BRST and Topological Gauge Theories

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and
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To Torbjörn and Mona,
with love and gratitude.

BRST and Topological Field Theories

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Abstract

In this thesis, the BRST-quantization of gauge theories is discussed. A detailed analysis of the BRST-quantization on inner product spaces is performed for a class of abelian models, including reparametrization invariant ones. General rules how to obtain physical wave functions and propagators are proposed. The gauge fixing fermion is seen to play a central role for an admissible choice of the specific state space representation.

Canonically equivalent solutions to the quantum master equation is found for a class of first order field theories, using a superfield formulation of the BV-framework. The analysis performed in $d = 4$ and $d = 6$ dimensions, shows that many master actions actually are canonically equivalent to simpler (minimal) master actions.

The geometrical framework of almost product structures (APS) is adopted in order to investigate the splitting of manifolds induced by for example Yang-Mills theories and Kaluza-Klein theories. The properties of the Riemann-tensor are analyzed via the APS ansatz. New curvature relations are found in terms of the so called Vidal- and adapted connections.

Keywords: Gauge theory, BRST-quantization, BV-quantization, topological field theory, superfield formulation, almost product structure (APS).

This thesis consists of an introductory text and the following four appended research papers, henceforth referred to as paper I-IV:

I. R. Marnelius and N. Sandström
Basics of BRST quantization on inner product spaces,
Int. J. Mod. Phys. **A15** (2000) 833-867, [`hep-th/9810125`].

II. M. Holm and N. Sandström,
Curvature relations in almost product manifolds,
[`hep-th/9904099`].

III. R. Marnelius and N. Sandström
Physical projections in BRST treatments of reparametrization invariant theories,
Int. J. Mod. Phys. **A16** (2001) 2909-2944, [`hep-th/006175`].

IV. L. Edgren and N. Sandström,
First order gauge field theories from a superfield formulation,
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Contents

1

INTRODUCTION

"The effort to understand the universe
is one of the very few things that lifts
human life a little above the level of farce,
and gives it some of the grace of tragedy."
Steven Weinberg

1.1 Physical Theories

The field of physics can, more than anything else, be described as an attempt in understanding the interactions responsible for the structures we observe in our universe. The increase in this understanding, brought about by classical theories such as Newton's theory of gravity, Maxwell's **Electrodynamics** and Einstein's **General Theory of Relativity**, is most adequately described by the word enormous. However, physicists soon realized that there existed several phenomena in Nature that these classical theories could not account for; particularly phenomena occurring at the smallest[1] distances found in Nature. The efforts to obtain a theory describing the Nature on small length-scales, led to the birth of the so called **modern physics**, some seventy years ago. Since then it has become clear that physics at the smallest distances is described by **quantum theory**. The quantum mechanical framework has today caused a revolutionary increase in our understanding of the fundamental interactions of Nature. There are roughly three different levels at which one can approach the understanding of physical phewnomena, namely: (i) the **macroscopic**-, (ii) the **mesoscopic**- and (iii) the **microscopic** levels.

(i) A macroscopic theory has no explicit reference to concepts that are inherently quantum mechanical, such as for example spin and expectation values, and it ignores details regarding the fundamental constituents in

[1]With small is here meant roughly 10^{-9} m and below.

Nature[2]. Such a theory is usually called a **classical theory** in the literature and it is deterministic by default. Typically, macroscopic theories are accurate on length scales very much larger than that of atoms and molecules. Examples of macroscopic theories are **Electrodynamics** and **General Relativity**.

(ii) A mesoscopic theory have some constituents that are inherently quantum mechanical and some others which are classical. Such a theory is sometimes referred to as a **semi-classical** theory[3]. Characteristic length scales for mesoscopic systems are intermediate to those of macroscopic and microscopic length-scales. The field of mesoscopic physics is vast and includes for example various theories concerning **molecular physics**, **bio-physics**, **solid state physics**, just to mention a tiny fraction of the disciplines involved.

(iii) A microscopic theory is inherently quantum mechanical and thus required for a detailed description of physics at length-scales characteristic for atoms, subatomic particles and below. Due to the fact that microscopic theories describes physics at the smallest length scales in Nature, they are the most fundamental ones. Well known examples of microscopic theories are the so called **Standard Model** and **String Theory** (**M-theory**). From a reductionist[4] point of view, all mesoscopic- and macroscopic theories must follow as certain limits of more fundamental, underlying, microscopic theories. The number one goal of **theoretical particle physics** is to find a microscopic theory that unifies all interactions, a so called "**theory of everything**". There exist serious indications that Nature is indeed unified in this sense at high enough energies. The success of the Standard Model provides strong such evidence, since it is a microscopic theory which describes the **strong**-, the **weak**- and the **electromagnetic interactions** in a unified context. However, the Standard Model does not give a complete description of the interactions found in Nature since it does not include gravity. Gravity persistently resisted all attempts of a microscopic description for a long time but since the dawn of String Theory, there is new hope in a consistent theory containing **quantum gravity**. Today, String Theory is a serious candidate for the unification of all interactions in a microscopic context, i.e. a potential "theory of everything"; it is at present stage of maturity, however, incapable of making any experimentally verifiable predictions.

[2]For example in fluid dynamics, the existence of atoms is ignored and replaced with a continuum.

[3]The word semi-classical is also often used in a different sense in particle physics in theories that contain anti commuting objects after the limit $\hbar \to 0$ is taken. In that case that limit is often called the semi-classical limit.

[4]To the best of our knowledge, a vast majority of the physicists adhere to the reductionist school.

1.2 Symmetry and Interactions

The Standard Model and General Relativity are so called **gauge theories** and their indisputable success in describing the fundamental interactions of Nature, has made it clear that **gauge symmetries** (**local symmetries**) and interactions are intimately linked: gauge symmetries gives rise to interactions. A gauge theory is, by definition, a theory which is invariant under some **local symmetry group** $\mathcal{G}$, called the **gauge group**. In **Chapter 2**, a short review of the apparatus needed for the understanding of classical gauge theories, is given. The Standard Model for example, is invariant under the compact semi-simple Lie-group $SU(3) \otimes SU(2) \otimes U(1)$ and this invariance gives rise to the strong-, the weak- and the electromagnetic interaction, respectively. The gauge group of General Relativity is not a compact Lie-group, but the group of diffeomorphisms $Diff(M)$ (local reparametrizations) of a given space-time manifold M. String Theory possesses both Lie-group- and reparametrization invariance. There, the gauge groups $SO(32)$, $E_8 \times E_8$ and *Diff(M)* plays distinguished roles and gauge theories in lower dimensions are obtained by compactifying the "superfluous" dimensions. Theories in which interactions are introduced by dimensional reduction of higher dimensional gravity goes under the name **Kaluza-Klein theories**.

From a mathematical point of view, gauge invariance in a theory always implies the existence of constraints in the **action functional** of the theory. This means in particular that not all the degrees of freedom used in formulating the theory are physical ones. One illustrative example of this fact is $d = 4$ pure **Maxwell theory** in which the connection 1-form A contains four degrees of freedom, but due to the U(1)-invariance $A \to A + dA$, there are only two physical degrees of freedom in A, corresponding to the two polarization states of the photon. The natural language of gauge theories is the framework of **principal fiber bundles**. The idea is that we are allowed to perform a gauge transformation at every point in the **base manifold**, e.g. space-time, without affecting the physical content of our theory. This can be described by objects constructed by attaching the gauge group, i.e. the fiber, to every point in the base manifold. Gauge transformations correspond to motions along the directions of the fibers which then constitute the unphysical degrees of freedom. There are many ways one can attach these fibers on the base-manifold and any non-trivial attachment corresponds to non-abelian gauge theories. Geometrically, the non-abelian case corresponds to the fact that the fiber bundle cannot globally be viewed as a direct product of the fiber and the gauge group. This is the case for most gauge groups, for example the gauge group of the Standard Model, given above. Pure Maxwell theory introduced above, however, corresponds to a trivial $U(1)$-principal fiber bundle.

A framework in terms of which gauge theories and Kaluza-Klein theories can be fruitfully studied is that of **almost product structures** (**APS**). The APS ansatz constitute a more general split of a manifold than do principal fiber bundles, which are only a special case of an almost product structure.

APS theory gives therefore a deeper understanding of both ordinary gauge theory and Kaluza-Klein theory. The foundation of APS is given in **chapter 8**, wherein an example of the relation between gauge theory and Kaluza-Klein theories is given. In **paper II** the properties of the Riemann-tensor is analyzed and, using the APS ansatz, different new curvature relations are derived in terms of the so called Vidal- and adapted connections.

1.3 Quantum Theory

As was said earlier, a consistent quantum theory is needed in an adequate description of physical processes in the microscopic regime. With consistent is in particular meant that it should be unitary (have a probabilistically meaningful interpretation), finite (i.e. possible to extract finite numbers for observable quantities) and Lorentz invariant (valid in the relativistic regime), but there are additional requirements. The Standard Model is a relativistic **quantum field theory** which is finite, at least perturbatively.

One might now ask how to construct quantum theories? Ideally, one would like to be able to just sit down and cook up a consistent quantum theory from first principles. As of today this is unfortunately impossible because our present understanding of quantum mechanics is not deep enough; we cannot in simple, well understood terms express the principles by which it works. So for now, we have to be content with less. What one can do is to start from a classical theory which one has a (relatively) good understanding of and somehow "make" it into a quantum theory. This brings us to the notion of **quantization**. With quantization is thus meant turning a classical theory into a quantum theory. It was in this way the Standard Model was constructed. From our previous discussion about the relation between gauge theories and interactions we realize that the quantization of gauge theories will be particularly interesting; in fact most of this thesis is concerned with questions concerning quantization of gauge theories. Now, the quantization problem can be formulated using two different formalisms: the **path-integral-** and the **operator formalism** :

(i) **Path-integral quantization**: The central object in this approach is the so called path- or **functional-integral** and the quantum mechanical amplitude of a function X is given by its (**correlation function**), $\langle X \rangle \sim \int \mathcal{D}\mu\, X\, e^{\frac{i}{\hbar}\int S}$. $\langle X \rangle$ corresponds to a weighted average of X over all paths in the space under consideration. This is similar to how one obtains expectation values of quantities in terms of the (classical) partition function in statistical physics. In order to define a consistent quantum theory the path integral must for example be properly gauge fixed and the measure $\mathcal{D}\mu$ free of anomalies, and perhaps also regularized. Additional problems might also occur. Path-integrals are used and discussed in various contexts in this thesis, see for example **chapter 3,4,5,7** and **paper III**.

(ii) **Operator quantization**: In this procedure one associates to every (classical) function, an hermitian operator in a **Hilbert space**, which in the case of gauge theories, in general has **indefinite metric**. The **physical state space** is then obtained as the subspace of the original Hilbert space, consisting of gauge invariant states with positive definite norms. It is in usual difficult to find the physical subspace for a given model. In addition to problems with regularization of singular operators and states, there usually also exist ordering problems and other difficulties. The operator formalism is used extensively in **chapter 3,4,5,7** and **papers I,III**.

Above we noted that one inevitable implication of gauge theories is that they contain unphysical degrees of freedom. Now, a reduction of the unphysical degrees of freedom will in general spoil the manifest covariance of the theory and this is undesirable since it in general makes many calculations much more difficult, particularly in perturbation theory. One way to keep manifest covariance is to keep all these extra degrees of freedom which is done at the price of losing unitarity in the theory; this is the origin of the indefinite metric discussed above.

There exist a powerful, general approach to quantization of gauge theories, termed **BRST quantization**, which features both manifest covariance and unitarity. In the BRST framework the gauge symmetry of the original model is replaced with a rigid symmetry called the **BRST symmetry** generated by a conserved charge Q, called the **BRST charge**. The problem of finding the gauge invariant states for a theory, can then be reformulated as the problem of solving the cohomology $H(Q)$ of the corresponding BRST-charge. There are two version of the BRST framework. One is Hamiltonian and called **BFV-BRST** quantization and the other is Lagrangian and is called **BV** (**field-antifield**) quantization.

In **chapter 3**, a short review of BFV-BRST framework is given and some results of BRST-quantization on inner product spaces are discussed. These results are used in the detailed investigation of BRST-quantization on inner product spaces for a general class of finite dimensional abelian models, performed in **paper I**. The results found in **paper I** are also applied to non-abelian models. In **chapter 4** the BRST-quantization of reparametrization invariant abelian systems with finite degrees of freedom is discussed, both from an operator and a path-integral point of view. The chapter serves as an introduction to **paper III**, wherein physical wave functions and physical propagators are derived as projections of solutions to a BFV-BRST quantization on inner product spaces. It turns out that only certain choices of the specific state spaces used, are allowed. A specific prescription for admissible choices of state spaces is given.

A particularly rich class of quantum theories originates from the so called **topological field theories**. This class of theories is very interesting both from a gauge- and quantum theoretic point of view. This is because they display the rare property of being fully interacting theories whose corresponding quantum theories are solvable. They encode valuable information about the topological sector as well as the non-perturbative regime of ordinary quantum field theories. The word topological is here to be taken literally in the sense that the correlation functions of topological quantum field theories are independent of any metric on the manifolds on which they are defined. In the BRST-framework this translates to the fact that their energy-momentum tensor is BRST exact, i.e. unobservable. At the classical level this is realized by the observation that these theories are formulated without any reference to any metric which implies that their energy-momentum tensors are trivially zero and for which reason they can not contain any physical excitations. The gauge invariance of these theories are always rich enough to enforce them to be void of any local degrees of freedom.

In **chapter 6**, a **superfield** formulation is introduced for the BV-quantization of a certain class of field theories. The framework presented in this chapter serves as a foundation for some parts of **chapter 7** and it is the formalism used in **paper IV**. In **chapter 7**, topological field theories are introduced and the structure of the so called **quantum master equation** is studied for a number of **topological gauge theories**, using the superfield formulation. In **Paper IV**, canonically equivalent solutions to the quantum master equation are found for a class of first order gauge field theories.

We close the introduction by mentioning that there are several indications of that topological field theories might be important for understanding quantum gravity. There have been several attempts at constructing a quantum gravity theory by the quantization of classical gravity; all those attempts have failed and it is quite clear by now that much more sophisticated approaches than just "simple" quantization must be used in order to get hold of a consistent theory of quantum gravity.

2

CLASSICAL GAUGE THEORY

In all sections in this chapter but section 2.5, we will for sakes of simplicity consider bosonic systems with a finite number of degrees of freedom.

2.1 Symplectic Geometry

There are two different but equivalent formalisms in terms of which one can define physical theories. A theory is said to be written in the Lagrangian or in the Hamiltonian formulation. We will start with the latter for the following reason: We are aiming at constructing a quantum theory from a classical gauge theory. This implies that the structure of the quantum theory is explicitly determined by the properties of the set of constraints. It is also the constraints that define the geometry of the theory in question. This geometry is most fruitfully studied in phase space in which coordinates and momenta are treated on the same footing with respect to the equations of motion, in contrast to their form in configuration space. As will be seen, some assumptions will be made in the beginning of our review and later on we will go back and start from a Lagrangian, pass on to the Hamiltonian and in that process see how all these assumptions are justified. The concepts discussed in this section should be seen as a framework in which we more effectively can discuss the constraint analysis, which is the topic of the next section.

Now consider the **symplectic manifold** $(T^*\mathbf{Q}, \omega)$ where $T^*\mathbf{Q}$ denotes the cotangent bundle of the configuration manifold $\mathbf{Q}$ and $\omega \in \Omega^2(T^*\mathbf{Q})$ the **symplectic two-form** or **symplectic structure** defined on $T^*\mathbf{Q}$. Darboux's theorem states that there always exists local canonical coordinates $\{q^\alpha, p_\alpha\}$ in terms of which the symplectic two-form can be written as

$$\omega = dq^\alpha \wedge dp_\alpha \tag{2.1}$$

where $\alpha \in \{1, ..., n\}$ and $dim\mathbf{Q} = n$. Later on we will see how the Poisson bracket can be expressed in terms of ω. In general every function $f : T^*\mathbf{Q} \to \mathbb{R}$

defines a vector field X_f on phase space in the following way

$$i_{X_f}\omega = df \tag{2.2}$$

where by definition, $X_f : T^*\mathbf{Q} \to T(T^*\mathbf{Q})$. Above, i_X denotes the interior product. Given a Hamiltonian this means in particular that we can write the Hamilton equations of motion as

$$i_{X_H}\omega = dH \tag{2.3}$$

Now, eq. (2.3) is only valid in an unconstrained space. In this space the symplectic two-form is obviously non-degenerate due to (2.1). In the case of independent constraints the constraint set is said to be **irreducible** (a complete theory of how to treat a **reducible** constraint set exists, but will not be discussed in this thesis). Now suppose that we are given a set of **first-class** constraints,

$$\phi_k = 0, \quad k \in \{1, ..., a\} \tag{2.4}$$

The set $\{\phi_k\}$ defines, via (2.4), a submanifold $\Sigma \subset T^*\mathbf{Q}$ called the constraint surface. The **first-class** property means that all the constraints satisfy

$$\omega(X_{\phi_k}, X_{\phi_{k'}}) = 0 \quad \forall k, k' \in \{1, ..., a\} \tag{2.5}$$

on the constraint surface Σ. Constraints for which eq. (2.5) does not hold are defined to be **second-class** constraints[1]. The presence of constraints in our theory implies that the symplectic two-form $\omega^* = i^*\omega$ induced on Σ is non-degenerate. Now, in the presence of the constraint set eq. (2.4) one might ask, how does eq.(2.3) get modified? By studying the embedding (inclusion map) $i^* : \Sigma \hookrightarrow T^*\mathbf{Q}$ it is easily shown that the Hamilton-Dirac equations can be written

$$d\phi_i(Z) = 0 \tag{2.6}$$

$$i_Z\omega - dH = f^i d\phi_i \tag{2.7}$$

Above the $\{f^i\}$ are arbitrary functions on Σ and Z is the vector field to be solved for. This explicitly shows in a geometrical fashion how the Lagrange multipliers, $\{f^i\}$, are naturally introduced into the theory. Eq. (2.6) simply imposes the constraints, by requiring that the vector-fields that solve eq. (2.7) are tangential to Σ. In fact, this reformulates the global definition of a first-class system (the first one was given by eq. (2.5)).

2.2 Constraint Analysis

Let us assume that we are given a theory defined by the Lagrangian $L : T\mathbf{Q} \to \mathbf{R}$, where $dim\mathbf{Q} = n$. The equations of motion (Euler-Lagrange equations) follow upon variation and can be written as

$$\mathcal{L}_X\theta_L = dL \tag{2.8}$$

[1] The classification of constraints into first- and second class is due to Dirac and was originally presented in the influential paper [1].

where $\theta : T\mathbf{Q} \to T^*(T\mathbf{Q})$ denotes the Liouville form, $\mathcal{L}$ the Lie-derivative and $X : T\mathbf{Q} \to T(T\mathbf{Q})$ a vector field. Locally, the Liouville form is given by $\theta_L = \frac{\partial L}{\partial \dot{q}^\alpha} dq^\alpha$ where $\alpha \in \{1, ..., n\}$. If one studies eq.(2.8) in terms of a coordinate system, one notices that it is *impossible* to express uniquely the accelerations in terms of the coordinates and velocities iff

$$det \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \dot{q}^{\alpha'}} \right) = 0 \tag{2.9}$$

Condition (2.9) is always satisfied for gauge theories since only then will it be possible to have arbitrary functions of time in the solutions to the equations of motion. In particular this means that the Legendre transformation $\mathcal{F}L : T\mathbf{Q} \to T^*\mathbf{Q}$, relating the Lagrangian with the Hamiltonian formalism, will be non-invertible. Locally this transformation is realized by defining the canonical momenta as

$$p_n = \frac{\partial L}{\partial \dot{q}^n} \tag{2.10}$$

where H and L are related as

$$H = \dot{q}^n p_n - L \tag{2.11}$$

i.e. we have locally

$$\mathcal{F}L[L] = H = \dot{q}^n p_n - L \tag{2.12}$$

Equation (2.11) tells us that $H : T^*\mathbf{Q} \to \mathbf{R}$. Via the inverse function theorem we see that the non-invertibility of $\mathcal{F}L$ corresponds to the fact that we cannot uniquely determine the velocities in terms of coordinates and momenta. This implies that all of the momenta cannot possibly be independent and conclusively we have a number of constraints between the phase space variables. These constraints follow directly from the definition of the momenta, without the use of the equations of motion. Therefore they are called **primary constraints** in the literature. Constraints that follow upon use of the equations of motion is called **secondary constraints**[2]. Now, we would would like to have a phase space action applicable to Hamilton's principle, i.e we would like to be able to vary the variables in the action independently without regard to any constraint. This is much more convenient since it is very difficult to explicitly impose the non-independence of the variables in the variation, especially when the constraints are complicated. From a purely *technical* point of view this can be achieved by introducing Lagrange multipliers into the theory (in the previous section it was noticed how these occurred naturally from geometrical considerations of the constraint surface). Let us label the constraint functions found so far by Φ_k, where $k = 1, ..., m$ and $\Phi_k : T^*\mathbf{Q} \to \mathbf{R}$. One can easily show that the phase space action that handles this business can be written as

$$\int dt \, \left(\dot{q}^n p_n - H - v^k \Phi_k \right) \tag{2.13}$$

[2]The classification of primary- or secondary constraints is also due to Dirac [1].

