Faculty of Science

Supersymmetric Quantum Mechanics with Applications in Mathematics

Andreas Ask
Jonas Helsen
Emil Molander
Theresa Nilsson
Susanne Petterson
Peter Svensson

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15 hec

UNIVERSITY OF GOTHENBURG

Abstract

This report presents the work done during a bachelor thesis project in supersymmetry quantum mechanics with applications in mathematics. The project has, from calculation and literature study, resulted in a text whose aim is to introduce the concept of supersymmetry in a simple and pedagogical manner to third year physics students. Thus the text assumes that the reader is familiar with concepts such as quantum mechanics, analytical mechanics, linear algebra and has some knowledge of tensor calculus. Due to the target group, the pedagogical aspect of the text is important, and compared to similar texts that treat this subject a lot of effort has been done to show detailed calculations. The text is presented in the main part of this report and consists of 6 chapters. Chapter 1 is an introduction to supersymmetry and an overview of content of the following chapters. Chapter 2 and 3 cover the theory of supersymmetry in flat space and in chapter 4 and 5 curvature is applied to the theory. The last chapter is a concluding part where the mathematical results are discussed and some ideas of further theories are briefly touched upon.

What is it indeed that gives us the feeling of elegance in a solution, in a demonstration? It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the ensemble and the details.

Henri Poincaré

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About the Project

Background

After hearing the grandiose name supersymmetry thrown around in the world of physics, as a student of physics one naturally gets curious. Thus in this report we finally get to investigate the theory behind the alluring name! The building blocks of Nature, the elementary particles, come in to distinct types: the fermions, which are the matter particles, and the bosons, which carry forces between the matter particles. For instance, the electron is a fermion and it interacts with other fermions via exchanges of photons, which carry the electromagnetic force. The idea of supersymmetry is to unify these two classes of particles, by postulating that they should all come in boson-fermion pairs with the same mass and charge. The ongoing experiments at the LHC at CERN in Geneva are searching for evidence of supersymmetry in Nature, but so far the predicted superpartners have not yet been found.

Nevertheless supersymmetry has proven to be a useful tool in many fields of study e.g. the most important to us, the connection to a topological invariant (the Euler characteristic) of the manifold one is working in. So in this report we will not focus on the possible phenomenological applications of supersymmetry, but rather study a certain supersymmetric version of quantum mechanics, which turns out to have fascinating applications in mathematics. So however we want to continue our studies in physics supersymmetric quantum mechanics will be a good tool in our physical toolboxes.

Aim

The aim of our report was to transform the complicated texts on supersymmetric quantum mechanics and bring them to a level suited for a physics student in his or her third year, like ourselves. The specific parts within supersymmetry we wished to elucidate were the methods which pave the way from supersymmetric quantum mechanics to solving differential equations, via de Rham cohomology and the Atiyah-Singer index theorem. As a third year student, we expect the reader to have some basic knowledge in linear algebra, analytical mechanics and quantum mechanics. In more detail, what we wish to explain in a pedagogical manner is:

1. Why the Witten index of a supersymmetric quantum mechanics is equal to the Euler characteristic of the cohomology of the Q-operator (supercharge).
2. Why the Witten index of a supersymmetric sigma model on a curved Riemannian manifold M computes the Euler characteristic of M. This is a stronger statement than the one given in 1.
3. (A bonus if we have time to spare). Show that our results can be used in solving differential equations, through a physical proof of the Atiyah-Singer index theorem.

An effective way of knowing if we have succeeded in our aim, namely to explain the theory in a pedagogical manner, would be to test it on a third year physics student. If there are no volunteers during spring, we will get our doom in the beginning of June from our opponents.

Problem Formulation

The aim was to explain aspects of supersymmetric quantum mechanics. This brought us to the task of explicitly calculating and exposing all the steps commonly skipped in texts on supersymmetry and present them in a pedagogical manner. In the course of this, and to be able to actually calculate the steps, we had to reach deeper into classical and quantum mechanics, where we e.g needed to learn the path integral formalism. We had to compute partition functions for different models and learn mathematics as ζ-function regularisation and Poisson resummation. We also had to study the concepts behind the words in the goals above, such as indices, manifolds, cohomology, Euler characteristic etc.

It was also necessary to keep our target audience in mind and figure out how to best present and explain the material we produced.

Method

Our bachelor project is a theoretical and mathematical investigation into supersymmetry together with a presentation. Our main sources have been:

- Mirror Symmetry; Vafa, Hori et al (Only ch 10 will be covered) [1].
- Gravitation, Gauge Theories and Differential Geometry; T. Eguchi, P.B. Gilkey, A.J. Hanson (mainly ch 2-3) [6].
- Topology and Geometry for Physicists; C. Nash, S. Sen [8].
- Supersymmetric harmonic oscillator; Per Salomonson [4].

The learning process has included lectures with the supervisors on relevant topics for the project. We have had weekly meetings with them, where we discussed and presented what we have learnt from literature and our own calculations. These calculations form a vital part of the project, as you can not really understand a subject until you have checked it for yourself. We have also written typed texts about the subjects we cover, where we express the theory in our own words. These texts have been made available to all of us via a *Dropbox* account that we share. Partially they have been included in the final report. It has been important for us to reach specific goals 1 and 2 together, meaning that everyone has more or less worked through all of the parts of the final report.

Limitations

This project is limited to supersymmetric quantum mechanics and we will not encounter quantum field theory in higher dimensions. Our research has been focused on supersymmetric quantum mechanics in flat space and to some extent supersymmetric quantum mechanics with non-trivial curvature.

Results

Our studies of the supersymmetric QM and its applications in mathematics have resulted in a report, that hopefully is suited as an introduction to the field for physics students.

We start our report by establishing some results in analytical mechanics and how they carry through to quantum mechanics. We also introduce the path integral technique, presumably not known to the reader. We carry on by applying these techniques to a supersymmetric system in flat one-dimensional space and find the Witten index. To proceed with more complicated manifolds we have a chapter dedicated to introducing the necessary mathematical techniques. After this we can find the Witten index of the supersymmetric theory, and its connection to the Euler characteristic through the de Rham cohomology in a more interesting manifold. We also have a number of appendices which concern some specific calculations and proofs, omitted in the report to not disrupt the flow.

So far our work has not been read by a fellow student, but our aim to understand and calculate missing steps of the literature on supersymmetry (especially Mirror Symmetry; Vafa, Hori et al) for our part has been successful, since we were able to finish the paper intended as our goal. So our paper reflects what we all have learned during the build up of the report, since we were almost ignorant of the subject before starting. Our decision not to split the work and let separate parts of the group handle different areas of the theory, has resulted in a good understanding of more or less the entire report, by each and every one of the group. In addition to theory we have all learned mathematical techniques which will be useful in our future study of physics.

Discussion and Conclusion

The character of this bachelor project, has been like a mix of doing research and taking a course. Even though we have not done any actual research ourselves, the working methods have shown many similarities to the methods used by a practising researcher. Unlike a regular course, where the literature often slowly introduces the subject to the reader, starting off with the most fundamental, basic knowledge, and gradually introducing new concepts, the texts studied in this project has often left much unanswered. Big leaps from one equation to another have been common, and the literature have often assumed a better prior knowledge of the subject that we might have had. As a consequence, a lot of time has been spent on filling out these missing steps in the literature, and include them in our own report. We would also like to mention, that our main focus during this project has been the learning process itself. One could argue that we should have spent more time actually writing on the report, instead of doing calculations, but we feel that the subject was so comprehensive that the time we spent on studying was necessary for the quality of the report.

Before we started this project, we discussed different approaches to it. Our report would probably have contained more material, if we had chosen to divide the subject amongst each other, letting everybody focus on their own part separately. However, we were convinced that this would deeply affect our understanding of supersymmetry. We think that the broader perspective, which is already lacking when you jump into a subject like this, is of big importance and would be hurt if you do not go through all the calculations yourself. Therefore, as much as possible, everyone has done all the calculations, and we only split up the work of explaining them in the final report.

In the beginning of the project we held a series of lectures to each other and our supervisors. They were a good exercise in presentation, and they made us aware of the hard work needed to really understand something good enough to be able to explain it to someone else. But since all of us had already done the calculations that were presented each week, we felt that the time with our supervisors could be spent more efficiently, and we stopped doing this a couple of months into the project.

One of the true advantages of doing a project of this kind, is the opportunity to repeat a lot of material covered in previous courses. Classical mechanics, quantum mechanics and linear algebra are all subjects that we have improved our understanding of, all at the same time as we have been learning something new. And it is not only supersymmetry itself that was new to us. On our way to understand supersymmetry, we have learned about differential geometry, tensors, manifolds and many other useful concepts, both in physics and mathematics, that we believe will be useful in further studies at master level. This makes us feel that this was a good end to our studies at bachelor level, and hopefully it will leave us well prepared for whatever awaits us in the future.

Part II

Report

Chapter 1

Introduction

1.1 Introduction

Symmetry is a fascinating concept. The search for mathematical descriptions of the physical world has always revolved around finding patterns and invariants which bring order and regularity to the universe. Some of these patterns express themselves as symmetries, i.e. things that seem to be unchanged when one looks at them in a different manner. A prime example of a symmetry would be the rotational invariance that a spherical object, like the earth we all live on, possesses. If you were to take a spherical object in your hands and rotate it around, every part of it would look identical to you, not a single point is different. This makes describing a spherical object very simple. This simplification is true in general for symmetries and it is the prime reason to incorporate symmetries when trying to construct a description of the world around us, i.e. a physical theory. Now, in constructing such theories the language used has since long been that of mathematics. This means we better have a mathematical description of symmetry too. This particular role was eventually filled by mathematical objects called groups. The groups that describe continuous symmetries, like rotational invariance or translational symmetry, are called Lie groups, after the mathematician Sophus Lie who developed the theory in the end of the 19th century.

For a long time it was thought that all symmetries that a physical theory could possibly have were known, but in the middle of the 20th century a new possible symmetry of nature was conceived and described in a mathematical framework. Dubbed supersymmetry, it embodied a radical new idea intimately linking the bosons, which in modern particle theory are the carriers of the fundamental forces (such as the photon) and fermions, which are the particles of matter (such as protons and neutrons). The idea was that a physical theory should be invariant under a special type of transformation that interchanges every boson with a fermionic particle, and every fermion with a bosonic particle, their respective superpartner. The search for the superpartners and evidence for an actual physical manifestation of supersymmetry as a theory, is an ongoing research project at CERN institute in Geneve. Though, no such particles have yet been found, which have made physicist starting to question whether or not the superpartners actually exists

Over the following years and decades theorists played around with incorporating supersymmetry in other sub fields of physics such as quantum field theory and string theory. At some point they also tried to work out how the older theory of quantum mechanics would look like if it incorporated supersymmetry. This effort proved successful because even though there seemed to be no direct connection between supersymmetry and physical particles in quantum mechanics it did prove to be a fruitful ground for theoretical considerations. It spawned numerous new methods that often were simpler and more elegant than earlier theories and sometimes problems that were difficult to calculate even approximately suddenly allowed for simple or even exact solutions using the ideas of supersymmetry. Theorists would also find a deep connection between supersymmetric quantum mechanics and the seemingly completely detached mathematical fields of differential geometry and topology. It is this connection, and more precisely the connection between the Witten index (after Edward Witten) and a topological invariant called the Euler characteristic that we will look into in this report. The

Witten index is a quantum mechanical operator that counts the relative amount of fermionic and bosonic states a quantum mechanical system possesses. It is connected to a concept in the seemingly detached field of topology called the Euler characteristic. This Euler characteristic is a topologically invariant number. This means that it is unchanged under continuous deformation of the topological space it is defined on. In our report this space will always be a smooth Riemannian manifold. The Euler characteristic is a measure of the relative amount of odd and even dimensional objects that 'make up' the manifold. A topological invariant can be very difficult to calculate using conventional techniques but it turns out that they sometimes have a natural interpretation in physics, in the case of the Euler characteristic it is the quantum mechanical Witten index. This means you can use physics to learn things about purely mathematical objects. This connection is surprising in a sense. One would at first not suspect that developing a physical theory would lead to discoveries in pure mathematics. Finding such a connection gives a sense of deep significance. It is really the melting together of two separate fields of scientific study, leading to fruitful cooperations and major advances that benefit both those fields and human knowledge in general.

