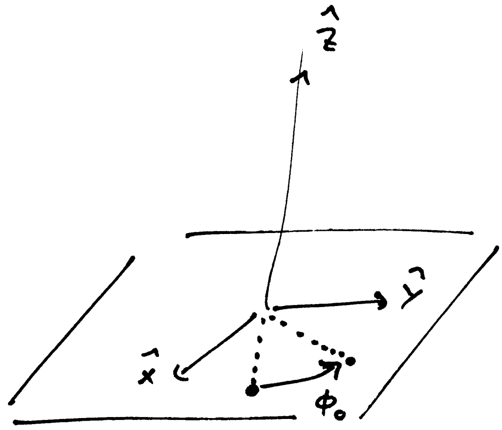
Theory of angular momentum



Review of last week:

2D rotation of a classical state:

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{R(\phi_0 \hat{z})} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} \xrightarrow{R(\phi_0 \hat{z})} \begin{pmatrix} p_x' \\ p_y' \end{pmatrix}$$

2D rotation of a **quantum state:**

$$U[R(\varepsilon \hat{z})] = \mathbb{1} - \frac{i\varepsilon}{\hbar} L_z$$

$\phi_0 = \varepsilon \ll 1$

$L_z = X P_y - Y P_x$

$$U[R(\phi_0 \hat{z})] : |\psi\rangle \longrightarrow |\psi_R\rangle$$

$$U[R(\phi_0 \hat{z})] \xrightarrow[\{|\rho,\phi\rangle\} \text{ basis}]{} \exp\left(-\phi_0 \frac{\partial}{\partial \phi}\right)$$

$$\exp\left(-\phi \frac{\partial}{\partial\phi}\right) \psi(\rho,\phi) = \psi(\rho, \phi - \phi_0)$$

Eigenvalue problem of L_z :

$$L_z |\ell_z\rangle = \ell_z |\ell_z\rangle$$

coordinate basis $\psi_{\ell_z}(\rho,\phi) \equiv \langle \rho\phi | \ell_z \rangle$

$$-i\hbar \frac{\partial}{\partial\phi} \psi_{\ell_z}(\rho,\phi) = \ell_z \psi_{\ell_z}(\rho,\phi)$$

EIGENVALUES $\ell_z = m\hbar$, $m = 0, \pm 1, \pm 2, \ldots$

EIGENFUNCTIONS $\psi_{\ell_z}(\rho,\phi) = R(\rho)\, e^{im\phi}$

Rotational invariant Hamiltonian H

$$[H, L_z] = 0 \Rightarrow$$ common eigenbasis

$$\Downarrow$$

$R(\rho)$ determined from the eigenvalue problem of H

$$\left\{ -\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2}\right) + V(\rho) \right\} R(\rho) = E R(\rho)$$

$$\Downarrow$$

specifies a unique solution

$$\psi_{E,m} = \mathcal{R}_{E,m}(\rho)\,\Phi_m(\phi)$$

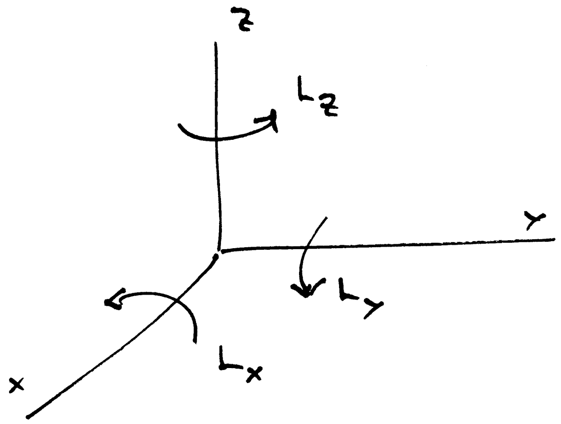
$$\mathcal{R}_{E,m}(\rho) \sim \sqrt{2\pi}\,R(\rho) \qquad \Phi_m(\phi) \sim \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