From which the equations of motion in the constrained case follows upon variation of q^α, p_α and the Lagrange multipliers v^k. This action is actually defined in an extended phase space in which the multipliers $\{v^k\}$ themselves and their momenta $\{\pi_k\}$ are introduced as coordinates, however as is obvious from above, the momenta satisfies $\pi_k = \frac{\partial L}{\partial v^k} = 0$, which implies that there is no dynamics in terms of these coordinates at this level. Now we require that the constraints Φ_k should be conserved in time, i.e be constants of motion. If the theory in question is supposed to describe fundamental interactions, the constraints should not change in time. This implies that all the constraints must commute with the following Hamilton function

$$\bar{H} = \mathcal{F}L[L] - u^k \Phi_k \tag{2.14}$$

in the Poisson bracket sense. This commutativity can be written in various equivalent ways,

$$\dot{\Phi}_k = \mathcal{L}_{X_{\bar{H}}} \Phi_k = \{\Phi_k, \bar{H}\} = -\omega(X_{\bar{H}}, X_{\Phi_k}) = 0 \tag{2.15}$$

This requirement either produces new **secondary** constraints or lead to restrictions on the Lagrange multipliers. Preservation in time is then imposed on all secondary constraints found and the procedure is iterated until no new constraints appear. This leaves us with a, possibly larger, complete[3] set of constraint functions,

$$\{\Phi_j\}, \ \ j \in \{1...m\}, \text{ where } m \geq a \tag{2.16}$$

where the restrictions on the lagrange multipliers can be written as

$$\omega(X_{\Phi_j}, X_{\bar{H}}) = 0 \tag{2.17}$$

Now one can show that each Lagrange multiplier v^k can be split into one part that is fixed by eqs. (2.17) above and one part which is a totally arbitrary function of time,

$$v^k = u^k + w^k \tag{2.18}$$

Above w^k denotes the arbitrary part in the Lagrange multiplier. A somewhat more detailed description of the constraint analysis is for instance given in refs. [2,3]. In the next section we will see how the totally arbitrary part of the Lagrange multipliers enters in the gauge transformations.

2.3 Gauge Generators

If we study the equations of motion[4]

$$\dot{D} = -\omega(X_{\bar{H}}, X_D) \tag{2.19}$$

[3] I.e. it contains all of the constraints in the theory.

[4] We stress once again that this equation should be interpreted as an equality on the constraint surface only.

for an arbitrary dynamical variable D, they imply that the change δD under an infinitesimal time translation will be

$$\delta D = -\delta w^p \, \omega(X_D, X_{\Phi_p}) \tag{2.20}$$

where the index p runs over all[5] first class constraints. Since the functions $\{w^k\}$ are totally arbitrary this must be a physically insignificant change, or in other words, first-class constraints generate gauge transformations. The group which has the first class constraints as generators in the sense of eq.(2.20) above is nothing but the gauge group $\mathcal{G}$ of the theory. The second class constraints will not be generators of some relevant transformations in the same sense as the first-class ones, since they do not preserve the constraints. In the next section it will be shown how one can eliminate the second-class constraints.

2.4 The Constraint Surface

Above, the constraint analysis gave us a set of constraint functions

$$\{\Phi_j\}, \quad j \in \{1, ..., m\} \tag{2.21}$$

Let us split this set into first- and second-class constraints

$$\{\phi_a, \chi_b\} \tag{2.22}$$

where $a \in \{1, ..., m-r\}$ and $b \in \{m-r+1, ..., m\}$. This set defines the submanifold Σ by

$$\phi_a = \chi_b = 0 \tag{2.23}$$

in which our dynamical system is forced to live, i.e. the constraint surface. Let us now pick a solution to the equations of motion for the canonical variables. Let us also pick some boundary conditions and a specific choice of the arbitrary functions of time which is included in the solution. Surely this uniquely defines a physical state. This choice represents a point x on Σ. Now let the $\mathcal{G}$ act on this point (we assume that $\mathcal{G}$ is non-empty), this represents another choice of the arbitrary functions of time, i.e. corresponds to another point on the constraint surface. In particular we can form the orbit $\mathcal{G}_x$, of x under $\mathcal{G}$. Since $\mathcal{G}$ is a continuous group, $\mathcal{G}_x$ constitutes a submanifold of Σ. This manifold corresponds to one physical state. So obviously in the case of first-class constraints, we still have some arbitrariness left in our space, represented by the absence of a one-to-one correspondence between physical states and points

[5] It is not at all evident that the secondary first-class constraints should be generators of gauge transformations too. The assumption that all first class constraints generate gauge transformations goes under the name of the Dirac conjecture, which has proven to hold in all physical applications so far, even though one can construct systems for which it is false. The reason for assuming all first class constraints to be gauge generators is that it will be very hard to quantize a theory in which the gauge generators does not form a complete set, i.e. a set in terms of which all gauge generators can be expressed.

on Σ. We are allowed to remove this arbitrariness by imposing what is called gauge conditions. These amount to imposing a set (equally many as there are first-class constraints) of extra constraints

$$C(q,p)_a = 0 \tag{2.24}$$

into the theory. These amount to fixing the choice of arbitrary functions of time. Geometrically, a good set of gauge conditions represents a surface which intersect all the gauge orbits once and only once. This in turn implies that the set $\{\phi_a, C_a\}$ constitutes a second class system, so there are no first class constraints left in our theory after complete gauge fixing. This gives us an idea of how to get rid of the second class constraints. It is possible to view every second class constraint as resulting from a first-class constraint together with a gauge condition. This means that we can reformulate a system in which second-class constraints are included, in terms of an "unfixed" first class system. Another method for treating second-class systems is by the so called conversion-mechanism introduced in [4]. In this thesis we deal only with the quantization of first class systems.

2.5 Generating Sets in Field Theory

This section contains a brief discussion about the generators of gauge transformations in field theories. In contrast to the other sections in this chapter, we will here consider relations valid for $\mathbb{Z}_2$-graded fields in the Lagrangian framework; this sets the stage for chapters 5, 6 and 7. We start by introducing the condensed notation of DeWitt [5]: every repeated discrete index are to be summed *and* integrated over. This implies for example that $\delta\Phi^i = R^i_A\epsilon^A$ should be interpreted as $\delta\Phi^i(x) = \int dy R^i_A(x,y)\epsilon^A(y)$. Consider an action S possessing some gauge invariance. This means that S is invariant under some transformations $\delta\Phi^i = R^i_A\epsilon^A$, where R^i_A are the generators of the gauge algebra and the ϵ^A are parameters of the gauge transformation. Due to self-consistency we must then have[6]

$$\delta_\epsilon S = S\frac{\overleftarrow{\partial}}{\partial\Phi^i}\delta\Phi^i = S\frac{\overleftarrow{\partial}}{\partial\Phi^i}R^i_A\epsilon^A = 0 \tag{2.25}$$

which implies the so called **Noether identities**,

$$S\frac{\overleftarrow{\partial}}{\partial\Phi^i}\,R^i_A = 0 \tag{2.26}$$

Obviously, the Noether identities states that the equations of motion are not independent - this means that the solutions to the equations of motion will contain arbitrary functions of time. As a consequence of the Noether identities, propagators do not exist. The surface Σ above is defined as the part of

[6]Since the fields are $\mathbb{Z}_2$-graded, we must make a distinction between left- and right derivatives.

function space in which the equations of motion holds. In order to treat the set of gauge transformations in a systematic way, one is forced to impose certain **regularity conditions** [6], the most important consequence of which is that if a function of the fields vanish on-shell, that function must be a linear combination of the equations of motion. This property is called the **completeness** of the equations of motion,

$$f(\phi)\Big|_{\Sigma} = 0 \Rightarrow f(\phi) = \frac{\overrightarrow{\partial}}{\partial\Phi^i} S_0 \lambda^i(\phi) \tag{2.27}$$

Given a complete set of invariances of a theory, the most general solution to the Noether identities is a gauge transformation,

$$\frac{\overrightarrow{\partial}}{\partial\Phi^i} S_0 \lambda^i = 0 \Leftrightarrow \lambda^i = R^i_A \chi^A + \frac{\overrightarrow{\partial}}{\partial\Phi^i} S_0 T^{ij} \tag{2.28}$$

where the tensor T is graded antisymmetric, $T^{ij} = -(-1)^{ij} T^{ji}$. The T-dependent part of (2.28) represents trivial gauge transformations that exist for every action (they do not imply conserved quantities). If the set of generators R^i_A are independent and m in number, we have rank $R^i_A|_\Sigma = m$; thus the total degrees of freedom in the system is equal to $n - m$. In the case when rank $R^i_A|_\Sigma < m$, the generators are dependent and constitute a **reducible** set of generators. This means that there exist a number of relations,

$$R^i_A R^A_{1A_1} = \frac{\overrightarrow{\partial}}{\partial\Phi^i} S_0 V^{ij}_{1A_1} \tag{2.29}$$

and so on depending on the degree of reducibility of the set of generators $\{R^i_A\}$. It is the degree of reducibility in a theory that determines how big the hierarchy of ghosts, ghost of ghosts etc. will be, in order to gauge fix the theory. For a thorough treatment of the reducible case see [2, 6]. Many important theories are reducible by default: BF-theories in higher dimensions, Topological Yang-Mills and string field theory, the last of which, is infinitely reducible. The Noether identities and the completeness of the generators, implies that the commutator of two gauge transformations δ_1 and δ_2 is given by

$$[\delta_1, \delta_2]\Phi^i = R^i_A T^A_{BC} \epsilon^B_1 \epsilon^C_2 - \frac{\overrightarrow{\partial}}{\partial\Phi^j} S_0 E^{ji}_{BC} \epsilon^B_1 \epsilon^C_2 \tag{2.30}$$

The coefficients T^A_{BC} and E^{ji}_{BC} are usually called **structure tensors** of the gauge algebra. The structure tensors possess definite parity- and symmetry properties; this follows directly from the algebra (2.30) and will not be discussed here. The nature of the coefficients T and E in the commutator (2.30) gives rise to a number of definitions which are commonly seen in the literature:

▶ **Open algebra (1):** An algebra of generators having $E \neq 0$ is termed an **open algebra**. Otherwise the algebra is termed a closed algebra.

We can see from the commutator (2.30) that this means that the algebra only

close on-shell.
▶ **Soft algebra (2):** When the function $E = 0$ and T depends on the fields, the algebra is termed a **soft algebra**.
▶ **Lie algebra (3):** If the function $E = 0$ and T is constant, the algebra of the generators reduces to a Lie algebra.
▶ **Non-linear algebra (4):** If $E = 0$ and T depends in a non-linear way on the fields Φ^i, the algebra of generators is termed a non-linear algebra.

The on-shell commutator between two gauge transformations is necessarily a new gauge transformation which in turn must obey the Noether identities; this implies that we can use completeness again by introducing new structure tensors. This game can be played over and over again until the process terminates, i.e. when the higher order structure tensors vanishes. The set of all the structure tensors produced in this way contain all the information about the gauge transformations of a theory. In [6] a detailed investigation of an irreducible algebra and a first stage reducible theory is given. All these relations are, however, easier obtained via the BV-framework by considering the master action for a given gauge theory. In **chapter 7**, we will see how the structure tensors and their consistency equations show up as a consequence of solving the master equation for a number of different models.

3

BRST QUANTIZATION

3.1 What is quantization ?

In brief, quantization of a theory means turning a classical theory into a quantum theory. Observe that with a classical theory we mean a theory which is allowed to have both Grassmann even(commuting) and odd(anticommuting) degrees of freedom, but on which none of the postulates of quantum mechanics have been imposed (Grassmann odd elements is necessary for the description of objects with half integer spin after quantization). Thus, the goal of any quantization scheme is to establish a relationship between a classical and a quantum system, and in the latter system, identify the physical states and operators. These observables must of course obey the postulates of quantum mechanics. Since there exist essentially two different formulations of quantum mechanics, the operator and the path-integral formulation, quantization methods can naturally be divided into two categories depending on which of the two formulations they are based on. One of the postulates of (operator) quantum mechanics says that an observable of a physical system is represented by a self-adjoint operator in a Hilbert space. More precisely this means that every admissible quantization method must provide a relation between functions on a symplectic manifold (M, ω) and operators in the corresponding Hilbert space $\mathcal{H}$. The operator method was first developed by Dirac and is called canonical quantization and it was extended to Grassmann odd objects by Berezin [7,8]. All operator quantization schemes are based on canonical quantization which stipulates the following correspondence between the canonical coordinates in phase space $\{Z^A\} = \{X^A, P_A\}$ and the corresponding operators $\{\hat{Z}^A\}$ in $\mathcal{H}$

$$i\hbar\{Z^A, Z^B\} = [\hat{Z}^A, \hat{Z}^B] \tag{3.1}$$

In the equation above, the Poisson and commutator bracket should be read in the graded sense, i.e. their respective generalizations, valid for both Grassmann even and odd elements. These are defined as

$$\{F_1(Z^A), F_2(Z^A)\} := (-1)^{\epsilon_{F_1}} \frac{\partial^L F_1}{\partial X^A} \frac{\partial^L F_2}{\partial P_A} - (-1)^{\epsilon_{F_1}\epsilon_{F_2}} \frac{\partial^L F_2}{\partial P^A} \frac{\partial^L F_1}{\partial X_A} \quad (3.2)$$

$$[F, G] := FG - (-1)^{\epsilon_F \epsilon_G} GF \quad (3.3)$$

where X^A denotes the coordinates and P_A the conjugate momenta in phase space and ∂^L left derivative. ϵ_X denotes the Grassmann parity of the function or operator X and it satisfies

$$\epsilon_X = \begin{cases} 0 & \text{for } X \text{ Grassmann even} \\ 1 & \text{for } X \text{ Grassmann odd} \end{cases} \quad (3.4)$$

Canonical quantization tells us what algebra the operators corresponding to the canonical coordinates in phase space must satisfy. Implicitly, the correspondence above associates to every classical function, an operator on some Hilbert space (modulo ordering problems). This is at the heart of the difficulties that have to be surmounted by a quantization scheme. How should one represent the corresponding operators on some state space, so that the resulting quantum theory makes sense and at the same time satisfies the postulates of quantum mechanics ? With "make sense" we mean for instance preserving, or displaying quantum mechanical counterparts of the symmetries of the underlying classical theory; if the classical (local) symmetries are violated at quantum level one talks about a **gauge anomaly**. Many of the problems related to quantization is solved by the BRST framework in which the gauge symmetry of the classical theory manifests itself as a rigid symmetry in the quantized theory. It is this rigid symmetry that is called the **BRST symmetry**. The quantum theory should also be unitary[1] in order to have a probabilistic interpretation. As was said earlier, most of what follows will be discussed from the operator version point of view. Even though, as we will see in the next section, the BRST symmetry was originally discovered within the path integral context.

3.2 Origin of the BRST Symmetry

Consider the path-integral (functional-integral) formulation of Yang-Mills theory (in which the gauge group is a Lie group) defined by the Lagrangian $\mathcal{L}$ depending on some field A,

$$\int \mathcal{D}A \, e^{i \int d^4x \, \mathcal{L}[A]} \quad (3.5)$$

The measure $\mathcal{D}A$ is given by $\mathcal{D}A = \prod_x \prod_{a,\mu} dA^a_\mu$, and where a and μ represent algebra- and space-time indices respectively. The density $\mathcal{L}$ defines a gauge theory which means that there are redundant degrees of freedom. This implies

[1] I.e., all states different from zero have positive norm.

that we integrate an infinite number of times over physically equivalent field configurations in the path integral above, i.e the path integral diverges. In order to be able to integrate over inequivalent field configurations only, L.D. Fadeev and V.N. Popov [9] introduced a trick in which they expressed the unit operator as

$$\mathbf{1} = \int \mathcal{D}\alpha(x)\delta(G(A^{\alpha}))\det\left(\frac{\delta G(A^{\alpha})}{\delta\alpha}\right) \tag{3.6}$$

where the non-degenerate $G(A^{\alpha})$ is a gauge fixing function, $A^{\alpha} = (A^{\alpha})^a_{\mu} = A^a_{\mu} + \frac{1}{g}D_{\mu}\alpha^a$ the gauge transformed field and D_{μ} the gauge covariant derivative. Inserted in (3.5) and performing the $\mathcal{D}\alpha$-integral we get (after dividing out the volume of the gauge group)

$$\int \mathcal{D}A \det\left(\frac{\delta G(A^{\alpha})}{\delta\alpha}\right) e^{i\int d^4x\ \mathcal{L}+\mathcal{L}_{gf}} \tag{3.7}$$

$\mathcal{L}_{gf}$ is a modification of the original Lagrangian which arises due to the functional integration over α. It is denoted with index gf, because it breaks the gauge invariance of $\mathcal{L}$ by fixing the gauge. L.D. Fadeev and V.N. Popov rewrote the determinant in (3.7) as a functional integral over anticommuting fields $\mathcal{C}$ and $\bar{\mathcal{C}}$. This is possible since a "gaussian" integral over odd Grassman elements is proportional to the determinant of the operator squeezed between them, in contrast to the commuting case in which the integral is proportional to the determinant raised to the power $-1/2$. This implies that the original functional integral now can be written

$$\int \mathcal{D}A\mathcal{D}\mathcal{C}\mathcal{D}\bar{\mathcal{C}}\ e^{i\int d^4x\ \mathcal{L}+\mathcal{L}_{gf}+\mathcal{L}_{gh}} \tag{3.8}$$

where $\mathcal{L}_{gh}$ denotes the ghost term. Thus the ghost fields can be interpreted as negative degrees of freedom, since they bring in positive powers of the functional determinant, which exactly cancel the determinants arising from the gauge degrees of freedom and which sit in the denominator. The effective Lagrangian above can thus be written

$$\mathcal{L}_{BRST} = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh} \tag{3.9}$$

where $\mathcal{L}_{gf}$ and $\mathcal{L}_{gh}$ denotes the gauge fixing- and the ghost part of $\mathcal{L}$, respectively. If written as simple as possible, the Lagrangian $\mathcal{L}_{BRST}$ describing the original Yang-Mills theory takes the form

$$\mathcal{L}_{BRST} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2\xi}(B^a)^2 + B^a\partial^{\mu}A^a_{\mu} + \bar{\mathcal{C}}^a(-\partial^{\mu}D^{ac}_{\mu})\mathcal{C}^c \tag{3.10}$$

The BRST Lagrangian $\mathcal{L}_{BRST}$ is not gauge invariant anymore, due to the gauge fixing and ghost terms added to the original $\mathcal{L}$. However Becchi, Rouet, Stora and, independently, Tyutin [10, 11] discovered a new rigid symmetry that $\mathcal{L}_{BRST}$ possesses, which contains a parameter which is anticommuting.