1.2 Reader's Guide

In chapter 2 we begin by reviewing certain aspects of quantum theory and classical mechanics. Some of the ideas and methods introduced here, like the Lagrangian and Hamiltonian formalisms and the quantum mechanical operator formalism, will probably be known to the reader. Others, such as the concept of path integral quantization, will be new. We work the calculations involved through thorougly, with plenty of examples and explanations. In chapter 3 we will, using the techniques from chapter 2, start to build our first supersymmetric versions of quantum mechanical systems (called sigma models, for historical reasons). These are relatively simple models, using simple target manifolds such as $\mathbb{R}$ and S^1 (the circle). However, a lot of the concepts we encounter while working these systems, such as the Witten Index and the supersymmetry charge operator Q, carry over to a more advanced setting. The most important fact we will bring with us from these systems is the connection between the Witten index and the zero states of the Hamiltonian. We will again provide plenty of explicit calculations and calculate the Witten index for some example systems in order to gain a good understanding of the behaviour of this object under different circumstances. A lot of the knowledge we gain here we will use when we start calculating more intricate sigma models. These sigma models are defined on manifolds of arbitrary curvature, so in order to successfully interpret systems like this we have to first introduce some new mathematical machinery. This we will do in chapter 4. This chapter develops the mathematical theory of differential manifolds. Differential manifolds are some of the most general mathematical objects that still possess enough structure to allow physics to develop on them. Topics that will be introduced include manifolds, differential forms, the Riemann curvature tensor and the de Rham cohomology. The de Rham cohomology is a very powerful mathematical construct that will prove to be the bridge linking supersymmetric quantum mechanics on a manifold and the topological properties of that manifold. We will provide a definition of the Euler index in terms of the de Rham cohomology through de Rham's theorem, which links the topological and differential structures of a manifold. In chapter 5 it is finally time to realise that connection. We construct a supersymmetric quantum mechanics on a compact Riemannian manifold of arbitrary curvature. We start doing this by first building a classical system using something called the superspace technique. Again we will carefully work through every calculation and provide ample explanations at every point. Once we have this classical system we can then proceed to quantum mechanics using the time-tested Dirac quantization scheme. The resulting quantum mechanical Hilbert space will prove to be intimately linked to the Hilbert space of the differential forms on our manifold. Using this link we can find an expression for the de Rham cohomology in terms of a cohomology generated by the supersymmetry charge Q. The last step of our journey consists of linking the Witten index to this Q-cohomology using its connection to the zero energy states of the sigma model. This will provide us with a final equality between the topological Euler index and the quantum mechanical Witten index.

1.3 References

As the main reference for this paper we used chapter 10 of the book 'Mirror Symmetries' [1] by the Clay Mathematical institute. Other important references, notably for the chapter on differential geometry, are the books 'Gravitation, Gauge Theories and Differential Geometry' [6] by Eguchi, Gilkey & Hanson and 'Topology and Geometry for Physicists' [8] by Nash & Sen.

1.4 Acknowledgements

We would like to acknowledge our advisers: Daniel Persson and Per Salomonson, for taking the time to answer all the questions we had and generally guiding us through the murky waters that this difficult subject sometimes seemed.

Chapter 2

Preliminaries on Quantum Mechanics

In this chapter, we encounter conserved charges in classical mechanics, and partition functions in quantum mechanics. A review of classical mechanics is necessary to make the leap to quantum mechanics surmountable. There are different ways to describe quantum mechanics. In section 2.2 we consider and compare operator formalism and path integral formalism in two examples. In the last of these examples, we will for the first time encounter a sigma model, a quantum mechanical system where the target space is by some means nontrivial. Our main references are 'Mirror Symmetry' [1] and the book of Nakahara [7].

2.1 Classical Mechanics

In the 17th century, the Newtonian mechanics were developed as a method to describe the motion of a system of particles. During the next century, an alternative method was developed, the *analytical mechanics*. In this view of physics, the concepts of energy and integrals are more important, rather than the concept of force. For advanced problems, the analytical approach is superior. We will in this section have a short look at this approach, derive Lagrange's equations of motion, and investigate the Hamiltonian formalism of mechanics. In the Hamiltonian formalism the Poisson bracket is introduced, which takes us from classical mechanics to quantum mechanics. Lagrange's equations are used to prove Noether's theorem, one of the most important theorems in physics. It relates a continuous symmetry to a conserved quantity, e.g. translation invariance in time leads to conservation of energy. In later chapters we will return to Noether's theorem to see what conserved quantities the supersymmetry gives rise to. The section ends with the Hamiltonian formulation of mechanics.

2.1.1 Lagrangian Mechanics

Consider the motion of a mechanical system in N dimensions. Introduce n *generalized coordinates* $q^\nu(t),\ \nu = 1, 2, \ldots, n$ to describe the system. Also introduce the generalised velocities $\dot{q}^\nu$ which are defined as

$$\dot{q}^\nu = \frac{d}{dt}\left(q^\nu\right). \tag{2.1}$$

Let $T(q, \dot{q})$ be the kinetic energy of the system and $V(q)$ be the potential energy (notably not depending on the generalised velocities). Then the *Lagrangian* of the system is defined as

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q). \tag{2.2}$$

Now, look at the evolution of the system during the time interval $[t_1, t_2]$, from one point $q_1^\nu = q^\nu(t_1)$ to another point $q_2^\nu = q^\nu(t_2)$. The system will take a definite path from q_1^ν to q_2^ν. This is the actual path, $q^\nu(t),\ t \in [t_1, t_2]$. Hypothetically, we could think of the system taking another path from q_1^ν to q_2^ν. Let this path be written as $q_\varepsilon^\nu(t) = q^\nu(t) + \delta q^\nu(t)$,where $\delta q^\nu(t)$ is small so that in every moment the hypothetical path is a small deviation from the real path. For some reason, $q_\varepsilon^\nu(t)$ is avoided in favour of $q^\nu(t)$. This fact is very fundamental, and often stated in terms of Hamilton's variational principle:

"The motion of a mechanical system between two points q_1^ν and q_2^ν is such that the time integral of the Lagrangian between q_1^ν and q_2^ν is stationary under infinitesimal variations of the path."

Due to this principle, it is natural to introduce a quantity S called the *action*, which is the time integral of the Lagrangian mentioned above

$$S = \int_{t_1}^{t_2} dt\ L(q, \dot{q}). \tag{2.3}$$

Hence, we may express Hamilton's variational principle in mathematical terms as $\delta S = 0$.

There is no proof for Hamilton's variational principle, except that no experiment has been performed so far that contradicts it. We may use it to derive the equations of motion of the system. In the following we use Einstein's summation convention: this implies that whenever an index (for example ν) appears once as a superscript and once as a subscript in a single term, it must be summed over. As a simple example we could for instance have

$$q^\nu p_\nu = q^1 p_1 + q^2 p_2 + \cdots + q^n p_n. \tag{2.4}$$

We will now introduce the aforementioned variation and demand that the action is stationary

$$\begin{aligned} 0 = \delta S = \delta \int dt\ L(q,\dot{q}) &= \int dt\ \delta L(q,\dot{q}) = \int dt\ L(q+\delta q, \dot{q}+\delta\dot{q}) - L(q,\dot{q}) \\ &= \int dt\ \left(\frac{\partial L}{\partial q^\nu}\delta q^\nu + \frac{\partial L}{\partial \dot{q}^\nu}\delta \dot{q}^\nu \right) = \int dt\ \frac{\partial L}{\partial q}^\nu \delta q^\nu + \frac{\partial L}{\partial \dot{q}^\nu}\frac{d}{dt}(\delta q^\nu) \\ &= \int dt\ \frac{\partial L}{\partial q^\nu}\delta q^\nu + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\nu}\delta q^\nu\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\nu}\right)\delta q^\nu \\ &= \int dt\ \left(\frac{\partial L}{\partial q^\nu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\nu}\right)\right)\delta q^\nu + \left[\frac{\partial L}{\partial \dot{q}^\nu}\delta q^\nu\right]_{t_1}^{t_2}. \end{aligned} \tag{2.5}$$

Since the endpoints of the path are assumed to be fixed, we have $\delta q_1^\nu = \delta q_2^\nu = 0$ at the endpoints. This means that the integrated term must be zero. Then we are left with the integral, which must be zero for all possible δq^ν. Then $\frac{\partial L}{\partial q_\nu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\nu}\right)$ must be identically zero for all ν. We call the equations we get *Lagrange's equations of motion*

$$\frac{\partial L}{\partial q^\nu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\nu}\right) = 0, \qquad \nu = 1, 2, \ldots, n. \tag{2.6}$$

2.1.2 Noether's Theorem

Noether's theorem, proved by the German mathematician Emmy Noether in 1915, states that for every continuous symmetry of the motion, there exists a quantity conserved in time. A continuous symmetry is an invariance of a system under a continuous transformation of the coordinates in that system. For instance, a system that has rotational symmetry (a sphere for example) is invariant under a continuous rotation of the coordinate system (a sphere looks the same no matter from what angle you look at it). Because we are talking about continuous symmetries it is enough to only work with infinitesimal coordinate transformations.

To prove Noether's theorem we consider such an infinitesimal transformation of the coordinates, characterised by: $q^\nu \to q'^\nu = q^\nu + \delta q^\nu$, where $\delta q^\nu = f^\nu(q, \dot{q})$. We again consider δq^ν to be very small (infinitesimally small). We can now posit some function $g(q, \dot{q})$ so that we are able to write the variation of the Lagrangian under this coordinate transformation as a total time derivative of this function

$$\delta L = (L(q', \dot{q}') - L(q, \dot{q})) =: \frac{d}{dt} g(q, \dot{q}). \tag{2.7}$$

For the rationale behind this we look at the variation of the action

$$\delta S = \int dt\ \delta L = \int dt\ \frac{d}{dt} g(q, \dot{q}) = g(q, \dot{q})\big|_{q_2^\nu}^{q_1^\nu} = 0. \tag{2.8}$$

So our action is manifestly invariant if we write δL this way. We can also look at δL in a different manner by means of the total differential. This gives us

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial q^\nu}\delta q^\nu + \frac{\partial L}{\partial \dot{q}^\nu}\delta \dot{q}^\nu \\ &= \left(\frac{\partial L}{\partial q^\nu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\nu}\right)\delta q^\nu + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\nu}\delta q^\nu\right) \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\nu}\delta q^\nu\right) \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\nu}f^\nu(q,\dot{q})\right),\end{aligned} \tag{2.9}$$

where we have used Lagrange's equations. If we subtract (2.7) and (2.9) we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\nu}f^\nu(q,\dot{q})\right) - \frac{d}{dt}\left(\, g(q,\dot{q})\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\nu}f^\nu(q,\dot{q}) - g(q,\dot{q})\right) = 0. \tag{2.10}$$

The quantity $Q = \frac{\partial L}{\partial \dot{q}^\nu}f^\nu(q,\dot{q}) - g(q,\dot{q})$ is then constant in time. We call Q the *conserved charge.*

We have proved Noether's theorem in classical mechanics. It generalizes to quantum field theory, but the proof is much more advanced and we will not pursue it here. Let us compute the conserved charges that follow from three symmetries.

- Example 1

  Consider a particle in $\mathbb{R}^3$ acted on by a conservative force $\mathbf{F} = -\nabla V$. If the coordinates are $x^i,\ i = 1,2,3$, then classically

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^i\dot{x}_i - V(x). \tag{2.11}$$

  If there is a symmetry under translation in time, $t \to t+\varepsilon$, where ε is some infinitesimally small number then we get the variations of the coordinates as

$$\delta x^i = x^i(t+\varepsilon) - x^i(t) = \frac{dx^i}{dt}\varepsilon = \varepsilon\dot{x}^i, \qquad f^i(x,\dot{x}) = \varepsilon\dot{x}^i. \tag{2.12}$$

  Note that ε has the dimension of time. We calculate the variation of the Lagrangian

$$\delta L = L(t+\varepsilon) - L(t) = \varepsilon\frac{dL}{dt} = \varepsilon\dot{L}, \qquad g = \varepsilon L(x,\dot{x}). \tag{2.13}$$

  This leads us to our conserved charge

$$\begin{aligned}\varepsilon Q &= \frac{\partial L}{\partial \dot{x}^i}\varepsilon\dot{x}^i - \varepsilon L(x,\dot{x}) = \varepsilon\left(m\dot{x}_i\dot{x}^i - \left(\frac{1}{2}m\dot{x}_i\dot{x}^i - V\right)\right) \\ &= \varepsilon(T+V) = \varepsilon E, \\ Q &= E,\end{aligned} \tag{2.14}$$

  where E is the total energy of the system. Translation symmetry in time thus results in conservation of energy.

- Example 2

  Now study a particle whose motion is described by the coordinates $q^\nu,\ \nu = 1,2,\ldots,n$. Suppose that L is invariant under translation of one of them, say q^1. Then

$$\delta q^\nu = \varepsilon\delta^\nu_1, \qquad f^\nu = \varepsilon\delta^\nu_1, \tag{2.15}$$

$$\delta L = L(q^1 + \delta q^1) - L(q^1) = 0, \qquad g = 0. \tag{2.16}$$

  If L is invariant under translation of q^1, then L does not depend on q^1 and $\delta L = 0$. The conserved charge becomes

$$\varepsilon Q = \frac{\partial L}{\partial \dot{q}_\nu}\delta q^\nu - 0 = \frac{\partial L}{\partial \dot{q}_1}\varepsilon = \varepsilon m\dot{x}_1, \qquad Q = m\dot{x}_1. \tag{2.17}$$

  This result also follows directly from Lagrange's equations of motion. We conclude that translation symmetry in space results in conservation of momentum.

