QUANTUM MECHANICS FKA 081 / FYN 190

SELECTED EXAM PROBLEMS 2000 - 01

SUGGESTIONS AND ANSWERS

DEC 12, 2000

1. (i)

$\lambda_{y_1} = +1$, $\lambda_{y_2} = -1$

Eigenkets $|\lambda_{y_1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $|\lambda_{y_2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\lambda_{z_1} = +1$, $\lambda_{z_2} = -1$

$P(\lambda_{y_1}) = P(\lambda_{y_2}) = P(\lambda_{z_1}) = P(\lambda_{z_2}) = 1/2$

(ii)

Most general state

$$|\phi\rangle = \frac{1}{\sqrt{2}} e^{i\delta_1} |\lambda_y = +1\rangle + \frac{1}{\sqrt{2}} e^{i\delta_2} |\lambda_y = -1\rangle$$

$$\langle \phi | \lambda_z | \phi \rangle = \cos(\delta_2 - \delta_1)$$

(iii)

$$|\phi\rangle \rightarrow e^{-iHt}|\phi\rangle = \frac{1}{\sqrt{2}}\left(e^{i(\delta_1 - \varepsilon t/t_1)}|\lambda_y = +1\rangle + e^{i(\delta_2 + \varepsilon t/t_1)}|\lambda_y = -1\rangle\right)$$

$$\langle\phi|\lambda_z|\phi\rangle = \cos(\delta_2 - \delta_1 \pm 2\varepsilon t/t_1)$$

depending on assignment $\pm\varepsilon$ to $|\lambda_y = \pm 1\rangle$

JAN 15, 2001

3. (a) $|\psi_p\rangle = |\uparrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z)$

(b) (i) No. In each case the two beams emerging from the SG device would contain equal number of atoms.

(ii) If the field is along the x-axis, all atoms in the P beam would appear in the "up" beam.

(c)
$$|\psi\rangle = \frac{1}{\sqrt{2}}(a_1|\uparrow\rangle_x + a_2|\downarrow\rangle_x)$$
$$= \frac{1}{\sqrt{2}}\Big(c_1\underbrace{(|\uparrow\rangle_z + |\downarrow\rangle_z)}_{\sim|\uparrow\rangle_x} + c_2\underbrace{(|\uparrow\rangle_z - |\downarrow\rangle_z)}_{\sim|\downarrow\rangle_x}\Big)$$

$\Rightarrow \quad a_1 = c_1 + c_2, \quad a_2 = c_1 - c_2$

$\Rightarrow \quad c_1 = (a_1 + a_2)/2, \quad c_2 = (a_1 - a_2)/2$

Relative number of atoms in the two emerging beams:

$$|c_1|^2 / |c_2|^2$$

$$|c_1|^2 = \frac{1}{4}\{\underbrace{|a_1|^2 + |a_2|^2}_{=1} + \underbrace{\langle a_1^* a_2\rangle + \langle a_2^* a_1\rangle}_{=0}\} = \frac{1}{2}$$

$$|c_2|^2 = \frac{1}{2}$$

$\Rightarrow |c_1|^2/|c_2|^2 = 1$ as predicted in b).

OCT 30, 2001

1. (a) $\langle X\rangle = \langle\Psi|X|\Psi\rangle$ max for

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

(b) $$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle\right)$$

(c) $$\langle X\rangle = \langle\Psi(t)|X|\Psi(t)\rangle$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2}\cos\omega t$$

use $X = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^+)$

2. See Sakurai, sec. 3.5

5. Sakurai, sec 2.2

JAN 18, 2002

1. (a)

$$\langle n | X^4 | n \rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \langle n | (a + a^\dagger)^4 | n \rangle$$

Expand! Keep only terms with two a's and two $a^\dagger$'s (why?) Read off the answer! (We did the analogous problem for X^2 in lecture!)

(b)

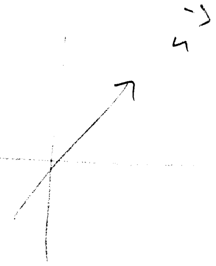
$$H = \underbrace{\frac{1}{2m} P^2 + \frac{1}{2} m\omega^2 X^2}_{H_{HO}} - \lambda X^4$$

$$\langle n | H_{HO} | n \rangle = (n + 1/2) \quad , \quad n = 0, 1, 2, \ldots$$

$$\langle n | \lambda X^4 | n \rangle = [\text{ANSWER IN (a)}]$$

$\Rightarrow$ Spectrum: $E_n = (n + 1/2) - \lambda [\text{ANSWER IN (a)}]$, $n = 0, 1, 2, \ldots$

SKETCH :



3. (a) $\vec{n} = (\sin\theta\cos\varphi)\,\hat{i} + (\sin\theta\sin\varphi)\,\hat{j} + (\cos\theta)\,\hat{k}$

$$\vec{n}\cdot\vec{S} = S_x \sin\theta\cos\varphi + S_y \sin\theta\sin\varphi + S_z\cos\theta$$

insert Pauli matrices $\Rightarrow$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta\end{pmatrix}$$

DIAGONALIZATION gives the secular equation

$$-\frac{\hbar^2}{4}(\cos\theta-\lambda)(\cos\theta+\lambda) - \frac{\hbar^2}{4}\sin^2\theta = 0 \quad\Rightarrow\quad \lambda = \pm\frac{\hbar}{2}$$

EIGENVECTORS

$\lambda = +\frac{\hbar}{2}$

$$\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}a\\ b\end{pmatrix} = |\lambda_+\rangle$$

$$\Rightarrow b = a\tan\left(\tfrac{1}{2}\theta\right)e^{i\varphi}$$

Normalization $a^2+b^2=1$ $\Rightarrow$ $\boxed{a = \cos\tfrac{1}{2}\theta\,,\quad b = e^{i\varphi}\sin\theta/2}$

Similarly, for $\left(\lambda = -\frac{\hbar}{2}\right)$

$$\boxed{a = -\sin(\theta/2)\,,\quad b = e^{i\varphi}\cos\theta/2} \rightsquigarrow |\lambda_-\rangle$$

(b)

$$|\lambda_+\rangle = \cos(\theta/2)\begin{pmatrix}1\\0\end{pmatrix} + e^{i\varphi}\sin(\theta/2)\begin{pmatrix}0\\1\end{pmatrix}$$

$$|\lambda_-\rangle = -\sin(\theta/2)\begin{pmatrix}1\\0\end{pmatrix} + e^{i\varphi}\cos(\theta/2)\begin{pmatrix}0\\1\end{pmatrix}$$

$$|\langle 1\,0|\lambda_-\rangle|^2 = \cos^2(\theta/2)$$