Damage spreading in small world Ising models

Pontus Svenson*
Institute for Theoretical Physics, Chalmers University of Technology and Göteborg University, SE-412 96, Gothenburg, Sweden

Desmond A. Johnston†
Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, Scotland

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We study damage spreading in the ferromagnetic Ising model on small world networks using Monte Carlo simulation with Glauber dynamics. The damage spreading temperature T_d is determined as a function of rewiring probability p for small world networks obtained by rewiring the two-dimensional square and three dimensional cubic lattices. We find that the damage for different values of p collapse onto master curves when plotted against a rescaled temperature and that the distance between T_d and the critical temperature T_c increases with p. We argue that when using the Ising model to study social systems, it is necessary to place the spins on a small world network rather than on a regular lattice.

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INTRODUCTION

The Ising model is one of the most important models of statistical mechanics. It and its generalizations have been used to model a variety of natural phenomena, ranging from biology to computer science and social science (e.g., [1–4]). For instance, many social systems can be modeled by letting spin up/down denote different opinions or preferences. In such models, a ferromagnetic interaction is interpreted as two people who prefer to agree, while an antiferromagnetic interaction means that they want to disagree. A magnetic field adds a bias that can be interpreted as ''prejudice'' or ''stubbornness,'' while the randomness induced by a finite temperature can be seen as ''free will.''

Damage spreading is a tool for studying the influence of perturbations on the equilibrium state of a system. It has been used to determine some properties of the energy landscape for disordered spin systems [5], and also has great uses for playing ''what if''-type scenarios in models of complex systems. For a voter model, for instance, damage spreading studies how much influence a (small) set of voters can have over the final outcome of the election. Damage spreading was first used by Kauffman [6] as a tool for studying biologically motivated dynamical systems, but has also since found widespread use in physics (e.g., [7]).

Damage spreading works by duplicating an equilibrium spin configuration of a system and changing a fraction d_0 of the spins. Both systems are then subjected to the same thermal noise and the distance between them is calculated. In Monte Carlo simulations, both systems are simulated simultaneously: the same spin is selected for spin flip in both systems, and the same random number (''thermal noise'') is used to determine whether an energy-raising flip should be performed.

After equilibrating both systems, the Hamming distance (the number of different spins) between the spin configurations S^α and S^β

$$h(S^\alpha,S^\beta)=\frac{1}{N}\sum_i\,(1-\delta_{S_i^\alpha}^{S_i^\beta}) \tag{1.1}$$

(where δ is the Kronecker delta function) is measured. The Hamming distance can also be expressed in terms of the Parisi overlap [8]

$$q=\frac{1}{N}\sum_i\,S_i^\alpha S_i^\beta=1-2h. \tag{1.2}$$

Most of the work on both spin models and damage spreading places the spins either on a finite-dimensional lattice or on a random graph. Here we instead use small world graphs [9,10] to study the ferromagnetic Ising model on graphs interpolating between two- and three-dimensional simple cubic lattices and random graphs with the same connectivity. The Hamiltonian of our model is

$$H=-\sum_{i<j}\,J_{ij}S_iS_j, \tag{1.3}$$

where J_{ij} is 1 if and only if there is an edge between spins i and j and 0 otherwise.

Small world graphs are intermediate between a regular lattice and a random graph; they have previously been used to study, e.g., computation, diffusion, and spreading of diseases. The original motivation for studying small worlds is that they possess both small diameters (like a random graph [11]) and a high degree of clustering (like a regular lattice). For examples of real world networks with small world characteristics and reviews of previous work, see, e.g., [10,12–14].