3D

$$L_x = YP_z - ZP_y$$

$$L_y = ZP_x - XP_z$$

$$L_z = XP_y - YP_x$$



L_x, L_y, L_z implement the infinitesimal rotations in Hilbert space

$$\Downarrow$$

ANGULAR MOMENTUM COMMUTATION RELATIONS
("SU(2) algebra")

$$\boxed{[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k \quad , \quad i,j,k = x,y \text{ or } z}$$

$$L^2 = |\vec{L}|^2 = L_x^2 + L_y^2 + L_z^2$$

$$[L^2, L_i] = 0 \quad , \quad i = x, y \text{ or } z$$

Rotational invariance in 3D

$$\langle \psi_R | H | \psi_R \rangle = \langle \psi | H | \psi \rangle$$

$$\Downarrow$$

$$U^{+}[R]\, H\, U[R] = H$$

$$\Downarrow$$

$$[H, L_i] = 0 \quad , \quad i = x, y \text{ or } z$$

$$\Downarrow \quad [L_i, L^2] = 0$$

$$[H, L^2] = 0$$

H, L_z, L^2 compatible

together specify a unique basis

Start with the **eigenvalue problem of L_z and L^2**

In 2D we used a coordinate basis: $|\psi\rangle \longrightarrow \psi(\rho,\phi) \equiv \langle\rho\phi|\psi\rangle$

Now, in 3D, let's try to work directly in the eigenbasis
(cf. Dirac's approach to the HO)

$$L^2|\alpha\beta\rangle = \alpha|\alpha\beta\rangle$$
$$L_z|\alpha\beta\rangle = \beta|\alpha\beta\rangle$$

$L_\pm \equiv L_x \pm i L_y$ **LADDER OPERATORS**

$[L_z, L_\pm] = \pm\hbar L_z \Rightarrow$ shifts the eigenvalue of L_z by $\pm\hbar$

$[L^2, L_\pm] = 0 \Rightarrow$ eigenvalues of L^2 insensitive to $L_\pm$

$\underline{Ex}$ $L_z(L_+|\alpha\beta\rangle) = (L_+L_z + [L_z, L_+])|\alpha\beta\rangle$
$= (L_+L_z + \hbar L_+)|\alpha\beta\rangle$
$= (\beta+\hbar)\underbrace{(L_+|\alpha\beta\rangle)}_{C_+(\alpha\beta)|\alpha,\beta+\hbar\rangle}$

$\Downarrow$

$$L_+|\alpha\beta\rangle = C_+(\alpha\beta)|\alpha,\beta+\hbar\rangle$$

$$L_-|\alpha\beta\rangle = C_-(\alpha\beta)|\alpha,\beta-\hbar\rangle$$

Let's have a closer look!

BLACKBOARD ! $\rightarrow$

$$\alpha = \beta_{max}(\beta_{max}+\hbar)$$

$$\beta_{max} = \frac{\hbar k}{2}, \quad k = 0, +1, +2, \ldots$$

$$-\beta_{max} \le \beta \le \beta_{max}$$

L_z can have half-integral eigenvalues !

$$\langle \alpha\beta | L^2 - L_z^2 | \alpha\beta \rangle = \langle \alpha,\beta | \underbrace{L_x^2 + L_y^2}_{\text{positive definite operators}} | \alpha,\beta \rangle \geq 0$$

$$\Rightarrow \quad \alpha - \beta^2 \geq 0 \quad \Rightarrow \quad \alpha \geq \beta^2$$

$$\Downarrow$$

$$\exists\, |\alpha,\beta_{max}\rangle \; : \; L_+ |\alpha,\beta_{max}\rangle = 0$$

$$|\alpha,\beta_{min}\rangle \; : \; L_- |\alpha\,\beta_{min}\rangle = 0$$

Start with the "top state"

$$0 = L_- L_+ |\alpha,\beta_{max}\rangle = (L^2 - L_z^2 - \hbar L_z) |\alpha\,\beta_{max}\rangle$$

