

Freezing in random graph ferromagnets

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Using $T=0$ Monte Carlo and simulated annealing simulation, we study the energy relaxation of ferromagnetic Ising and Potts models on random graphs. In addition to the expected exponential decay to a zero energy ground state, a range of connectivities for which there is power law relaxation and freezing to a metastable state is found. For some connectivities this freezing persists even using simulated annealing to find the ground state. The freezing is caused by dynamic frustration in the graphs, and is a feature of the local search nature of the Monte Carlo dynamics used. The implications of the freezing on agent-based complex system models are briefly considered.

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The way a physical system approaches equilibrium is a subject of interest to both physicists and mathematicians. In order to measure thermodynamical properties of systems it is important to be certain that the system really is in equilibrium. To ensure this, in computer simulations using the Monte Carlo dynamics it is necessary to first run the simulation for a long time before measuring. The way that various properties (e.g., the energy) of the system change during equilibration is also interesting in itself, e.g., in studies of how an epidemic disease or an opinion spreads in a model of social agents.

Here we study the relaxation of the energy of ferromagnetic Ising and Potts models on random graphs using Monte Carlo simulations with the Metropolis dynamics. We find an interesting transition as the connectivity of the graph is varied. For very small connectivities, the energy relaxes exponentially fast, for intermediate connectivities the system freezes in a local minimum (with power law relaxation to it), and for graphs with large connectivities there is again exponentially fast decay.

The model under consideration is the standard Ising model with ferromagnetic interactions but with spins placed on a random graph. In graph theory terminology [1], the ensemble used is $\mathcal{G}(N, M)$, which consists of all graphs with N vertices and $M = \frac{1}{2} \gamma N$ randomly selected edges. On average, each node is connected to γ others; γ is the connectivity or average degree of the graph. Each edge in the graph is a ferromagnetic interaction between the two linked spins, and the energy of the model can be taken to be

$$\epsilon = \frac{1}{N} \sum_{i < j} J_{ij} (1 - \delta_{s_i}^{s_j}) = - \frac{1}{2N} \sum_{i < j} J_{ij} s_i s_j + \frac{1}{4} \gamma, \quad (1)$$

where exactly M of the J_{ij} 's are nonzero and equal to 1. Thus, ϵ counts the number of edges linking spins with different values. Note that this differs from the standard ferromagnetic Hamiltonian by a γ -dependent term. Similar models on random graphs have been used to study many different systems in biology and social science as well as in physics (e.g., Refs. [2,3]).

The model can also be viewed as a constraint satisfaction problem. Each of the M edges in the graph corresponds to a constraint that the two linked spins should be equal. A natural interpretation of this is a model of a social system where there are N agents choosing from two different opinions or activities. A link between two agents would mean that the two prefer to agree.

By relaxation of a model, we mean the behavior of the energy after a quench from a high temperature disordered spin configuration. The Monte Carlo method [4] tries to decrease the energy of the system by changing the configuration of spins locally. In the Glauber dynamics [5] used in this paper, the change is accomplished by attempting to flip a randomly selected spin. If the new spin configuration has lower energy than the old, it is accepted. If the energy is raised Δ units by the change, the new configuration is accepted with probability $\exp[-\beta\Delta]$, where $\beta = 1/T$ is inverse temperature (this is the Metropolis [4] algorithm). In temperature $T=0$ simulation, no changes that raise the energy are accepted. In most of the simulations reported here, the Mitchell-Moore additive generator (see, e.g., Ref. [6]) was used to generate random numbers. Some runs were also performed using the standard C library's `drand48()` generator; these gave the same results.

For the standard two-dimensional (2D) Ising model, with $J_{ij} = 1$ if and only if spins i and j are nearest neighbors on the square lattice, two behaviors of the relaxation are possible. If the order parameter is conserved by the dynamics, so that the magnetization of the system does not change, $\epsilon \sim t^{-1/3}$, while $\epsilon \sim t^{-1/2}$ if single spin flip dynamics are used. These behaviors can be understood by considering domains of up and down spins [7].