The small world is constructed by considering in turn all the edges (i,j) of a lattice and with some probability p re-

*Present address: Department of Data and Information Fusion, Division of Command and Control Warfare Technology, Swedish Defence Research Agency, SE-172 90, Stockholm, Sweden. Electronic address: tfkps@fy.chalmers.se

†Electronic address: des@ma.hw.ac.uk

placing it with a random edge (i,k). The rewiring parameter p thus determines how many of the links are removed and can be used to interpolate between the regular lattice and a random graph. Note that the small world for $p=1$ differs slightly from a random graph, since all nodes are guaranteed to have a local connectivity of at least $z/2$, where z is the connectivity of the regular lattice. The distribution of connectivities is broader for the small world with $p=1$. We chose to a the small world model where links are rewired and not one where they are added because we wanted to keep the average connectivity of the graphs the same for all p.

The use of small world graphs to study physical models has so far been limited. Barrat and Weigt [15] and Gitterman [16] have used them to study the crossover from one-dimensional (1D) to mean-field behavior for the ferromagnetic Ising model, finding a disorder-order transition at a finite temperature $T_c(p)$ for any $p>0$, provided that the system size is large enough.

Most of the work on small world networks has started by rewiring a 1D ring lattice, but here we instead use the 2D square and 3D simple cubic lattices. One reason for doing this is that while the 1D Ising model is trivial and disordered for all finite temperatures, the 2D and 3D versions are ordered below a critical temperature T_c. The 2D model can be solved exactly, while for 4D and higher dimensions, mean-field theory explains the phase transition (see, e.g., [17]). An important concept in the study of phase transitions and critical phenomena is that of universality class. Models displaying the same behavior close to T_c are said to be in the same universality class, and it turns out that there are many fewer universality classes than models. Putting spin models on small world graphs provides an opportunity to study the crossover from a finite-dimensional universality class to mean-field behavior. Here we restrict ourselves to determining T_c, but it would also be interesting to see how the critical exponents change as p is increased.

It should be noted that the small world networks used here differ from those obtained by rewiring a ring lattice in one respect: their clustering coefficient does not display the same threshold behavior as a function of p: it starts at 0 for $p=0$ (since the regular lattices used are bipartite) and then grows to the random graph value. The graphs used here are, however, still clustered in the sense that if j and k are neighbors of i, then there is a short path between them that does not pass through i.

While the emphasis in the present work is on the damage spreading behavior of the model, we also determined the critical temperature T_c for the order-disorder transition. This was done primarily in order to compare it with the damage spreading temperature T_d; the numerical accuracy of T_c is smaller than that for T_d.

The Monte Carlo (MC) method used was the standard single spin-flip Metropolis [18] algorithm. In each time step, N spin flips are attempted. For each flip attempt, a spin is randomly selected and the energy change ΔH if it is flipped is calculated. If the change in energy is negative, the spin is always flipped, otherwise it is flipped with probability $e^{-\Delta H/T}$, where T is the temperature. We also did some runs using different MC procedures (heat-bath algorithm, spin exchange, using an ordered update instead of a random). We found that using the heat-bath algorithm caused the damage to heal at temperatures close to and above T_c, while for the spin-exchange dynamics with the Metropolis algorithm the damage spreads for all temperatures. Updating the spins in order instead of randomly gives a smaller damage for all temperatures. These results agree with the results of Vojta [19–21] for the standard Ising model.

In most of the simulations, we used the Mitchell-Moore additive random number generator (see, e.g., [22] for a description). We also did some runs with the standard C library's `drand48()` generator and found the same behavior. All simulations were averaged over N_l different rewiring procedures, and for each small world graph an average over N_r independent Monte Carlo runs was performed. Typical values were $N_l=N_r=10$, but this was varied for some runs in order to check self-averaging. No significant differences in behavior were found.

Our simulation procedure was simple. After equilibrating the system (using simulated annealing), a copy is made and d_0N spins in it are flipped. Both systems are then simulated using the same random numbers to determine which spin to select and whether or not to flip it. After equilibrium has been reached again, we start measuring the damage as well as other quantities such as the magnetization and energy and their standard deviations. We used $d_0=0.01$ in all of the simulations presented here; none of the results presented are sensitive to the exact value of d_0. In order to check the dependence on initial conditions, we also performed some runs damaging a nonequilibrated system; these gave the same results.