This symmetry is today accordingly called the BRST symmetry. In eq.(3.10) the first two terms on the right hand side constitutes the original Lagrangian, the following two are gauge fixing terms and the last one the ghost term. The commuting auxilliary field B was introduced only to make the BRST symmetry manifest off shell (i.e. without the use of the equations of motion). B is obviously not dynamical. $\mathcal{L}_{BRST}$ is invariant under the BRST transformations,

$$d_{{}_{BRST}} A^a_\mu = \varepsilon D^{ac}_\mu \mathcal{C}^c \tag{3.11}$$
$$d_{{}_{BRST}} \psi = ig\varepsilon\, \mathcal{C}^a t^a \psi \tag{3.12}$$
$$d_{{}_{BRST}} \mathcal{C}^a = -\frac{1}{2} g\varepsilon f^{abc} \mathcal{C}^a \mathcal{C}^c \tag{3.13}$$
$$d_{{}_{BRST}} \bar{\mathcal{C}} = \varepsilon B^a \tag{3.14}$$
$$d_{{}_{BRST}} B^a = 0$$

In the equations above ε denotes the anticommuting parameter. One important observation to be made from these equations is that the BRST transformations above are nilpotent. Thus the original gauge invariance has been traded against a global nilpotent, Grassmann odd, symmetry. The above discussion illustrates how the BRST-symmetry was originally discovered. Due to the invariance of $\mathcal{L}_{BRST}$ under the BRST transformations one can construct the corresponding conserved quantity which will be nothing but the BRST charge Q, the generator of the symmetry. Now, the transition amplitude for the Yang-Mills theory is given by a BRST-invariant path integral. This translates in the operator formalism to the fact that the Hamiltonian commutes with the BRST-charge $[H, Q]$. In the extended formalism, i.e with the addition of the ghosts, Q defines the physical subspace. The Fadeev-Popov method used above essentially only works for Yang-Mills gauge theories, so one cannot expect to find the charge Q by first constructing the effective Lagrangians for more complicated gauge theories because that is notoriously difficult. It is the reversed "attitude" that led to the development of the general BRST method; one requires that every quantum theory based on some gauge theory should possess the BRST symmetry - and the physical states in that theory are the ones that are annihilated by the nilpotent BRST charge. In fact, one can construct Q solely on the knowledge of the constraints in the theory. This construction is non-trivial and will be discussed in the next section. The introduction of the BRST symmetry in order to quantize gauge theories has been shown to be a fundamental ingredient, applicable to all gauge theories. The BRST quantization method has been successfully applied to both supergravity and string theory, see for example [12, 13] and [14], respectively on these matters. The rigorous mathematical construction of BRST, which came some years after the discovery of the symmetry is the topic of the section below.

3.3 The BRST Construction

Any reduction of degrees of freedom runs counter to the manifest realization of symmetries. Therefore it is not desirable to gauge fix a system before quantization is made. This is because one must then explicitly check that no anomalies occur in the quantum theory - calculations that might be very difficult to perform in general. On the other hand, by not gauge fixing - there will be unphysical variables in the theory which might spoil unitarity. This happens for example in the quantization of relativistic theories in which, due to the negative sign in the metric, states with negative inner product is allowed. These states must be removed from the physical spectrum and such a reduction might be very difficult to achieve in practice. The advantage of the BRST method is that it keeps all the variables in the original phase space, thereby guaranteeing the manifest covariance. Moreover, it adds further degrees of freedom; e.g. ghosts and antighosts which kills the original unphysical degrees of freedom when we impose BRST invariance of the physical states in the theory so the resulting theory will be unitary. Herein lies the main advantages of the BRST quantization method, towards the construction of which we now turn. Let us assume that we are given a gauge theory containing a set of first class bosonic constraint functions

$$\{\phi_k\}, \quad k = \{1, ..., n\} \tag{3.15}$$

This set is derived along the lines described in the first part of this thesis. The constraint surface $\Sigma \subset T^*(\mathbf{Q})$ is defined by the relations $\phi_k = 0$, where $T^*(\mathbf{Q})$ denotes the original phase space. Provided the following two conditions are satisfied

$$\begin{aligned} \exists\, \Sigma \quad &\hookrightarrow \quad T^*(\mathbf{Q}) && (3.16)\\ [X_{\phi_k}, X_{\phi_{k'}}] \quad &\subset \quad \mathrm{span}\{X_{\phi_k}\}, && (3.17)\end{aligned}$$

the theory of homological resolutions of algebras $[2, 15]$ implies the existence of a differential d_{BRST} in extended phase space $\mathfrak{P}$, to be explained later, such that for arbitrary elements $x, y \in \mathfrak{P}$ and function F on $\mathfrak{P}$,

$$\begin{aligned} d^2_{BRST} &= 0 && (3.18)\\ d_{BRST}(xy) &= x d_{BRST} y + (-1)^{\epsilon_y}(d_{BRST})xy && (3.19)\\ d_{BRST} &= \delta + d + \text{"more"} && (3.20)\\ d_{BRST} F &= \{F, Q\} && (3.21)\\ H^0(d_{BRST}) &= \{\text{gauge invariant functions on } \Sigma\} && (3.22)\end{aligned}$$

The resolution of a given algebra requires that one extends the phase space $T^*(\mathbf{P})$ with extra degrees of freedom. One has the liberty of choosing these to be equally many as twice the number of constraints. This is crucial, since only then can one divide the extra variables into pairs, one of which is the conjugate of the other. Moreover each conjugate pair must be of opposite

Grassman parity as the constraint it corresponds to. The introduction of these extra generators makes possible the construction of extended phase space $\mathfrak{P}$, the structure of which will be more fruitfully discussed after some explaining remarks about the assumptions made above and the properties of d_{BRST}. First of all, conditions (3.16) and (3.17), simply means that we should have an embedding of the constraint surface. In our case it was noticed in chapter 2 that the inclusion map into the original phase space provided that for us. Moreover since it also was showed that $\omega(X_{\phi_k}, X_{\phi_{k'}}) = 0$ on Σ, the fields X_k must be integrable, which in turn is a result of the first class property of $\{\phi_k\}$. Equation (3.18) states the nilpotency of the BRST differential, which was shown in the previous section to be the case for Yang-Mills theories. This property expresses the gauge invariance of the theory and makes it possible to form the cohomology of d_{BRST}. This nilpotency, also implies the nilpotency of the BRST charge, $Q^2 = \{Q, Q\} = 2Q^2 = 0$. Property (3.19) says that d_{BRST} is a graded derivation, i.e. it obeys the graded Leibniz rule. Relation (3.20) gives an expansion of d_{BRST} in terms of **antighost number** ($antigh_\#$) (which is the grading of δ), telling that the first two terms are the Koszul-Tate differential δ and the exterior derivative d along the gauge orbits respectively. Since for observables f,

$$\mathcal{L}_{X_{\phi_k}} f = df(X_{\phi_k}) = 0, \tag{3.23}$$

d identifies gauge invariant functions. The part "more" in (3.20) corresponds to higher order terms with respect to $antigh_\#$, and they are exact so they decouple from the cohomology. Even though there is great freedom in those terms, they must be chosen as to make the d_{BRST} nilpotent. Equation (3.21) says that it is possible to impose a symplectic structure on $\mathfrak{P}$, in terms of which the BRST charge Q is the generator of the canonical transformations. This is an important fact since it says that the bracket structure is preserved under BRST transformations. Without this property one could never get in touch with quantum mechanics since there would be no way of giving commutation rules for the quantum mechanical operators. Since the BRST charge Q generates d_{BRST}, it contains just as much information about the system under study. In fact, the charge Q will be the central object in BRST theory. Equation (3.22) implies that when we form the cohomology we get the classical observables which lives in the $gh_\# = 0$ sector. This is important to note, because it will be required of the quantum observables as well.

3.4 Extended Phase Space

In order to examine the structure of the state space, we must analyze the expression of the BRST differential a little closer. In the expansion,

$$d_{BRST} = \delta + d + \text{"more"}, \tag{3.24}$$

above, d and δ imposes the necessary restrictions when passing to the cohomology. The Koszul-Tate differential acts in the graded algebra $C^\infty(\mathcal{P}) \otimes \mathbb{C}[\mathcal{P}_k]$

and imposes the restriction to the constraint surface via the resolution,

$$H_0(\delta) = C^\infty(\Sigma) \tag{3.25}$$
$$H_k(\delta) = 0 \quad k \neq 0 \tag{3.26}$$

Above, each generator $\mathcal{P}_k$ corresponds to one constraint but with opposite Grassmann parity ϵ

$$\epsilon(\mathcal{P}_k) = \epsilon(\phi_k) + 1 \tag{3.27}$$

As was stated before the set $\{X_{\phi_k}\}$ is assumed to be linearly independent. This means that we can construct a basis of 1-forms which is dual to the vector fields,

$$\{\vartheta^k : \vartheta^k(X_l) = \delta^k_l\} \tag{3.28}$$

It is obvious from the way they are defined that the ϑ^k's are forms along the gauge orbits. These 1-forms will be denoted $\mathcal{C}^k$ hereafter and called ghosts. Each $\mathcal{C}^k$ will also have the opposite Grassman parity as the corresponding constraint

$$\epsilon(\mathcal{C}^k) = \epsilon(\phi_k) + 1 \tag{3.29}$$

This is all consistent because if we have a Grassmann odd constraint the corresponding form will be Grassmann even and vice versa. It is on these forms that the exterior derivative along the gauge orbits d is defined to act. More specifically d acts in the graded algebra $C^\infty(T^*(\mathbf{P})) \otimes \mathbb{C}[\mathcal{P}_k]$. Since we have equally many generators in the two algebras, which fulfill $\epsilon(\mathcal{C}^k) = \epsilon(\mathcal{P}_k)$, these can be defined to be conjugate to each other

$$\{\mathcal{C}^k, \mathcal{P}_l\} = \delta^k_l \tag{3.30}$$

This is achieved by lifting the action of both d and δ to total phase space $\mathfrak{P}$

$$\mathfrak{P} := T^*(\mathbf{P}) \otimes \mathcal{P}_k \otimes \mathcal{C}^k \tag{3.31}$$

with (super)functions living in

$$C^\infty(T^*(\mathbf{P})) \otimes \mathbb{C}[\mathcal{P}_k] \otimes \mathbb{C}[\mathcal{C}^k] \tag{3.32}$$

We see that the extension of the original phase space is considerable. This is the price paid for unwinding the, in general very complex, geometry of the constraint surface. A collective label for the original phase space variables will be used

$$\Pi^A = \{q^i, p_i\} \tag{3.33}$$

Above, the gradings in the extended phase space, are given by

$$gh_\# \mathcal{C}^k = 1 \tag{3.34}$$
$$gh_\# \mathcal{P}_k = -1 \tag{3.35}$$
$$gh_\# \Pi^A = 0$$

The ghost number $gh_{\#}$ grading is actually a linear combination of the gradings that exist in the two algebras related to δ and d above. The value of the ghost number lies in the fact that it represents a conserved quantity and it will therefore be used to label states in the quantum theory. It also possesses a canonical action which measures the ghost content of a function in extended phase space

$$N := \mathcal{C}^k \mathcal{P}_k, \quad \{N, f\} = gh_{\#} f \tag{3.36}$$

With this definition N is purely imaginary

$$N^* = -N \tag{3.37}$$

Note that eq.(3.36) implies that

$$\begin{aligned} \{N, \mathcal{C}^k\} &= \mathcal{C}^k && (3.38)\\ \{N, \mathcal{P}^k\} &= -\mathcal{P}_k && (3.39)\\ \{N, \Pi^A\} &= 0 && (3.40)\\ \{N, Q\} &= Q && (3.41)\\ & && (3.42) \end{aligned}$$

in agreement with the definitions above. From now on the $\mathcal{P}_k$'s will be called the ghost momenta. Moreover, all of the variables in extended phase space will be chosen to be real, since the constraints may always be chosen to be real. But since we consider a space with indefinite metric, this will not imply real eigenvalues for the corresponding operators.

3.5 Construction of Q

Since $d_{BRST} F = \{F, Q\}$ one can show from the way that the Koszul-Tate differential acts on the momenta, that Q must look like

$$Q = \mathcal{C}^k \phi_k + \text{"more"} \tag{3.43}$$

to lowest order in ghost momenta (ghost momenta have antigh$_{\#} = 1$ and $gh_{\#} = -1$). Consider now the expansion of Q with respect to antighosts

$$Q = \sum_{n \geq 0} Q_n = \left\{ \begin{array}{lcl} Q_0 & = & \mathcal{C}^k \phi_k \\ Q_n & = & U^{a_1 \ldots a_n} \mathcal{P}_{a_1} \ldots \mathcal{P}_{a_n} \end{array} \right. \tag{3.44}$$

where

$$\text{antigh}_{\#} Q_n = n \tag{3.45}$$

and the expansion coefficients may depend on the rest of the variables in the theory. Now the nilpotency property,

$$\{Q, Q\} = 0 \tag{3.46}$$

implies that

$$\sum_{m,n}^{\infty}\{Q_m, Q_n\} = 0 \tag{3.47}$$

It has been shown that the eqs.(3.18-3.22) together with (3.47) uniquely determines the charge Q, modulo a canonical transformation. The last equation can be solved iteratively for Q, using as initial value $Q_0 = \mathcal{C}^k \phi_k$. In the next section we turn to the quantum theory. This is done by canonically quantizing the variables in extended phase space $\mathfrak{P}$ in the graded sense[2], i.e.

$$i\{Z^A, Z^B\} = [\hat{Z}^A, \hat{Z}^B] \tag{3.48}$$

The left bracket is the graded Poisson bracket in $\mathfrak{P}$ and the right bracket is the graded commutator in state space. Z^A is a collective label for all the canonical variables in $\mathfrak{P}$ and $\hat{Z}^A$ denotes their corresponding operators in state space.

3.6 The State Space

One important aim of every physical model is to determine all possible states in which the system can be found. Previously it was established that the physical states in the classical regime are given by the set of smooth functions on the reduced phase space, $C^\infty(\Sigma/\mathcal{G})$. We will now study the properties of the "corresponding" quantum state space, seen from the BRST perspective. We begin quite generally, by studying the BRST-algebra and its representations

$$\begin{aligned} [Q, Q] &= 0 \\ [N, N] &= 0 \\ [N, Q] &= Q \end{aligned} \tag{3.49}$$

where

$$Q = Q^\dagger, \quad N = -N^\dagger \tag{3.50}$$

Above, N denotes the conserved charge stemming from the invariance of the action under rescaling of the ghosts. N is referred to as the ghostnumber operator. Now, all genuine physical states[3] have finite norms, which implies that it is relevant to study the algebra (3.49) on an inner product space, or more specifically, on a **non-degenerate**, **inner product space** V. The last assumption makes this study much easier and is always valid since the "fully" degenerate states decouple from every other state (even from themselves) and can thereby be discarded from the theory at the beginning, without harming the physical content. In order to obtain a complete description of the physics, the irreducible representations of the algebra (3.49) on V must be obtained. The nilpotency of the BRST-charge implies that we can have at most

[2] In this correspondence $\hbar = 1$.

[3] "Genuine physical" states in the mundane sense, i.e the physics that we can measure is always finite.

2-dimensional representations of the algebra. In ref. [16] it was shown that only three different types of representations exist, namely, **singlets** $|s\rangle \in V_S$, **singlet-pairs** $|sp\rangle \in V_{SP}$ and **doublets**[4] $|q\rangle \in V_D$. Thus the state space V decomposes as

$$V = V_S \oplus V_{SP} \oplus V_D \tag{3.51}$$

Since N is a conserved charge, each of these states can be chosen to be classified according to $|k, n\rangle$, where n denotes the ghostnumber $gh_{\#}$ satisfying

$$N|k, n\rangle = n|k, n\rangle \tag{3.52}$$

and where k is a collective label for all the other quantum numbers whose corresponding operators commute with N. In terms of this classification we have for all states in V

- $V_S = \{v \in V | \forall u \in V : Qv = 0 \wedge v \neq Qu \ \wedge \ gh_{\#}v = 0)\}$
- $V_{SP} = \{v \in V | \forall u \in V : Qv = 0 \wedge v \neq Qu \ \wedge \ gh_{\#}v \neq 0)\}$
- $V_D = \{(u, v) \in V | u = Qv \neq 0\}$

Theories with singlet pairs are inconsistent since they lead to a physical subspace with indefinite metric. They are not present in Yang-Mills theories [16–18]. Thus, for all consistent theories the state space is decomposed as

$$V = V_S \oplus V_D, \tag{3.53}$$

which will be assumed in the following. At this point it is natural to ask, what kind of space is V and what properties does it possess? Well, we observe that the BRST-charge satisfies nilpotency, $Q^2 = 0$ and is hermitian, $Q^\dagger = Q$. Those properties are impossible to realize non-trivially on a positive-definite, non-degenerate space. Thus we must use a pseudo-Hilbert space as our state space, i.e a space in which we have an indefinite metric. This is achieved automatically since the ghosts satisfy unphysical commutation relations. Of course, we must require the **true**[5] physical subspace to be positive definite. From the classical analysis (3.22) we saw that all the observables had $gh_{\#} = 0$. This is not at all guaranteed in the quantum theory. In fact it might happen that none of the physical states fulfills $gh_{\#} = 0$[6]. This is a property which one would like to be fulfilled for the quantum mechanical states also. This brings us to the concept of **nonminimal sector**. One can show that if one introduces extra variables α^a and β^a in our theory such that

$$[Q, \alpha^a] \ = \ \beta^a \tag{3.54}$$
$$[Q, \beta^a] \ = \ 0 \tag{3.55}$$

[4] These are also called quartets in the literature - since they actually consists of pairs of doublets.

[5] In the BRST formalism one defines every state $|x\rangle$ verifying $Q|x\rangle = 0$ to be a physical one. But as we shall see later on this does not imply that $|x\rangle$ is a true physical state.

[6] This can depend on what representation one uses for the operators. See for instance [2] on this issue.

it is possible to bring all cohomology to $gh_{\#} = 0$. Specifically, this is achieved by introducing dynamical Lagrange multipliers v^a and corresponding antighosts $\bar{\mathcal{C}}_a$ together with their momenta, π_a and $\bar{\mathcal{P}}^a$ respectively, into the theory. We introduced the Lagrange multipliers before but then they were not dynamical since they did not have any momenta that could generate some time development via the Hamiltonian. From now on the extended phase space $\mathfrak{P}$ always refer to

$$T^*(\mathbf{P}) \otimes \mathcal{P}_k \otimes \mathcal{C}^k \otimes \bar{\mathcal{P}}^a \otimes \bar{\mathcal{C}}_a \tag{3.56}$$

These new generators are chosen to be real and are graded as

$$\begin{aligned} \mathrm{gh}_{\#}\bar{\mathcal{C}}_a &= -1 & (3.57)\\ \mathrm{gh}_{\#}\bar{\mathcal{P}}_a &= 1 & (3.58) \end{aligned}$$

From the digression above we conclude that all physical states $|\phi\rangle$ verify,

$$Q|\phi\rangle = 0 \;\;,\;\; gh_{\#}|\phi\rangle = 0 \tag{3.59}$$

This implies, due to the nilpotency of Q, that the physics always will be contained in the zeroth cohomology group

$$\{|\phi\rangle\} \in H^0(Q) = \ker Q/\mathrm{Im\ Q} \tag{3.60}$$

Above we saw that the doublet states in V_D was either BRST-exact, or did not possess $gh_{\#} = 0$. This implies that the physical state space is isomorphic to V_S. In other words

$$V_S = H^0(Q) = \{|\phi\rangle\} \tag{3.61}$$

In this section we have extracted as much information as possible about the state space, solely from representation theory. For a given theory, however, this is not enough to fully characterize the inner product space V_S. In the construction above, V_S was defined to be a subspace of an inner product space V. So far that definition is only a formal statement which needs to be completed by the explicit construction of an inner product on V. Only then will the formal developments above make sense.

3.7 The Physical Subspace

Let us sum up where we stand at this point. We have postulated the existence of a space $V_S \subseteq V$, where V is an inner product space. From representation theory it was then showed that the cohomology $\mathrm{Ker}Q/\mathrm{Im}Q$ of the states in V with respect to Q, is isomorphic to V_S. The problem is that if we are given a BRST-charge and solve for the states $|\phi\rangle$ in $\mathfrak{P}$ which satisfy the physical state conditions

$$\begin{aligned} Q|\phi\rangle &= 0 & (3.62)\\ gh_{\#}|\phi\rangle &= 0 & (3.63) \end{aligned}$$

not all the states $|\phi\rangle$ will be inner product states, i.e. not all the $|\phi\rangle$'s belong to V_S. To see how this comes about let us assume that we expand an arbitrary state in V_S in terms of ordinary Dirac-states and the ghosts,

$$|\phi\rangle = |\phi\rangle_0 + |\phi\rangle_{ab}{}^{cd}\mathcal{C}^a\bar{\mathcal{C}}^b\mathcal{P}_c\bar{\mathcal{P}}_d + ... \tag{3.64}$$

Obviously the ghost part of the state above consists of all possible combinations of ghosts, verifying $gh_{\#} = 0$ (the coefficients contain the rest of the variables in the theory). We see for example that states like (only one fermionic ghost pair assumed for simplicity)

$$\begin{aligned} |\phi\rangle &= |q\rangle|\pi\rangle_0 \otimes \mathcal{C}\bar{\mathcal{C}} \\ |\phi'\rangle &= |q'\rangle|\pi\rangle_0 \otimes \mathcal{C}\bar{\mathcal{C}} \end{aligned} \tag{3.65}$$

are physical for an abelian model in which $Q = \mathcal{C}p + \bar{\mathcal{P}}\pi$. But the usual Dirac "inner-product" between those states is

$$\langle\phi|\phi'\rangle = \delta(q-q')\cdot\delta(0)\cdot 0 = \delta(q-q')\cdot 0\cdot\infty \tag{3.66}$$

The zero and the infinity come from the fermionic and bosonic delta functions respectively. So, obviously this "naive" inner product cannot be the physical one since we need well defined states. Earlier we concluded that the BRST charge Q should be hermitian, which is a property which is defined in relation to an inner product. One important question is thus, in relation to which inner product(s) can that property be defined - or less ambitiously, what does an admissible inner product look like? That question will be answered in the next section. Observe that we at this stage only talk about finding a well defined inner product on V. Once it is found, it is possible to find out which states have positive norms and which have not, i.e. to determine the true physical states.

