P(\omega_i)$ of measuring ω_i :

$$P(\omega_i) \sim |\langle\omega_i|\psi\rangle|^2 = \langle\psi|\omega_i\rangle\langle\omega_i|\psi\rangle$$

$$= \langle\psi|\mathbb{P}_{\omega_i}|\psi\rangle$$





$[\Omega, \Lambda] \neq 0$

INCOMPATIBLE OBSERVABLES CANNOT BE MEASURED SIMULTANEOUSLY!

$\Downarrow$

HEISENBERG UNCERTAINTY RELATION

$$[\Omega, \Lambda] = i\Gamma$$

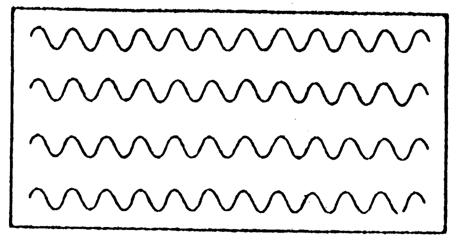
$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}\langle\psi|\{\bar{\Omega}, \bar{\Lambda}\}|\psi\rangle^2 + \frac{1}{4}\langle\psi|\Gamma|\psi\rangle^2$$

Ω, Λ CANONICALLY CONJUGATE $\Rightarrow \Gamma = \hbar$

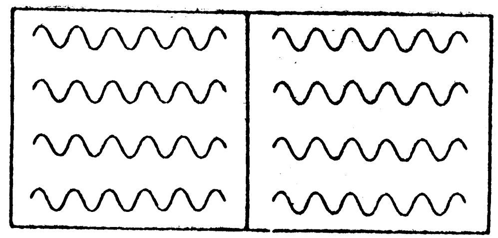
$$\Rightarrow (\Delta\Omega)(\Delta\Lambda) \geq \hbar/2$$

SAKURAI SEC 1.7 OR MY SCANNED LECTURE NOTES, WEEK 39 (COURSE HOME PAGE)

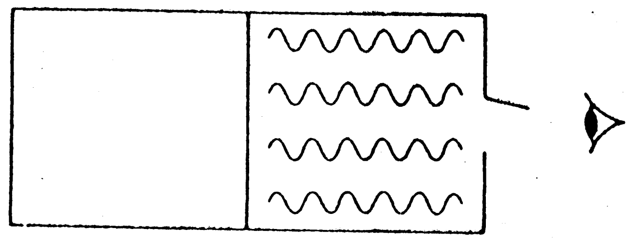
Collapse of a quantum wave. (a) When a single quantum particle is confined to a box its associated wave is spread uniformly throughout the interior. (b) A screen is inserted, dividing the box into two isolated chambers. (c) An observation reveals the particle to be in the right-hand chamber. Abruptly the wave in the other chamber, which represents the probability of finding the particle there, vanishes.



(a)

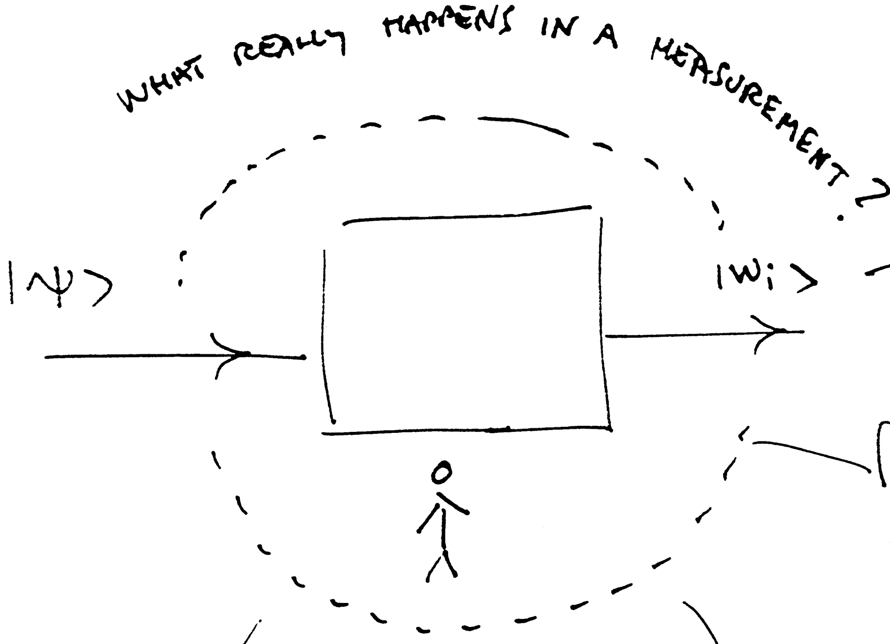


(b)