Figure 1 shows the end damage as a function of temperature for p ranging from 0 to 1. The rewired lattice in this figure is the 2D square with $N=10^4$ spins. We tested some different system sizes and found that this seems to be a large enough number of spins that finite-size effects are minimized. The data were averaged over $N_l=10$ graphs and for each graph the Monte Carlo simulation was restarted $N_r=10$ times in order to improve numerical accuracy. Error bars for the damage in this and the following figures were determined to be at most on the order of 0.01 and in almost all cases considerably smaller. Note though that the errors increase with p, as should be expected since the averaging becomes more important for large p.

Figure 2 shows the corresponding data for the 3D lattice. The system size here is $N=8000$ and $N_l=N_r=10$ as for the 2D data.

We can define a damage spreading temperature $T_d(\epsilon)$ as the lowest temperature for which the damage d is larger than some (small) ϵ,

$$T_d(\epsilon)=\min\{T:d(T)>\epsilon\}. \tag{1.4}$$

In the limit as $\epsilon\to 0$, our $T_d(\epsilon)$ converges to the standard T_d, which is defined as the lowest temperature for which the damage is nonzero. We use a nonzero ϵ in Eq. (1.4) when determining T_d from our data because using a ϵ smaller than the error bar for the damage would lead to noise in T_d. Figures 1 and 2 clearly show that T_d increases with p, as is to

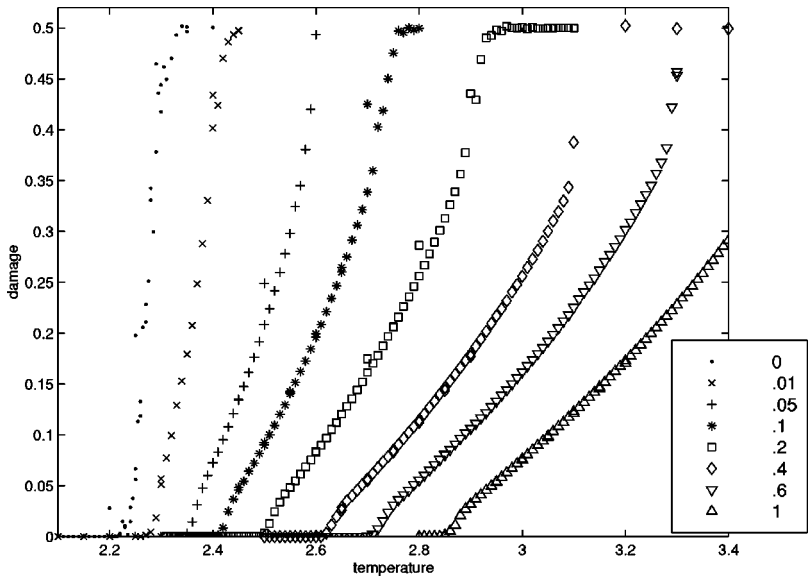

FIG. 1. The damage as a function of temperature for small world graphs obtained by rewiring a 100×100 2D square lattice with (from left to right) $p=0$, 0.01, 0.05, 0.1, 0.2, 0.4, 0.6, and 1. For each p, an average over 10 graphs and 10 restarts per graph was performed. The location of T_d shifts to higher temperatures as p is increased, and the slope of $d(T)$ decreases.

be expected. In order to quantify this, Fig. 3 compares T_d to the order-disorder transition temperature T_c for the 2D data. Figure 3 also shows T_d for $\epsilon=10^{-4}$, 10^{-3}, 10^{-2}, and 10^{-1}; it is clear that the definition of T_d is independent of ϵ for small enough ϵ's. The temperature where the damage attained its maximum value of 0.5 seems to approach T_c; this is in agreement with previous work [23]. The critical temperature T_c was determined as the temperature at which the Binder's cumulant

$$c=\frac{\langle m^4\rangle}{\langle m^2\rangle^2} \tag{1.5}$$

curves for large system sizes cross. For the 2D lattice, the largest system simulated consisted of 10^4 spins, while in the 3D case shown in Fig. 4, system sizes up to $21^3=9261$ were used to determine T_c. The error bars for T_c are larger than for T_d; note that the mean-field value for (regular) random graphs with coordination number z is $T_c=z$. The value of T_d obtained for $p=1$ here is in reasonable agreement with normal random graphs.