$$= (\alpha - \beta_{max}^2 - \hbar\beta_{max}) |\alpha\beta_{max}\rangle = 0$$

$$\Rightarrow \quad \alpha = \beta_{max}(\beta_{max} + \hbar) \qquad (1)$$

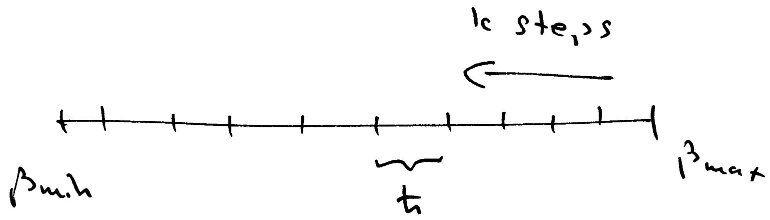
Now operate k times with L_- till we reach $|\alpha\,\beta_{min}\rangle$.
Use that $L_- |\alpha,\beta_{min}\rangle = 0$

$$\Rightarrow \quad 0 = L_+ L_- |\alpha\beta_{min}\rangle = (L^2 - L_z^2 + \hbar L_z) |\alpha\,\beta_{min}\rangle$$

$$\Rightarrow \quad \alpha = \beta_{min}(\beta_{min} - \hbar) \qquad (2)$$

(1) & (2) $\Rightarrow$ $\beta_{min} = -\beta_{max}$ $\Rightarrow$ $2\beta_{max} = \beta_{max} - \beta_{min}$

$= \hbar k$



$$\Downarrow \quad \boxed{-\beta \leq \beta \leq \beta_{max}}$$

$$\beta_{max} = \frac{\hbar k}{2}, \quad k = 0, 1, 2, \ldots$$

$$\boxed{\alpha = (\beta_{max})(\beta_{max} + \hbar) = \hbar^2 \frac{k}{2}\left(\frac{k}{2} + 1\right)}$$

L_z can have half-integral eigenvalues!

cf. to the 2D analysis:

$$\beta = \ell_z = m\hbar \quad , \quad m = 0, \pm 1, \pm 2, \ldots$$

?

In the 3D analysis we relied only on the commutation relations

$$[L_z, L_\pm] = \pm\hbar L_z \quad , \quad [L^2, L_\pm] = 0$$

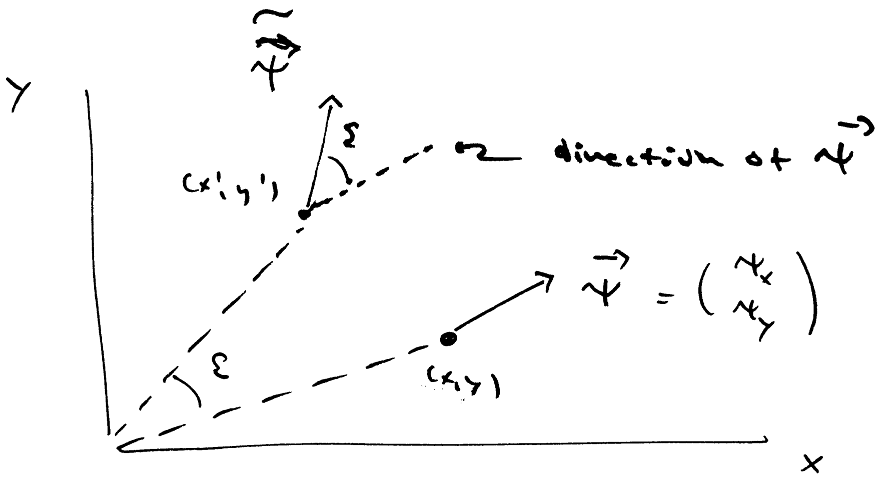
no projection onto a coordinate basis

with only scalar wave functions

$$\psi(x, y, z) = \langle xyz | \psi \rangle$$

The half-integral eigenvalues of L_z reflect the possibility of having **multicomponent wave functions.** The components get shuffled around when doing a rotation. The "shuffling around" is governed by a **spin operator.** *blackboard!*