(c)

THE MEASUREMENT PROBLEM



DECOHERENCE THEORY

ZUREK 1980s

"CONSISTENT HISTORIES APPROACH"

GRIFFITHS, GELL-MANN 1980s

COPENHAGEN INTERPRETATION

"METAPHYSICS"! DON'T ASK!

BOHR, HEISENBERG, PAULI, ... 1920s

"HIDDEN VARIABLES"

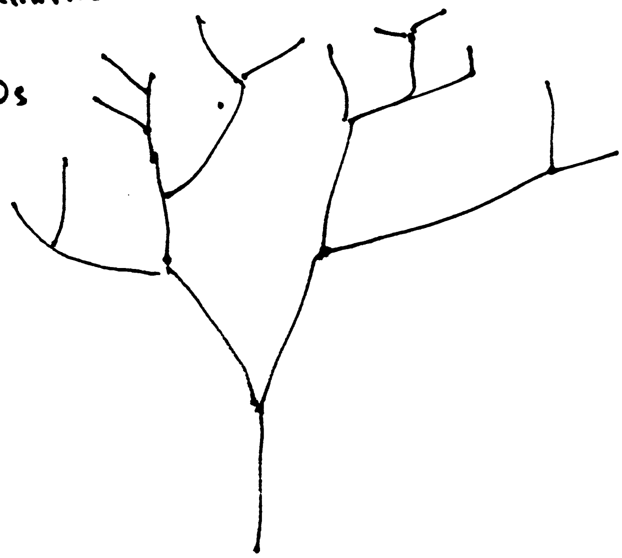
"THIS WILL BE ANSWERED IN THE FUTURE BY A (CLASSICAL) DETERMINISTIC THEORY OF WHICH QUANTUM MECHANICS IS ONLY AN APPROXIMATION"

BOHM 1950s

MANY-WORLDS INTERPRETATION

"THE UNIVERSE SPLITS SO THAT ALL POSSIBLE OUTCOMES OF AN OBSERVATION ARE REALIZED"

EVERETT 1950s



THE FOURTH POSTULATE

TIME EVOLUTION

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$ SCHRÖDINGER EQUATION

HAMILTONIAN

ANSWERS THE QUESTION

"GIVEN $|\psi(0)\rangle$, WHAT IS $|\psi(t)\rangle$?"

EXAMPLES OF HAMILTONIANS

HARMONIC OSCILLATOR

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \xrightarrow{p \to P,\ x \to X} H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

PARTICLE IN AN E.M. FIELD

$$\mathcal{H} = \frac{|p^2 - (\frac{q}{c})\vec{A}(\vec{r},t)|^2}{2m} + q\phi(\vec{r},t)$$

$$\Big\downarrow \begin{array}{l} p \to P,\ x \to X,\ y \to Y,\ z \to Z \\ \text{SYMMETRIZE!} \end{array}$$

$$H = \frac{1}{2m}\left(\vec{P}^2 - \frac{q}{c}(\vec{P}\cdot\vec{A} + \vec{A}\cdot\vec{P}) + (\frac{q}{c})^2\vec{A}^2\right)$$

$$+ q\phi(X,Y,Z) \qquad \vec{A} \equiv \vec{A}(X,Y,Z)$$

HOW DOES IT WORK?

ASSUME H TIME-INDEPENDENT (COMMON!)

GIVEN $|\psi(0)\rangle$, WHAT IS $|\psi(t)\rangle$?

SOLVE $i\hbar|\dot{\psi}\rangle = H|\psi\rangle$! ✡

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

"TIME-EVOLUTION OPERATOR"
= "PROPAGATOR"

$U(t)$?