Table I shows the values for T_d for different p for small worlds obtained by rewiring the 2D square and 3D cubic lattices. For $p=0$, we get values in agreement with those reported in the literature [24,25,20].

Scaling plots are used to combine data from runs with different values of some parameter into one curve. In our case, we can make the data for different p fall onto the same curve by plotting the damage as a function of a rescaled temperature

$$\tilde{T}=\frac{T-T_d}{\Delta(p)}. \tag{1.6}$$

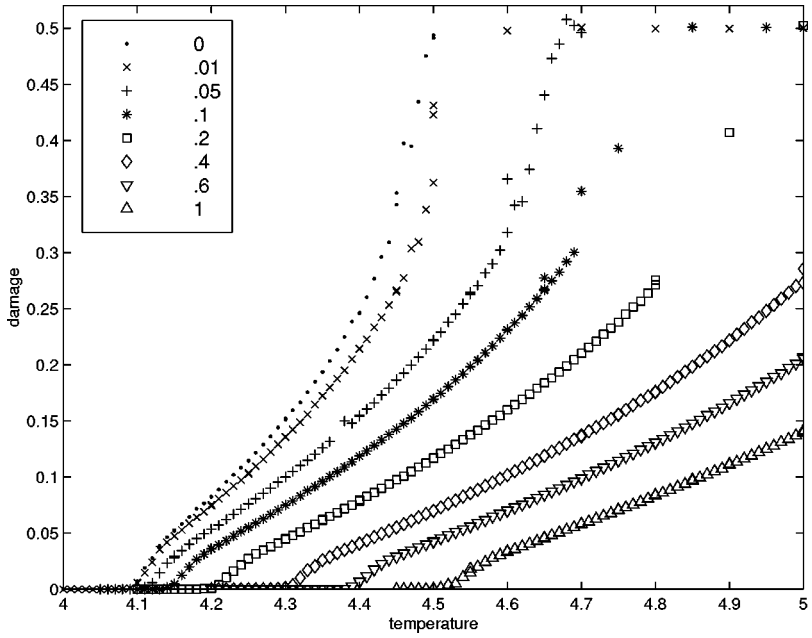

FIG. 2. The damage as a function of temperature for small world graphs obtained by rewiring a $20\times20\times20$ 3D cubic lattice with (from left to right) $p=0$, 0.01, 0.05, 0.1, 0.2, 0.4, 0.6, and 1. For each p, an average over 10 graphs and 10 restarts per graph was performed. As in the 2D case, the location of T_d shifts to higher temperatures as p is increased, and the slope of $d(T)$ decreases.