- Example 3

  Suppose a particle moves in space $\mathbb{R}^3$ under influence of a central force $\mathbf{F} = -\nabla V(r)$. We write most easily the Lagrangian in spherical coordinates

$$L = \frac{1}{2}mv^2 - V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\ \dot{\phi}^2) - V(r). \tag{2.18}$$

  We note that no ϕ appears in the Lagrangian. This means that we have translational invariance in the $\mathbf{e}_\phi$ direction. Let us look at the following coordinate transformation:

$$\delta r = 0, \qquad \delta\theta = 0, \qquad \delta\phi = \varepsilon. \tag{2.19}$$

  Since L does not depend on ϕ

$$\delta L = L(\phi + \varepsilon) - L(\phi) = 0. \tag{2.20}$$

  Then we see that

$$\varepsilon Q = \frac{\partial L}{\partial \dot{\phi}}\delta\phi - 0 = mr^2\sin^2\theta\ \dot{\phi}\ \varepsilon, \qquad Q = mr^2\sin^2\theta\ \dot{\phi} \tag{2.21}$$

  We may rewrite Q as $Q = m(r\sin\theta)(r\sin\theta\ \dot{\phi})$, where $r\sin\theta$ is the distance from the z-axis and $r\sin\theta\ \dot{\phi}$ is the speed around the same axis. Thus Q is the classical angular momentum in the z direction. If L is invariant under translation in ϕ, that is, if we have isotropy in this thin ring of space defined by the azimuthal angle ϕ, then it leads to conservation of angular momentum in the direction around which ϕ circles.

We have seen some standard examples showing the connection between symmetry and conserved charge. In the special case of supersymmetry, we will get conserved *supercharges*. They will be an important topic from chapter 3 and onwards.

2.1.3 The Hamiltonian Formalism and Poisson Brackets

There is another way of looking at classical mechanics called the Hamiltonian formulation. As we saw before the Lagrangian formalism leads to N second order differential equations for a system with n degrees of freedom (generalized coordinates q^ν). The Hamiltonian formalism will eventually lead to $2n$ first order differential equations. This can make certain systems easier to solve and understand. The Hamiltonian formalism also leads to the notion of a Poisson bracket, a mathematical object that serves as a gateway between classical mechanics and quantum theory. We first define the *generalised conjugate momentum* to be

$$p_\nu := \frac{\partial L}{\partial \dot{q}^\nu}.$$

If we take q to be the standard coordinate x, then p is the standard momentum in the x direction, hence the name. Now we define an object called the Hamiltonian to be

$$H(p, q, \dot{q}) := p_\nu \dot{q}^\nu - L(q, \dot{q})$$

where L is the Lagrangian of the system. The Hamiltonian is a function of both the generalised coordinates and the conjugate momenta. Now consider an infinitesimal variation of H

$$\begin{aligned}
\delta H &= \dot{q}^\nu \delta p_\nu + p_\nu \delta\dot{q}^\nu - \frac{\partial L}{\partial q^\nu}\delta q^\nu - \frac{\partial L}{\partial \dot{q}^\nu}\delta\dot{q}^\nu \\
&= \dot{q}^\nu \delta p_\nu + p_\nu \delta\dot{q}^\nu - \frac{\partial L}{\partial q^\nu}\delta q^\nu - p_\nu\delta\dot{q}^\nu \\
&= \dot{q}^\nu \delta p_\nu - \frac{\partial L}{\partial q^\nu}\delta q^\nu \\
&= \dot{q}^\nu \delta p_\nu - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\nu}\delta q^\nu \\
&= \dot{q}^\nu \delta p_\nu - \dot{p}_\nu \delta q^\nu,
\end{aligned} \tag{2.22}$$

where we used Lagrange's equations of motion. But on the other hand we also have

$$\delta H = \frac{\partial H}{\partial q^\nu}\delta q^\nu + \frac{\partial H}{\partial p^\nu}\delta p^\nu. \tag{2.23}$$

Equating (2.22) and (2.23) gives us a set of $2N$ linear differential equations, namely

$$\begin{aligned} \dot{q}^\nu &= \frac{\partial H}{\partial p^\nu} \\ \dot{p}^\nu &= -\frac{\partial H}{\partial q^\nu}. \end{aligned} \tag{2.24}$$

These are called *Hamilton's equations of motion.* Now consider a physical variable A which is a function of p^ν and q^ν. If we look at the time dependence of $A(p,q)$ we get some interesting results

$$\begin{aligned} \frac{d}{dt}A(p,q) &= \frac{\partial A}{\partial p^\nu}\dot{p}^\nu + \frac{\partial A}{\partial q^\nu}\dot{q}^\nu \\ &= \frac{\partial A}{\partial q^\nu}\frac{\partial H}{\partial p^\nu} - \frac{\partial A}{\partial p^\nu}\frac{\partial H}{\partial q^\nu} \\ &=: \{A, H\}_P. \end{aligned} \tag{2.25}$$

We call this last object, defined this way, the *Poisson bracket.* It is a very important tool for elucidating the correspondence between classical and quantum physics. As an example we will calculate the Poisson bracket between the standard x coordinate and its standard momentum. This gives

$$\{x, p\}_P = \frac{\partial x}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial x}{\partial p}\frac{\partial p}{\partial x} = 1. \tag{2.26}$$

2.2 Two Ways to Approach Quantum Mechanics

The reader's first encounter with quantum mechanics was probably through the concept of operators. These correspond to classical quantities, e.g. $-i\hbar\nabla$ corresponds to the momentum of a particle in three dimensions. This *operator formalism* was developed in the 1920's. Later, another way of approaching quantum mechanics using *path integrals* was introduced, which generalizes the concept of action in classical mechanics to quantum mechanics. In this section these two formalisms will be introduced, and then visualized by working through two examples, the quantum harmonic oscillator, and the sigma model on a circle. In both examples, we get the same partition function of the system, independently of which of the two formalisms we use.

2.2.1 Operator Formalism

The principle of Hamiltonian quantization, also sometimes called the Dirac quantization scheme, essentially consists of replacing all the physical variables with operators on a Hilbert space (which might be infinite dimensional!). The calculational rules for these operators are obtained by calculating all the Poisson brackets of the variables of the system and replacing them with commutators or anti-commutators of the operator equivalents of these variables times $i\hbar$, where i is the imaginary unit and $\hbar$ is the reduced Planck constant or Dirac constant. In symbols, for given variables A and B

$$\{A, B\}_P \to i\hbar[\hat{A}, \hat{B}]. \tag{2.27}$$

The commutator and anti-commutator between two operators $\hat{A}$ and $\hat{B}$ are defined as

$$\begin{aligned} \left[\hat{A}, \hat{B}\right] &:= \hat{A}\hat{B} - \hat{B}\hat{A}, \\ \left\{\hat{A}, \hat{B}\right\} &:= \hat{A}\hat{B} + \hat{B}\hat{A}. \end{aligned} \tag{2.28}$$

The hats on $\hat{A}$ and $\hat{B}$ serve to make it clear that they are now operators. Later we will drop the hats when it cannot cause confusion. For clearer presentation we will from now on use natural units, i.e. we set the Dirac constant equal to unity and we also set the mass $m = 1$.

Example: QM harmonic oscillator in operator formalism

Let us try the operator approach to calculate the partition function for the quantum mechanical version of the simple harmonic oscillator. We define the Hamiltonian for the QM harmonic oscillator to be

$$H = \frac{p^2}{2} + \frac{x^2}{2}. \tag{2.29}$$

Here x is the position operator and p is the corresponding momentum operator (in the classical scheme this was $\dot{x}$). Using the Dirac quantization scheme together with the Poisson bracket in section 2.1.3 we obtain the following commutation rule

$$[x, p] = i. \tag{2.30}$$

Using this property we can rewrite the Hamiltonian in a form that will prove more tangible down the road

$$\begin{aligned} \frac{p^2}{2} + \frac{x^2}{2} &= \frac{1}{2}\left(p^2 + x^2 + 1 - 1\right) \\ &= \frac{1}{2}\left(p^2 + x^2 + 1 + i[x,p]\right) \\ &= \frac{1}{2}\left(p^2 + x^2 + ixp - ipx + 1\right) \\ &= \frac{1}{2}\left(p + ix\right)\left(p - ix\right) + \frac{1}{2}. \end{aligned} \tag{2.31}$$

We now define two new operators

$$a = \frac{1}{\sqrt{2}}(p - ix), \tag{2.32}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(p + ix). \tag{2.33}$$

These are called the lowering and raising operators, the reason for these names will become apparent soon. We can now rewrite our Hamiltonian as

$$H = a^\dagger a + \frac{1}{2}. \tag{2.34}$$

The commutator between a and $a^\dagger$ is

$$\begin{aligned} [a, a^\dagger] = aa^\dagger - a^\dagger a &= \frac{1}{2}\left((p - ix)(p + ix) - (p + ix)(p - ix)\right) \\ &= \frac{1}{2}\left(p^2 + x^2 - ixp + ipx - p^2 - x^2 + ipx - ixp\right) \\ &= -i[x, p] = 1. \end{aligned} \tag{2.35}$$

It is also of use to calculate the commutators between the raising and lowering operators and the Hamiltonian

$$\begin{aligned} [H, a] &= Ha - aH \\ &= a^\dagger a a + \frac{1}{2}a - a a^\dagger a - \frac{1}{2}a \\ &= [a^\dagger, a]a = -[a, a^\dagger]a = -a, \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} [H, a^\dagger] &= Ha^\dagger - a^\dagger H \\ &= a^\dagger a a^\dagger + \frac{1}{2}a^\dagger - a^\dagger a^\dagger a - \frac{1}{2}a^\dagger \\ &= a^\dagger [a, a^\dagger] = a^\dagger. \end{aligned} \tag{2.37}$$

Now we posit a set of eigenstates $|\psi\rangle$ for H, so

$$H|\psi\rangle = E|\psi\rangle. \tag{2.38}$$

Here E is a real number that corresponds to the energy of the state $|\psi\rangle$, which we know is real because our Hamiltonian operator is Hermitian. We can now try and find out how the lowering (or raising) operator affects such a state $|\psi\rangle$. We get

$$\begin{aligned} Ha|\psi\rangle &= ([H,a]+aH)\,|\psi\rangle \\ &= (-a+aH)|\psi\rangle \\ &= a(H-1)|\psi\rangle = (E-1)a|\psi\rangle. \end{aligned} \tag{2.39}$$

Now we see why it makes sense to call a the lowering operator, as it effectively lowers the energy of a state by one unit of energy. A similar calculation for the raising operator gives us

$$Ha^\dagger|\psi\rangle = (E+1)a^\dagger|\psi\rangle.$$

The raising operator can be used repeatedly to reach higher energy states without limit. But the lowering operator cannot lower the energy in an infinite number of steps, since the energy always must be positive. Why is that? We see in (2.34) that H and $a^\dagger a$ only differ by a constant. Thus, they have the same eigenfunctions, which is obviuos from

$$(a^\dagger a + \frac{1}{2})|\psi\rangle = E|\psi\rangle. \tag{2.40}$$

Let λ be the eigenvalue of $a^\dagger a$. Then

$$a^\dagger a|\psi\rangle = \lambda|\psi\rangle, \qquad E = \langle\psi|H|\psi\rangle = \langle\psi|\left(a^\dagger a + \frac{1}{2}\right)|\psi\rangle = \left(\lambda + \frac{1}{2}\right). \tag{2.41}$$

Since $a^\dagger$ and a are complex conjugates of each other, we have that

$$\langle\psi|a^\dagger a|\psi\rangle = \langle a\psi|a\psi\rangle \quad\Longrightarrow\quad \lambda\langle\psi|\psi\rangle = \langle a\psi|a\psi\rangle. \tag{2.42}$$

But the norm of the Hilbert space, $\mathcal{H}$, of our ψ functions is positive, hence λ is non-negative. But then $E = \lambda + \frac{1}{2}$ is positive. Therefore, we can define a ground state $|0\rangle$, which is annihilated by a, written

$$a|0\rangle = 0.$$

The ground state energy is then

$$\begin{aligned} H|0\rangle &= (a^\dagger a + \frac{1}{2})|0\rangle \\ &= \frac{1}{2}|0\rangle. \end{aligned} \tag{2.43}$$

Using the raising operator, we can iteratively define the n-th energy state as

$$|n\rangle = (a^\dagger)^n|0\rangle.$$

This gives us

$$H(a^\dagger)^n|0\rangle = \left((a^\dagger)^n H + [H,(a^\dagger)^n]\right)|0\rangle$$

here we use $[H,(a^\dagger)^n] = n(a^\dagger)^n$, that gives

$$\begin{aligned} &= ((a^\dagger)^n H + n(a^\dagger)^n)|0\rangle \\ &= (a^\dagger)^n(H+n)|0\rangle \\ &= (a^\dagger)^n(n+\frac{1}{2})|0\rangle \\ &= (n+\frac{1}{2})|n\rangle. \end{aligned} \tag{2.44}$$

Now the energy states $|n\rangle$ define a basis in which H is diagonal (it is an eigenbasis for H). This means we can calculate the *partition function*, expressed as $Z(\beta) = \mathrm{Tr}\ \exp(-\beta H)$, with relative ease in this basis. Maybe the reader is familiar with the partition function from statistical mechanics, where it simply was the sum of Boltzmann factors for all quantum states. In fact, the partition function is a more general mathematical concept, which plays an important role in quantum field theory. It is useful when computing probabilities and expectation values etc. To conclude the section of the operator formalism we compute the partition function for the QM harmonic oscillator

$$
\begin{aligned}
Z(\beta) = \text{Tr } \exp(-\beta H) &= \sum_{n=0}^{\infty} \exp[-\beta(n+\frac{1}{2})] \\
&= \exp(-\beta/2) \sum_{n=0}^{\infty} (\exp[-\beta])^n \\
&= \frac{\exp(-\beta/2)}{1-\exp(-\beta)} = \frac{1}{2\sinh(\beta/2)}.
\end{aligned} \tag{2.45}
$$

2.2.2 Path Integral Formalism

The action $S(X)$ is a function of the coordinate X, which we will assume only depends on time t, so that $X = X(t)$. If $L(X)$ is the Lagrangian of the system, then in the non-relativistic case with mass $m = 1$ we get

$$
S(X) := \int dt\ L(X, \dot{X}) = \int dt\ T(\dot{X}) - V(X) = \int dt\ \frac{1}{2}\left(\frac{dX}{dt}\right)^2 - V(X). \tag{2.46}
$$

We define the partition function for this system in the path integral formalism to be

$$
Z(X_2, t_2; X_1, t_1) := \int_{X_1}^{X_2} DX(t)\ e^{iS(X(t))}, \tag{2.47}
$$

where, $X_1 = X(t_1)$ and $X_2 = X(t_2)$. The integration is over all possible paths that may be taken from X_1 to X_2, that is, from time t_1 to t_2. We do not know how to interpret DX yet, but we state the definition like this and hope it will be fruitful.