Since a random graph is locally treelike, it is natural to approximate the behavior of the random graph model with that of the same model on a tree. Johnston and Plecháč [8] have shown that the thermodynamical behavior of the ferromagnetic Ising model is independent of the presence of loops in a graph: it is the same on a random regular graph and on a Bethe tree with the same connectivity. Da Silva and Silva [9] studied relaxation in the Ising ferromagnet on Cayley trees, and showed that mean-field theory predicts exponential relaxation. Exponential relaxation can also be argued for easily by writing a mean-field equation for the time dependence

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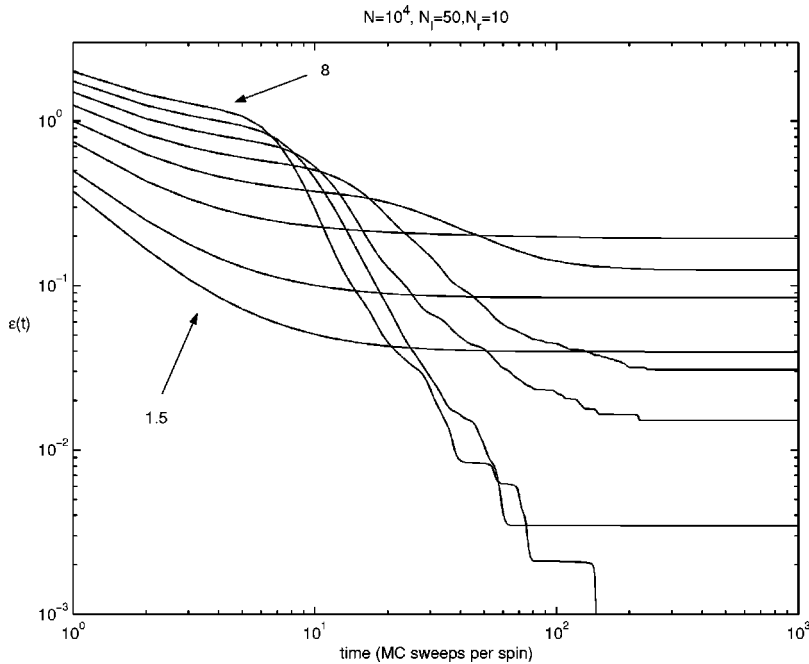


FIG. 1. The relaxation in a ferromagnetic Ising model on a random graph with 10^4 vertices, averaged over 50 graphs and 10 restarts per graph. For small (not shown) and large γ , there is a fast exponential relaxation, while the behavior for $\gamma=2$ and 3 is a power law $\epsilon = \epsilon_0 + t^{-\nu}$, with $\nu \approx 1.3$. The lower arrow indicates the curve for $\gamma=1.5$ and the upper one the $\gamma=8$ data. In between them are data for $\gamma=2, 3, 4, 5, 6$ and 7.

of a spin in terms of its nearest neighbors. Glassiness in the Cayley tree ferromagnet has been studied by Mélin *et al.* [10], who find a crossover temperature that scales inversely with the logarithm of the number of surface sites. For the random graph model considered here, this is 0, since there are no surface sites.

Figure 1 shows the relaxation behavior of ϵ for $\gamma=1.5$ to 8 and graphs of size 10^4 . All data were averaged over 50 different graphs, and the Monte Carlo (MC) algorithm was restarted in 10 different initial spin configurations for each graph. In order to check self-averaging, we also made runs with averages over 5 graphs and 100 initial configurations, and 500 graphs and 1 initial configuration, and found no differences. Error bars were determined to be on the order of 10^{-3} or smaller. The figure shows that large γ 's cause very fast relaxation to the ground state, while the system freezes for intermediate values of γ . For very small γ 's, the relaxation is of course still fast (not shown in the figure). The behavior for intermediate values of γ is thus different from the treelike models. We have also obtained similar results using ferromagnetic k state Potts models on random graphs.

This behavior can be explained qualitatively by noting that even though the ferromagnetic models always have a ground state with zero energy, it is possible for the $T=0$ Monte Carlo algorithm (and all other local search methods) to get stuck in a local minimum. The simplest case where this can happen is when there is a link between two nodes that have different values and each of the nodes have two neighbors with the same value, see Fig. 2. Because there is only one path between the up and down domains in this figure, it is not possible to lower the energy by flipping a spin. Thus, even though the model itself is solvable and contains no frustration, the dynamics gives rise to dynamical frustration for local search methods. (Very recently, Spirin *et al.* [11] have found freezing to a blinker state in the 3D Ising model. Blinkers will appear in the random graphs stud-

ied here too, but because of the relatively small connectivities at which the freezing appears it is more likely that it is caused by subgraphs such as those shown in Fig. 2.)