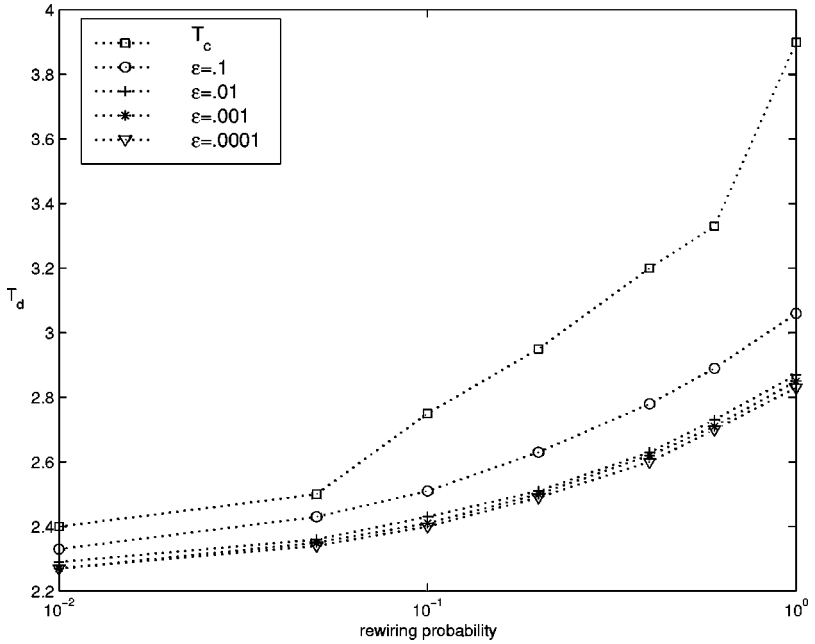

FIG. 3. For the same data as in Fig. 1, this figure shows the p dependence of T_c (squares) and T_d for some different ϵ. Note the logarithmic scale of the p axis in this plot. It is clear that T_d is independent of ϵ for small enough ϵ's.

Our scaling ansatz is that the damage can be written as

$$D(T,p)=f\left(\frac{T-T_d(p)}{\Delta(p)}\right), \tag{1.7}$$

for some f which is independent of p. In Eq. (1.7), $\Delta(p)$ is determined by the inverse of the rate at which the damage develops for different p

$$\frac{dD}{dT}(T=T_d)=\frac{1}{\Delta(p)}\frac{df}{d\tilde{T}}(\tilde{T}=0). \tag{1.8}$$

Δ is an increasing function of p; physically it tells us how much more we must increase the temperature in order to get the same increase in damage for different p:

$$\Delta T\propto\Delta(p)\Delta D. \tag{1.9}$$

The values for $\Delta(p)$ determined from the data in Figs. 1 and 2 are shown in Table II. We found a reasonable scaling $\Delta(p)\sim p^{\alpha}$ with $\alpha\approx 0.35$ for the 2D data and $\alpha\approx 0.2$ for the 3D data. The function f turns out to be linear.

Figure 5 plots the damage as a function of $\tilde{T}$ for the 2D case. A very good collapse is obtained for all $p>0$. The data for $p=0$ cannot be made to fall onto the same curve. Note

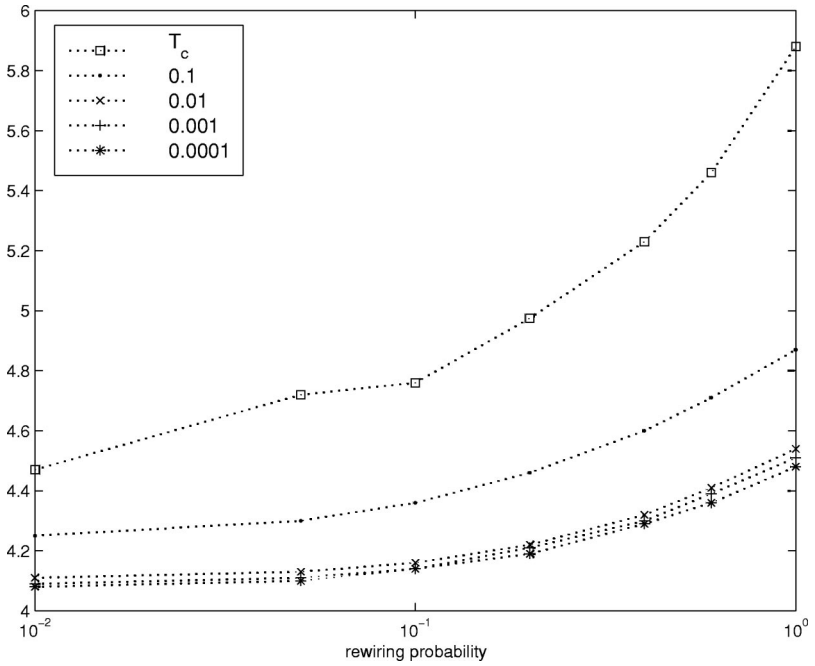

FIG. 4. T_c (squares) and T_d as a function of p for some different ϵ for the 3D case. Here, too, the values for T_d are independent of the exact value of ϵ, provided that it is small enough.