The imaginary unit i was not present in statistical mechanics. It is here now since X is a variable in quantum mechanics with probability amplitude interpretation. We do not want to integrate, i.e. to sum complex numbers with different phases like this, since it might make the integral divergent. A way out of this would be to *Euclideanize* it. If we rotate the time variable an angle $\frac{\pi}{2}$ counterclockwise in the complex plane, we get a new imaginary time coordinate τ. That is, $t \to it =: \tau$. This rotation yields the identities

$$
\begin{aligned}
t &= -i\tau, \\
dt &= -id\tau, \\
\frac{dX}{dt} &= \frac{1}{-i}\frac{dX}{d\tau},
\end{aligned} \tag{2.48}
$$

and then for the action we get

$$
S(X(\tau)) = \int -i\ d\tau \left(\frac{1}{2}\left(\frac{1}{-i}\right)^2 \left(\frac{dX}{d\tau}\right)^2 - V(X)\right) = i\int d\tau \frac{1}{2}\left(\frac{dX}{d\tau}\right)^2 + V(X). \tag{2.49}
$$

Now define the Euclidean action S_E as

$$
S_E := \int d\tau\ \frac{1}{2}\left(\frac{dX}{d\tau}\right)^2 + V(X). \tag{2.50}
$$

We see that $S(X) = iS_E(X)$ and the *Euclidean partition function* Z_E becomes

$$
Z_E = Z_E(X_2, \tau_2; X_1, \tau_1) = \int_{X(\tau_1)}^{X(\tau_2)} DX(\tau)\ e^{-S_E(X)}. \tag{2.51}
$$

Example: QM harmonic oscillator in path integral formalism

We now turn to the specific example of the quantum harmonic oscillator. If the spring constant is $k = 1$, we have $V(X) = \frac{1}{2}X^2$ and (2.50) becomes

$$S_E = \frac{1}{2}\int d\tau \left(\frac{dX}{d\tau}\right)^2 + X^2. \tag{2.52}$$

If $V(X)$ is a positive semi-definite function of X (as is the case here), then S_E is a non-negative number, and the convergence of (2.51) seems probable. So far, we have only considered integration from X_1 to X_2 where those are different points in a space. We could also carry out the integration on a circle in time, with time of revolution β. Then, $X(\tau+\beta) = X(\tau)$, since we come back to the same point. Using this, the action may be written in a nice way.

To find this new form we start by defining an operator $\Theta = -\frac{d^2}{d\tau^2} + 1$. Then we compute

$$X\Theta X = X\left(-\frac{d^2}{d\tau^2} + 1\right)X = -X\frac{d^2X}{d\tau^2} + X^2. \tag{2.53}$$

Using the fact that

$$\frac{d}{d\tau}\left(X\frac{dX}{d\tau}\right) = \left(\frac{dX}{d\tau}\right)^2 + X\frac{d^2X}{d\tau^2}, \tag{2.54}$$

we can write

$$X\Theta X = \left(\frac{dX}{d\tau}\right)^2 - \frac{d}{d\tau}\left(X\frac{dX}{d\tau}\right) + X^2. \tag{2.55}$$

This enables us to express the action in a simple way

$$\begin{aligned}\int_{X(\tau)}^{X(\tau+\beta)} d\tau \left(\frac{dX}{d\tau}\right)^2 + X^2 &= \int_{X(\tau)}^{X(\tau+\beta)} d\tau\ X\Theta X + \frac{d}{d\tau}\left(X\frac{dX}{d\tau}\right) \\ &= \int_{X(\tau)}^{X(\tau+\beta)} d\tau\ X\Theta X + \left[X\frac{dX}{d\tau}\right]_{\tau}^{\tau+\beta} \\ &= \int_{X(\tau)}^{X(\tau+\beta)} d\tau\ X\Theta X,\end{aligned} \tag{2.56}$$

since $X(\tau+\beta) = X(\tau)$ and $\frac{dX}{d\tau}\big|_{\tau+\beta} = \frac{dX}{d\tau}\big|_{\tau}$. Thus we end up with the following expression for the partition function

$$Z_E = \int_{X(\tau)}^{X(\tau+\beta)} DX(\tau)\ e^{-\frac{1}{2}\int d\tau\ X\Theta X}. \tag{2.57}$$

To be able to calculate this complicated integral, a change of variables is useful. We find orthonormal eigenfunctions f_n to the operator Θ

$$\int d\tau\ f_m^*(\tau)f_n(\tau) = \delta_{n,m}, \qquad \Theta f_n = \lambda_n f_n, \qquad \left(-\frac{d^2}{d\tau^2} + 1\right)f_n = \lambda_n f_n, \tag{2.58}$$

$$\frac{d^2 f_n(\tau)}{d\tau^2} + (\lambda_n - 1)f_n(\tau) = 0 \tag{2.59}$$

with solutions in two modes

$$f_n(\tau) = A\cos(\sqrt{\lambda_n - 1}\ \tau) + B\sin(\sqrt{\lambda_n - 1}\ \tau). \tag{2.60}$$

We have the same boundary condition for f_n as we had for X, that is: $f_n(\tau+\beta) = f_n(\tau)$. This gives

$$\begin{aligned}f_n(\tau+\beta) &= A\cos(\sqrt{\lambda_n - 1}\ (\tau+\beta)) + B\sin(\sqrt{\lambda_n - 1}\ (\tau+\beta)) \\ &= A\cos(\sqrt{\lambda_n - 1}\tau + \sqrt{\lambda_n - 1}\beta) + B\sin(\sqrt{\lambda_n - 1}\tau + \sqrt{\lambda_n - 1}\beta).\end{aligned} \tag{2.61}$$

The condition is fulfilled whenever $\sqrt{\lambda_n - 1}\beta = 2\pi n$, where n is a non-negative integer. The eigenvalues become

$$\lambda_n = 1 + \left(\frac{2\pi n}{\beta}\right)^2. \tag{2.62}$$

We have now found a basis of orthonormal eigenfunctions $\{f_n\}$ for our space. If we express X as $X = \sum_n c_n f_n$, where the c_n are real numbers reaching from $-\infty$ to ∞, we get

$$\begin{aligned}\int d\tau\ X\Theta X &= \int d\tau\ \sum_m c_m f_m \Theta \sum_n c_n f_n = \int d\tau\ \sum_m c_m f_m \sum_n c_n \lambda_n f_n \\ &= \sum_m \sum_n \lambda_n c_m c_n \int d\tau\ f_m f_n = \{f_n \text{ are real}\} = \sum_m \sum_n \lambda_n c_m c_n \delta_{mn} \\ &= \sum_n \lambda_n c_n^2.\end{aligned} \tag{2.63}$$

And the partition function becomes

$$Z_E(\beta) = \int DX(\tau)\ e^{-\frac{1}{2}\sum_n \lambda_n c_n^2} = \int DX(\tau)\ \prod_n e^{-\frac{1}{2}\lambda_n c_n^2}. \tag{2.64}$$

We now have an integrand expressed in the variables c_n. Then we need to express DX in terms of c_n as well. We note that

$$\int_{-\infty}^{\infty} dc_n\ e^{-\frac{1}{2}\lambda_n c_n^2} = \sqrt{\frac{2\pi}{\lambda_n}}, \qquad \int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\lambda_n c_n^2} = \frac{1}{\sqrt{\lambda_n}}. \tag{2.65}$$

The factor $\sqrt{2\pi}$ was shuffled to the LHS to make the RHS as simple as possible. We have a product of n such integrands. Then it is very natural to interpret the path integral variable as $DX = \prod_n \frac{dc_n}{\sqrt{2\pi}}$. In such a case

$$\begin{aligned}Z_E(\beta) &= \int \prod_n \frac{dc_n}{\sqrt{2\pi}}\ \prod_n e^{-\frac{1}{2}\lambda_n c_n^2} = \prod_n \int \frac{dc_n}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\lambda_n c_n^2} = \prod_n \frac{1}{\sqrt{\lambda_n}} \\ &= \frac{1}{\sqrt{\det \Theta}}.\end{aligned} \tag{2.66}$$

When $n = 0$ we have $\lambda_n = 1$ and $f_0 =$ constant. For n=1,2,3, ..., f_n consists of two linearly independent functions, $\sin x$ and $\cos x$. Thus we have two modes and the multiplicity of each eigenvalue is 2. Thus, rather than multiplying $\lambda_1\lambda_2\lambda_3\ldots$ we should multiply $\lambda_1^2\lambda_2^2\lambda_3^2\ldots$ Then

$$Z_E(\beta) = 1\prod_{n=1}^{\infty}\frac{1}{\lambda_n} = \prod_{n=1}^{\infty}\frac{1}{1+\left(\frac{2\pi n}{\beta}\right)^2} = \prod_{n=1}^{\infty}\left(\frac{2\pi n}{\beta}\right)^{-2}\prod_{n=1}^{\infty}\frac{1}{1+\left(\frac{2\pi n}{\beta}\right)^{-2}}. \tag{2.67}$$

We are now left with the mathematical issue to calculate these products. The second one is the easiest. As Euler showed in the 18th century, the elementary functions have *infinite product representations*. For $\sinh z$ we have the product

$$\sinh z = z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2 n^2}\right). \tag{2.68}$$

This identity can be shown in many ways. A proof using residue calculus is found in Appendix B.1. Then, by just changing variables in (2.68) to $z = \frac{\beta}{2}$, we arrive at the expression

$$\prod_{n=1}^{\infty}\frac{1}{1+\left(\frac{\beta}{2\pi n}\right)^2} = \frac{\beta}{2\sinh\left(\frac{\beta}{2}\right)}. \tag{2.69}$$

The other product, $\prod_{n=1}^{\infty}\left(\frac{2\pi n}{\beta}\right)^{-2}$ is much more difficult. Here we must use the method of *ζ function regularisation*. Instead of looking at the product, let us look at the corresponding sum

$$\zeta_1(s) = \sum_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2s}. \tag{2.70}$$

If we now differentiate the sum with respect to s, we get

$$\zeta_1'(s) = \sum_{n=1}^{\infty} \ln\left(\frac{2\pi n}{\beta}\right)(-2)\left(\frac{2\pi n}{\beta}\right)^{-2s}, \qquad \zeta_1'(0) = \sum_{n=1}^{\infty} \ln\left(\frac{2\pi n}{\beta}\right)^{-2}. \tag{2.71}$$

Then we take the exponential of $\zeta_1'(0)$ and find a useful expression

$$e^{\zeta_1'(0)} = e^{\sum_{n=1}^{\infty} \ln\left(\frac{2\pi n}{\beta}\right)^{-2}} = \prod_{n=1}^{\infty} e^{\ln\left(\frac{2\pi n}{\beta}\right)^{-2}} = \prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2}. \tag{2.72}$$

Our definition of $\zeta_1(s)$ is very similar to the common Riemann ζ function, which is simply $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, when $s > 1$. Expressed in this function, our $\zeta_1(s)$ becomes

$$\zeta_1(s) = \left(\frac{2\pi}{\beta}\right)^{-2s} \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = \left(\frac{2\pi}{\beta}\right)^{-2s} \zeta(2s). \tag{2.73}$$