3.8 Regularized Inner Product

In [19–21] it was proven that provided one work with a non-minimal Q, and provided one can find a decomposition of the BRST charge satisfying

$$Q = \delta + \delta^\dagger, \quad \delta^2 = 0, \quad [\delta, \delta^\dagger] = 0 \tag{3.67}$$

then the states $|ph\rangle$ invariant under $\delta, \delta^\dagger$,

$$\delta|ph\rangle = \delta^\dagger|ph\rangle = 0 \tag{3.68}$$

will be inner product solutions. Furthermore $|ph\rangle$ must always be possible to express as a linear combination of states$\in$ Eig N, where Eig denotes the Eigenspace. This decomposition has been proved always to exist for Lie-group gauge theories [21, 22] in which case it leads to the explicit form,

$$|ph\rangle = e^{[Q,\psi]}|\phi\rangle \tag{3.69}$$

of the inner product states. ψ denotes the gauge fixing fermion for which $gh_{\#}\psi = -1$. The state $|\phi\rangle \in \text{span}(\text{Eig N})$ is BRST-closed and subject to some simple conditions described in [21, 22]. It must be emphasized at this point that in order for states of the form (3.69) to be inner product states, the model must be supplemented with certain quantization rules, which were derived in [21, 23]. They read explicitly

▲ **Quantization rules {1}** The unphysical degrees of freedom, represented by ghosts and antighosts as well as Lagrange multipliers and gauge degrees of freedom are to be quantized with opposite metric signature.

How these rules work for abelian models is investigated in detail in **paper I**, included in this thesis. In the same article it is also shown that the choice of gauge fixing fermion ψ determines these quantization rules. Even though this was only done for abelian models it is believed to be true for general models due to the abelianization theorem[7]. Another approach towards the construction of inner product states was taken in [25]. In that approach one considers the true physical states, i.e. the singlet states $|s\rangle \in H^0(Q)$

$$Q|s\rangle = 0 \quad |s\rangle \neq 0 \tag{3.70}$$

For every charge Q that have the BFV-form[8] they were shown to be given by,

$$|ph\rangle = e^{[Q,\psi]}|\phi\rangle_s, \tag{3.71}$$

for *arbitrary reducible* gauge theories. Above, the gauge fixing fermion is the same as in eq.(3.69) but in constrast to the $|\phi\rangle$ states above, the $|\phi\rangle_s$ states are determined by conditions

$$D_r|\phi\rangle_s = 0 \tag{3.72}$$

where

$$\begin{aligned} \{D_i\} &= \{B_i, C_i\} && (3.73)\\ B_i &:= [Q, C_i] && (3.74) \end{aligned}$$

and $\{D_i\}$ constitutes a maximal set of independent hermitian BRST doublet operators in involution such that

$$D_i' = e^{[Q,\psi]} D_i e^{-[Q,\psi]} \tag{3.75}$$

satisfy

$$\det([D_i', (D_j')^\dagger]) \neq 0 \tag{3.76}$$

[7]The abelianization theorem states that we always can find *local* coordinates in terms of which the constraints are abelian [24].

[8]The BFV form allows for a very general class of theories, so general in fact, that one believes that all gauge theories can be described by a BRST charge, written in that form. See references [26–28] for more details on this matter.

The last conditions determines a legitimate gauge fixing. With "operator doublets in involution" is simply meant that $\{D_i\}$ satisfies a closed algebra

$$[D_i, D_j] = C_{ij}{}^k D_k \tag{3.77}$$

An explicit example on how this construction is carried out, is given in paper I for an abelian model. Due to the relations,

$$C_i|\phi\rangle_s = 0 \tag{3.78}$$

the states $|\phi\rangle_s$ are gauge fixed states with $gh_{\#}|\phi\rangle_s = 0$. The set $\{|\phi\rangle_s\}$ is thus a gauge fixed version of $|\phi\rangle$. It is important to note that even in the last approach we have to supplement the quantization rules in order for the singlet states $|s\rangle$ to be inner product states. The two ways in which inner product states were derived above, have also been shown to be related for some models, for example in section 4 of paper I, an explicit relation between the δ-operator and the B_i-operators is given. Whether or not it is possible to give such a relation for all type of models is not at all clear, since the requirement of the existence of a decomposition $Q = \delta + \delta^\dagger$ verifying eq.(3.68) is very restrictive. The conclusion of this section is, that there is strong evidence for, that whenever inner product solutions exists they can for an arbitrary gauge theory be written in the form [25]

$$|ph\rangle = e^{[Q,\psi]}|\phi\rangle, \tag{3.79}$$

where the state $|\phi\rangle$ is BRST-closed and always obeys some simple conditions. The strength of the form (3.79) lies in the fact that it is relatively easy to determine the states $|\phi\rangle$, in particular since the prescription does not restrict them to be inner product states.

4

REPARAMETRIZATION INVARIANT THEORIES

In a **reparametrization invariant (RI)** theory the dynamical variables depends on parameters in such a way, that the theory is invariant under arbitrary changes (reparametrizations) of those parameters. This means in particular that the parameters in question do not possess any physical relevance. RI-theories are also often called **generally covariant theories** in the literature. Many important theories are formulated in a RI-form, for example: gravity, string theory and the relativistic particle. The reparametrization invariance can be viewed as a gauge symmetry and in this chapter we will give a brief review of the BRST-quantization of RI-systems with finite degrees of freedom on inner product spaces, in which case we have only one parameter. More details regarding this issue can be found in **paper III**. We concentrate on how one can extract the physical states and propagators from their BRST-invariant counterparts.

Including time and its conjugate momenta among the dynamical variables leads to a Hamiltonian that is pure gauge (i.e. vanish on-shell) and this implies that the time evolution can be interpreted as a gauge transformation. A lot of the material presented in this chapter is intimately connected to the discussion about quantization on inner product spaces in chapter 3.

4.1 Preliminaries

Given any regular[1] theory, $S[q(t), \dot{q}(t)] = \int dt L(q(t), \dot{q}(t))$, it is always possible to reformulate it as a reparametrization invariant theory [2, 29, 30]. This is

[1] A regular theory is a theory without any non-trivial gauge invariance, i.e. a theory for which the Hessian is invertible: det $\frac{\partial^2 L(t)}{\partial \dot{q}^i \dot{q}^j} \neq 0$.

easily seen by considering the time t as a function of an arbitrary parameter τ,

$$\int dt\, L(t) = \int d\tau\, \dot{t}\, L(t(\tau)) := \int d\tau\, L'(\tau) \tag{4.1}$$

Obviously, the action $\int d\tau\, \dot{t}\, L(t(\tau))$ is invariant under arbitrary reparametrizations $\tau \to \tau'(\tau)$. The theory defined by $L'(\tau)$ is singular which reflects the fact that we deal with a constrained dynamical system. There is only one constraint and it is given by $\pi + H(t) = 0$. Note that the degrees of freedom in the systems defined by L and L' are the same; for the latter theory we have added one dynamical degree of freedom, namely $t(\tau)$. In order to be able to discuss BRST operator quantization (BFV), we need to rewrite $\int d\tau\, L'$ in first order phase space form

$$L_P(\tau) = p_i \dot{q}^i + \pi \dot{t} - v(\pi + H(t)) \tag{4.2}$$

The theory defined by (4.2) possesses the two constraints $\pi + H(t) = 0$ and $\pi_v = 0$. The constraints are first-class and their algebra is abelian. Note that the Hamiltonian H_P of (4.2) is pure gauge since $H_P = v(\pi + H(t))$.

4.2 Formal BRST Solutions on Inner Product Spaces

Using the BFV operator version of BRST, one may derive formal solutions on inner product spaces for the models described by (4.2). These solutions are formal in the sense that one also need to supplement them with an explicit representation of the extended phase space and only then will the solutions derived in this section be true inner product states.

Consider now the phase space $T^*(\mathbf{Q})$ of the model L_P with coordinates $z^k = (q^i, p_i, t, \pi, v, \pi_v)$ where $i \in \{1, ..., n\}$ and $k \in \{1, ..., 2n+4\}$. The quantization is performed by promoting the phase space variables to hermitian even operators, $z^k \to Z^k$, and adding a ghost $\mathcal{C}$, an antighost $\bar{\mathcal{C}}$ and their conjugate momenta, $\mathcal{P}$ and $\bar{\mathcal{P}}$ respectively. All the added ghost operators are odd and hermitian. The set of all operators define the extended phase space $\mathfrak{P}$, discussed in chapter 3, in the sense, $\mathfrak{P} = T^*(\mathbf{Q}) \otimes \mathcal{C} \otimes \mathcal{P} \otimes \bar{\mathcal{C}} \otimes \bar{\mathcal{P}}$. The non-trivial operator algebra is given by ($\hbar$=1)

$$[Q^i, P_i] = i, \delta^i_j, \quad [T, \Pi] = i \quad [V, \Pi_V] = i, \quad [\mathcal{C}, \mathcal{P}] = 1, \quad [\bar{\mathcal{C}}, \bar{\mathcal{P}}] = 1 \tag{4.3}$$

In terms of the operators in extended phase space, the BRST charge for the theory defined by L_P is given by,

$$Q = \mathcal{C}(\Pi + H(T)) + \bar{\mathcal{P}}\Pi_V \tag{4.4}$$

As was explained at the end of chapter 3, consistent solutions[2] on the form $|ph\rangle = e^{[Q,\psi]}|\phi\rangle$ requires us in this case to find four operators in involution[3]

$$D_i = (B_i, C_i), \; i \in \{1, 2\} \tag{4.5}$$

[2]See eqs. (3.73), (3.72) and (3.76) for some more details on this issue.

[3]With involution is meant that $B_i = [Q, C_i]$.

such that

$$B_i|\phi\rangle = C_i|\phi\rangle = 0 \tag{4.6}$$

and

$$\det([D'_i, (D'_j)^\dagger]) \neq 0 \tag{4.7}$$

where $D'_i = e^{[Q,\psi]}D_i e^{-[Q,\psi]}$. Two different allowed choices are given by,

$$\begin{aligned} \Pi_V|\phi\rangle &= \mathcal{C}|\phi\rangle = 0 & (4.8)\\ \bar{\mathcal{P}}|\phi\rangle &= (\Pi + H(T))|\phi\rangle = 0 & (4.9) \end{aligned}$$

It should be mentioned that not all choices leading to $Q|\phi\rangle = 0$ are allowed; some examples of non-allowed choices are given in **paper III**. An admissible form of gauge fixing fermion ψ for the regulator $e^{[Q,\psi]}$, which is valid for both (4.8) and (4.9), is given by $\psi = \mathcal{P}\Lambda(V) + \bar{\mathcal{C}}\chi(T)$ [31]. The condition (3.76) implies that the functions χ and Λ must have non-vanishing derivatives. The states $|ph\rangle = e^{[Q,\psi]}|\phi\rangle$ are only unique modulo zero norm states, and in order to obtain the BRST singlets $|s\rangle$ we must impose extra gauge fixing conditions on the states $|\phi\rangle$ [16, 18]. In the case of (4.8) and (4.9), two allowed choices are given by $\chi(T)|\phi\rangle = \bar{\mathcal{C}}|\phi\rangle = 0$ and $\Lambda(V)|\phi\rangle = \mathcal{P}|\phi\rangle = 0$ respectively.

4.3 Wave Function Representations

Below, we take a look at various wave function representations of the states introduced in the previous section. The formal solutions given there must be supplemented with an explicit choice of the state space and we will see below that this choice corresponds to pinpointing which of the unphysical operators that possess imaginary (complex) eigenvalues. The wave function representation of the state $|\phi\rangle$ in (4.8), with the gauge fixing fermion chosen as $\psi = \mathcal{P}\Lambda(V)$, is given by

$$\langle q, t, v, \mathcal{P}, \bar{\mathcal{P}}|\phi\rangle = \delta(\chi(T))\varphi(q, t) \tag{4.10}$$

where $(q, t, v, \mathcal{P}, \bar{\mathcal{P}})$ now denote the eigenvalues of the corresponding operators. The quantization rules presented at the end of chapter 3, dictate that half of the unphysical variables of our theory, namely $\{t, \pi, v, \pi_v, \mathcal{C}, \bar{\mathcal{C}}, \mathcal{P}, \bar{\mathcal{P}}\}$ must span an indefinite metric state space. This means that the corresponding hermitian operators have imaginary (or complex) eigenvalues. Which of the operators that have imaginary eigenvalues is not specified a priori and the possible choices are governed by the gauge fixing fermion ψ [32]. In (4.10) for example, the argument of the delta function must be real in order for the solution to be consistent. The wave function representation of the corresponding singlet state $|s\rangle = e^{[Q,\psi]|\phi\rangle}$ was in **paper III** derived to be

$$\begin{aligned} \Psi(q, t, v, \mathcal{P}, \bar{\mathcal{P}}) &= \langle q, t, v, \mathcal{P}, \bar{\mathcal{P}}|s\rangle \\ &= e^{-i\Lambda'(v)\bar{\mathcal{P}}\mathcal{P}}\delta(\chi(t - i\Lambda(v)))e^{\Lambda(v)(-i\partial_t + H_S(t))}\varphi(q, t) \end{aligned} \tag{4.11}$$

Above, H_S is the Schrödinger representation of the operator $H(T)$. The solution (4.11) admits the choices (i) t real, v imaginary and $\Lambda(v)$ imaginary or (ii) t imaginary, v real and $\chi(t)$ imaginary. For the fermions we may choose $\mathcal{P}$ real and $\bar{\mathcal{P}}$ imaginary. One can show that the inner product $\langle s|s\rangle$ is only positive definite for $\Lambda' < 0$ which means that we must rule out positive Λ'. Obviously these two choices are not connected by a unitary transformations since such a transformation preserves the norm. The inner product of the singlet state $|s\rangle$ is independent of the choice of gauge parameters which makes it possible to define the physical wave function ϕ by

$$\phi(q,t_0) := \frac{1}{|\dot{\chi}(t_0)|}\varphi(q,t_0) \tag{4.12}$$

In the preceding equation t_0 is the unique solution to $\chi(t) = 0$. Different values of t_0 can be reached by unitary transformations of the form $U(a) = e^{a[Q,\mathcal{P}]}$, where a is a real constant. A general transformed state $\phi'(q,t)$ is then given by

$$\phi'(q,t) = e^{-(t-t_0)(\partial_t + iH_S(t))}\phi(q,t_0), \tag{4.13}$$

which imply that $\phi'(q,t)$ satisfies the Schrödinger equation with respect to t. Moreover, $\phi'(q,t)$ coincides with $\phi(q,t_0)$ at time $t = t_0$ and this tells us that $\phi'(q,t)$ is the **gauge invariant extension** of the wave function $\phi(q,t_0)$.

4.4 Physical Projections of BRST Singlets

In **paper III** we proposed how one should define physical wave functions, and this proposition was based on studying inner products of the singlet states $\langle s|s\rangle$. Here we address the question of how one can derive physical wave functions and propagators by certain projections of the singlet states. Intuitively, these projections should correspond to getting rid of the unphysical degrees of freedom in the theory. Consider the wave function representation of the singlet state (4.11). We observe that

$$\phi(q,t_0) = \int du d\mathcal{P} d\bar{\mathcal{P}}\, \Psi(q,t_0,iu,\mathcal{P},i\bar{\mathcal{P}}), \tag{4.14}$$

which suggests that we should define the projection of singlets to physical wave functions as

$$\phi_{\text{phys}}(q,t_0) = \Psi|_{b_i=c_i=0} \tag{4.15}$$

where b_i and c_i are the eigenvalues, or equivalently, the Weyl symbols of B_i and C_i respectively. The projection (4.15) actually connects the conditions imposed on $|\phi\rangle$ with the boundary conditions imposed on the wave function representation of a singlet state Ψ, because it can be restated as

▲ **Projection {2}** Impose boundary conditions determined by the conditions on $|\phi\rangle$ on the wave function representation of Ψ of the BRST singlets $|s\rangle$.

The preceding projection does not produce gauge invariant wave functions, but one can show in a similar way as above that if we impose all conditions on Ψ that is imposed on $|\phi\rangle$, except the gauge fixing ones, the result is the gauge invariant extension of $\phi_{\text{phys}}(q, t_0)$ above. More succintly, the projection rule can be stated as follows:

▲ **Projection {3}** In order to obtain gauge invariant physical wave functions, impose the conditions on $|\phi\rangle$, except for the gauge fixing conditions, as boundary conditions on the wave function representation of $|s\rangle$.

The physical wave function obtained by using the projection {2} on Ψ can be obtained directly from the wave function representation of $|\phi\rangle$, by using the following projection rule

▲ **Projection {4}** The physical wave functions may be obtained from the wave function of the $|\phi\rangle$-state by imposing the dual conditions to the gauge fixing conditions to the gauge fixing conditions on $|\phi\rangle$ as boundary conditions.

By studying the norm of singlet states one can derive the projection for the physical propagator. The norm $\langle s|s\rangle$ is given by

$$\begin{aligned}\langle s|s\rangle &= \langle\phi|e^{\gamma[Q,\psi]}|\phi'\rangle \\ &= \int d^{n+4}R''\, d^{n+4}R'\ \langle\phi'|R''^{*}\rangle\langle R''|e^{\gamma[Q,\psi]}|R'^{*}\rangle\langle R'|\phi\rangle,\end{aligned} \tag{4.16}$$

where the collective label $R = \{q^i, t, iu, \mathcal{P}, i\bar{\mathcal{P}}\}$ has been introduced. Above, γ denotes a real parameter. From the expression for R it is also clear which state space representation we are using in this case. In equation (4.16) one can show that

$$\int du' du'' d\mathcal{P}'' d\bar{\mathcal{P}}'' d\mathcal{P}' d\bar{\mathcal{P}}' \langle R''|e^{\gamma[Q,\psi]}|R'^{*}\rangle, \tag{4.17}$$

is the physical propagator, and by looking at the integrations we can invoke the following projection rule for physical propagators,

▲ **Projection {5}** Impose as boundary conditions on the extended propagator $\langle R''|e^{\gamma[Q,\psi]}|R'^{*}\rangle$ the conditions on $|\phi\rangle$ except for the gauge fixing conditions.