TABLE I. T_d for the small world starting from 2D and 3D lattices.

	2D									3D								
p	0	0.01	0.05	0.1	0.2	0.4	0.6	1		p	0	0.01	0.05	0.1	0.2	0.4	0.6	1
T_d	2.24	2.28	2.34	2.40	2.49	2.60	2.70	2.83		T_d	4.08	4.08	4.10	4.14	4.19	4.29	4.36	4.48

TABLE II. $\Delta(p)$ for the 2D and 3D rewired lattices.

	2D									3D								
p	0	0.01	0.05	0.1	0.2	0.4	0.6	1		p	0	0.01	0.05	0.1	0.2	0.4	0.6	1
$\Delta(p)$	0.07	0.1	0.18	0.25	0.3	0.38	0.43	0.5		$\Delta(p)$	0.28	0.3	0.36	0.41	0.5	0.55	0.61	0.7

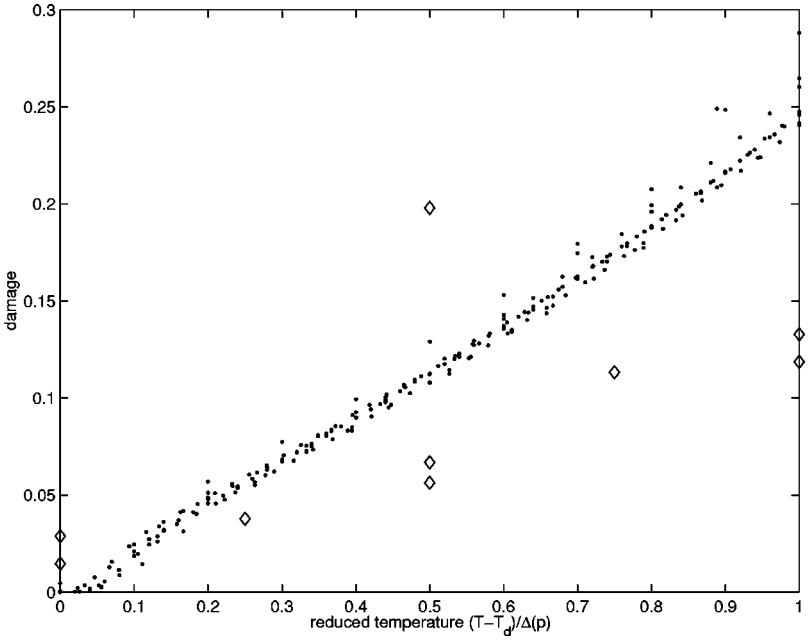

FIG. 5. The same data as in Fig. 1, plotted as a function of a rescaled temperature. By plotting the damage as a function of a reduced temperature $\tilde{T}=(T-T_d)/\Delta(p)$, it is possible to get collapse for all p except $p=0$ (small squares), which does not follow the same functional form as the other curves.