If we differentiate the first and last expressions above with respect to s we get

$$\zeta_1'(s) = \ln\left(\frac{2\pi}{\beta}\right)^{-2} \left(\frac{2\pi}{\beta}\right)^{-2s} \zeta(2s) + \left(\frac{2\pi}{\beta}\right)^{-2s} 2\zeta'(2s) \tag{2.74}$$

$$\zeta_1'(0) = \ln\left(\frac{2\pi}{\beta}\right)^{-2} \zeta(0) + 2\zeta'(0). \tag{2.75}$$

But what are $\zeta(0)$ and $\zeta'(0)$? So far we have just used Riemann's ζ function for $s > 1$. For example, by using Fourier series one proves that $\zeta(2) = \frac{\pi^2}{6}$. For $s = 1$ the function coincides with the harmonic series, which is certainly divergent. But by doing an analytic continuation in the complex plane, one can find that $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$. Proofs are found in Appendices B.2 and B.3. With the equation above we simply get

$$\zeta_1'(0) = \ln\left(\frac{2\pi}{\beta}\right)^{-2}\left(-\frac{1}{2}\right) + 2\left(-\frac{1}{2}\ln(2\pi)\right) = -\ln\beta, \qquad \prod_{n=1}^{\infty}\left(\frac{2\pi n}{\beta}\right)^{-2} = e^{-\ln\beta} = \frac{1}{\beta}. \tag{2.76}$$

Then (2.67), the partition function, becomes

$$Z_E(\beta) = \frac{1}{\beta}\frac{\beta}{2\sinh\left(\frac{\beta}{2}\right)} = \frac{1}{2\sinh\left(\frac{\beta}{2}\right)}. \tag{2.77}$$

We see that this is the same result for the partition function as we got in the operator formalism, at least if the not Euclideanized partition function is the same thing as the Euclideanized partition function. Why can we actually go from real time to complex time in (2.48)? A simple answer is that $t \to it =: \tau$ is just an ordinary change of variables in the complex plane, and integration in general is independent of the choice of parameter. A more involved answer demands deeper insights in quantum field theory and path integrals. However, we got the same result using both formalisms in the example, so in this case it worked.

2.2.3 Sigma Model on a Circle

We will now proceed by introducing a *sigma model*, and we do this by using a simple example on a circle. The circle is an example of a nontrivial target space. In contrast to the real line the circle is closed, which makes it an interesting target space. We use the same approach here as we did for the harmonic oscillator above, i.e. first we calculate the partition function using operator formalism and then we compare it to the partition function calculated using path integral formalism. In later chapters, we will apply the sigma model to more complex spaces.

Operator formalism

As mentioned above our target space is a circle S^1_R, with the circumference R. We set the potential $V(X) = 0$. The coordinate X is periodic in R, i.e. $X \sim X + R$. The Hamiltonian for our system is given by

$$H = \frac{p^2}{2} = \frac{1}{2}\left(-i\frac{d}{dX}\right)\left(-i\frac{d}{dX}\right) = -\frac{1}{2}\frac{d^2}{dX^2}. \tag{2.78}$$

Let us calculate the eigenfunctions ψ_n using the Hamiltonian H given in (2.78)

$$H\psi_n = E_n\psi_n \quad \Longrightarrow \quad -\frac{1}{2}\left(\frac{d^2}{X^2}\right)\psi_n = E_n\psi_n \ , n \in \mathbb{Z}. \tag{2.79}$$

Solving the above equation yields

$$\psi_n = \psi_n(X) = Ae^{i\sqrt{2E_n}X}, \tag{2.80}$$

where A is a constant. For simplicity, we set $A = 1$. While X is a periodic variable, we have $\psi_n(0) = \psi_n(R)$, which gives us

$$\begin{aligned} \psi_n(0) = 1 \quad \Longrightarrow \quad \psi_n(R) &= e^{i\sqrt{2E_n}R} \\ &= e^{i2\pi n}. \end{aligned} \tag{2.81}$$

From (2.81) we get the expression for the eigenvalues E_n accordingly

$$i\sqrt{2E_n}R = i2\pi n \quad \Longrightarrow \quad E_n = \frac{2\pi^2 n^2}{R^2}. \tag{2.82}$$

Using (2.82) we can rewrite the expression for the eigenfunctions ψ_n,

$$\psi_n(X) = e^{i2\pi nX/R}, \tag{2.83}$$

where we have expressed our variable in terms of the circumference R. This provides us with the partition function

$$Z(\beta) = \mathrm{Tr}e^{-\beta H} = \sum_{-\infty}^{\infty} e^{-\beta 2\pi^2 n^2/R^2}. \tag{2.84}$$

Path integral formalism

Now we calculate the partition function using path integral formalism. We start with the expression

$$Z(\beta) = \int DXe^{-S_E(X)} \tag{2.85}$$

where $S_{E(X)}$ is the Euclidean action given by

$$S_E(X) = \int \left(\frac{1}{2}\left(\frac{dX}{d\tau}\right)^2 + V(X)\right) d\tau. \tag{2.86}$$

As for the operator formalism, $V(x) = 0$ which gives us the partition function

$$Z(\beta) = \int DXe^{-\int_0^\beta \frac{1}{2}\left(\frac{dX}{d\tau}\right)^2 d\tau}. \tag{2.87}$$

Integration over all maps S^1_β to S^1_R requires introducing the *winding number* m. To get a picture of what the winding number is, let us imagine the situation in figure 2.1, where a person stands in the middle of the red path representing the motion of the black particle. As the particle follows the path around, the person in the middle follows it with his eyes. The winding number m increases by one for every round the person has to take in his reference frame (represented by the blue path) while keeping his eyes on the particle. In this case

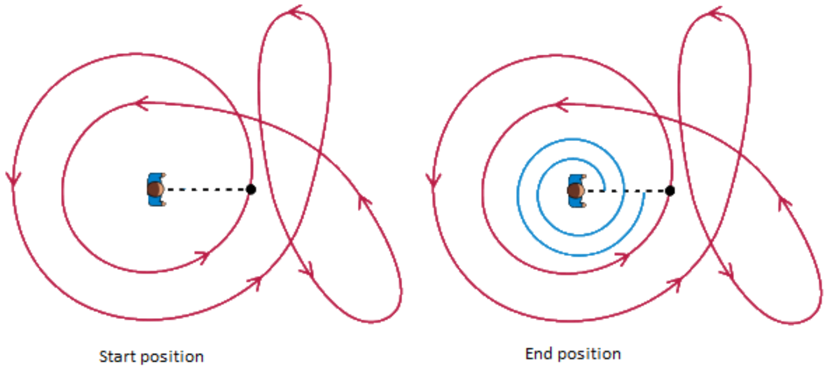


Figure 2.1: A visual representation of the winding number m, [17].

$m = 2$. All of these winding sectors will have to be considered. Thus, the path integral becomes

$$Z(\beta) = \sum_{m=-\infty}^{\infty} \int DX_m e^{-S_E(X_m)}, \tag{2.88}$$

where $X_m(\beta) = X(0) + Rm$ is a variable that for every m represents a map. We can express $X_m(\tau)$ as $X_m(\tau) = \frac{m\tau R}{\beta} + X_0(\tau)$. Here, $X_0(\tau)$ is a periodic function. Using the expression for X_m we can rewrite the action accordingly

$$\left(\frac{dX_m}{d\tau}\right)^2 = \left(\frac{mR}{\beta} + \frac{d}{d\tau}X_0(\tau)\right)^2 \quad \Rightarrow \tag{2.89}$$

$$\begin{aligned} S_E(X_m) &= \frac{1}{2}\int_0^\beta \left[\frac{m^2R^2}{\beta^2} + \frac{2mR}{\beta}\frac{d}{d\tau}X_0(\tau) + \left(\frac{d}{d\tau}X_0(\tau)\right)^2\right] d\tau \\ &= \frac{m^2R^2}{2\beta} + \frac{1}{2}\int_0^\beta \left(\frac{d}{d\tau}X_0(\tau)\right)^2 d\tau. \end{aligned} \tag{2.90}$$

In the middle step, the integration of $\frac{2mR}{\beta}\frac{d}{d\tau}X_0(\tau)$ yields zero, because $X_0(\tau)$ is a periodic function (start point and end point are the same). Following the same procedure as in the case of the harmonic oscillator we define

$$\int d\tau \dot{X}^2 = \frac{1}{2}\int d\tau X\Theta X, \tag{2.91}$$

where $\Theta = -\frac{d^2}{d\tau^2}$. The eigenfunctions f_n have eigenvalues $\lambda_n = \left(\frac{2\pi n}{R}\right)^2$. We can now rewrite the partition function once more,

$$Z(\beta) = \sum_{m=-\infty}^{\infty} e^{-\frac{m^2R^2}{2\beta}} \int DX_0 e^{-\frac{1}{2}\sum_n c_n^2\lambda_n} = \sum_{m=-\infty}^{\infty} e^{-\frac{m^2R^2}{2\beta}} \int DX_0 \prod_n e^{-\frac{1}{2}c_n^2\lambda_n}. \tag{2.92}$$

In the expression above $DX_0 = \prod_n \frac{dc_n}{\sqrt{2\pi}}$, as in the example of the QM harmonic oscillator. We now have two cases, $n = 0$ and $n \neq 0$, needed to be treated separately

$$Z(\beta) = \sum_{m=-\infty}^{\infty} e^{-\frac{m^2R^2}{2\beta}} \underbrace{\int DX_0 e^{-\frac{1}{2}c_0^2\lambda_0}}_{n=0} \underbrace{\int DX_0 \prod e^{-\frac{1}{2}c_n^2\lambda_n}}_{n\neq 0}. \tag{2.93}$$

For $n = 0$ we have $\lambda_0 = 0$, so the part of the expression where $n = 0$ becomes $\int DX_0$. From this mode we get a normalization constant $1/\sqrt{\beta}$ accordingly

$$f_n(t) = A\cos(\sqrt{\lambda_n}t) + B\sin(\sqrt{\lambda_n}t) \quad \Rightarrow \quad f_0(t) = A\cos(0) + B\sin(0) = A, \tag{2.94}$$

$$1 = \int f_0^* f_0 dt = A^2 \int_0^\beta dt = A^2\beta \quad \Rightarrow \quad f_0(t) = 1/\sqrt{\beta}. \tag{2.95}$$

We have to take the normalization constant under consideration when we set the limits in the integration. The limits for the zero mode is thus [0, R$\sqrt{\beta}$], and we get

$$\int_0^{R\sqrt{\beta}} DX_0 = \int_0^{R\sqrt{\beta}} \frac{dc_0}{\sqrt{2\pi}} = \frac{R\sqrt{\beta}}{\sqrt{2\pi}}. \tag{2.96}$$

For $n \neq 0$ we do the substitution,

$$y^2 = c_n^2\lambda_n, \qquad \frac{dc_n}{dy} = \frac{1}{\sqrt{\lambda_n}}. \tag{2.97}$$

The above relations provide us with the expression for the part where $n \neq 0$

$$\int_{-\infty}^{\infty} \prod_{n\neq 0} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} \frac{dy}{\sqrt{\lambda_n}} = \prod_{n\neq 0} \frac{1}{\sqrt{\lambda_n}} = \frac{1}{\sqrt{\det' \Theta}}, \tag{2.98}$$

where $\det' \Theta$ is the determinant with all λ_n except λ_0. Multiplying both parts ($n = 0$ and $n \neq 0$) provides us with

$$\frac{R\sqrt{\beta}}{\sqrt{2\pi}} \frac{1}{\sqrt{\det' \Theta}} = \frac{R\sqrt{\beta}}{\sqrt{2\pi}} \frac{1}{\sqrt{\det' \left(-\frac{d^2}{d\tau^2}\right)}}. \tag{2.99}$$

The expression for the determinant is given by

$$\det' \left(-\frac{d^2}{d\tau^2}\right) = \prod_{n\neq 0} \left(\frac{2\pi n}{\beta}\right)^2 = \beta^2, \tag{2.100}$$

where the last step is done by ζ function regularisation as in section 2.2.2. So we get the path-integral

$$Z(\beta) = \frac{R}{\sqrt{2\pi\beta}} \sum_{m=-\infty}^{\infty} e^{-\frac{m^2R^2}{2\beta}}. \tag{2.101}$$

This does not look the same as (2.84), the partition function via the operator formalism, but with the use of a technique called *Poisson resummation* we can write them in the same form, in effect turn $\sum_{m=-\infty}^{\infty} e^{-m^2R^2/2\beta}$ into $\sum_{n=-\infty}^{\infty} e^{-\beta 2\pi^2 n^2/R^2}$. We start the Poisson resummation with the identity

$$\sum_{n=-\infty}^{\infty} \delta(x + 2\pi n) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{imx}. \tag{2.102}$$

We can see that this identity is correct by finding the Fourier series for the delta-function

$$f(x) = \sum_{n=-\infty}^{\infty} \delta(x + 2\pi n) = \sum_{m=-\infty}^{\infty} c_m e^{imx}. \tag{2.103}$$

The coefficients c_m are found in the usual Fourier series manner

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-imx} dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \delta(x + 2\pi n) e^{-imx} dx = \frac{1}{2\pi} \cdot 1 = \frac{1}{2\pi}. \tag{2.104}$$

In the integration interval $[0, 2\pi]$ only one of the delta functions of the summation contributes, when $n = 0$. If $g(x) = e^{-imx}$ and $n = 0$, the delta-function picks out the value of $g(0) = e^{-im\cdot 0} = 1$, making the whole sum and integral equal to 1, proving the identity. Now we multiply the identity by $e^{-\frac{\alpha}{2}x^2}$ and integrate over x

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x+2\pi n)e^{-\frac{\alpha}{2}x^2}dx = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{imx-\frac{\alpha}{2}x^2}dx. \tag{2.105}$$

Using the definite integral $\frac{1}{2}\sqrt{\frac{\pi}{a}}\int\limits_{-\infty}^{\infty} e^{2bx-ax^2}dx = \sqrt{\frac{\pi}{a}}e^{b^2/a}$ on the right hand side, we get the relation

$$\sum_{n=-\infty}^{\infty} e^{-\frac{\alpha}{2}(-2\pi n)^2} = \frac{1}{\sqrt{2\pi\alpha}}\sum_{m=-\infty}^{\infty} e^{-\frac{1}{2\alpha}m^2}. \tag{2.106}$$

When we set $\alpha = \frac{\beta}{R^2}$ this yields

$$\sum_{n=-\infty}^{\infty} e^{-\frac{\beta 2\pi^2 n^2}{R^2}} = \frac{R}{\sqrt{2\pi\beta}}\sum_{m=-\infty}^{\infty} e^{-\frac{m^2R^2}{2\beta}}, \tag{2.107}$$

as we wanted. Now we clearly see that the partition function of the operator formalism (2.84) and path integral formalism (2.101) again are equal!