If there are sufficiently many edges between the up and down domains, the relaxation will be fast. No dynamical frustration will occur and one dominant value will spread quickly through the graph. If there are few edges, this will take longer, and different values will dominate different parts of the graph. This makes it plausible that introducing a metric and adding a restriction to the range of the edges could cause changes in the relaxation. This conjecture is confirmed by simulations of a model where the spins are arranged on a chain and edges only allowed between spins whose distance is less than αN , where α is independent of N . The large γ behavior is now power law relaxation and freezing. This is similar to the behavior of the antiferromagnetic Ising and Potts models on a random graph [12]. This is a model for the graph coloring problem, a combinatorial optimization problem that is NP complete [13], meaning that its worst case instances in all likelihood require exponential time to solve on a deterministic Turing machine.

Returning to the model with no restrictions on the edges, Fig. 3 shows the value of ϵ after 10^3 MC steps per spin as a function of γ and for system sizes ranging from 50 to 10^4 ,

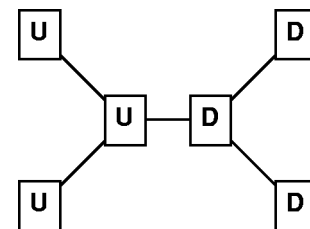


FIG. 2. An example of a configuration that can cause freezing due to insufficient clustering. U and D denote up and down spins, respectively.

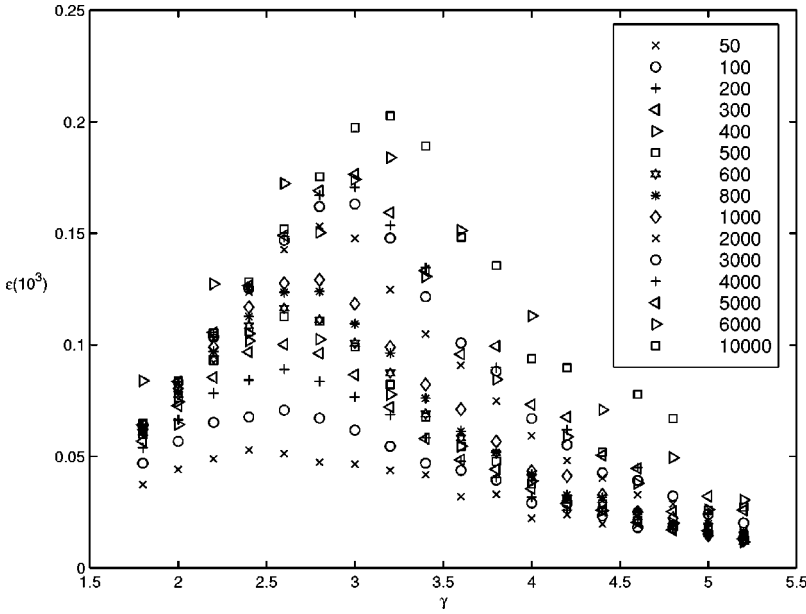


FIG. 3. This figure shows ϵ after 10^3 MC steps per spin as a function of γ for system sizes N ranging from 50 to 10^4 .

determined using about 100 different graphs and about 50 runs for each graph. The freezing region can be seen clearly. The error bars of the results shown in this and the other figures were small, typically on the order of or smaller than the symbols used to plot the data.

For large N , it is possible to fit all the data from Fig. 3 on a universal curve. Figure 4 shows the energy for large N , rescaled so that the maximum is 1, as a function of a rescaled parameter

$$\hat{\gamma} = \frac{\gamma - \gamma_0}{\Delta^\pm}, \quad (2)$$

where γ_0 is the location of the maximum and Δ^\pm was calculated so that $\hat{\gamma} = \pm 1$ marks the points where the energy attains half its maximum value. Note that the original data

are nonsymmetric around their maxima: when calculating $\hat{\gamma}$ we divided by different factors right and left of the maximum. To determine the locations of the maximum and half-maximum points, a cubic spline fit of the data was used; the plot shows the original data points.

Since the energy barrier surrounding the local minimum shown in Fig. 2 is small, it is likely that finite temperature MC simulations would not show the same behavior. To test this, we have also tried simulated annealing [14] on the problem. Simulated annealing starts at a high temperature and then gradually decreases it during the simulation. We used a linear decrease in temperature, $T(t) = T_1 - kt$, where k was chosen so that the simulation ends at zero temperature.