EXAMPLE



$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{R} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

SCALAR WAVE FUNCTION

$$U[R] : \Psi(x,y) \to \tilde{\Psi}(x,y) , \quad \tilde{\Psi}(x',y') = \Psi(x,y)$$

2-COMPONENT WAVE FUNCTION TRANSFORMING AS A VECTOR

$$\begin{pmatrix} \Psi_x(x,y) \\ \Psi_y(x,y) \end{pmatrix} \xrightarrow[\text{OF } \vec{\Psi}]{\text{ROTATE THE COMPONENTS}} \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} \Psi_x(x,y) \\ \Psi_y(x,y) \end{pmatrix}$$

$$\underset{\text{ROTATION}}{\overset{\text{INFINITESIMAL}}{\approx}} \begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} \Psi_x(x,y) \\ \Psi_y(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} \Psi_x(x,y) - \varepsilon \Psi_y(x,y) \\ \varepsilon \Psi_x(x,y) + \Psi_y(x,y) \end{pmatrix} \xrightarrow[\text{OF } \Psi_x, \Psi_y]{\text{ROTATE THE ARGUMENTS}}$$

p 5 Lecture Notes last week $\rightarrow$

$$\begin{pmatrix} \psi_x(x+y\varepsilon, y-x\varepsilon) - \varepsilon\psi(x+y\varepsilon, y-x\varepsilon) \\ \varepsilon\psi_x(x+y\varepsilon, y-x\varepsilon) + \psi_y(x+y\varepsilon, y-x\varepsilon) \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{\psi}_x(x,y) \\ \tilde{\psi}_y(x,y) \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} \tilde{\psi}_x(x,y) \\ \tilde{\psi}_y(x,y) \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i\varepsilon}{\hbar}\begin{pmatrix} L_z & 0 \\ 0 & L_z \end{pmatrix} - \frac{i\varepsilon}{\hbar}\overbrace{\begin{pmatrix} 0 & -i\hbar \\ i\hbar & 0 \end{pmatrix}}^{S_z} \right\} \times$$

$$\times \begin{pmatrix} \psi_x(x,y) \\ \psi_y(x,y) \end{pmatrix}$$

$$\Rightarrow \quad \left(\mathbb{1} - \frac{i\varepsilon}{\hbar} J_z\right)\begin{pmatrix} \psi_x(x,y) \\ \psi_y(x,y) \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_x(x,y) \\ \tilde{\psi}_y(x,y) \end{pmatrix}$$

W.R.T. COMPONENTS

$$J_z = \underbrace{L_z^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)}}_{\text{W.R.T. TO ARGUMENTS}} \otimes S_z^{(2)}$$

Comments

1. Origin? Relativity, Dirac equation ...
2. S_z off-diagonal ?! But this is a vector wave function we are looking at, not a spinor... Also, the basis is not diagonal w.r.t. S_z (obviously!)
3. Spin is not like a top spinning around its axis...

more about this later...

$$\underbrace{\vec{J}}_{\text{total angular momentum}} = \underbrace{\vec{L}}_{\text{orbital angular momentum}} + \vec{S} \leftarrow \text{spin ("intrinsic angular momentum")}$$

$$\boxed{\begin{aligned} [J_i, J_k] &= i\hbar\, \varepsilon_{ijk} J_k \\ [L_i, S_j] &= 0 \end{aligned}}$$

$i, j, k = x, y$ or z

$$\Rightarrow \quad [S_i, S_j] = i\hbar\, \varepsilon_{ijk} S_k$$

particles with spin

↓

wave functions more complicated than scalars

spinors, vectors, ...

electron   W, Z, ...

Back to the eigenvalue problem of L_z and L^2!