SOLVE ✡ !

blackboard

HP4

$$U(t) = \sum_E |E\rangle\langle E|\, e^{-iEt/\hbar} \stackrel{\downarrow}{=} e^{-iHt/\hbar}$$

- WHAT IF THE SPECTRUM OF H IS <u>DEGENERATE</u>?

BRING IN A COMPATIBLE OBSERVABLE Λ!

$$\Lambda |E,\lambda\rangle = \lambda |E,\lambda\rangle$$

$$U(t) = \sum_{\lambda}\sum_{E} |E,\lambda\rangle\langle E,\lambda| e^{-iEt/\hbar}$$

- WHAT IF THE SPECTRUM IS CONTINUOUS?

$$U(t) = \int dE \, |E\rangle\langle E| e^{-iEt/\hbar}$$

$|E(t)\rangle = |E\rangle e^{-iEt/\hbar}$ ARE CALLED "STATIONARY STATES"

SINCE ANY OBSERVABLE IS TIME-INDEPENDENT IN SUCH A STATE

$$P(\omega,t) = |\langle\omega|E(t)\rangle|^2 = |\langle\omega|E\rangle e^{-iEt/\hbar}|^2 = |\langle\omega|E\rangle|^2 = P(\omega,0)$$

- WHAT IF H IS TIME-DEPENDENT?

LATER ..... SEE SAKURAI p 72f & chapter 5

~

HOW TO SOLVE THE TIME-INDEPENDENT SCHRÖDINGER EQ.

$$H|\psi\rangle = E|\psi\rangle$$

blackboard

H IS A GENERATOR OF TIME TRANSLATIONS

$$U(t) = e^{-iHt/\hbar} \underset{t=\Delta t \ll 1}{\approx} 1 - \frac{i}{\hbar} H t + \cancel{\mathcal{O}(t^2)}$$

H HERMITIAN $\longrightarrow$ U UNITARY

$\Downarrow$

TIME EVOLUTION = "ROTATION" IN HILBERT SPACE

HEISENBERG PICTURE.

blackboard

FREE PARTICLE IN 1D

$$i\hbar|\dot{\psi}\rangle = H|\psi\rangle = \frac{P^2}{2m}|\psi\rangle$$

feed in!

STATIONARY STATES $|\psi\rangle = |E\rangle e^{-iEt/\hbar}$

$$\Downarrow$$

$$H|E\rangle = \frac{P^2}{2m}|E\rangle = E|E\rangle$$

$$P|p\rangle = p|p\rangle \Rightarrow \frac{p^2}{2m}|p\rangle = E|p\rangle$$

$$\Rightarrow \left(\frac{p^2}{2m} - E\right)|p\rangle = 0$$

$$\Downarrow \quad |p\rangle \neq |0\rangle$$

$$p = \pm\sqrt{2mE}$$

$$\left\{ \begin{array}{l} |E,+\rangle = |p = \sqrt{2mE}\rangle \\ |E,-\rangle = |p = -\sqrt{2mE}\rangle \end{array} \right. \longrightarrow |p\rangle$$

$$|E\rangle = \alpha|p = \sqrt{2mE}\rangle + \beta|p = -\sqrt{2mE}\rangle$$

IS ALSO AN EIGENSTATE

PROPAGATOR

$$U(t) = \int_{-\infty}^{\infty} |p\rangle\langle p| \, e^{-iE(p)t/\hbar} \, dp = \int_{-\infty}^{\infty} |p\rangle\langle p| \, e^{-ip^2 t/2m\hbar}$$

$$\langle x|U(t)|x'\rangle = U(x,t;x') = \int_{-\infty}^{\infty} \langle x|p\rangle\langle p|x'\rangle \, e^{-ip^2t/2m\hbar} \, dp$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ip(x-x')/\hbar} \, e^{-ip^2t/2m\hbar} \, dp$$