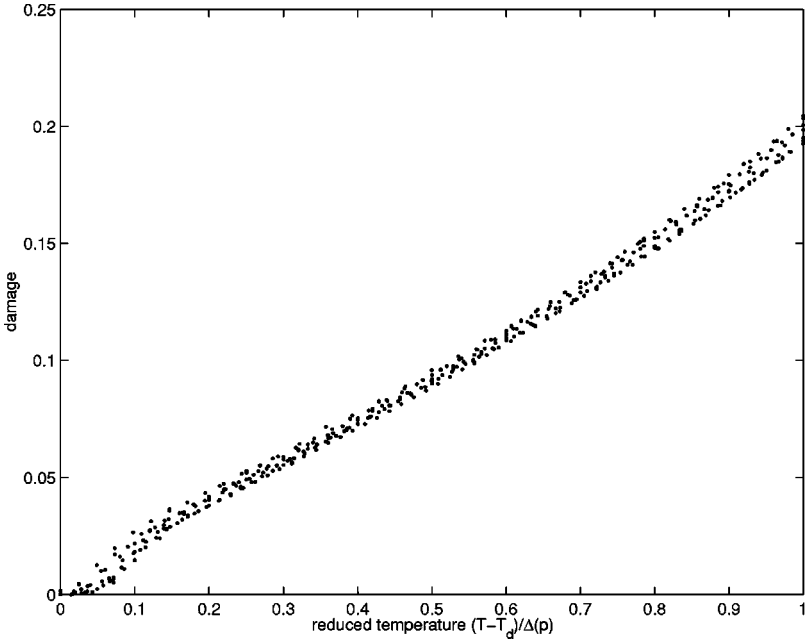

FIG. 6. In contrast to the 2D case, by plotting the damage as a function of a reduced temperature $(T-T_d)/\Delta(p)$, it is possible to get collapse for all p for the 3D data.

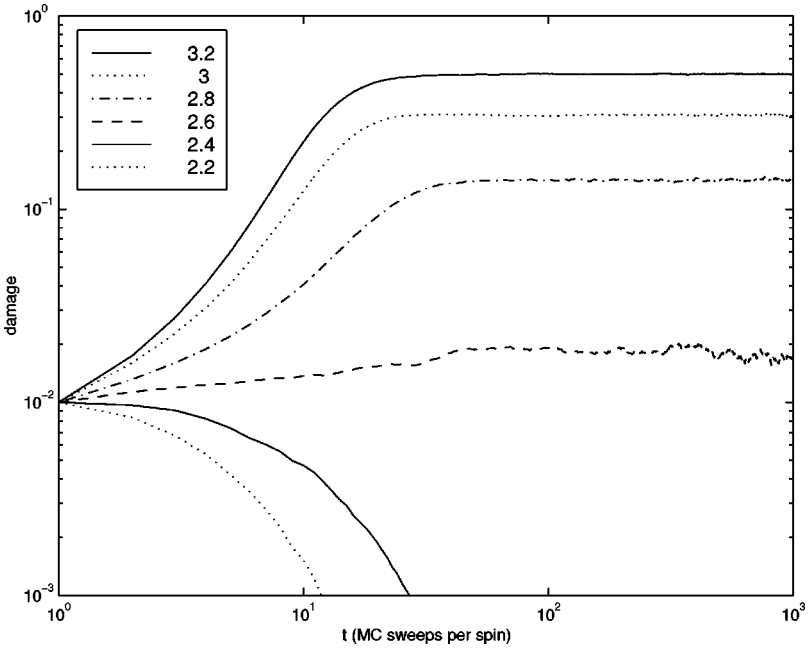

FIG. 7. The time dependence of the damage for the 2D model with $p=0.4$ and $T=2.2$, 2.4, 2.6, 2.8, and 3.0. The relaxation is exponential below T_d, and displays a power law for a short interval for $T>T_d$. The damage spreading transition takes place at $T_d \approx 2.6$.

that the distance between the master curve and the $p=0$ data is larger than the estimated error bars.

Figure 6 shows that, in contrast to the 2D case, the 3D data do collapse onto one curve for all p, including the $p=0$ (i.e., simple cubic lattice) case.

This shows some qualitative differences between the 2D and 3D lattices. The way that damage spreads in the model can be seen as a form of generalized random walk; we speculate that the difference between the 2D $p=0$ data and the other data might be related to the differences (in, e.g., return time) between random walks on 2D and 3D/random lattices [26].