Summary of what we've found:

$$J^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle \quad , \quad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

$$J_z |jm\rangle = m\hbar |jm\rangle \quad , \quad m = -j, -j+1, \ldots, j$$

$$\beta = m\hbar \quad , \quad \alpha = j(j+1)\hbar^2$$

If no spin $\Rightarrow \vec{J} = \vec{L}$

$$L^2 |\ell m\rangle = \ell(\ell+1)\hbar^2 |\ell m\rangle \quad , \quad \ell = 0, 1, 2, \ldots$$

$$L_z |\ell m\rangle = m\hbar |\ell m\rangle \quad , \quad m = -\ell, -\ell+1, \ldots, \ell$$

Recall: $L_\pm |\alpha\beta\rangle = C_\pm(\alpha\beta) |\alpha, \beta \pm \hbar\rangle$

$\alpha \to j$, $\beta \to m$ $\Downarrow$ **add spin**

$$J_\pm |jm\rangle = C_\pm(jm) |j, m \pm 1\rangle$$

$\downarrow$ some algebra (see Sakurai, p. 191 f)

$$C_\pm(jm) = \hbar\sqrt{(j \mp m)(j \pm m + 1)}$$

Matrix elements of J_x, J_y, J_z in the eigenbasis of J_z and J^2

(Sakurai, p. 192 f)

$$\langle j'm' | J_x | jm\rangle = \langle j'm' | \tfrac{1}{2}(J_+ + J_-) | jm\rangle = \ldots\ldots$$

$$\langle j'm' | J_y | jm\rangle = \langle j'm' | \tfrac{1}{2i}(J_+ - J_-) | jm\rangle = \ldots\ldots$$

$$\langle j'm' | J_z | jm\rangle = m\hbar\, \delta_{mm'} \delta_{jj'}$$

$$\langle j'm' | J^2 | jm\rangle = j(j+1)\hbar^2 \delta_{mm'} \delta_{jj'}$$

MATRIX REPRESENTATION IN THE J_z - J^2 EIGENBASIS

J_z and J^2 diagonal.

J_x

$$J_x \rightarrow \begin{bmatrix} \boxed{0} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \hbar/2 & 0 & 0 & 0 & \\ 0 & \hbar/2 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \hbar/2^{1/2} & 0 & \\ 0 & 0 & 0 & \hbar/2^{1/2} & 0 & \hbar/2^{1/2} & \\ 0 & 0 & 0 & 0 & \hbar/2^{1/2} & 0 & \\ \vdots & & & & & & \ddots \end{bmatrix}$$

J_y

$$J_y \rightarrow \begin{bmatrix} \boxed{0} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -i\hbar/2 & 0 & 0 & 0 & \\ 0 & i\hbar/2 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -i\hbar/2^{1/2} & 0 & \\ 0 & 0 & 0 & i\hbar/2^{1/2} & 0 & -i\hbar/2^{1/2} & \\ 0 & 0 & 0 & 0 & i\hbar/2^{1/2} & 0 & \\ \vdots & & & & & & \ddots \end{bmatrix}$$

J_x and J_y block diagonal!

$$[J_i^{(j)}, J_k^{(j)}] = i\hbar \varepsilon_{ijk} J_k^{(j)}, \quad j = 0, \tfrac{1}{2}, 1, \ldots$$

satisfied within each (2j+1)x(2j+1) block

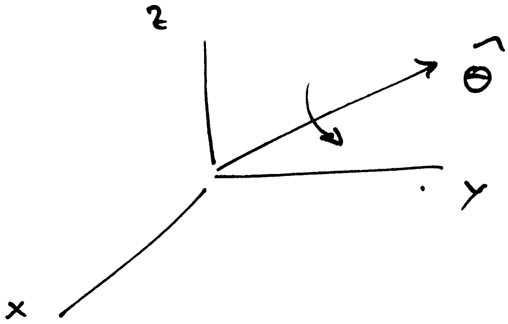
irreducible rep of $\vec{J}$

The rotation matrices characterized by a definite **j** form a unitary group: **SU(2)**

$\exists$ INVERSE, IDENTITY ELEMENT, PRODUCT RULE (matrix algebra)