GAUSSIAN INTEGRAL

$$= \left(\frac{m}{2\pi\hbar i t}\right)^{1/2} e^{im(x-x')^2/2\hbar t} \qquad *$$

$$\psi(x,t) = \int U(x,t;x') \, \psi(x',0) \, dx' \qquad **$$

SPECIAL CASES

(1) start off with the particle localized at x_0'

$$\psi(x') = \delta(x' - x_0') \Rightarrow \psi(x,t) = U(x,t;x_0')$$

"TRANSITION AMPLITUDE"

THE PROPAGATOR (IN THE $|x\rangle$ BASIS) IS THE AMPLITUDE THAT A PARTICLE STARTING OUT AT $(x_0', t'=0)$ ENDS UP AT (x,t)

wave packet at $t'=0$

(2) $$\psi(x') = e^{ip_0x'/\hbar}\,\frac{e^{-x'^2/2\Delta^2}}{(\pi\,\cancel{*}\,\Delta^2)^{1/4}}$$

↑ correction!

$\langle X\rangle = 0$

$\Delta X = \Delta/\sqrt{2}$

$\langle P\rangle = p_0$

$\Delta P = \hbar/\sqrt{2}\Delta$

Sakurai p. 57 f

(*) & (**)

$\Downarrow$

$$\psi(x,t) = \left[\sqrt{\pi}\left(\Delta + \frac{i\hbar t}{m\Delta}\right)\right]^{-1/2} \exp\left\{\frac{-(x-p_0t/m)^2}{2\Delta^2(1+i\hbar t/m\Delta^2)}\right\}$$
$$\times \exp\left\{\frac{ip_0}{\hbar}\left(x-\frac{p_0t}{2m}\right)\right\}$$

$\Downarrow$

"THE CLASSICAL EQUATIONS OBEYED BY DYNAMICAL VARIABLES HAVE COUNTERPARTS IN Q.M. AS RELATIONS AMONG EXPECTATION VALUES"

1) $$\langle X\rangle = \frac{p_0t}{m} = \frac{\langle P\rangle t}{m}$$

Cf. Ehrenfest's theorem

2) $$\Delta X = \left(\langle(X-\langle X\rangle)^2\rangle\right)^{1/2} = \frac{\Delta}{\sqrt{2}}\left(1+\frac{\hbar^2t^2}{m^2\Delta^4}\right)^{1/2}$$

increasing uncertainty in position

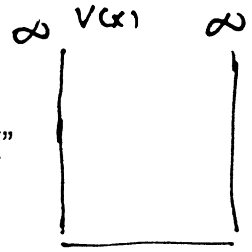
That's basically all there is to quantum mechanics...

the rest are applications!

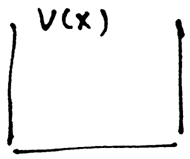
SIMPLE PROBLEMS IN 1D

FREE PARTICLE

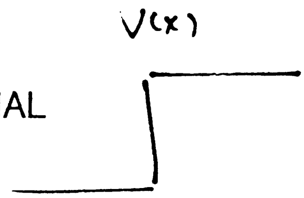
"PARTICLE IN A BOX"



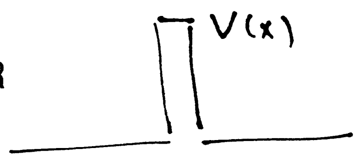
POTENTIAL WELL



SINGLE STEP POTENTIAL



POTENTIAL BARRIER



and variations on this theme....

A SPECIAL PROBLEM

THE HARMONIC OSCILLATOR

"...the most important problem in quantum physics"

J. Schwinger

ALL OTHER PROBLEMS
(October - December!)

The harmonic oscillator

Why bother?

Any system fluctuating by small amounts near a configuration of stable equilibrium may be described by an oscillator or a system of decoupled harmonic oscillators..... and the problem can be solved exactly!

Classical Hamiltonian $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

HAMILTON'S EQUATIONS

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \quad , \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -m\omega^2 x^2$$

equations of motion

$$\ddot{x} + \omega x^2 = 0$$

$$\Downarrow$$

$$x(t) = A\cos\omega t + B\sin\omega t = x_0 \cos(\omega t + \phi)$$

Quantum

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle \quad , \quad H = \mathcal{H}(x \to X, p \to P)$$
$$= \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2$$

As always: the complete dynamics is coded by the propagator U(t) which can be expressed in terms of the eigenvectors and eigenvalues of H. Find these!