Figure 5 compares the values of ϵ after 10^3 MC steps per spin for the $T=0$ MC algorithm with those obtained using simulated annealing with start temperature $T_1 = 0.1, 0.2$, and

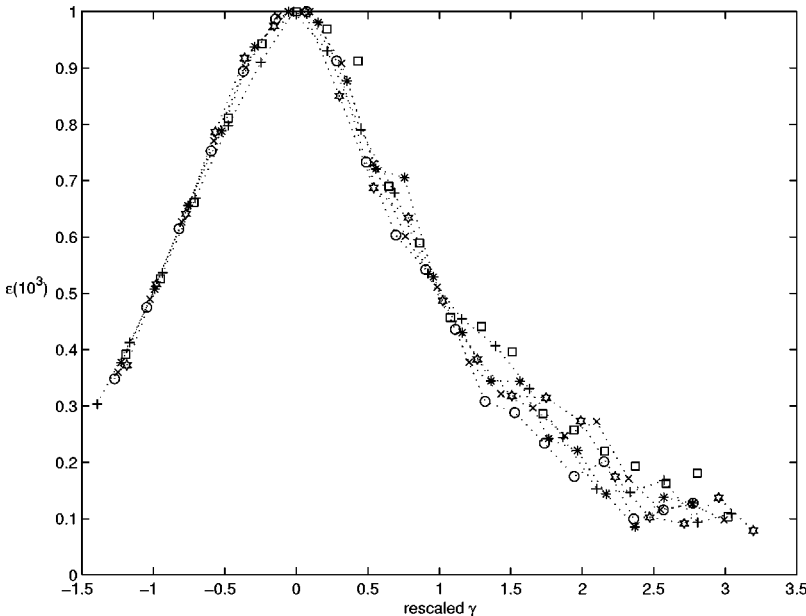


FIG. 4. Some of the data from Fig. 3, rescaled by the maximum energy for each N and plotted as a function of a rescaled parameter $\hat{\gamma}$ described in the text.

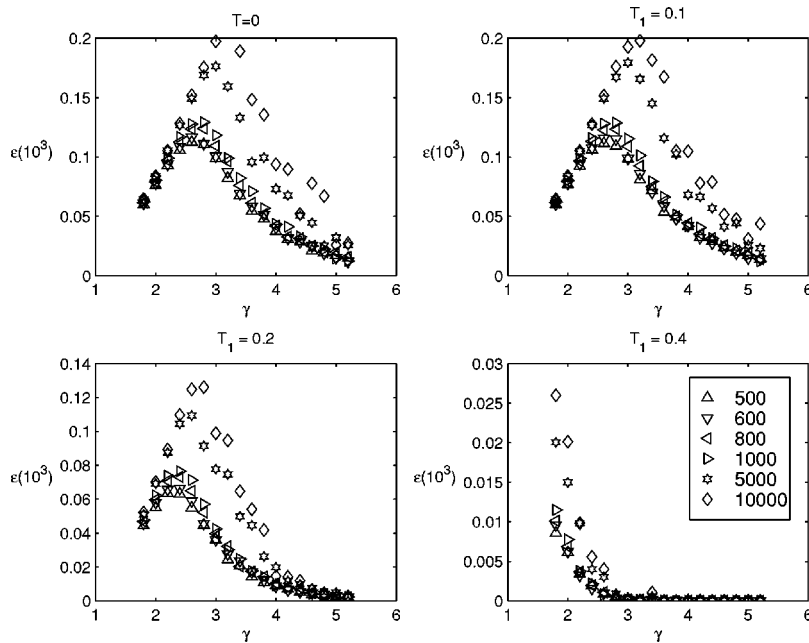


FIG. 5. This figure shows the energy after 10^3 MC steps per spin for system sizes from 500 to 10^4 using $T=0$ MC simulation (top left) and simulated annealing with start temperature $T_1 = 0.1, 0.2,$ and 0.4 (top right, bottom left, bottom right) and end temperature 0. Most of the freezing effect disappears using simulated annealing, but even for the $T_1=0.4$ runs the algorithm was unable to find the ground state for $\gamma \sim 2$.

0.4 and the same averaging as in the previous runs. Most of the freezing disappears in these runs, but there is a region of γ 's for which it remains. Note that the percolation threshold for random graphs is at $\gamma=1$, well below the freezing.

In conclusion, we presented results from Monte Carlo and simulated annealing studies of the ferromagnetic Ising model on random graphs. We find different regions of behavior of $\epsilon(t)$ —the expected exponential relaxation but also some regions where there is power law relaxation. More importantly, freezing was found in the model. The freezing persisted even for some simulated annealing runs, but almost disappears for large start temperatures. The freezing is a feature of the local search and hill-climbing characteristics of the MC method

used. This has implications for the study of models of choice-making agents on random graphs: for some connectivities it is not possible to reach a consensus or the globally most effective solution by using only local information. There are intriguing similarities and differences between this model and the corresponding antiferromagnetic model studied elsewhere; these could be studied further by examining the model where there are mostly ferromagnetic bonds but with some probability p of instead having an antiferromagnetic bond.

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