What about **finite rotations?**

$$U[R(\vec{\theta})] = \lim_{N \to \infty} \Big(\mathbb{1} - \frac{i}{\hbar} \underbrace{\left(\frac{\theta}{N}\right)}_{\varepsilon} \hat{\theta} \cdot \vec{J} \Big)^N = e^{-i\vec{\theta}\cdot\vec{J}/\hbar}$$



ROTATION WITH AN ANGLE θ ABOUT $\hat{\theta}$

J_i block diagonal $\Rightarrow \hat{\theta}\cdot\vec{J}$ block diagonal

$\Rightarrow U[R(\vec{\theta})]$ block diagonal

j^{th} block

$\mathcal{D}^{(j)}[R]$

To rotate $|m_j\rangle$ we only need to use $\mathcal{D}^{(j)}[R]$ $\quad\lightning\quad$ "invariant subspaces"

More generally: if $|\Psi\rangle$ has components only in $V_0, V_1, V_2, \ldots\ldots V_j$
(V_j spanned by $|m=j, j\rangle, \ldots |m=-j, j\rangle$), we only need the first j+1
matrices

$$\mathcal{D}^{(0)}[R], \ldots, \mathcal{D}^{(j)}[R]$$

$$\mathcal{D}^{(j)}[R(\theta)] = \exp\left(-i\,\frac{\vec{\theta}\cdot\vec{J}^{(j)}}{\hbar}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\theta}{\hbar}\right)^n \left(\hat{\theta}\cdot\vec{J}^{(j)}\right)^n$$

$$= \sum_{n=0}^{2j} f_n(\theta)\left(\hat{\theta}\cdot\vec{J}^{(j)}\right)^n$$

can be calculated
Sakurai, sec. 3.8
(not included in the course)

Within each invariant subspace V_j , H has the same eigenvalue E_j, since $[H, J_\pm] = 0$

$\Downarrow$

all states of a given j are degenerate in energy

Now... let's look at the

Angular momentum eigenfunctions in the coordinate basis

$\Downarrow$ Sakurai, 3.6

summary

$$L_\pm \equiv L_x \pm i L_y \xrightarrow[\{|r\theta\phi\rangle\}]{} \pm \hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta \frac{\partial}{\partial\phi}\right)$$

spherical coordinate basis

Start with the "highest-weight state" $|m=\ell, \ell\rangle$

$$L_+ |\ell\ell\rangle = 0$$

$$\langle r\theta\phi| L_+ |\ell\ell\rangle = \left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) \overbrace{\psi_\ell^\ell(r\theta\phi)}^{\langle r\theta\phi|\ell\ell\rangle} = 0$$

ψ_ℓ^ℓ eigenfunctions of L_z

$\Leftarrow$

$$\psi_\ell^\ell(r,\theta,\phi) = U_\ell^\ell(r,\theta)\, e^{i\ell\phi}$$

FEED BACK IN !

$$\Rightarrow \left(\frac{\partial}{\partial\theta} - \ell\cot\theta\right) U_\ell^\ell = 0$$

$$\Rightarrow \frac{dU_\ell^\ell}{U_\ell^\ell} = \ell\,\frac{d(\sin\theta)}{\sin\theta}$$

$$\Rightarrow U_\ell^\ell(r,\theta) = R(r)\,(\sin\theta)^\ell$$

$\hat{\llcorner}$ ARBITRARY NORMALIZABLE FUNCTION

Suppose there's no r-dependence:

$\Downarrow$ NORMALIZATION

$$R(r) = (-1)^{\ell} \left(\frac{(2\ell+1)!}{4\pi} \right)^{1/2} \frac{1}{2^{\ell} \ell!}$$

EIGENFUNCTION TO L_z AND L^2 (m = l)

$$Y_{\ell}^{m=\ell}(\theta, \phi) = (-1)^{\ell} \left(\frac{(2\ell+1)!}{4\pi} \right)^{1/2} \frac{1}{2^{\ell} \ell!} (\sin\theta)^{\ell} e^{i\ell\theta}$$

$$\int Y_{\ell}^{\ell *}(\theta,\phi)\, Y_{\ell}^{\ell}(\theta,\phi)\, d\Omega = 1 \qquad \text{CHECK !}$$