We also studied the approach to equilibrium of the damaged system. Figure 7 shows the relaxation of the damage as a function of the number of complete Monte Carlo sweeps after the damage is introduced. The figure also shows data for the 2D model with $p=0.4$; the relaxation behavior for other values of p as well as for the 3D case is similar. It is clearly seen that there is a power law for a short interval above T_d.

The data can be very approximately fitted to a form $d(t) \sim t^a$ with $a \approx 1.5 \pm 0.1$ for T considerably larger than T_d and for all $p>0$. The exponent for $p=0$ is significantly different, $a \approx 1.1$.

In conclusion, we found that the damage for different small worlds falls onto a universal curve when plotted as a function of a rescaled temperature. The distance between T_d and T_c increases as a function of rewiring probability p, i.e., the range in temperature where the model is ordered but small perturbations are important increases. This is important for models of social systems, where we can interpret the temperature as a form of (random) ''free will.''

We believe that putting spin models on small world graphs provides an ideal method not only for studying social models more realistically but also for testing hypotheses regarding spin models. For instance, it is an interesting open question of how to accurately describe the ground state and low-lying excitations of the 3D $\pm J$ spin glass model. By putting this model on a small world graph and studying the crossover to the $p=1$ mean-field behavior, it might be possible to learn more about this.

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[1] P. W. Anderson, in *Emerging Syntheses in Science*, edited by D. Pines (Addison-Wesley, Reading, MA, 1984), p. 17.
[2] W. Li, in *Pattern Formation in the Physical and Biological Sciences* , edited by H. F. Nijhout, L. Nadel, and D. L. Stein (Addison-Wesley, Reading, MA, 1997), pp. 189–200.
[3] R. Dickman, e-print cond-mat/0012079.
[4] S. Galam, Physica A **238**, 66 (1997).
[5] M. Heerema and F. Ritort, Phys. Rev. E **60**, 3646 (1999).
[6] S. A. Kauffman, J. Theor. Biol. **22**, 437 (1969).
[7] H. E. Stanley, D. Stauffer, J. Kertész, and H. J. Herrmann, Phys. Rev. Lett. **59**, 2326 (1987).
[8] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
[9] D. J. Watts and Steven H. Strogatz, Nature (London) **393**, 440 (1998).
[10] D. J. Watts, *Small Worlds* (Princeton University Press, Princeton, NJ, 1999).
[11] B. Bollobas, *Random Graphs* (Academic, New York, 1985).

[12] M. E. J. Newman, J. Stat. Phys. **101**, 819 (2000).
[13] R. Albert and A-L. Barabási, e-print cond-mat/0106096.
[14] S. N. Dorogovtsev and J. F. F. Mendes, e-print cond-mat/0106144.
[15] A. Barrat and M. Weigt, Eur. Phys. J. B **13**, 527 (2000).
[16] M. Gitterman, J. Phys. A **33**, 8373 (2000).
[17] J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman, *The Theory of Critical Phenomena* (Oxford University Press, Oxford, 1992).
[18] N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller, J. Chem. Phys. **21**, 1087 (1953).
[19] T. Vojta, J. Phys. A **31**, 6595 (1998).
[20] T. Vojta and M. Schreiber, Phys. Rev. E **58**, 7998 (1998).
[21] T. Vojta, Phys. Rev. E **55**, 5157 (1997).
[22] D. E. Knuth, *Seminumerical Algorithms*, 3rd ed., The Art of Computer Programming, Vol. 2 (Addison-Wesley, Reading, MA, 1997).
[23] H. Hinrichsen, E. Domany, and D. Stauffer, J. Stat. Phys. **91**, 807 (1998).
[24] P. Grassberger, J. Phys. A **28**, L67 (1995).
[25] P. Grassberger, Physica A **214**, 547 (1995).
[26] G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes* (Oxford University Press, Oxford, 1992).