RECIPE:

START IN THE ENERGY BASIS (EIGENBASIS TO H)

$$U(t) = \sum_E |E\rangle\langle E| \, e^{-iEt/\hbar}$$

$$\underbrace{\left(\frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2\right)}_{H}|E\rangle = E|E\rangle$$

PROJECT ONTO THE X-BASIS : $X \to x$, $P \to -i\hbar\frac{d}{dx}$, $|E\rangle \to \psi_E(x)$

$$\Downarrow$$

TIME-INDEPENDENT SCHRÖDINGER EQUATION

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right)\psi_E(x) = E\psi_E(x)$$

FIND ALL SOLUTIONS TO THIS EQUATION THAT LIE IN THE (PHYSICAL) HILBERT SPACE (OF FUNCTIONS NORMALIZABLE TO UNITY OR THE DIRAC DELTA FUNCTION)

$$\downarrow$$

a long and winding road...

$E_n = (n+\frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$

Hermite polynomials

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right]$$

$$U(x,t;x',t') = \sum_{n=0}^{\infty} A_n^2 \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n(x) \exp\left(-\frac{m\omega}{2\hbar}x'^2\right)$$

$$\times H_n(x') \exp\left[-i(n+\tfrac{1}{2})\omega(t-t')\right]$$

Some observations:

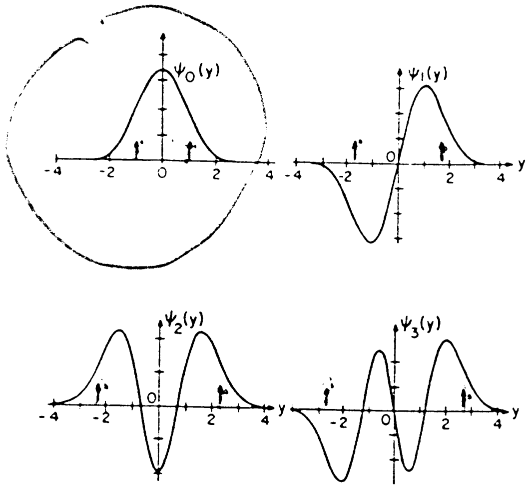
1) Quantization from the requirement of a (physical) Hilbert space

2) Equally spaced energy levels ⟶ think of "quanta" (fictitutious particles) of energy $\hbar\omega$ (cf Einstein, 1907)

$$E_n = (n + 1/2)\hbar\omega \,, \; n = \cdot \; \ldots$$

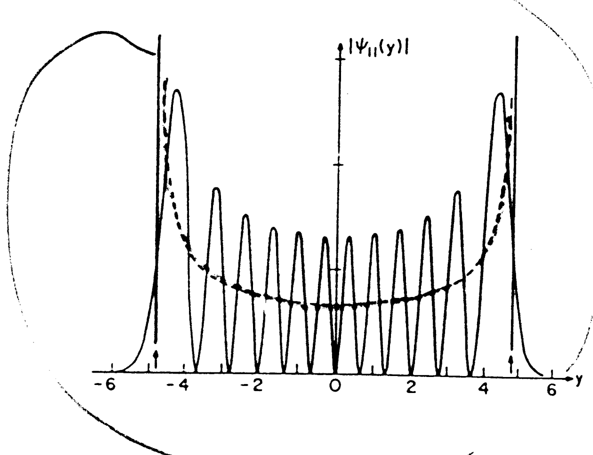
3) Zero point energy $\hbar\omega/2$

4) Eigenfunctions are even or odd



5) Excursions beyond the classical turning point possible!

6) The probability distribution is very different from the classical case



However, at large n (cf. Bohr's "correspondence principle") the quantum distribution wiggles so rapidly (on a scale set by the classical amplitude) that only its mean can be detected, and this agrees with the classical case.

A smarter way to handle the harmonic oscillator....

Could we work directly in the energy basis without knowing ahead of time the operators X and P in this basis?

Dirac (1928): "Yes, use the fact that the canonical commutation relation $[X, P] = i\hbar$ **is basis independent!"**

...

BLACKBOARD