Apply L_- (l-m) times to $|\ell,\ell\rangle$

EIGENFUNCTIONS FOR $m \geq 0$ $\Downarrow$ repeat the analysis

$$Y_{\ell}^{m}(\theta,\phi) = (-1)^{\ell} \left(\frac{(2\ell+1)!}{4\pi} \right)^{1/2} \frac{1}{2^{\ell} \ell!} \left(\frac{(\ell+m)!}{(2\ell)!\,(\ell-m)!} \right)^{1/2}$$

$$\times\, e^{im\phi} (\sin\theta)^{-m} \frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}} (\sin\theta)^{2\ell}$$

SPHERICAL HARMONICS

For m < o, use $\quad Y_{\ell}^{-m} = (-1)^{m} \left(Y_{\ell}^{m} \right)^{*}, \quad m \geq 0$

But suppose there ***is*** an r-dependence!

Large degeneracy! How to "nail down" R(r) ?

Bring in H ! Works fine if H is rotationally invariant...

$$[H, L_z] = [H, L^2] = 0$$

Eigenvalue problem for H ("time-independent Schrödinger eq.")

$$H|E m l\rangle = E|E m l\rangle$$

$= |m l\rangle$ "split by E"

$\{|r\theta\phi\rangle\}$ basis $\Downarrow$

ROTATIONAL INVARIANCE $\Rightarrow$ NO θ, ϕ IN V

$$\left\{ -\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) + V(r)\right\}$$

$$\times\ \psi_{Eml}(r,\theta,\phi) = E\,\psi_{Eml}(r,\theta,\phi)$$

$$\psi_{Eml} = \langle r\theta\phi|Eml\rangle$$

FEED IN $\psi_{Eml}(r,\theta,\phi) = R_{Eml}(r)\,Y_l^m(\theta,\phi)$

$$\Downarrow$$

" **RADIAL EQUATION** "

effective centrifugal potential from the orbital angular momentum

$$\left\{-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{\ell(\ell+1)}{r^2}\right)+V(r)\right\}R_{E\ell}=E R_{E\ell}$$

$$\Downarrow \quad U_{E\ell}\equiv r R_{E\ell}$$

$$\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+V(r)+\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)U_{E\ell}=E U_{E\ell}$$

use that D_ℓ is Hermitian, and require $U_{E\ell}$ to be normalizable

$$\Downarrow$$

BOUND STATE

$$U_{E\ell}\xrightarrow{r\to 0}0, \quad U_{E\ell}\xrightarrow{r\to\infty}\begin{cases}0, & E<0\\ e^{ikr}, & E>0\end{cases}$$

UNBOUND STATE

Paradigm cases

Free particle $H = -\frac{\hbar^2}{2\mu}\nabla^2$

$$\psi_E(x,y,z) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}\cdot\vec{r}/\hbar}, \quad E = \frac{p^2}{2\mu} = \frac{(\hbar k)^2}{2\mu}$$

spherical coordinates

$$\psi_{E\ell m}(r,\theta,\phi) = j_\ell(kr)\, Y_\ell^m(\theta,\phi), \quad E = \frac{(\hbar k)^2}{2\mu}$$

SPHERICAL BESSEL FUNCTION OF ORDER ℓ

(Isotropic) harmonic oscillator

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{2}\mu\omega^2(x^2+y^2+z^2)$$

$$\psi_{E\ell m}(r,\theta,\phi) = \frac{u_{E\ell}(r)}{r} Y_\ell^m(\theta,\phi)$$

$$y \equiv \left(\frac{\mu\omega}{\hbar}\right)^{1/2} r$$

$$u_{E\ell}(y) = e^{-y^2/2} v(y)$$

$$v(y) = y^{\ell+1}\sum_{n=0}^{\infty} C_n y^n$$

can be determined recursively