If we want to change our target space to the real line, we can still use the sigma model here developed but with a little trick. This is neat because there arises some problems when doing a straightforward calculation on the real line. Consider an example of a system without a potential, $V(X) = 0$. The action is then

$$S = \int \frac{1}{2}\dot{X}^2 dt \tag{2.108}$$

with the Hamiltonian given by $H = \frac{1}{2}p^2 = -\frac{1}{2}\frac{\partial^2}{\partial X^2}$. When $p = k$, the wave number, we have the plane-wave solution $\Psi_k = e^{ikX}$. The energy for each eigenstate is

$$E_k = \frac{1}{2}k^2. \tag{2.109}$$

The wave-functions Ψ_k still obey the orthogonality relation

$$\int \Psi_k^*(X)\Psi_{k'}dX = 2\pi\delta(k-k') \tag{2.110}$$

but are no longer square normalizable, because of their non-localized nature. And, due to $V(X) = 0$, the spectrum becomes continuous so the partition function $Z(\beta) = \text{Tr } e^{-\beta H}$ is no longer well defined. The trick to overcome these difficulties is to use the sigma model on S^1_R but let $R \to \infty$, then by (2.101) we get the partition function

$$Z(\beta) = \lim_{R\to\infty}\frac{R}{\sqrt{2\pi\beta}}\sum_{m=-\infty}^{\infty} e^{-\frac{m^2R^2}{2\beta}} = \frac{\lim_{R\to\infty} R}{\sqrt{2\pi\beta}}. \tag{2.111}$$

Chapter 3

Supersymmetric Quantum Mechanics - Part I

Supersymmetric QM is a mathematical construct that connects every fermion to a bosonic superpartner and every boson to a fermionic superpartner. It harbours conserved supercharges Q which are the supersymmetric equivalents to the conserved charges we saw in Noether's theorem. By using the supercharges it is possible to map back and forth between fermionic and bosonic states in Hilbert space,

$$Q|\text{boson}\rangle = |\text{fermion}\rangle.$$

In this chapter we will begin to explore supersymmetric QM by applying the methods from chapter 2. We start with an introduction of Grassmann numbers that will be used throughout the rest of the paper. If already familiar with this algebra, the reader may skip section 3.1 without loosing any of the supersymmetry concepts. We proceed with the Lagrangian formulation and apply supersymmetric transformations to find an expressions for the supercharges. After quantization of the system we can use our supercharges to study the properties of supersymmetry. We will come across the *Witten index* which gets us halfway to our goal of connecting supersymmetric QM to topological invariants of the manifold we are working in. We will end the chapter with two examples computing the Witten index for two different systems. Our main references for this chapter are 'Mirror Symmetry' [1] and 'Constraints on Supersymmetry Breaking' [2].

3.1 Grassmann numbers

To be able to study supersymmetric quantum mechanics we must use bosonic and fermionic operators. Bosonic operators commute with each other but fermionic operators anti-commute and to be able to do functional integration of fermionic fields, a new kind of numbers must be introduced. These numbers are called *Grassmann numbers* and are different from real and complex numbers.

Let θ_i, θ_2 be two independent Grassmann numbers that are not composed out of different Grassmann numbers, we call them basic Grassmann numbers. Basic Grassmann numbers are odd and have the property that they anti-commute with each other.

$$\begin{aligned} &\{\theta_i, \theta_j\} = \theta_i\theta_j + \theta_j\theta_i = 0 \\ &\Rightarrow \theta_i\theta_j = -\theta_j\theta_i, \end{aligned} \tag{3.1}$$

which makes the square of a basic Grassmann number zero,

$$\theta_i^2 = 0$$

since it is the only way the condition $\theta_i\theta_i = -\theta_i\theta_i$ can be fulfilled. Basic Grassmann numbers commute with real and complex numbers. In symbols, let $x \in \mathbb{R}$, a$\in \mathbb{C}$, and then

$$\theta_i x = x\theta_i, \qquad \theta_i a = a\theta_i. \tag{3.2}$$

The differentiation of a basic Grassmann number is defined in the following way

$$
\begin{aligned}
&\frac{\partial \theta_j}{\partial \theta_i} = \delta_{ij} \\
&\frac{\partial (a\theta_j)}{\partial \theta_i} = a\frac{\partial \theta_i}{\partial \theta_j} = a\delta_{ij} \\
&\frac{\partial (\theta_i \theta_j)}{\partial \theta_i} = \theta_j \\
&\frac{\partial (\theta_i \theta_j)}{\partial \theta_j} = -\theta_i,
\end{aligned}
\tag{3.3}
$$

whereas integration may be defined as

$$
\begin{aligned}
&\int d\theta_i \ a = 0 \\
&\int d\theta_i \ a\theta_k = a\delta_{ik} \\
&\int d\theta_i \ \theta_j \theta_i = -\theta_j \\
&\int d\theta_i \theta_i \theta_j = \theta_j.
\end{aligned}
\tag{3.4}
$$

We also have the double integrals defined as

$$
\begin{aligned}
&\int d\theta_i d\theta_j (-i\theta_i \theta_j) = 1 \\
&\int d\theta_i d\theta_j (\theta_i) = 0 \\
&\int d\theta_i d\theta_j a = 0.
\end{aligned}
\tag{3.5}
$$

The factor of i in (3.5) is not necessary, but this is the definition we will use later on in section 5.1.2 for convenience in calculations.

Let us now study a real Grassmann algebra, where we have an arbitrary number of real numbers $r_1, r_2, ... r_n$ and an arbitrary number of basic Grassmann numbers $\theta_1, \theta_2, ... \theta_n$. A Grassmann number is the arbitrary sum of an arbitrary product of real numbers and basic Grassmann numbers. So a Grassmann number z, can be expressed as

$$
z = z_r + z_g, \tag{3.6}
$$

where z_r is a real number and z_g a general Grassmann number (also called Grassmann variable). The set of all Grassmann numbers is called a real Grassmann algebra. A general Grassmann number can be expressed as a sum of even and odd Grassmann numbers

$$
z_g = \sum_{n=1}^{N} \theta_1 \theta_2 ... \theta_n. \tag{3.7}
$$

Where θ_1 is an odd Grassmann variable and $\theta_1 \theta_2$ is an even Grassmann variable since it is composed of the basic Grassmann numbers $\theta_1, \theta_2 ... \theta_n$ which are odd. So any product of an even number of basic Grassmann number will result in an even Grassmann variable and any product of an odd number of basic Grassmann number will result in an odd Grassmann variable. Odd Grassmann variables follow the same rules as the basic Grassmann number since both have the properties of being odd. That is, odd Grassmann variables anti-commute with each other but commute with real and complex numbers, and the square of such a variable is zero. Even Grassmann numbers on the other hand commute with each other

$$
\begin{aligned}
[\theta_1\theta_2, \theta_3\theta_4] &= \theta_1\theta_2\theta_3\theta_4 - \theta_3\theta_4\theta_1\theta_2 = \theta_1\theta_2\theta_3\theta_4 + \theta_3\theta_1\theta_4\theta_2 \\
&= \theta_1\theta_2\theta_3\theta_4 - \theta_1\theta_3\theta_4\theta_2 = \theta_1\theta_2\theta_3\theta_4 - \theta_1\theta_2\theta_3\theta_4 = 0
\end{aligned}
\tag{3.8}
$$

where $\theta_1\theta_2$ and $\theta_3\theta_4$ are two even Grassmann variables. The even Grassmann numbers does also commute with real and complex numbers. Odd and even Grassmann variables commute with each other

$$[\theta_1, \theta_2\theta_3] = \theta_1\theta_2\theta_3 - \theta_2\theta_3\theta_1 = 0. \tag{3.9}$$

The square of an even Grassmann number does not have to be zero but it is possible

$$(\theta_1\theta_2)^2 = \theta_1\theta_2\theta_1\theta_2 = -\theta_1\theta_2\theta_2\theta_1 = -\theta_1\theta_2^2\theta_1 = 0 \tag{3.10}$$

since $\theta_2^2 = 0$ still holds. But if one for instance has the even Grassmann variable $\theta_1\theta_2 + \theta_3\theta_4$ the square is non-zero

$$(\theta_1\theta_2 + \theta_3\theta_4)^2 = \theta_1\theta_2\theta_1\theta_2 + \theta_1\theta_2\theta_3\theta_4 + \theta_3\theta_4\theta_1\theta_2 + \theta_3\theta_4\theta_3\theta_4 = 2\theta_1\theta_2\theta_3\theta_4 \tag{3.11}$$

So the squares of even Grassmann variables behave in different ways. The Grassmann number z can always be written as the sum of an even and an odd part

$$z = z_e + z_o, \tag{3.12}$$

where z_e is the even part and z_o the odd part. They can then be expressed as

$$\begin{aligned} z_e &= z_r + \sum_{n=1}^{N/2} \theta_1\theta_2 ... \theta_{2n}, \\ z_o &= \sum_{n=1}^{N/2} \theta_1\theta_2 ... \theta_{2n-1}. \end{aligned} \tag{3.13}$$

The even part z_e is commutating and the odd part z_o is anti-commutating. The fermionic variables are described by odd Grassmann numbers and the bosonic variables are even Grassmann valued.

3.2 The Lagrangian and Supersymmetry Transformations

In this section we will first show that a given Lagrangian, for a general potential theory of one variable, is supersymmetric by expressing it as a total time derivative. By then invoking Neothers theorem we find the supercharges. We end the section by quantizing the system.