We will close this chapter by considering a path integral representation of the propagator $\langle R''|e^{\gamma[Q,\psi]}|R'^{*}\rangle$ and apply the projection {5} to obtain the physical propagator. The norm of the singlet state $|s\rangle$ in (4.16) is given by,

$$\langle s|s\rangle = \int d^{n+4}R''\, d^{n+4}R'\, \langle\phi'|R''^{*}\rangle\langle R''|e^{\gamma[Q,\psi]}|R'^{*}\rangle\langle R'|\phi\rangle \tag{4.18}$$

By time slicing the propagator $\langle R''|e^{\gamma[Q,\psi]}|R'^*\rangle$ we obtain its corresponding path integral representation,

$$\langle R''|e^{\gamma[Q,\psi]}|R'^*\rangle = \int \mathcal{DRDP}\exp\left\{i\int_{\tau'}^{\tau''} d\tau\,(P\dot{R} - i[Q,\psi]_W)\right\} \tag{4.19}$$

This is the BFV-form of the propagator with the only difference that we have $[Q,\psi]_W$ instead of the Poisson-bracket expression $\{Q,\psi\}_W$. Above we have defined $\gamma = \tau'' - \tau'$ and $[Q,\psi]_W$ denotes the Weyl symbol of $[Q,\psi]$ (a definition of the Weyl symbol can be found in **paper III**, but we will not consider its explicit definition here). $[Q,\psi]_W$ might differ by $\hbar$-terms from the Poisson bracket expression $\{Q,\psi\}_W$. The measure $\mathcal{DRDP}$, is given by

$$\prod_{\tau} \frac{d^n q\, dt du d\bar{\mathcal{C}} d\mathcal{C} d^n p\, d\pi d\pi_u d\mathcal{P} d\bar{\mathcal{P}}}{(2\pi)^{n+2}} \tag{4.20}$$

The explicit state space representation is chosen as $R = \{q^i, t, v{=}iu, \mathcal{C}, -i\bar{\mathcal{C}}\}$ and $P = \{p_i, \pi, \pi_v = -i\pi_u, \mathcal{P}, i\bar{\mathcal{P}}\}$. Choosing the gauge fixing fermion as $\psi = \mathcal{P}\Lambda(V)$ implies that the propagator can be written as

$$\langle R''|e^{\gamma[Q,\psi]}|R'^*\rangle = \int \mathcal{D}R\,\mathcal{D}P\exp\left\{i\int_{\tau'}^{\tau''} d\tau\, L_{\text{eff}}(\tau)\right\} \tag{4.21}$$

where the effective Lagrangian L_{eff} is given by

$$\begin{aligned} L_{\text{eff}}(\tau) &= pq + \pi\dot{t} + \pi_u\dot{u} + i\mathcal{P}\dot{\mathcal{C}} + i\bar{\mathcal{P}}\dot{\bar{\mathcal{C}}} - \lambda(\pi + H(T)) & (4.22)\\ &+ i\lambda'\bar{\mathcal{P}}\mathcal{P} & (4.23)\end{aligned}$$

With our specific choice of state space, L_{eff} is real. Note also that $\lambda(u) = i\Lambda(iu)$ is real, since $\Lambda(iu)$ is forced to be imaginary in order for the wave function representation to exist. According to the projection {5}, the physical propagator $K(q'', t''|q', t')$ is given by

$$K(q'', t''|q', t') = \int du' du'' d\mathcal{P}' d\bar{\mathcal{P}}' d\mathcal{P}'' d\bar{\mathcal{P}}''\,\langle R''|e^{\gamma[Q,\psi]}|R'^*\rangle \tag{4.24}$$

The result of the integrations is

$$K(q'', t''|q', t') = \text{sign}((\tau'' - \tau')\lambda')\,\langle q'', t''|q', t'\rangle \tag{4.25}$$

where $\langle q'', t''|q', t'\rangle$ denotes the correct physical propagator obtained from ordinary quantum mechanics. Note that $\text{sign}((\tau'' - \tau')\lambda')$ determines the sign of the norm of the singlet states [31] and this information is thus encoded in the projected propagator.

5

BV-FORMALISM

The historical development towards the **BV-formalism**, also called the field-antifield formalism, can be understood in terms of the gradually increasing success in gauge fixing ever more complex gauge theories. The gauge fixing is directly connected to the (local) invariances of the theory and thus determined by the structure of the gauge algebra. For example, in the case of Maxwell theory with $U(1)$-gauge group, Faddeev and Popov [9] showed how one could gauge fix by introducing ghost fields. When the gauge generators satisfy a general Lie algebra, the Faddeev-Popov method is not powerful enough and this led to the development of the BRST quantization method [10, 11]. The investigation of various supergravity theories during the 70's introduced new difficulties with respect to gauge fixing because these theories displayed certain types of open algebras. An extended version of the BRST formalism was developed [13, 33] for supergravity theories and later for general open algebras [34]. This generalization of BRST suffered from a major drawback; the BRST-generator Q was only nilpotent on-shell, which implies that the cohomology only exist on-shell. Besides the complications caused by the open algebras, there are other difficulties in gauge fixing which arise when the set of gauge generators are reducible. Reducibility occurs in many important theories, as for example antisymmetric tensor fields (for $d \geq 4$) [35], the GS superstring [36, 37], which is infinitely reducible and open string field theory [38] which is also infinitely reducible.

A general quantization method, capable of handling all the difficulties above was developed by Batalin and Vilkovisky in the beginning of the 80's. By introducing so called antifields in addition to the (ordinary) fields, Batalin and Vilkovisky constructed a Lagrangian generalization of the BRST method, capable of quantizing arbitrary open algebras and which provide for an off-shell nilpotent BRST-operator [39]. It was also clarified in [40] that the BV-framework enables a gauge fixing of arbitrary reducible gauge theories while at the same time maintaining manifest locality and covariance. Important references regarding the BV-formalism are, [34, 39–45] and for some good reviews, see [2, 6, 46, 47].

5.1 The Classical Master Equation

In this section we briefly review the classical aspects of the field-antifield formalism. Let us start from the BRST framework and consider the gauge fixed action,

$$S_q = S_0[\Phi] + \int d^n x \, \{Q, \Psi\} \tag{5.1}$$

Suppressing explicit reference to ghosts and multiplier fields we write the gauge fixing fermion as $\Psi = \Psi(\Phi)$. Since the Q-variation of Ψ is given by $\delta_Q \Psi = \delta_Q \Phi^P \frac{\vec{\partial}}{\partial \Phi^P} \Psi$, we can write S_q as

$$S_q = S_0 + \int \delta_Q \Phi^P \frac{\vec{\partial}}{\partial \Phi^P} \Psi = S_0 + \int \delta_Q \Phi^P \Phi^*_P \Big|_{\Phi^*_P = \left[\frac{\vec{\partial}}{\partial \Phi^P} \Psi\right]} \tag{5.2}$$

In the line above we have introduced the additional field Φ^*_P. S_q can now be written as the restriction of the action,

$$S_M[\Phi, \Phi^*] := S_0 + \int \delta_Q \Phi^P \Phi^*_P \tag{5.3}$$

to the surface Σ_Ψ defined by $\Phi^*_P = \frac{\vec{\partial}}{\partial \Phi^P} \Psi$ in the space of fields and antifields. This is the reason one always gauge fix in BV-theory by setting the antifields equal to the derivative of some gauge fixing fermion. Since $\epsilon(\Psi) = 1$ and $gh_{\#}(\Psi) = -1$, it follows that,

$$\epsilon(\Phi^*_P) + \epsilon(\Phi^P) = 1 \tag{5.4}$$

$$gh_{\#}(\Phi^*_P) + gh_{\#}(\Phi^P) = -1 \tag{5.5}$$

From now on we will term the fields Φ^*, the **antifields** and the action S_M defined in (5.3), the **masteraction**. It follows that the Q-variation of the fields is given by

$$\delta_Q \Phi^P = \frac{\vec{\delta}}{\delta \Phi^*_P} S_M \tag{5.6}$$

Notice that the last equation tells us that the antifields act as sources for the BRST transformations of the fields. The Q-variations of the antifields are similarly defined as,

$$\delta_Q \Phi^*_P = S_M \frac{\overleftarrow{\delta}}{\delta \Phi^P} (-1)^{P+1} \tag{5.7}$$

Let us now introduce a bracket structure on the space of fields and antifields. The mapping $(\ ,\ ) : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is termed **antibracket** and is defined as,

$$(A, B) = A \frac{\overleftarrow{\delta}}{\delta \Phi^P} \frac{\vec{\delta}}{\delta \Phi^*_P} B - (-1)^{(A+1)(B+1)} B \frac{\overleftarrow{\delta}}{\delta \Phi^P} \frac{\vec{\delta}}{\delta \Phi^*_P} A \tag{5.8}$$

Above, $\mathcal{F}$ denotes the space of all functionals endowed with a ghost number- and $\mathbb{Z}_2$-grading and $A, B \in \mathcal{F}$. In terms of the antibracket we get the following symmetric form on the BRST-variations of the field and antifields,

$$\delta_Q \Phi^P = (\Phi^P, S_M) \tag{5.9}$$
$$\delta_Q \Phi^*_P = (\Phi^*_P, S_M) \tag{5.10}$$

In terms of the antibracket, the $BRST$-invariance of the action S_M can now be expressed as,

$$(S_M, S_M) = 0 \tag{5.11}$$

which is the **classical master equation**. Note that in contrast to the graded version of the Poisson-bracket, the antibracket between identical (Grassmann) even objects is nontrivial. Requiring that (5.11) is a non-trivial equation for S_M and that $S_M[\Phi, \Phi^*]|_{\Phi^*=0} = S_0[\Phi]$ implies that we must have,

$$\epsilon(S_M) = 0 \tag{5.12}$$
$$gh_{\#}(S_M) = 0 \tag{5.13}$$

Formally, the solution to the master equation can be written as a power series expansion in terms of the antifields in the following way,

$$S_M[\Phi, \Phi^*] = S_0[\Phi] + \sum_{s=0}^{L} \mathcal{C}^*_{s-1,\alpha_{s-1}} R^{\alpha_{s-1}}{}_{s\alpha_s} \mathcal{C}^{\alpha_s}{}_s + \text{higher order terms in } \mathcal{C}^* \tag{5.14}$$

The expansion incorporates the correct boundary condition, namely that the master action should reduce to the classical action when the antifields are set to zero, $S_M[\Phi, \Phi^*]|_{\Phi^*=0} = S_0[\Phi]$. In (5.14), $\mathcal{C}^*_{s-1,\alpha_{s-1}}$ denotes the antifield of $\mathcal{C}^{\alpha_s}_s$ and $\mathcal{C}^*_{-1,\alpha_{-1}} := \Phi^*_i$ and $\mathcal{C}^{\alpha_{-1}}_{-1} := \Phi^i$. The master equation enforces the functionals $R^{\alpha_{s-1}}{}_{s\alpha_s}$ and their higher order analogs in (5.14) to satisfy the gauge structure equations discussed in section (2.5). Hence, it follows that the master action contains all the information about the gauge structure of a theory[1]; the existence of a solution to the master equation guarantees that the corresponding theory described by S_0 is consistent as a classical gauge theory.

We end this section by listing some important properties of the antibracket. Consider $A, B, C \in \mathcal{F}$, then the ghost number- and ($\mathbb{Z}_2$)-gradings are given by,

$$\epsilon[(A,B)] = \epsilon(A)+\epsilon(B)+1 \tag{5.15}$$
$$gh_{\#}[(A,B)] = gh_{\#}A+gh_{\#}B+1 \tag{5.16}$$

The antibracket possesses graded antisymmetry with respect to the parities $(A+1)$ and $(B+1)$,

$$(A,B) = -(-1)^{(A+1)(B+1)}(B,A), \tag{5.17}$$

[1] That is, all the information about the Noether identities, closure of the gauge transformations and the higher order gauge identities.

and satisfies the following graded version of the Jacobi-identity,

$$\sum_{cyclic} ((A, B), C)(-1)^{(A+1)(C+1)} = 0 \tag{5.18}$$

The antibracket is also a graded derivation of the pointwise product between functionals,

$$(A, BC) = (A, B)C + (-1)^{BC}(A, C)B \tag{5.19}$$
$$(AB, C) = A(B, C) + (-1)^{AB}B(A, C) \tag{5.20}$$

The last relation implies trivially that the BRST operator δ_Q is also a graded derivation of the pointwise product,

$$\delta_Q(AB) = A\,\delta_Q B + (-1)^B \delta_Q A\,B \tag{5.21}$$

It follows from the Jacobi-identity (5.18) that δ_Q is also a graded derivation of the product defined by the antibracket,

$$\delta_Q(A, B) = ((A, B), S) = (A, \delta_Q B))(-1)^{B+1} + (\delta_Q A, B)(-1)^{B+1} \tag{5.22}$$

5.2 Quantum Master Equation

Below, we recapitulate how one may derive the quantum theory analog of the classical master equation. Some possible obstacles toward constructing a well-defined quantum theory will also be touched upon. Now, the path integral expression of an arbitrary correlation function is

$$\mathcal{I}_\Psi = \int \mathcal{D}\Phi\mathcal{D}\Phi^*\,\delta\Big(\Phi^*_P - \frac{\overrightarrow{\partial}}{\partial\Phi^P}\Psi\Big)\exp\Big(\frac{i}{\hbar}W[\Phi, \Phi^*]\Big)X[\Phi, \Phi^*] \tag{5.23}$$

Above, $X[\Phi, \Phi^*]$ denotes an arbitrary correlation function and $W[\Phi, \Phi^*]$ the **quantum master action**. In order for the quantum master action to be the analog of S_M, we must have $\lim_{\hbar\to 0} W = S_M$. A power expansion of the quantum master action W in terms of $\hbar$ can then be considered, $W = S_M + \hbar W_1 + \hbar^2 W_2 + ...$, where the higher order terms can be viewed as modifications of the path integral measure. Note that the δ-function above imposes the gauge fixing condition $\Phi^*_P = \frac{\partial\Psi}{\partial\Phi^P}$ on the antifields. Consider now the integrand in (5.23),

$$\bar{I}[\Phi, \Phi^*] := \exp\Big(\frac{i}{\hbar}W[\Phi, \Phi^*]\Big)\,X[\Phi, \Phi^*] \tag{5.24}$$

The change of the quantity $\mathcal{I}_\Psi$ under infinitesimal deformations of Ψ is easily evaluated as,

$$\delta_\Psi\mathcal{I} = \int \mathcal{D}K\ \Delta\bar{I}\,\delta\Psi + \mathcal{O}\big((\delta\Psi)^2\big), \tag{5.25}$$

where the Δ operator is defined by [39, 40],

$$\Delta := \frac{\overleftarrow{\partial}}{\partial\Phi^P}\frac{\overleftarrow{\partial}}{\partial\Phi^*_P}(-1)^{P+1} \tag{5.26}$$

One takes the independence of infinitesimal deformations in Ψ as the definition of a good integrand,

$$\Delta \bar{I} = 0 \tag{5.27}$$

We must require that the partition function is gauge independent; this corresponds to the case $X = 1$ in (5.23) and it requires (to second order in $\hbar$),

$$\Delta \exp\left(\frac{i}{\hbar}W\right) = \exp\left(\frac{i}{\hbar}W\right)\left(\frac{i}{\hbar}\Delta W - \frac{1}{\hbar^2}(W,W)\right) \tag{5.28}$$

We have thus arrived at the **quantum master equation**,

$$\frac{1}{2}(W,W) = i\hbar\Delta W \tag{5.29}$$

Given a quantum master action W that satisfies (5.29), a functional $X[\Phi,\Phi^*]$ must satisfy,

$$(X,W) = i\hbar\Delta X, \tag{5.30}$$

in order to produce gauge independent correlation functions. The BRST-operator measures the failure of a quantity to fulfill the master equation, and from (5.30) we see that the quantum BRST-transformation $\delta_{\hat{Q}}$ should be defined as,

$$\delta_{\hat{Q}} := (X,W) - i\hbar\Delta X \tag{5.31}$$

The quantum master equation implies the nilpotency of $\delta_{\hat{Q}}$ which in turn guarantees the existence of a cohomology $H^n(\delta_{\hat{Q}})$ at ghost number n. The **quantum observables** are defined as the elements of $H^n(\delta_{\hat{Q}})$. The solution to the quantum master equation is unique modulo a canonical transformation, the effect of which is a BRST-exact modification of the untransformed quantity. When violation of the quantum master equation occurs, one speaks of a gauge anomaly [48]. The existence of a gauge anomaly implies that not all of the classical gauge symmetries survive quantization. An important observation is that, contrary to its classical counterpart, $\delta_{\hat{Q}}$ is not a graded derivation. This follows from,

$$\delta_{\hat{Q}}(AB) = A(\delta_{\hat{Q}}B) + (-1)^B\delta_{\hat{Q}}(A)B - i\hbar(-1)^B(A,B) \tag{5.32}$$

This implies that quantum observables do not constitute an algebra; given two observables X_1 and X_2, there is no longer any guarantee that X_1X_2 is an observable. There are however important special cases where the set of quantum observables still defines an algebra; these cases will be discussed later on in this section and in chapter 6.

Let us now discuss in some more detail, the properties and subtleties of the operator Δ. The $\mathbb{Z}_2$- and ghost number gradings of Δ follow directly from the definition (5.26) and the gradings of the fields and antifields (5.4),

$$\epsilon(\Delta) = 1 \tag{5.33}$$

$$gh_{\#}(\Delta) = 1 \tag{5.34}$$

The operator Δ is nilpotent, $\Delta^2 = 0$, and the antibracket can be written in terms of Δ in the following way,

$$(A, B) = \Delta(AB)(-1)^{A} - (\Delta A)B(-1)^{A} - A(\Delta B) \tag{5.35}$$

Note that the antibracket measures the failure of Δ to be a derivation of the ordinary pointwise product. Δ is however a derivation of the product defined by the antibracket, since

$$\Delta(A, B) = (\Delta A, B) + (A, \Delta B)(-1)^{A+1} \tag{5.36}$$

As is seen from (5.26), Δ will in general be singular when acting on local functionals[2]. This implies that one needs a suitable[3] regularization scheme. Two interesting cases of solutions to (5.29) exist when: (i) The classical action S_0 does not possess any gauge symmetries. This implies that a proper solution to the master equation is given by S_0 itself and obviously we then have $\Delta S_0 = 0$. (ii) If one can find a regularization scheme such that $\Delta S_M = 0$ and (S_M, S_M)=0 separately; in this case the classical master action S_M is also a solution to the quantum master equation, $S_M = W$. Case (ii) is the situation for all of the models formulated in the BV-framework in this thesis; the existence of a valid regularization scheme of the bosonic part of (5.26) is assumed [49, 50].

It should be pointed out that a solution to the quantum master equation is not a sufficient condition for a sensible quantum theory; there may for example still remain issues such as renormalization, unitarity and locality. These problems will not be addressed in this thesis. For articles on regularization and renormalization in the BV-framework, we refer to [51–56].

It should also be mentioned that the space of fields and antifields equipped with the antibracket, possesses a rich geometrical structure. The investigation of this structure was initiated by Witten in [57] and has since then been extended to curved supermanifolds with an odd symplectic structure [58] and the development of the theory of the so called **Q-P manifolds** and their intimate relation with topological quantum field theory [59].

5.3 Consistent Interactions as Deformations

Consider the action of a given gauge theory, depending on some fields. If we add a term depending of the same set of fields to that action, we have performed a so-called deformation of the original action. Not all possible terms that we can add are admissible, some might lead to an inconsistent gauge theory, others are admissible but do not deform the gauge stucture; for example adding a total derivative will not affect the gauge structure. The field of studying how deformations affect the gauge structure of a given theory is called

[2]This is the case since, being a second order functional differential operator, it will produce at least delta functions and in most cases also derivatives of delta functions.

[3]I.e. a regularization scheme that respects the quantum BRST symmetry.

deformation theory. The fundamental question regarding deformations of a theory is: Given an action $\overset{(0)}{S_0}[\Phi]$ with the gauge symmetries,

$$\delta_\epsilon \Phi^A = \overset{(0)}{R^A}{}_\alpha \epsilon^\alpha, \tag{5.37}$$

in what ways can we deform $\overset{(0)}{S_0}[\Phi]$,

$$\overset{(0)}{S_0} \to S_0 = \overset{(0)}{S_0} + g \overset{(1)}{S_0} + g^2 \overset{(2)}{S_0} + \dots \tag{5.38}$$

in a *consistent* manner? The parameter g above, is called the **deformation parameter** or coupling constant. The indices above the S_0's in the expansion, simply denotes the order of the deformation. In order to explain what we mean with consistent deformations and to set up a general framework for the discussion that follows, some definitions are needed:

► **Consistent deformations (5):** With a consistent deformation[4] of an action $\overset{(0)}{S_0}[\Phi]$ is meant a deformation of the type (5.38) such that the deformed gauge generators,

$$\overset{(0)}{R^A}{}_\alpha \to R^A_\alpha = \overset{(0)}{R^A}{}_\alpha + g \overset{(1)}{R^A}{}_\alpha + g^2 \overset{(2)}{R^A}{}_\alpha + \dots \tag{5.39}$$

are the gauge symmetries of the deformed action S_0,

$$\frac{\delta(\overset{(0)}{S_0} + g \overset{(1)}{S_1} + g^2 \overset{(2)}{S_2} + \dots)}{\delta \Phi^A} (\overset{(0)}{R^A}{}_\alpha + g \overset{(1)}{R^A}{}_\alpha + g^2 \overset{(2)}{R^A}{}_\alpha + \dots) = 0 \tag{5.40}$$

The last equation must be satisfied order by order in g.

The preceding definition is a natural one, since it says that consistent deformed gauge transformations should close on-shell.

► **Trivial deformations (6):** A deformation that is due to a field redefinition,

$$\Phi^A \to \Phi^A + g \overset{(1)}{\Phi^A} + g^2 \overset{(2)}{\Phi} + \dots \tag{5.41}$$

is called trivial.