3.2.1 Supersymmetric Lagrangian

To show that a given Lagrangian is supersymmetric, we begin our journey with a Lagrangian retrieved from Mirror Symmetry [1]

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(h'(x))^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h''(x)\bar{\psi}\psi \tag{3.14}$$

where ψ is a complex fermionic parameter (with complex conjugate $\bar{\psi}$) and the superpartner of x. $h(x)$ is the superpotential that enters in the bosonic and fermionic potential energy terms. It is the symmetry between the bosonic and fermionic variables that creates the supersymmetry and it will emerge through the following transformation of the parameters

$$\begin{aligned} \delta x &= \epsilon\bar{\psi} - \bar{\epsilon}\psi, & \delta\dot{x} &= \epsilon\dot{\bar{\psi}} - \bar{\epsilon}\dot{\psi} \\ \delta\psi &= \epsilon(i\dot{x} + h'(x)), & \delta\dot{\psi} &= \epsilon(i\ddot{x} + \frac{d}{dt}h'(x)) \\ \delta\bar{\psi} &= \bar{\epsilon}(-i\dot{x} + h'(x)), & \delta\dot{\bar{\psi}} &= \bar{\epsilon}(-i\ddot{x} + \frac{d}{dt}h'(x)). \end{aligned} \tag{3.15}$$

Here ϵ is a complex fermionic parameter and has the complex conjugate $\bar{\epsilon}$. These transformations are specific to the Lagrangian and found through trial and error. We compute the variation of the Lagrangian using the supersymmetric transformations,

$$\begin{aligned} \delta L &= \frac{1}{2}\delta(\dot{x})^2 - \frac{1}{2}\delta(h'(x))^2 + \frac{i}{2}\delta(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \delta(h''(x)\bar{\psi}\psi) \\ &= \dot{x}\delta(\dot{x}) - h'(x)\delta(h'(x)) + \frac{i}{2}\delta\bar{\psi}\dot{\psi} + \frac{i}{2}\bar{\psi}\delta\dot{\psi} - \frac{i}{2}\delta\dot{\bar{\psi}}\psi - \frac{i}{2}\dot{\bar{\psi}}\delta\psi \\ &\quad - \delta h''(x)\bar{\psi}\psi - h''(x)\delta\bar{\psi}\psi - h''(x)\bar{\psi}\delta\psi. \end{aligned} \tag{3.16}$$

We know that $\delta h'(x) = h''(x)\delta x$ which gives

$$\delta h''(x)\bar{\psi}\psi = h'''(x)\delta x\bar{\psi}\psi = h'''(x)(\epsilon\bar{\psi} - \bar{\epsilon}\psi)\bar{\psi}\psi = 0, \tag{3.17}$$

since ψ is an odd Grassmann number and $\psi^2 = 0$. This gives

$$\begin{aligned}
\delta L &= \dot{x}(\epsilon\dot{\bar{\psi}} - \bar{\epsilon}\dot{\psi}) - h'(x)h''(x)(\epsilon\bar{\psi} - \bar{\epsilon}\psi) + \frac{i}{2}\bar{\epsilon}(-i\dot{x} + h'(x))\dot{\psi} + \frac{i}{2}\bar{\psi}\epsilon(i\ddot{x} + \frac{d}{dt}h'(x)) \\
&\quad - \frac{i}{2}\bar{\epsilon}(-i\ddot{x} + \frac{d}{dt}h'(x))\psi - \frac{i}{2}\dot{\bar{\psi}}\epsilon(i\dot{x} + h'(x)) - h''(x)\bar{\epsilon}(-i\dot{x} + h'(x))\psi - h''(x)\bar{\psi}\epsilon(i\dot{x} + h'(x)) \\
&= \epsilon\dot{\bar{\psi}}\dot{x} - \bar{\epsilon}\dot{\psi}\dot{x} - \epsilon\bar{\psi}h'(x)h''(x) + \bar{\epsilon}\psi h'(x)h''(x) + \frac{1}{2}\bar{\epsilon}\dot{\psi}\dot{x} + \frac{i}{2}\bar{\epsilon}\dot{\psi}h'(x) \\
&\quad - \frac{1}{2}\bar{\psi}\epsilon\ddot{x} + \frac{i}{2}\bar{\psi}\epsilon\frac{d}{dt}h'(x) - \frac{1}{2}\bar{\epsilon}\psi\ddot{x} - \frac{i}{2}\bar{\epsilon}\psi\frac{d}{dt}h'(x) \\
&\quad + \frac{1}{2}\dot{\bar{\psi}}\epsilon\dot{x} - \frac{i}{2}\dot{\bar{\psi}}\epsilon h'(x) + i\bar{\epsilon}\psi\dot{x}h''(x) - \bar{\epsilon}\psi h''(x)h'(x) - i\bar{\psi}\epsilon\dot{x}h''(x) - \bar{\psi}\epsilon h''(x)h'(x) \\
&= \frac{1}{2}\epsilon\dot{\bar{\psi}}\dot{x} - \frac{1}{2}\bar{\epsilon}\dot{\psi}\dot{x} + \frac{i}{2}\bar{\epsilon}\dot{\psi}h'(x) + \frac{i}{2}\bar{\epsilon}\psi\frac{d}{dt}h'(x) - \frac{i}{2}\dot{\bar{\psi}}\epsilon h'(x) - \frac{i}{2}\bar{\psi}\epsilon\dot{x}h'(x) + \frac{1}{2}\epsilon\bar{\psi}\ddot{x} - \frac{1}{2}\bar{\epsilon}\psi\ddot{x}.
\end{aligned} \tag{3.18}$$

Using the fact that $\frac{d}{dt}h'(x(t)) = h''(x)\dot{x}$ we write the variation of the Lagrangian as a total time derivative which shows that it is supersymmetric

$$\delta L = \frac{d}{dt}\left(\frac{1}{2}[\epsilon\bar{\psi}\dot{x} + \psi\bar{\epsilon}\dot{x} + i\bar{\epsilon}\psi h'(x) - i\bar{\psi}\epsilon h'(x)]\right). \tag{3.19}$$

3.2.2 Supercharges

If the variation of the Lagrangian δL can be written as a total time derivative there exists a conserved charge by Noether's theorem, as we saw in chapter 2. We will now express the total time derivative in two ways, subtract them and thereby find the supercharges. We have already found one way (3.19) in the previous section using the supersymmetric transformations directly, and now we will find another through differentiation of the Lagrangian. We get

$$\delta L = \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \delta\psi\frac{\partial L}{\partial \psi} + \delta\dot{\psi}\frac{\partial L}{\partial \dot{\psi}} + \delta\bar{\psi}\frac{\partial L}{\partial \bar{\psi}} + \delta\dot{\bar{\psi}}\frac{\partial L}{\partial \dot{\bar{\psi}}}. \tag{3.20}$$

When we differentiate functions of the odd Grassmann variables ψ and $\bar{\psi}$, we need to be very careful with the partial derivatives. We have introduced the convention that $\frac{\partial}{\partial\psi_1}(\psi_1\psi_2) = \psi_2$ and $\frac{\partial}{\partial\psi_2}(\psi_1\psi_2) = -\psi_1$. Consider for example a function $H_F := \bar{\psi}\psi$. Differentiating this gives $\delta H_F = \delta(\bar{\psi}\psi) = (\delta\bar{\psi})\psi + \bar{\psi}\delta\psi$. If we then naively write $\delta H_F = \frac{\partial H_F}{\partial\bar{\psi}}\delta\bar{\psi} + \frac{\partial H_F}{\partial\psi}\delta\psi$, and then by using the differentiation convention we get $\delta H_F = \psi\delta\bar{\psi} - \bar{\psi}\delta\psi$, which is not equal to the δH_F received by differentiating directly (a minus sign differs). If we instead write $\delta H_F = \delta\bar{\psi}\frac{\partial H_F}{\partial\bar{\psi}} + \delta\psi\frac{\partial H_F}{\partial\psi}$ we get the correct result. Therefore, the convention used for the fermionic variables here, and throughout the text, is

$$\delta\psi\frac{\partial F(\psi)}{\partial\psi}, \qquad \frac{\partial}{\partial\psi_i}(\psi_i\psi_j) = \psi_j \tag{3.21}$$

where F is an arbitrary function of ψ. Putting the variation to the right of the bosonic variables is mainly to display the difference compared to the fermionic ones. One could equally put them to the left.

Back to our calculation of δL, the Lagrangian equation of motion

$$\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\delta x = (0)\delta x = 0, \tag{3.22}$$

and the mathematical trick

$$\frac{\partial L}{\partial \dot{x}}\delta\dot{x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x, \tag{3.23}$$

provide us with the second expression for the variation

$$\delta L = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\delta x + \delta\psi\frac{\partial L}{\partial\dot{\psi}} + \delta\bar{\psi}\frac{\partial L}{\partial\dot{\bar{\psi}}}\right]. \tag{3.24}$$

Inserting the supersymmetric transformations

$$\frac{\partial L}{\partial \dot{x}}\delta x = \dot{x}(\epsilon\bar{\psi} - \bar{\epsilon}\psi), \qquad \delta\psi\frac{\partial L}{\partial\dot{\psi}} = -\frac{i}{2}\epsilon(i\dot{x} + h'(x))\bar{\psi}, \qquad \delta\bar{\psi}\frac{\partial L}{\partial\dot{\bar{\psi}}} = -\frac{i}{2}\bar{\epsilon}(-i\dot{x} + h'(x))\psi \tag{3.25}$$

we get

$$\delta L = \frac{d}{dt}\left(\dot{x}(\epsilon\bar{\psi} - \bar{\epsilon}\psi) - \frac{i}{2}\epsilon(i\dot{x} + h'(x))\bar{\psi} - \frac{i}{2}\bar{\epsilon}(-i\dot{x} + h'(x))\psi\right). \tag{3.26}$$

Now we use the two expressions (3.19) and (3.26) of δL to calculate the supercharges Q and $\bar{Q}$. As the two variations are equivalent they subtract to zero and we get

$$\begin{aligned}
0 &= \frac{d}{dt}\left[\underbrace{\frac{1}{2}[\epsilon\bar{\psi}\dot{x} + \psi\bar{\epsilon}\dot{x} + i\bar{\epsilon}\psi h'(x) - i\bar{\psi}\epsilon h'(x)]}_{\alpha}\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\delta x + \delta\psi\frac{\partial L}{\partial\dot{\psi}} + \delta\bar{\psi}\frac{\partial L}{\partial\dot{\bar{\psi}}}\right] \\
&= \frac{d}{dt}\left[\alpha - \dot{x}(\epsilon\bar{\psi} - \bar{\epsilon}\psi) + \frac{i}{2}\epsilon(i\dot{x} + h'(x))\bar{\psi} + \frac{i}{2}\bar{\epsilon}(-i\dot{x} + h'(x))\psi\right] \\
&= \frac{d}{dt}\left[\alpha - \dot{x}\epsilon\bar{\psi} + \dot{x}\bar{\epsilon}\psi - \frac{1}{2}\dot{x}\epsilon\bar{\psi} + \frac{i}{2}\epsilon\bar{\psi}h'(x) + \frac{1}{2}\dot{x}\bar{\epsilon}\psi + \frac{i}{2}\bar{\epsilon}\psi h'(x)\right] \\
&= \frac{d}{dt}\left[-\dot{x}\epsilon\bar{\psi} + \dot{x}\bar{\epsilon}\psi + i\bar{\epsilon}\psi h'(x) + i\epsilon\bar{\psi}h'(x)\right] \\
&= \frac{d}{dt}\left[\bar{\epsilon}(\dot{x} + ih'(x))\psi + \epsilon(-\dot{x} + ih'(x))\bar{\psi}\right] \\
&= \frac{d}{dt}\left[-i\bar{\epsilon}\psi(-i\dot{x} + h'(x)) - i\epsilon\bar{\psi}(i\dot{x} + h'(x))\right] \\
&= \frac{d}{dt}\left[-i\bar{\epsilon}\bar{Q} - i\epsilon Q\right].
\end{aligned} \tag{3.27}$$

Now we have found the longed for conserved supercharges

$$\begin{aligned}
Q &= \bar{\psi}(i\dot{x} + h'(x)), \\
\bar{Q} &= \psi(-i\dot{x} + h'(x)).
\end{aligned} \tag{3.28}$$

3.2.3 Hamiltonian Quantization

So far, we have just considered the classical theory, where x has been a simple number, and ψ and $\overline{\psi}$ have been odd Grassmann numbers. Now, we want to quantize the system so that these variables become operators. To be able to quantize the system, we need the correct commutators and anti-commutators. We find these by just multiplying their classical analogue, the Poisson bracket, with $i\hbar$.

Consider a specific example where the Lagrangian L is the difference between kinetic and potential energy, $L = T - V$. The Hamiltonian H is the sum of kinetic and potential energy, $H = T + V$. Then $H + L = 2T$. For one simple variable in one dimension, $T = \frac{1}{2}\dot{x}p$, and p is the (generalised) momentum $p = \frac{\partial L}{\partial \dot{x}}$. We get $2T = \dot{x}\frac{\partial L}{\partial \dot{x}}$ in this simple case, as we saw in section 2.1.3. But in the present discussion, we also have two other variables, ψ and $\overline{\psi}$. Thus, the Hamiltonian becomes

$$\begin{aligned}
H = 2T - L &= \dot{\psi}\frac{\partial L}{\partial\dot{\psi}} + \dot{\bar{\psi}}\frac{\partial L}{\partial\dot{\bar{\psi}}} + \dot{x}\frac{\partial L}{\partial\dot{x}} - L \\
&= -\frac{i}{2}\dot{\psi}\bar{\psi} - \frac{i}{2}\dot{\bar{\psi}}\psi + \dot{x}\dot{x} - \frac{1}{2}\dot{x}^2 + \frac{1}{2}(h'(x))^2 - \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) + h''(x)\bar{\psi}\psi \\
&= \frac{1}{2}\dot{x}^2 + \frac{1}{2}(h'(x))^2 + h''(x)\bar{\psi}\psi \\
&= \frac{1}{2}\dot{x}^2 + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)(\bar{\psi}\psi - \psi\bar{\psi}).
\end{aligned} \tag{3.29}$$

We have chosen to rewrite the last term $h''(x)\bar{\psi}\psi$ in H as two terms, using the anti-commutativity of the odd Grassmann numbers. It will be obvious later why we do this.