Above, $\overset{(k)}{\Phi}$, denotes functions of the fields Φ and the antifields Φ^*. Trivial deformations correspond to (anti)canonical transformations in the BV-formatlism, and due to

$$\begin{aligned} \overset{(0)}{S_0} \to S_0 &= \overset{(0)}{S_0} [\Phi^A + g \overset{(1)}{\Phi^A} + g^2 \overset{(2)}{\Phi} + \dots] &(5.42)\\ &= \overset{(0)}{S_0} + g \frac{\delta \overset{(0)}{S_0}}{\delta \Phi^A} \overset{(1)}{\Phi^A} + \dots &(5.43) \end{aligned}$$

[4] There might also be additional requirements; for example if the undeformed gauge transformations are reducible one should also require the deformed set of gauge transformations to be reducible; that case will not be considered here.

we see that they vanish on-shell; they simply corresponds to a rescaling of the coupling constant of the undeformed action.

$$\overset{(0)}{S_0} \rightarrow S_0 = (1 + k_1 g + k_2 g^2 + \ldots) \overset{(0)}{S_0} \tag{5.44}$$

▶ **Rigid theory (7):** A theory is called rigid if the only consistent deformations are proportional to $\overset{(0)}{S_0}[\Phi]$ modulo field redefinitions (canonical transformations).

The rigidity of a theory means that one cannot change the gauge structure of that theory by adding any further consistent interaction terms. The requirement of consistency in combination with locality is such a severe constraint that a given theory can in general only be deformed in a very limited number of ways. Finding consistent deformations of a given action by deforming both the original gauge generators and the original action, such that (5.40) is verified, is often hard to do since the equation must hold order by order in g. An equivalent reformulation [60, 61], which guarantees consistency is to consider the deformation of the corresponding master action $\overset{(0)}{S}_M$ instead,

$$\overset{(0)}{S_M} \rightarrow S_M = \overset{(0)}{S}_M + g \overset{(1)}{S}_M + g^2 \overset{(2)}{S}_M + \ldots \tag{5.45}$$

The master equation $(S_M, S_M) = 0$ of the deformed theory will then impose consistency on S_0 and R^A_α. The antibracket (,) induces a map in the cohomology of the BRST operator Q, called the **antibracket map** [60]:

$$(. , .) : H^p(Q) \times H^q(Q) \longrightarrow H^{p+q+1}(Q) \tag{5.46}$$

Above, p and q denote ghost numbers. We should need the following result in what follows [60],

▲ **Antibracket map {6}** The antibracket map is trivial, i.e. the antibracket of two BRST-closed functions is BRST-exact.

As we will see below, this result implies that there are no obstructions to consistent deformations unless locality is imposed on the theory. Due to the next theorem, the deformation of a theory is formulation-independent:

▲ **Auxilliary fields {7}** The BRST cohomologies $H^*(Q)$ and $H^*(Q')$ associated with two formulations of a theory differing in the auxiliary field content are isomorphic. Furthermore the isomorphism i: $H^*(Q) \longrightarrow H^*(Q')$ commutes with the antibracket map.

Proofs of the two theorems above are given in [62] and [60], respectively. Consider now the expansion of the solution of $(S, S) = 0$ in terms of the deformation parameter g,

$$\begin{aligned}
(\overset{(0)}{S}_M, \overset{(0)}{S}_M) &= 0 \quad \mathcal{O}(g^0) && (5.47)\\
2(\overset{(0)}{S}_M, \overset{(1)}{S}_M) &= 0 \quad \mathcal{O}(g^1) && (5.48)\\
2(\overset{(0)}{S}_M, \overset{(2)}{S}_M) + (\overset{(1)}{S}_M, \overset{(1)}{S}_M) &= 0 \quad \mathcal{O}(g^2) && (5.49)\\
&\vdots
\end{aligned}$$

These equations imply that there are no obstructions to construct consistent deformations of a given action $\overset{(0)}{S}_M$, unless we impose additional restrictions on the master action. This is realized if we study the systems of equations above order by order in g: Equation (5.47) is valid by default since the undeformed theory is assumed to be consistent. Equation (5.49) says that $\overset{(1)}{S}_M \in H^0(Q_0)$ (an exact $\overset{(1)}{S}_M$ corresponds to a trivial deformation). Equation (5.48) is also fulfilled by default since the antibracket map is trivial and this is so for all higher order equations. We conclude that there are no obstructions to consistent deformations and the non-trivial deformations are classified by $H^0(Q_0)$. The cohomology group $H^0(Q_0)$ is in general non-empty since it is isomorphic to the space of observables for the free theory [2]. So far, the discussion about deformations has been with respect to the space of arbitrary functionals; if we require $\overset{(k)}{S}_M$ to be local functionals[5], the consistent deformations will be severely restricted [60].

Let us now assume that each deformation term $\overset{(k)}{S}_M$ is given by $\overset{(k)}{S}_M = \int \overset{k}{\mathcal{L}}$, where $\overset{k}{\mathcal{L}}$ is a local n-form. The antibracket then induces a local antibracket $(\ ,\ )_l$ via $(F_1, F_2) = \int (f_1, f_2)_n$, where $F_i = \int f_i$. From the locality it follows that equations between local functionals are only equalities modulo a total derivative: consider $F_1 = \int f_1 = 0$, then $f_1 = df_{n-1}$, where $\oint f_{n-1} = 0$ (f_{n-1} being a local $(n-1)$-form). The local form of the consistency equations (5.47)-(5.49) can now be written

$$\begin{aligned}
Q_0 \overset{(0)}{\mathcal{L}} &= d\mathcal{N}_0 && (5.50)\\
2Q_0 \overset{(1)}{\mathcal{L}} &= d\mathcal{N}_1 && (5.51)\\
Q_0 \overset{(2)}{\mathcal{L}} + (\overset{(1)}{\mathcal{L}}, \overset{(1)}{\mathcal{L}})_l &= d\mathcal{N}_2 && (5.52)\\
&\vdots && (5.53)
\end{aligned}$$

Note that above, $\overset{(k)}{\mathcal{L}}$ denotes the integrand of $\int \overset{k}{\mathcal{L}}$ and $\mathcal{N}_p$ denotes a local n-form. The non-trivial local deformations are now governed by the cohomology $H^0(Q_0|d)$. As we saw earlier $(\overset{(1)}{S}_M, \overset{(1)}{S}_M)$ is always cohomologically trivial but

[5] There might also be further restrictions such as manifest Lorentz covariance.

it is *not* in general a BRST variation of a local functional [60]. This means that $(\overset{(1)}{\mathcal{L}}, \overset{(1)}{\mathcal{L}})_l$ is not necessarily BRST-exact modulo d. The requirement of BRST-exactness modulo d for consistent deformations will then impose constraints on the coefficients in $\overset{(1)}{\mathcal{L}}$, and these constraints are in general very restrictive. For example, the only possible consistent non-trivial deformations for abelian- Yang-Mills and Chern-Simons are their non-abelian counterparts respectively [7, 60, 63, 64]. A consistent deformation can naturally be divided into three different categories, depending on what it does to the gauge algebra of the undeformed theory: Category (I) - the deformation amounts to adding terms to the action which are invariant under under gauge transformations of the undeformed action; obviously such a deformation does not deform the gauge transformations. Examples are field strengths and covariant derivatives thereof. Category (II) - the deformation alters the gauge transformations, but the additional terms added to the gauge transformations are invariant under the gauge transformations of the undeformed action. As a result the gauge algebra is not deformed to first order in the coupling constant. This is the case for the Freedman-Townsend model [65]. Category (III) The additional terms in the undeformed gauge transformations are not invariant under the original gauge transformations; this implies that the gauge algebra is deformed. This happens for instance in the case mentioned above, namely by deforming abelian- Chern-Simon or Yang-Mills into their non-abelian versions. For further results on rigidity and no-go theorems regarding consistent deformations of various models we refer to [61, 66–71].

6

A SUPERFIELD FORMULATION OF FIRST ORDER GAUGE THEORIES

In this chapter we discuss a class first order gauge field theories in the context of a superfield formulation [49, 50, 59, 72–80]. The material presented in this chapter is intimately connected to **paper IV** and parts of **chapter 7**.

6.1 Superfield Action and n-Bracket

In [50, 80], Batalin an Marnelius presented the following first order superfield action:

$$\Sigma[K^P, K^*_P] = \int_{\mathcal{M}^{2n}} d^n u d^n \tau \mathcal{L}_n(u,\tau) \tag{6.1}$$

where the Lagrangian density is given by,

$$\mathcal{L}_n[u,\tau] = K^*_P(u,\tau) D K^P(u,\tau)(-1)^{Pn} - S[K^*_P(u,\tau), K^P(u,\tau)], \tag{6.2}$$

$\mathcal{M}^{2n}$ denotes a (n,n)-supermanifold coordinatized by (even,odd) coordinates (u^k, τ^k) and where $k \in \{1, ..., n\}$. D denotes the nilpotent **de Rham** operator and S the **local master action**. The master action Σ is defined to have ghost numbers, $gh_{\#}\Sigma = 0$ and $\mathbb{Z}_2$-grading, $\epsilon(\Sigma) = 0$. It is emphasized that S only depends on the fields and not derivatives thereof. The superfields K^P and associated[1] superfields K^*_P have the following $\mathbb{Z}_2$-gradings,

$$\epsilon(K^P) = P \tag{6.3}$$

$$\epsilon(K^*_P) = P+n+1 \tag{6.4}$$

[1] The term antisuperfields is inadequate since the component expansion of an associated field contain both ordinary BV-fields and antifields.

The odd coordinates carry one unit of ghost number, $gh_{\#}(\tau^k) = 1$, and for the measure we have $gh_{\#}\, d^n\tau = -n$. The de Rham operator also carries one unit of ghost numbers $gh_{\#}D = 1$; this is manifest in the local representation $D := \tau^k \frac{\partial}{\partial u^k}$. From the ghost numbers grading of the master action it now follows that,

$$gh_{\#}K^P + gh_{\#}K^*_P = n - 1 \tag{6.5}$$

In what follows, we will always choose as a convention, $gh_{\#}K^*_P \geq gh_{\#}K^P$. As a consequence of the chosen ghost numbers- and $\mathbb{Z}_2$-gradings above, it also follows that,

$$\epsilon(S) \;=\; n \tag{6.6}$$

$$gh_{\#}S \;=\; n \tag{6.7}$$

Now that the stage is set, let us talk a little more about the master action Σ. The antibracket,

$$(A, B) = \int A(u,\tau)\frac{\overleftarrow{\delta}}{\delta K^P}(-1)^{Pn} d^n u\, d^n\tau \frac{\overrightarrow{\delta}}{\delta K^*_P} B(u,\tau) - (A \leftrightarrow B)(-1)^{(A+1)(B+1)} \tag{6.8}$$

induces a local bracket, called the **n-bracket** $(\,,)_n$, between local functionals,

$$(F, G)_n = F\frac{\overleftarrow{\partial}}{\partial K^P}\frac{\overrightarrow{\partial}}{\partial K^*_P}G - (F \leftrightarrow G)(-1)^{(F+n+1)(G+n+1)} \tag{6.9}$$

Note that the n-bracket works as an ordinary antibracket in even dimensions and as a graded Poisson bracket in odd dimensions. It possesses a number of properties similar to the ordinary antibracket. It is graded antisymmetric with respect to $F{+}n{+}1$ and $G{+}n{+}1$,

$$(F, G)_n = -(-1)^{(F+n+1)(G+n+1)}(G, F)_n, \tag{6.10}$$

it satisfies a graded version of the Jacobi-identity,

$$\sum_{cyclic}((F, G)_n, H)_n(-1)^{(F+n+1)(G+n+1)} = 0 \tag{6.11}$$

and it obeys the Leibniz rule,

$$\begin{aligned}(F, GH)_n &= (F, G)_n H + G(F, H)_n(-1)^{G(F+n+1)} \\ (FG, H)_n &= F(G, H)_n + (F, H)_n G(-1)^{G(H+n+1)}\end{aligned} \tag{6.12}$$

The n-bracket also carries $1{-}n$ units of ghost numbers and $1{-}n$ units of parity,

$$gh_{\#}(F, G)_n \;=\; gh_{\#}F + gh_{\#}G + 1 - n \tag{6.13}$$

$$\epsilon((F, G)_n) \;=\; F{+}G{+}n{+}1 \tag{6.14}$$

The equations of motion that follow from the master action (6.1) are given by,

$$\begin{aligned}DK^P &= (S, K^P)_n \\ DK^*_P &= (S, K^*_P)_n\end{aligned} \tag{6.15}$$

and they endow the de Rham operator D with a BRST-charge interpretation. Note that this form is a little bit different than the standard BV-form in (5.9) and the reason for this difference is how one defines the parities sitting in the action; Σ for example, is defined with a factor $(-1)^{np}$ multiplying the kinetic term and this will manifest itself in the equations of motion.

It is shown in [50] that Σ solves the quantum master equation $\frac{1}{2}(\Sigma, \Sigma) = \Delta\Sigma$, provided the local master action S satisfies a classical local master equation in terms of the n-bracket,

$$(S, S)_n = 0 \tag{6.16}$$

and that the boundary condition $\int d^n u\, d^n\tau\ D\mathcal{L}_n = 0$ is fulfilled. The "Laplacian" Δ used in our discussion of the superfield formalism is defined differently from the one usually used in a conventional BV-treatment (section 5.2); we refer to **paper IV** for the explicit expression of Δ for superfields. A solution for the classical master action S thus defines a consistent quantum master action Σ. There is a big advantage in working with S, since the equation (6.16) has a much simpler structure than the full quantum master equation. Note that equations (6.15) are consistent with the nilpotency of D as a result of the local master equation (6.16) for S and the Jacobi-identity (6.11). In terms of deformation theory, the master equation Σ describes all possible deformations of abelian BF-theories [60, 64, 69, 75, 81–84]. The local master action S determines the gauge structure of the original model to which Σ corresponds (how one extracts the original model from a given Σ is described below). If for example $S = 0$, the gauge structure of the corresponding original theory is that of an abelian BF-theory. Recall, that the local master action S was defined to depend on the superfields only, and not derivatives thereof[2]. From a deformation theoretic perspective, this is sufficient, since terms with derivatives will be proportional to the equations of motion of the undeformed theory and therefore such terms does not modify the gauge structure of the theory. By observing that the fields in the classical model are the ghost numbers zero components of the superfields K^P and K^*_P, one is led to the following rules for extracting the n-dimensional classical field theory, corresponding to a given master action Σ:

$$\begin{array}{rcl}
d^n u d^n \tau & \rightarrow & 1 \\
D & \rightarrow & \text{exterior derivative } d \\
K^P : gh_{\#} K^P = k \geq 0 & \rightarrow & \text{k-form field } k^P \text{where,} \\
 & & \epsilon(k^P) = \epsilon_P + k \\
K^*_P : gh_{\#} K^*_P = (n-1-k) \geq 0 & \rightarrow & (n{-}1{-}k)\text{-form field } k^*_P \text{ where,} \\
 & & \epsilon(k^*_P) = \epsilon_P + k \\
\text{all other superfields} & \rightarrow & 0 \\
\text{pointwise multiplication} & \rightarrow & \text{wedge product.}
\end{array} \tag{6.17}$$

[2]With derivatives, we mean de Rham derivatives.

6.2 Canonical Generators for the n-Bracket

A **canonical generator** for the n-bracket can be found by observing that since $(\ ,\ )_n$ carries $1-n$ units of ghost numbers and $n+1$ units of parity we must have,

$$gh_{\#}\Gamma = n-1 \tag{6.18}$$
$$\epsilon(\Gamma) = n+1 \pmod 2 \tag{6.19}$$

It is easy to see that Γ preserves the n-bracket up to second order in the parameter γ under the transformation $X \to X + \gamma(X, \Gamma)$. That is, given fields K^P and K^*_P, the transformed fields $\tilde{K}^P = K^P + \gamma(K^P, \Gamma)$ and $\tilde{K}^* = K^*_P + \gamma(K^*_P, \Gamma)$ satisfy, $(\tilde{K}^P, \tilde{K}^*_{P'}) = \delta^P{}_{P'} + \mathcal{O}(\gamma^2)$. Letting γ be an infinitesimal parameter and iterating the transformation above gives us the group transformations connected to the identity. Thus given an object F, we obtain the canonically transformed object F_Γ as,

$$F_\Gamma = e^{\mathrm{ad}\Gamma} F \tag{6.20}$$

with the adjoint action, $\mathrm{ad}\Gamma = (\ ,\Gamma)_n$. In terms of chapter (5), γ is the deformation parameter, even though canonical transformations correspond to trivial deformations. A special class of canonical generators on the form $\Gamma = K^*_{P_1} K^*_{P_2} \ldots K^*_{P_n} \Gamma^{P_1 P_2 \ldots P_n}{}_{P_{1'} P_{2'} \ldots P_{n'}} K^{P_{1'}} K^{P_{2'}} \ldots K^{P_{n'}}$ where $\forall\ P_i, P_{i'}\ :\ P_i \neq P_{i'}$ was used in **paper IV**. Such generators generate transformations of the fields which are first order in the deformation parameter γ. In **paper IV**, it was shown that the solution to the master equation for the model studied in [80],

$$\begin{aligned} S \;=\; & \frac{1}{2} T^*_{E_1} T^*_{E_2} \omega^{E_1 E_2} + \frac{1}{2} T^*_{E_1} \omega^{E_1}{}_{E_2 E_3} T^{E_2} T^{E_3} + \\ & + \frac{1}{24} \omega_{E_1 E_2 E_3 E_4} T^{E_1} T^{E_2} T^{E_3} T^{E_4}, \end{aligned} \tag{6.21}$$

is canonically equivalent to the simpler model,

$$S = T^*_{E_1} T^*_{E_2} \omega^{E_1 E_2}. \tag{6.22}$$

This means that (6.21) and (6.22) have the same gauge structure. The canonical equivalence can be shown by using the canonical generator

$$\Gamma = \frac{1}{3} \gamma\, \gamma_{E_1 E_2 E_3} T^{E_1} T^{E_2} T^{E_3} \tag{6.23}$$

in terms of which the solution to the master equation for (6.21) can be written: $\omega^{E_1}{}_{E_2 E_3} = -4\gamma \omega^{E_1 E} \gamma_{E E_2 E_3}$ and $\omega_{E_1 E_2 E_3 E_4} = 24 \gamma^2 \gamma_{E_1 E_2 E} \omega^{E E'} \gamma_{E' E_3 E_4}$. If the parities of the fields are chosen so that $\gamma_{E_1 E_2 E_3}$ is totally antisymmetric, we observe that an invertible $\omega^{E_1 E_2}$ implies that $\omega^{E_1}{}_{E_2 E_3}$ can be identified with the structure coefficients of some semi-simple Lie algebra. Canonically equivalent master actions for several other models, particularly in $d = 6$, were also analyzed in **paper IV**.

6.3 Gauge Transformations

A fruitful approach to study the gauge structure of certain field theories is to start from their master action and derive the gauge transformations of the original model[3]. By calculating the Σ-variations of the fields we get the following local relations,

$$\begin{aligned}\delta_\Sigma K^P &= (\Sigma, K^P) = (-1)^n (DK^P - (S, K^P)_n), \\ \delta_\Sigma K^*_P &= (\Sigma, K^*_P) = (-1)^n (DK^*_P - (S, K^*_P)_n).\end{aligned} \tag{6.24}$$

The gauge transformations of the original model can be obtained by the following procedure (i) calculate the Σ-variations, (ii) use the rules (6.17), (iii) replace each k-form field in every term of the Σ-variations by a $k-1$-form gauge parameter, one at a time; in case of scalar fields, i.e. fields having ghost numbers zero, set the corresponding gauge parameter to zero. Several explicit examples of this procedure are given in **chapter 7.**.

[3] I.e. the model obtained by using the rules (6.17).

7

TOPOLOGICAL GAUGE THEORIES

The developments in theoretical physics during the last three decades, can adequately be characterized as a strong interplay between advanced ideas in mathematics and physics and we think it is fair to say that this is particularly true for topological field theory (**TFT**). For example, the developments launched by the discovery of magnetic monopoles [85, 86] in Georgi-Glashow $SU(2)$ gauge theory and the study of the classical Yang-Mills and instanton equations [87], led to a tremendous increase in the understanding of the topology and geometry of manifolds. Subsequent results that followed, established the fact that there also exists deep relations between topology and quantum theory. The field of study of those relations is known as **topological quantum field theory**. The first example of a topological quantum field theory (**TQFT**) was due to Schwarz, who showed that a certain topological invariant, the Ray-Singer torsion, could be obtained from the partition function of a specific TQFT [88]. A couple of years later, Witten in his study of Morse theory in terms of supersymmetric quantum mechanics [89], constructed a different type of topological field theory. In the years that followed, Witten also proved the existence of several other TQFT-representations of different topological invariants, most notably the Donaldson-invariants [90] and the Jones Polynomials [91]. The work of Schwarz and Witten triggered an extensive activity in the field of TQFT's and many other important results have been generated by a large number of people since then. At present, the two different constructions mentioned above due to Schwarz and Witten, encompasses all known topological field theories.

7.1 The Relevance of Topological Field Theories for Physics

Despite the fact that topological field theories do not possess any local degrees of freedom, i.e. do not allow any propagating modes in the theory, they may be very important for the understanding of "real" physical theories. The lack of local degrees of freedom in topological field theories means that every TFT possesses a large enough set of local symmetries, enforcing the local degrees of freedom to be zero. Thus, given an "ordinary" non-topological field theory, we can always consider a TFT which contains the symmetries of the non-topological theory as a subset. In this sense it is clear that we can always embed a TFT into a corresponding "ordinary" theory [92]. This means for example that if we consider a non-topological quantum field theory and the set of its observables, some subset of these will possess all of the symmetries of the TFT embedded into this quantum field theory. It is obvious then that a topological version of a theory gives information about the topological sector of the corresponding physical theory, and this sector contain information about the non-perturbative properties of the theory into which it is embedded [92]. Furthermore, if we add terms to the action of a TFT that break the topological invariance, we get a theory with local degrees of freedom, an observation which leads to the possibility that TFT's may represent unbroken phases of the corresponding physical theories. Indeed, the origin of the notion of a measure of distance in nature is not explained by any theory today, where one always is forced to insert a metric in an ad-hoc fashion. There does not exist, however, any good explanation for how the phase transition from a topological to a non-topological phase should commence. Another thing that makes TFT's interesting from a physical point of view, is the fact that they constitute a class of interacting field theories which are exactly solvable and they might therefore give information about new approaches to quantum field theory.

7.2 Topological Field Theories

In this section we review the construction of topological field theories and we will see that the BRST-BV framework plays a major role for this construction. Before getting to the formal definition of a TFT, though, let us start by considering the vacuum expectation value of a general observable $\mathcal{O}$,

$$\langle \mathcal{O} \rangle \;=\; \int \mathcal{D}\Phi \, e^{-S_q} \, \mathcal{O}(\Phi) \tag{7.1}$$

Above, Φ is a collective label for all the fields and S_q denotes the **quantum action**,

$$S_q = \int_{\mathcal{M}^n} dx^n \mathcal{L} = \int_{\mathcal{M}^n} dx^n (\mathcal{L}_{cl} + \mathcal{L}_{gh} + \mathcal{L}_{gf}) \tag{7.2}$$

where $\mathcal{L}_{cl}$, $\mathcal{L}_{gh}$ and $\mathcal{L}_{gf}$ denotes the classical part, the ghost part and the gauge fixing part of S_q respectively. $\mathcal{M}^n$ denotes an n-dimensional Riemannian manifold with metric g, possibly endowed with some additional structure. The energy-momentum tensor of the theory defined by S_q is as usual given by the response of the action under infinitesimal deformations of the metric,

$$\delta_g S_q = \frac{1}{2}\int_{\mathcal{M}^n} dx^n \sqrt{g}\;\delta g^{\alpha\beta}\,T_{\alpha\beta} \tag{7.3}$$

Now, assuming no metric anomalies in the measure of the path integral, the g deformation of $\langle\mathcal{O}\rangle$ is given by,

$$\delta_g\langle\mathcal{O}\rangle = \int \mathcal{D}\Phi\, e^{-S_q}(\delta_q\mathcal{O} - \delta_q S_q\,\mathcal{O}) \tag{7.4}$$

Given the BRST-charge Q for the model S_q and the BRST-invariance of the vacuum, we observe that under the assumptions (i) that the (BRST) observable $\mathcal{O}$ has unobservable dependence of the metric, $\delta_g\mathcal{O} = \{Q, R\}$ and (ii) the BRST-exactness of the energy-momentum tensor $T = \{Q, V\}$, we have

$$\delta_g\langle\mathcal{O}\rangle = 0 \tag{7.5}$$

In other words, $\mathcal{O}$ constitutes a topological invariant in the sense that its expectation value is independent of the metric. Note that the g-independence of $\langle\mathcal{O}\rangle$ is a highly non-trivial result, since a legitimate gauge fixing $\mathcal{L}_{gf}$ must include a choice of metric on $\mathcal{M}^n$. It was the observation described above that Witten used in his monumental work considering several different TFT's at the end of the 80's. His results led to the present day BRST-BV framework for studying TFT's and the formal definition of a topological theory frequently found in the literature [90, 93]:

▶ **Topological field theory (8):** The fundamental objects in a topological field theory are: (I) A set of Grassmann $\mathbb{Z}_2$-graded fields $\{\Phi\}$ defined on a Riemannian manifold $(\mathcal{M}, g)$. (II) A Grassmann-odd, metric independent, nilpotent operator Q. (III) Physical states are defined by the Q-cohomology. (IV) The energy momentum tensor T of the theory is Q-exact: $T = \{Q, f(\Phi, g)\}$.

Now, there are some words that should be said about this definition. First of all it is a "rough" one in the sense that there are theories which are topological but which violates some of the conditions in the definition. This is in general the case for models with a more involved gauge structure, such as for example topological sigma models [93] and $d \geq 4$ non-abelian BF theories [94, 95]. In section 7.6 we will study how some of the conditions in definition(1) are violated for the higher dimensional non-abelian BF theories as a consequence of the on-shell reducibility of the gauge symmetry of those models. Also notice that in definition(1) there is no *a priori* identification of the operator Q as the BRST-charge but at present for all known models, Q has a BRST-charge interpretation and will be identified with such a charge in the following. Given a theory that verifies definition(1), we note that

$$\langle ph|H|ph'\rangle = \langle ph|\int T_{00}\ |ph'\rangle \tag{7.6}$$
$$= \langle ph|\int \{Q, V_{00}\}\ |ph'\rangle \tag{7.7}$$
$$= 0 \tag{7.8}$$

Thus, the energy of every physical state of a topological field theory is zero, which implies that there are no physical excitations. From the classical analysis regarding reparametrization invariant theories this is to be expected if the action S_q is invariant under diffeomorphisms and we integrate over all metrics. The local invariances of a TFT can be divided into two classes, depending on their nature and to that effect one can classify the set of TFT's into two categories. This classification will be the topic of the next section.

7.3 Classification of Topological Field Theories

Let us start this section with the definition of a Witten type TFT:

► **Witten type[Cohomological[1]] (9):** A topological field theory is of type Witten if the quantum action S_q is BRST-exact:

$$S_q[\Phi, g] = \{Q, V(\Phi, g)\} \tag{7.9}$$

modulo addition of topological terms.

There are two kind of symmetries in a type Witten theory: (i) the so called **topological shift symmetry** of the form $\delta\Phi = \Lambda$ for some of the fields[2], and (ii) additional local symmetries. The construction of the charge Q thus represents a combination of the two different kind of symmetries present in type Witten theories. It is the topological shift symmetry that makes it possible to interpret the BRST-charge also as generator of supersymmetry interpretation. Due to $\delta\Phi = \Lambda$, we see that to each such field corresponds another field, namely Λ which has opposite Grassmann parity. Since physical states are annihilated by Q, these corresponding fields can be interpreted as ghosts; thus to every physical field corresponds an unphysical field, thus imposing zero degrees of freedom in our theory (the explicit count of degrees of freedom will be shown explicitly for topological Yang-Mills below). Given a metric independent BRST-charge Q the BRST-exactness of the energy-momentum tensor follows from definition(2) since $T_{\mu\nu} = \{Q, \frac{2}{\sqrt{g}}\frac{\delta V}{\delta g^{\mu\nu}}\}$. Some important examples of topological field theories of Witten type are: topological Yang-Mills [90] and topological sigma models [93, 96–98], and super BF-theories. [94, 95, 99].

[1]Theories of Witten type is frequently also called, cohomological theories in the literature.
[2]Λ being a local parameter.

Let us now turn to the definition of Schwarz type theories:

► **Schwarz type[quantum[3]] (10):** A topological field theory is of type Schwarz if the quantum action S_q can be written as,

$$S_q[\Phi, q] = S_{cl}[\Phi] + \{Q, V(\Phi, q)\} \tag{7.10}$$

where the classical action S_{cl} is non-trivial, i.e. different from a total derivative and metric independent.

In contrast to Witten type theories, the BRST-charge Q corresponds to ordinary gauge symmetries in Schwarz type theories. Since one cannot interpret the ghost fields as superpartners to the physical fields in this case, one cannot establish the fact that type Schwarz theories have no local degrees of freedom as easy as for the type Witten theories. In this case one can perform a constraint analysis or calculate the number of bosonic Laplacians in the partition function, in order to obtain the number of local degrees of freedom. Thus, for type Schwarz theories the action contains enough gauge symmetries (first class constraints) as to gauge away all the degrees of freedom. In analogy to the type Witten case, given the g-independence of the BRST charge, the energy momentum tensor is easily seen to be BRST-exact: $T_{\mu\nu} = \{Q, \frac{2}{\sqrt{g}} \frac{\delta V}{\delta g^{\mu\nu}}\}$. This is so because the classical part S_{cl} of $S_q[\Phi, q] = S_{cl}[\Phi] + \{Q, V(\Phi, q)\}$ is independent of any metric. The most well known examples of type Schwarz theories are given by $d = 3$ Chern-Simons theories [88, 91, 100] and BF theories [94, 99, 101–103].

7.4 Topological Gauge Theories

In the next two sections, we will take a look at an important class of TFT's, namely topological gauge theories. With a **topological gauge theory** (**TGT**) is meant a topological field theory that contains a Lie algebra[4] (Yang-Mills) gauge symmetry as a subset of its (local) symmetries. This Lie-algebra will be denoted $\mathcal{G}$ in the following. Many of the most well known TFT's are topological gauge theories; Chern-Simons, Topological Yang-Mills and BF-theories.

7.5 Topological Yang-Mills in superfield formalism

Let us, as an example, look at the most well known TFT of Witten type, namely topological Yang-Mills (**TYM**) [90, 104],

$$S[A] = \frac{1}{4} \int d^4x \; \mathrm{Tr}\, F \wedge F \tag{7.11}$$

[3] Theories of Schwarz type is frequently also called, quantum theories in the literature.

[4] For most of the models discussed below, the gauge group can be written as $\mathcal{G} = \mathcal{G}_0 \times \mathcal{G}_1$ where $\mathcal{G}_0$ is a product of $U(1)$ factors and $\mathcal{G}_1$ is compact and semi-simple - this guarantees that the standard Yang-Mills kinetic term is positive definite.

The gauge fixed partition function generates, among other invariants, the Donaldsons polynomials [104]. The 2-form field strength is given by $F = dA + A^2$, in terms of the $\mathcal{G}$-valued connection 1-form A. The action (7.11) possesses the ordinary Yang-Mills symmetry,

$$\delta A = d_A \Lambda \tag{7.12}$$

and the topological shift symmetry,

$$\delta A = \Lambda' \tag{7.13}$$

The parameters Λ and Λ' are also $\mathcal{G}$-valued. The gauge transformations (7.12) and (7.13) are off-shell dependent since we can always choose $\Lambda' = d_A \Lambda''$. This implies that the theory is reducible. Since the gauge parameter Λ' is a 1-form field with its own gauge invariance, we must introduce the $gh_{\#}$=1 1-form odd ghost field $\mathcal{C}^{a\mu}$ and $gh_{\#}$=2 0-form even ghost-of-ghost field η^a. Due to Λ, we introduce $gh_{\#} = 1$ 0-form odd ghost field $\mathcal{C}^a$. By counting the ghost[5] degrees of freedom, we can calculate the total degrees of freedom in TYM (algebra indices suppressed below):

$$\begin{array}{rcl} \#_{dof}[\mathcal{C}^\mu] & = & -4 \\ \#_{dof}[\eta] & = & +1 \\ \#_{dof}[\mathcal{C}] & = & -1 \end{array} \tag{7.14}$$

which brings us to a total of $-4+1-1 = -4$ degrees of freedom, which exactly cancel the 4 degrees of freedom in A. Note that the action can be written as a total derivative,

$$S[A] = \frac{1}{4} \int d^n x \; d\mathrm{Tr}[\, A dA + \frac{2}{3} A^3] \tag{7.15}$$

since the $\mathrm{Tr}\, A^4$-term in (7.11) obviously vanish. In combination with BRST-exact ghost- and gauge fixing terms we verify that TYM is a theory of Witten type.

Let us now analyze this theory in terms of the superfield formalism discussed in chapter 6. The discussion below is directly related to [80] and **paper IV**. We start by first writing the action (7.11) in first order form by introducing the auxiliary Lie-algebra valued 2-form field V in addition to the usual 1-form gauge connection A.

$$S[V, A] = \int d^n x \; \mathrm{Tr}[V dA - \frac{1}{4} V^2 + V A^2] \tag{7.16}$$

A superfield action that has (7.16) as a "classical" limit is given by,

$$\Sigma = \int d^4 u \, d^4 \tau \; (T^*_E D T^E (-1)^E - \frac{1}{4} T^*_{E_1} T^*_{E_2} \omega^{E_1 E_2} + T^*_{E_1} \omega^{E_1}{}_{E_2 E_3} T^{E_2} T^{E_3}) \tag{7.17}$$

[5]Meaning the entire hierarchy of ghost fields: ghosts, ghost-of-ghosts and so on.

provided we choose the parities and ghost numbers of the fields as,

$$\begin{aligned}\epsilon(T^E) &= 1 \\ gh_{\#}(T^E) &= 1 \\ \epsilon(T^*_E) &= 0 \\ gh_{\#}(T^*_E) &= 2 \end{aligned} \tag{7.18}$$

The reason for these choices is that we want the field T to be an even 1-form after the reduction (6.17). These choices are to be understood in the rest of this section. The interaction term of (7.17) can be read off as,

$$S = \frac{1}{4}T^*_{E_1}T^*_{E_2}\omega^{E_1E_2} - T^*_{E_1}\omega^{E_1}{}_{E_2E_3}T^{E_2}T^{E_3} \tag{7.19}$$

The master action $(S,S)_n = 0$ gives us the following conditions on the coefficients ω,

$$\omega^{E_1}{}_{(E_2|E_3}\omega^{E_3}{}_{|E_4E_5)} = 0 \tag{7.20}$$

$$\omega^{(E_1|}{}_{E_2E3}\omega^{E_3|E_4)} = 0 \tag{7.21}$$

Equation (7.20) imposes the Jacobi identity on the coefficients $\omega^{E_1}{}_{E_2E_3}$. The Jacobi identity together with the antisymmetry of the lower indices, $\omega^{E_1}{}_{E_2E_3} = -\omega^{E_1}{}_{E_3E_2}$ allow us to identify $\omega^{E_1}{}_{E_2E_3}$ as structure coefficients of a Lie algebra. If we require that $\omega^{E_1E_2}$ is invertible, equation (7.21) implies a group metric interpretation for $\omega^{E_1E_2}$. In that case we have $\omega_{E_1E}\omega^{E}{}_{E_2E_3} = \omega_{E_1E_2E_3}$, where $\omega_{E_1E_2E_3}$ are totally antisymmetric in all indices, which implies that we can identify $\omega^{E_1}{}_{E_2E_3}$ as the structure coefficients of a semi-simple Lie algebra. The Σ-variations (6.24) of the fields are given by,

$$\delta_\Sigma T^E = DT^E + (\frac{1}{2}T^*_{E_1}\omega^{EE_1} - \omega^{E}{}_{E_2E_3}T^{E_2}T^{E_3}) \tag{7.22}$$

$$\delta_\Sigma T_E = DT^*_E + 2T^*_{E_1}\omega^{E_1}{}_{E_2E}T^{E_2} \tag{7.23}$$

Using the rules (6.17) $T^E \to t^E$ and $T^*_E \to t^*_E$, where t^E is an even 1-form field and t^*_E is an even 2-form field. t^E and t^*_E are the ghost number zero components of the superfields T and T^* respectively. The gauge transformations thus read,

$$\delta t^E = d\tilde{t}^E - 2\omega^{E}{}_{E_2E_3}t^{E_2}\tilde{t}^{E_3} + \frac{1}{2}\tilde{t}^*_{E_1}\omega^{EE_1} \tag{7.24}$$

$$\delta t^*_E = d\tilde{t}^*_E + 2\tilde{t}^*_{E_1}\omega^{E_1}{}_{E_2E}t^{E_2} + 2t^*_{E_1}\omega^{E_1}{}_{E_2E}\tilde{t}^{E_2} \tag{7.25}$$

Above, $\tilde{t}^E$ and $\tilde{t}^*_E$ are even 0-form and 1-form gauge parameters respectively. Looking at the right hand side of (7.24), we identity the first two terms as the ordinary Yang-Mills gauge covariant derivative with connection t,

$$d\tilde{t}^E - 2\omega^{E}{}_{E_2E_3}t^{E_2}\tilde{t}^{E_3} \leftrightarrow d_t\tilde{t} \tag{7.26}$$

and the last one as the topological shift symmetry,

$$\frac{1}{2}\tilde{t}^*_{E_1}\omega^{EE_1} \leftrightarrow \tilde{t}^* \tag{7.27}$$

Thus, from the superfield formalism we can conclude that the gauge transformation of the connection 1-form t^E is a combination of the Yang-Mills and the topological shift symmetries. If we were to eliminate the fields t^*_E by means of their equations of motion, we would have to insert the equations of motion into the gauge transformations (7.24) and (7.25), but that would not change the form of the gauge transformation for t^E, since t^*_E is not present in (7.24).

7.6 BF-theories in Superfield Formalism

The action of abelian BF theory and non-abelian BF theory is given by,

$$S_{Cl} = \int_{\mathcal{M}^n} B_p \wedge dA_{n-p-1} \tag{7.28}$$

and

$$S_{Cl} = \int_{\mathcal{M}^n} \mathrm{Tr}\ B_{n-2} \wedge F_A \tag{7.29}$$

respectively. Above, $\mathcal{M}^n$ denotes some closed orientable n-dimensional manifold. All the fields in the actions are differential forms $\in \Omega^k(\mathcal{M}, g)$, and the field strength F_A is the curvature of some flat principal $\mathcal{G}$-bundle over $\mathcal{M}$. Let us start by discussing abelian BF theory shortly. The action (7.28) is invariant under the abelian symmetries,

$$\begin{aligned} \delta B_p &= d\Lambda_{(1)p-1} &(7.30)\\ \delta A_{n-p-1} &= d\Lambda_{(2)n-p-2} &(7.31)\end{aligned}$$

and the shifts,

$$\begin{aligned} \delta B_p &= \Gamma_{(1)p-1} &(7.32)\\ \delta A_{n-p-1} &= \Gamma_{(2)n-p-2} &(7.33)\end{aligned}$$

We see from (7.30) and (7.31) that the theory is reducible for $n \geq 4$. The equations of motion are,

$$\begin{aligned} dB_p &= 0 &(7.34)\\ dA_{n-p-1} &= 0 &(7.35)\end{aligned}$$

which imply that the solution space, modulo gauge transformations, is given by the finite dimensional vector space: $\mathcal{N} = H^p_d(M) \bigoplus H^{n-p-1}_d$. The action (7.28) thus gives a field theoretic description of the de Rham complex on $\mathcal{M}$. The gauge fixing of the symmetries is described in [95] and the quantum action takes the usual BRST form: $S_q = S_{cl} + \{Q, \Psi\}$, where Q is metric independent and off-shell nilpotent. From the earlier discussion above, this establishes the fact that abelian BF is a topological field theory. The partition function Z was shown by Schwarz [88] to be related to the Ray-Singer torsion. From a superfield perspective, n-dimensional abelian BF theories are described by the kinetic terms in the masteraction: $\Sigma = \int d^n u\, d^n \tau\, K^*_P D K^P (-1)^{P+n}$. This follows from the classical limit (6.17). Abelian BF theories thus corresponds to the case when the interaction term, S, vanishes.

Let us now discuss non-abelian BF in some greater detail and see how it can be analyzed, using the superfield formalism. The action,

$$S_{Cl} = \int_{\mathcal{M}^n} \mathrm{Tr}\; B_{n-2} \wedge F_A \tag{7.36}$$

possesses the symmetries,

$$\delta A = d_A \Lambda_0 \tag{7.37}$$
$$\delta B_{n-2} = d_A \Lambda_{n-3} + [B_{n-2}, \Lambda_0] \tag{7.38}$$

and can be viewed as the zero coupling limit of Yang-Mills since,

$$\frac{1}{g^2} \int \mathrm{Tr}[F \wedge *F] = \int \mathrm{Tr}[B \wedge F - \frac{1}{2} g^2 B \wedge *B] \tag{7.39}$$
$$\xrightarrow{g \to 0} \int \mathrm{Tr}[B \wedge F] \tag{7.40}$$

The bracket in (7.38) is the ususal $\mathcal{G}$-valued bracket for differential forms. The equations of motion are given by,

$$F_A = 0 \tag{7.41}$$
$$d_A B_{n-2} = 0 \tag{7.42}$$

The fact that $n < 4$ non-abelian BF are topological can be shown straightforwardly, since the BRST-charge Q is metric independent in this case and the quantum action is a sum of the classical part and a BRST-exact part. However, in higher dimensions ($n \geq 4$), the equations of motion imply that the theory is on-shell reducible. The on-shell reducibility leads to a number of complications when it comes to establishing the fact, that non-abelian BF theories are topological in nature [94,105,106]: (i) Q is only on-shell nilpotent, (ii) Q is metric independent, (iii) the quantum action $S_q \neq S_{cl} + \{Q, \Psi\}$, (iv) the quantum action contains ghost interactions which are cubic and therefore in general metric dependent. In spite of these difficulties one can nevertheless prove the topological nature of non-abelian BF models for all dimensions [95]. Let us now go back to the action (7.36) and rewrite it in terms of the connection 1-form A,

$$S = \int \mathrm{Tr}\, B \wedge F = \int \mathrm{Tr}\; B \wedge (dA + A^2) = \int \mathrm{Tr}\; [BdA + BA^2] \tag{7.43}$$

A master action whose limit (6.17) is given by (7.43), is given by,

$$\Sigma = \int d^n u d^n \tau \; K^*_P dK^P (-1)^{P+n} + K^*_{P_1} \omega^{P_1}{}_{P_2 P_3} K^{P_2} K^{P_3} \tag{7.44}$$

Above, $gh_\# K^P$=1 and $gh_\# K^*_P$=$n-2$. We also choose the $\mathbb{Z}_2$-grading of the fields to be odd, $\epsilon(K^P)$=1. That choice guarantees that K^P is Grassmann even in the classical action obtained by reducing (7.44) according to (6.17). The interaction term is obviously on the form $S = -K^*_{P_1} \omega^{P_1}{}_{P_2 P_3} K^{P_2} K^{P_3}$ in this case and the master equation $(S,S)_n = 0$ enforces the Jacobi-identities $\omega^{P_1}{}_{(P_2|P_3} \omega^{P_3}{}_{|P_4 P_5)}$=0 on the coefficients $\omega^{P_1}{}_{P_2 P_3}$. As was said earlier, this gives them an interpretation as the structure coefficients of a Lie-algebra. In this case, though, the group metric is not supplied by the model itself and it is not in general possible to add such a term other than for $n = 2$ or $n = 4$ (see **paper IV**). The trace requires a metric and we have to consider the subset of the solutions to $(S,S)_n$=0, consisting of semi-simple Lie-algebras, if we want the master action to describe true BF-theories. The full solution to the master equation can thus be regarded as a generalized BF-theory. If we calculate the Σ-variations (6.24) we get,

$$\delta_\Sigma K^P = (-1)^n (DK^P + (-1)^P \omega^P{}_{P_2 P_3} K^{P_2} K^{P_3}) \tag{7.45}$$

$$\delta_\Sigma K^*_P = (-1)^n (DK^*_P - 2K^*_{P_1} \omega^{P_1}{}_{P_2 P} K^{P_2}) \tag{7.46}$$

which gives the following gauge transformations after the limit (6.17),

$$(-1)^n \delta k^P = d\tilde{k}^P - 2\omega^P{}_{P_1 P_2} k^{P_1} \tilde{k}^{P_2} \tag{7.47}$$

$$(-1)^n \delta k^*_P = d\tilde{k}^*_P - 2\tilde{k}^*_{P_1} \omega^{P_1}{}_{P_2 P} k^{P_2} - 2k^*_{P_1} \omega^{P_1}{}_{P_2 P} \tilde{k}^{P_2} \tag{7.48}$$

The even 1-form and $(n-2)$-form fields k^P and k^*_P are the ghost number zero components of the superfields K^P and K^*_P. $\tilde{k}^P$ and $\tilde{k}^*_P$ denote 0-form and (n-3)-form gauge parameters respectively. Looking at equation (7.47) we see that it can be written $(-1)^n \delta k^P = d_k \tilde{k}$, where d_k is the gauge covariant derivative with connection k; this is exactly the equation (7.37). The second equation (7.48) can be identified as, $(-1)^n \delta k^P = d_k \tilde{k}^* + [k^*, \tilde{k}]$ - this agrees exactly with the symmetry (7.38).

8

ALMOST PRODUCT STRUCTURES

8.1 Introduction

From a geometrical point of view, gauge theories are naturally described in the context of principal fiber bundles. However there exist more general structures in terms of which fiber bundle theory is only a special case. In this chapter we will study such a generalization, namely almost product structures (**APS**). The idea that all interactions are described by gauge theories and that a theory unifying all forces must necessarily be higher dimensional (i.e. more than four space-time dimensions) are paramount to present day research in theoretical particle physics. These two properties are included in Kaluza-Klein theories, the structure of which can be found in for example string and M-theory. The gauge theory included in Kaluza-Klein theory arises as the result of a compactification over some manifold. APS lets one study the geometrical properties of these theories without having to perform the compactification. Thus APS shed new light on both gauge- and Kaluza-Klein theories which are just special cases of APS. In this chapter we will exclusively be interested in APS on Riemannian manifolds, even though some of the definitions and theorems hold regardless of the existence of any metric. Now, the aim of this chapter is to discuss the geometry of gauge theories and Kaluza-Klein theories seen trough APS glasses. In order to do so it will be very fruitful to have some important concepts from principal bundles in mind. A principal fiber bundle is defined as the quadruple (M, P, G, π). M denotes the base space and in physics this is usually the space-time manifold. G is playing a dual role, constituting both the structure group and the fibers on M, which means that it relates different charts (coordinate frames) with each other. In physical models this is the gauge group. If one attach the group manifold to each point of the base space the result is a new manifold, which is called the total space

P. We can project from the fibers (G) down to the base manifold (physical space) via the projection π, which thus acts as a projector down to the physical space. In terms of this framework a connection on P represents the gauge field or, equivalently, the twisting of the gauge group as one moves around in M. Non-abelian (Yang-Mills) gauge field corresponds to non-trivial twisting of G on M. The existence of curvature in the base manifold is represented as the non-commutativity of covariant derivatives in P. It will also be clear from the context below, that APS provides a natural setting in which foliations and integrability are intimately connected to the gauge field, curvature and torsion. We will also notice how these quantities are expressed in terms of the fundamental constituents of APS theory. The material presented below, is a short review of the machinery used in deriving the curvature relations, presented in **paper II**. Some references relevant to APS theory are given by [107–109].

8.2 Foundation of APS

Let us start right away by looking at the definition of an almost product manifold.

▶ **Almost Product Manifold (11):** With an almost product manifold is understood a triple $(\underline{\mathcal{M}}, I, g)$, where $\underline{\mathcal{M}}$ is a manifold, I is an endomorphism on the tangent bundle, $I : T\underline{\mathcal{M}} \to T\underline{\mathcal{M}}$ which squares to one, $I^2 = \mathbf{1}$. I is called an **almost product structure** on $\underline{\mathcal{M}}$. The metric g is compatible with the almost product structure in the sense $g(X,Y) = g(IX, IY)$.

In terms of "ordinary" gauge theory, the connection splits uniquely the tangent bundle in horizontal and vertical parts respectively, $TP = H \oplus V$, in which V_x is tangent to the fiber at point x in the base manifold. The definition above induces a similar, but more general split of the tangent bundle of some manifold $\underline{\mathcal{M}}$. This can be seen by studying the following induced distributions (a k-distribution is simply a subset of k vectors from the tangent bundle $T\underline{\mathcal{M}}$).

▶ **Induced distributions (12):** On $\underline{\mathcal{M}}$, I defines two distributions $\mathcal{D}$ and $\mathcal{D}'$ in $T\underline{\mathcal{M}}$. Let,

$$\mathcal{D}_x := X \in T_x\underline{\mathcal{M}} : IX = X \tag{8.1}$$

$$\mathcal{D}'_x := X \in T_x\underline{\mathcal{M}} : IX = -X \tag{8.2}$$

then $\mathcal{D}$ and $\mathcal{D}'$ is defined by,

$$\mathcal{D} := \bigcup_{x \in \underline{\mathcal{M}}} \mathcal{D}_x, \quad \mathcal{D}' := \bigcup_{x' \in \underline{\mathcal{M}}} \mathcal{D}'_x \tag{8.3}$$

Obviously this implies the following split of $T\underline{\mathcal{M}}$

$$T\underline{\mathcal{M}} = \mathcal{D} \oplus \mathcal{D}' \tag{8.4}$$

It is because of this split that we denoted the manifold $\underline{\mathcal{M}}$ from the beginning; the bar is there to emphasize that there are two natural directions in $\underline{\mathcal{M}}$, the primed and unprimed ones. The split is generic for every relevant object in the theory. In particular it implies that there is a natural split of every tensor defined on $\underline{\mathcal{M}}$. Courtesy of the property $I^2 = \mathbf{1}$, we can define projection operators onto these two directions

$$\begin{aligned} \mathcal{P} &:= \frac{1}{2}(\mathbf{1} + I) \\ \mathcal{P}' &:= \frac{1}{2}(\mathbf{1} - I) \end{aligned} \tag{8.5}$$

These projection operators implements this split in practice. Below, it is given for the metric, but the extension to other objects is obvious

$$\underline{g}(X,Y) = g(X,Y) + g'(X,Y) \tag{8.6}$$

where the induced metrics on each distribution is given by

$$\begin{aligned} g(X,Y) &:= \underline{g}(\mathcal{P}X, \mathcal{P}Y) \\ g'(X,Y) &:= \underline{g}(\mathcal{P}'X, \mathcal{P}'Y) \end{aligned} \tag{8.7}$$

The deformation tensor H is fundamental to APS since it measures the failure of the split of the tangent bundle to split the entire manifold. This means in particular that it measures the non-integrability of the respective distributions. Below it is defined together with its irreducible parts.

▶ **Fundamental Tensors (13):** Let $\mathcal{D}$ be a k-distribution with projection P on a Riemannian manifold $\underline{M}$ with non-degenerate metric g. Let ∇ be the Levi-Civita connection with respect to g and let $\mathcal{P}' := \mathbf{1} - \mathcal{P}$ be the co-projection of $\mathcal{D}$. The following tensors can now be defined

$$H(X,Y) := \mathcal{P}'\nabla_{\mathcal{P}X}\mathcal{P}Y \tag{8.8}$$

$$L(X,Y) := \frac{1}{2}(H(X,Y) - H(Y,X)) \tag{8.9}$$

$$K(X,Y) := \frac{1}{2}(H(X,Y) + H(Y,X)) \tag{8.10}$$

$$^{\sharp}\kappa := \mathrm{tr}H \tag{8.11}$$

$$W(X,Y) := K(X,Y) - \frac{1}{k}{}^{\sharp}\kappa g(X,Y) \tag{8.12}$$

$$\tag{8.13}$$

Above H, L, K and W denotes the **deformation-**, **twisting-**, **extrinsic curvature-** and **conformation** tensor respectively. The ranks of which is given by the following characteristics,

$$H, L, K \ : \ \Lambda^1_{\mathcal{D}} \times \Lambda^1_{\mathcal{D}} \to \Lambda^1_{\mathcal{D}'} \tag{8.14}$$

$$^{\sharp}\kappa \ : \ \Lambda^1_{\mathcal{D}'} \to \mathbb{R} \tag{8.15}$$

Above $\Lambda^1_{\mathcal{D}}$, $\Lambda^1_{\mathcal{D}'}$ denotes the set of vectors in $\mathcal{D}$, $\mathcal{D}'$ respectively.
These definitions are only made for the $\mathcal{D}$ distribution but the extension to $\mathcal{D}$ is obvious. From the previous definition H can be decomposed into its irreducible parts,

$$H(X,Y) = L(X,Y) + W(X,Y) + \frac{1}{k}{}^{\sharp}\kappa g(X,Y) \tag{8.16}$$

In terms of this decomposition all distributions can be classified and it turns out that only eight different distributions exist, corresponding to the following combinations of vanishing twisting-, extrinsic curvature- and conformation tensors

Name	$L=0$	$W=0$	$\kappa=0$	Notation
Distribution				D
Minimal Distribution			x	MD
Umbilic Distribution		x		UD
Geodesic Distribution		x	x	GD
Foliation	x			F
Minimal Foliation	x		x	MF
Umbilic Foliation	x	x		UF
Geodesic Foliation	x	x	x	GF

(8.17)

Following this classification we see for example that ordinary gauge theory in terms of principal bundles correspond to (GF, GD). This is because the elements in the fibers are integrable by definition, since they form a closed group, and thereby generate a submanifold, i.e. a foliation. The tangent bundle of the base manifold on the other hand is in general not integrable, unless the gauge group is trivially twisted over it. This non-triviality is indicated by the non-vanishing of the twisting tensor L in the GD case. In the nice case of both integrable distribution and co-distribution (GF, GF), the almost product manifold actually represents a direct product of manifolds. There are three important connections associated with an almost product manifold, namely the Levi-Civita, adapted and Vidal connections. The Levi-Civita connection is defined by its action on a 1-form φ

$$\nabla\varphi(X,Y) := \frac{1}{2}(d\varphi(X,Y) + \mathcal{L}_{\sharp\varphi}g(X,Y) \tag{8.18}$$

The adapted connection $\tilde{\underline{\nabla}}$ and the Vidal connection $\tilde{\tilde{\underline{\nabla}}}$ are related to the Levi-Civita connection as

$$\tilde{\underline{\nabla}}_X Y \;:=\; \underline{\nabla}_X Y + A(X,Y), \;\; A(X,Y) := \frac{1}{2} I \underline{\nabla}_X I(Y) \tag{8.19}$$

$$\tilde{\tilde{\underline{\nabla}}}_X Y \;:=\; \tilde{\underline{\nabla}}_X Y + B(X,Y), \;\; B(X,Y) := \frac{1}{4}(\underline{\nabla}_{IY} I + I\underline{\nabla}_Y I)(X) \tag{8.20}$$

In the oriented basis (in which the notation should be obvious)

$$\underline{E}_{\bar{a}} = (\underline{E}_a, \underline{E}_{a'}) \tag{8.21}$$

satisfying

$$[E_{\bar{a}}, E_{\bar{b}}] = C_{\bar{a}\bar{b}}{}^{\bar{c}} E_{\bar{c}} \tag{8.22}$$

the connection one-forms corresponding to the three connections can be written as

$$\underline{\omega} = \left[\begin{pmatrix} \omega & H \\ -H^t & \Omega \end{pmatrix}, \begin{pmatrix} \Omega' & H' \\ -H'^t & \omega' \end{pmatrix} \right] \tag{8.23}$$

$$\tilde{\underline{\omega}} = \left[\begin{pmatrix} \omega & 0 \\ 0 & \Omega \end{pmatrix}, \begin{pmatrix} \Omega' & 0 \\ 0 & \omega' \end{pmatrix} \right] \tag{8.24}$$

$$\tilde{\tilde{\underline{\omega}}} = \left[\begin{pmatrix} \omega & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} C' & 0 \\ 0 & \omega' \end{pmatrix} \right] \tag{8.25}$$

The curvature- and the torsion tensor for an arbitrary connection is given by

$$R(X,Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \tag{8.26}$$

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \tag{8.27}$$

The explicit components of the Ricci tensor and the curvature scalar, expressed in the irreducible parts, are given in **paper II** - wherein some very useful identities among these parts, are also derived. The nice thing about the adapted- and Vidal connections, is that they commute with the almost product structure I; this can be seen directly from the expressions of the connection one-forms above

$$\tilde{\underline{\nabla}}_X I = \tilde{\tilde{\underline{\nabla}}}_X I = 0 \tag{8.28}$$

The price paid for this property is of course that they both contain nonzero torsion in general. Moreover, the Vidal connection is not metric - that is the price for having a torsion which equals the Nijenhuis tensor

$$\frac{1}{4} N_I(X,Y) = \tilde{\tilde{\underline{T}}} \tag{8.29}$$

Thus the torsion of the Vidal connection measures the non-integrability of the distributions $\mathcal{D}, \mathcal{D}'$. In terms of any endomorphism I, not only APS, one can define the I bracket as

$$[X,Y]_I := [IX,Y] + [X,IY] - I[X,Y] \tag{8.30}$$

In terms of this bracket the Nijenhuis tensor can be defined as

$$N_I[X,Y] := I([X,Y]_I) - [I(X), I(Y)] \tag{8.31}$$

which means that it measures the deviation of the I-bracket from being a Lie bracket. Let us finish this chapter with an illustrating example. Assume that we are given a Kaluza-Klein theory [110] in the (GF,GD) case

$$S = \int d^m x \sqrt{\underline{g}}\,\underline{R} = \int d^k x\, d^{k'} y \sqrt{g}\sqrt{g'} \left(\tilde{\tilde{R}} + \tilde{\tilde{R}}' + L^2 \right) \tag{8.32}$$

The last equality follows from the fact that the mean curvature- and the conformation tensor vanish in the GD-case (the primed distribution is considered to be integrable). One can now show that the gauge field strength F is given by the twisting tensor

$$F(X,Y) = \mathcal{P}'[X,Y] = 2L(X,Y) \tag{8.33}$$

which obviously imply that the L^2 term in the action above is nothing but a Yang-Mills term. Note that the field strength F above equals the Nijenhuis-tensor. Not surprisingly one identity for L, derived in **paper II**, gives now the Bianchi identity for the gauge field

$$(\underline{\tilde{\tilde{\nabla}}}_{[a}F)^i_{bc]} = 0 \tag{8.34}$$

Moreover the Einsteins equations

$$\underline{R}_{ab'} = 0 \tag{8.35}$$

imply the equations of motion for the gauge field

$$(\underline{\tilde{\tilde{\nabla}}}_c F)^{ic}_a = 0 \tag{8.36}$$

This example indicates that both gauge theory and Kaluza-Klein theory is contained within APS. With the APS framework one can even treat generalized Kaluza-Klein theories in which no restrictions on the fibers are made.

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Notation

$$
\begin{aligned}
T\mathbf{M} &= \text{Tangent bundle of } \mathbf{M} \\
T^*\mathbf{M} &= \text{Co-tangent bundle of } \mathbf{M} \\
\mathcal{L} &= \text{Lie-derivative} \\
\complement &= \text{Complement} \\
\omega &= \text{Symplectic two-form} \\
Q &= \text{BRST charge} \\
\mathrm{Eig} &= \text{Eigenspace} \\
\mathrm{Ker} &= \text{Kernel} \\
gh_{\#} &= \text{Ghost number} \\
\hookrightarrow &= \text{Inclusion map} \\
\Omega^k(\mathbf{M}) &= \text{Space of all k-forms on } \mathbf{M} \\
i &= \text{Interior product} \\
\mathcal{C} &= \text{Ghost} \\
\mathcal{P} &= \text{Ghost momenta} \\
\bar{\mathcal{C}} &= \text{Antighost} \\
\bar{\mathcal{P}} &= \text{Antighost momenta} \\
X_f &= \text{Vector field induced by function } f \\
\mathcal{F}L &= \text{Legendre transformation} \\
\Sigma &= \text{Constraint surface, Masteraction} \\
\mathcal{G} &= \text{Group} \\
\mathcal{G}_x &= \text{Orbit of } x \text{ under group } G \\
\epsilon(X) &= \text{Grassman parity of object X} \\
{}^* &= \text{Complex conjugate} \\
{}^\dagger &= \text{Hermitian conjugate} \\
\mathrm{span} &= \text{The span of objects} \\
\mathrm{antigh}_{\#} &= \text{Antighost number} \\
\mathfrak{P} &= \text{Extended phase space} \\
\Pi^a &= \{q^k, p_k\}
\end{aligned}
$$

ψ	$=$	Gauge fixing fermion
∇	$=$	Levi-Civita connection
$\tilde{\nabla}$	$=$	adapted connection
$\underline{\tilde{\tilde{\nabla}}}$	$=$	Vidal connection
N_I	$=$	Nijenhuis tensor
I	$=$	Almost product structure
$\blacktriangleright$	$=$	Definition
$\blacktriangle$	$=$	Theorem, Proposal
$\frac{\overrightarrow{\partial}}{\partial}$	$=$	Left derivative
$\frac{\overleftarrow{\partial}}{\partial}$	$=$	Right derivative

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Paper I-V