We now turn to the Hamiltonian formulation of analytical mechanics. Instead of Lagrange's equation of motion in configuration space, which are of second order, we get Hamilton's equations in phase space, which are differential equations of first order. As usual, we start with the action S,

$$\begin{aligned} S &= \int dt\ L = \int dt \left(\dot{\psi}\frac{\partial L}{\partial \dot{\psi}} + \dot{\bar{\psi}}\frac{\partial L}{\partial \dot{\bar{\psi}}} + \dot{x}\frac{\partial L}{\partial \dot{x}} - H(x, \dot{x}, \psi, \bar{\psi}) \right) \\ &= \int dt \left(\dot{x}\dot{x} + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - H(x, \dot{x}, \psi, \bar{\psi}) \right) \\ &= \int dt \left(p\dot{x} + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - H(x, p, \psi, \bar{\psi}) \right). \end{aligned} \tag{3.30}$$

In the last step we have written p instead of $\dot{x}$, and the mass $m = 1$. Remembering the correct order of variation and partial derivative for the odd Grassmann variables, we may write the variation of the action

$$\begin{aligned} \delta S &= \int dt\ \delta L \\ &= \int dt\ \Big[(\delta p)\dot{x} + p\delta\dot{x} + \frac{i}{2}\left((\delta\bar{\psi})\dot{\psi} + \bar{\psi}\delta\dot{\psi} - (\delta\dot{\bar{\psi}})\psi - \dot{\bar{\psi}}\delta\psi \right) \\ &\quad - \frac{\partial H}{\partial x}\delta x - \frac{\partial H}{\partial p}\delta p - \delta\psi\frac{\partial H}{\partial \psi} - \delta\bar{\psi}\frac{\partial H}{\partial \bar{\psi}} \Big] \\ &= \int dt\ \Big[\left(\dot{x} - \frac{\partial H}{\partial p} \right)\delta p + p\frac{d}{dt}(\delta x) + \frac{i}{2}(\delta\bar{\psi})\dot{\psi} + \frac{i}{2}\bar{\psi}\frac{d}{dt}(\delta\psi) - \frac{i}{2}\frac{d}{dt}(\delta\bar{\psi})\psi - \frac{i}{2}\dot{\bar{\psi}}\delta\psi \\ &\quad - \frac{\partial H}{\partial x}\delta x - \delta\psi\frac{\partial H}{\partial \psi} - \delta\bar{\psi}\frac{\partial H}{\partial \bar{\psi}} \Big] \\ &= \int dt\ \Big[\left(\dot{x} - \frac{\partial H}{\partial p} \right)\delta p + \frac{d}{dt}(p\delta x) - \dot{p}\delta x + \frac{i}{2}(\delta\bar{\psi})\dot{\psi} + \frac{i}{2}\frac{d}{dt}(\bar{\psi}\delta\psi) - \frac{i}{2}\dot{\bar{\psi}}\delta\psi \\ &\quad - \frac{i}{2}\frac{d}{dt}\left((\delta\bar{\psi})\psi \right) + \frac{i}{2}(\delta\bar{\psi})\dot{\psi} - \frac{i}{2}\dot{\bar{\psi}}\delta\psi - \frac{\partial H}{\partial x}\delta x - \delta\psi\frac{\partial H}{\partial \psi} - \delta\bar{\psi}\frac{\partial H}{\partial \bar{\psi}} \Big] \\ &= \int dt\ \Big[\left(\dot{x} - \frac{\partial H}{\partial p} \right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial x} \right)\delta x + \left(-i\dot{\bar{\psi}} + \frac{\partial H}{\partial \psi} \right)\delta\psi + \left(-i\dot{\psi} + \frac{\partial H}{\partial \bar{\psi}} \right)\delta\bar{\psi} \Big] \\ &\quad + \Big[(p\delta x) + \frac{i}{2}(\bar{\psi}\delta\psi) - \frac{i}{2}\left((\delta\bar{\psi})\psi \right) \Big]_{\text{start point}}^{\text{end point}} \\ &= 0 \end{aligned} \tag{3.31}$$

with the integrated term equal to zero. The remaining integral shall be zero for all super-symmetric variations. Then each term in the brackets must be identically zero. Thus we obtain Hamilton's equations of motion

$$\dot{x} = \frac{\partial H}{\partial p}; \qquad \dot{p} = -\frac{\partial H}{\partial x}; \qquad \dot{\psi} = -i\frac{\partial H}{\partial \bar{\psi}}; \qquad \dot{\bar{\psi}} = -i\frac{\partial H}{\partial \psi}. \tag{3.32}$$

Now we are ready to find an expression for the Poisson bracket that we met in (2.25). By using Hamilton's equations in the following expression for the time derivative of the action $S = S(x, p, \psi, \bar{\psi})$ we reach our goal

$$\begin{aligned} \dot{S} &= \frac{dS}{dt} = \frac{\partial S}{\partial p}\dot{p} + \frac{\partial S}{\partial x}\dot{x} + \frac{\partial S}{\partial \psi}\dot{\psi} + \frac{\partial S}{\partial \bar{\psi}}\dot{\bar{\psi}} = -\frac{\partial S}{\partial p}\frac{\partial H}{\partial x} + \frac{\partial S}{\partial x}\frac{\partial H}{\partial p} - i\frac{\partial S}{\partial \psi}\frac{\partial H}{\partial \bar{\psi}} - i\frac{\partial S}{\partial \bar{\psi}}\frac{\partial H}{\partial \psi} \\ &= S\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\partial}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\partial}{\partial x} - i\left(\frac{\overleftarrow{\partial}}{\partial \psi}\frac{\partial}{\partial \bar{\psi}} + \frac{\overleftarrow{\partial}}{\partial \bar{\psi}}\frac{\partial}{\partial \psi} \right) \right) H =: \{S, H\}_P. \end{aligned} \tag{3.33}$$

We define $\overleftarrow{\frac{\partial}{\partial x}}$ as a partial differentiation to the factor on the left. Let us compute some Poisson brackets, using the definition given in (3.33) above

$$\{x,x\}_P = 0; \quad \{p,p\}_P = 0; \quad \{x,p\}_P = 1; \quad \{\psi,\psi\}_P = 0; \quad \{\bar{\psi},\bar{\psi}\}_P = 0; \quad \{\psi,\bar{\psi}\}_P = -i. \tag{3.34}$$

We now leave the classical physics behind and turn wholeheartedly to quantum mechanics. We multiply the Poisson brackets with $i\hbar$ and get commutators [,] in the case of bosonic variables (x and p), and anti-commutators { , } in the case of fermionic variables (ψ and $\bar{\psi}$). We skip the convention to write hat ˆ over the variables which now have become operators. When setting $\hbar = 1$ we get

$$[x,x] = 0; \quad [p,p] = 0; \quad [x,p] = i; \quad \{\psi,\psi\} = 0; \quad \{\bar{\psi},\bar{\psi}\} = 0; \quad \{\psi,\bar{\psi}\} = 1 \tag{3.35}$$

Let us restate the Hamiltonian we defined before

$$H = \frac{1}{2}p^2 + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)(\bar{\psi}\psi - \psi\bar{\psi}). \tag{3.36}$$

Notice that the terms of H are operators on a Hilbert space from now on.

3.3 General Structure of the Hilbert Space and the Witten Index

The purpose of this section is to introduce the Witten index $\mathrm{Tr}(-1)^F$. To reach this goal we first have to study the structure of the Hilbert space. We will see how the Hilbert space decomposes into fermionic and bosonic subspaces, and with this decomposition we will find the Witten index.

3.3.1 Supersymmetric Hilbert Space

In this section we will investigate some of the properties which we can ascribe the Hilbert space. We note the important identity following from the anti-commutator between x and p

$$[p, f(x)] = -if'(x) \tag{3.37}$$

where f is an arbitrary (albeit analytic) function of x. The proof of this identity can be found in appendix A.1.

We now introduce an operator F, defined as

$$F = \bar{\psi}\psi. \tag{3.38}$$

This operator is generally called the *Fermion number operator*, we will see later on why this name is chosen. F satisfies the following commutation relations

$$\begin{aligned}
[F,\psi] &= \bar{\psi}\psi^2 - \psi\bar{\psi}\psi = -(\{\psi,\bar{\psi}\} - \bar{\psi}\psi)\psi \\
&= -\psi \\
[F,\bar{\psi}] &= \bar{\psi}\psi\bar{\psi} - \bar{\psi}^2\psi = \bar{\psi}(\{\psi,\bar{\psi}\} - \bar{\psi}\psi) \\
&= \bar{\psi}.
\end{aligned} \tag{3.39}$$

This works because the square of a fermionic (odd Grassmann) variable is always zero. We now define a state $|0\rangle$ as

$$\psi|0\rangle = 0. \tag{3.40}$$

In order to assure ourselves that this definition makes sense let us see what ψ does to a randomly chosen state $|v\rangle$. There are two options here

$$\begin{aligned}
\psi|v\rangle &= 0 \\
\psi|v\rangle &\neq 0.
\end{aligned} \tag{3.41}$$

In the first case we can set $|v\rangle = |0\rangle$, in the second case we can set $\psi|v\rangle = |0\rangle$ because

$$\psi(\psi|v\rangle) = \psi^2|v\rangle = 0|v\rangle = 0. \tag{3.42}$$

So $\psi|v\rangle$ then satisfies our defining condition. We can think of ψ and $\bar{\psi}$ as analogous to the raising and lowering operators used in the treatment of the simple harmonic oscillator (section 2.2.1). This means that we can use the 'raising' operator $\bar{\psi}$ to build up a 'tower' of states from $|0\rangle$. But because $\bar{\psi}^2 = 0$ the tower contains only two vectors, namely

$$|0\rangle, \bar{\psi}|0\rangle. \tag{3.43}$$

These vectors span a two-dimensional space. We impose a coordinate system by setting

$$|0\rangle := (1,0)^T \quad \text{and} \quad \psi|0\rangle := (0,1)^T. \tag{3.44}$$

In this basis ψ and $\bar{\psi}$ are represented by

$$\psi := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{\psi} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.45}$$

This is only one part of the total Hilbert space on which the full Hamiltonian operates. The other part is spanned by the eigenvectors of the position operator (or the momentum operator: same space, different representation). This space is the space of square normalizable complex functions of a real variable (think wave functions). This space is denoted by $L^2(\mathbb{R}, \mathbb{C})$ for short. The full Hilbert space is then given by the tensor product of these two spaces

$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}) \otimes \text{Space}(|0\rangle, \bar{\psi}|0\rangle) \tag{3.46}$$

or equivalently

$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})|0\rangle \oplus L^2(\mathbb{R}, \mathbb{C})\bar{\psi}|0\rangle. \tag{3.47}$$

We now call the first and second component of $\mathcal{H}$ the space of bosonic states and the space of fermionic states, respectively. We write this as

$$\begin{aligned} \mathcal{H}^B &= L^2(\mathbb{R}, \mathbb{C})|0\rangle \\ \mathcal{H}^F &= L^2(\mathbb{R}, \mathbb{C})\bar{\psi}|0\rangle. \end{aligned} \tag{3.48}$$

Note that the operator F is the zero operator in the bosonic space and the identity operator in the fermionic space, which makes the choice of name apparent. We introduce an operator $(-1)^F$ that is the identity operator in the bosonic space and minus the identity in the fermionic space. This induces a $\mathbb{Z}_2$-grading on the space $\mathcal{H}$. A $\mathbb{Z}_p$-graded vector space is a space that can be decomposed into a direct sum of subspaces indexed by the elements of $\mathbb{Z}_p$. In our case $p = 2$ and the two subspaces are $\mathcal{H}^B$ and $\mathcal{H}^F$. We now use the charges Q and $\bar{Q}$ we defined earlier. We repeat them for clarity

$$\begin{aligned} Q &= \bar{\psi}\left(ip + h'(x)\right), \\ \bar{Q} &= \psi\left(-ip + h'(x)\right). \end{aligned} \tag{3.49}$$

Let us evaluate the commutator of Q with the Hamiltonian

$$\begin{aligned}
[H, Q] &= \left[\frac{1}{2}p^2 + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)(\bar{\psi}\psi - \psi\bar{\psi}), \bar{\psi}\left(ip + h'(x)\right)\right] \\
&= \frac{\bar{\psi}}{2}\left[p^2, ip\right] + \frac{\bar{\psi}}{2}\left[p^2, h'(x)\right] + \frac{\bar{\psi}}{2}\left[h'(x)^2, ip\right] + \frac{\bar{\psi}}{2}\left[h'(x)^2, h'(x)\right] + \frac{1}{2}\left[h''(x)\bar{\psi}\psi, \bar{\psi}\left(ip + h'(x)\right)\right] \\
&\quad - \frac{1}{2}\left[h''(x)\psi\bar{\psi}, \bar{\psi}\left(ip + h'(x)\right)\right] \\
&= \frac{\bar{\psi}}{2}\left[p^2, h'(x)\right] + \frac{\bar{\psi}}{2}\left[h'(x)^2, ip\right] + \frac{\bar{\psi}}{2}h''(x)\psi\bar{\psi}\left(ip + h'(x)\right) + \frac{\bar{\psi}}{2}\left(ip + h'(x)\right)\psi\bar{\psi}h''(x) \\
&= \frac{\bar{\psi}}{2}pph'(x) - \frac{\bar{\psi}}{2}h'(x)pp - \frac{\bar{\psi}}{2}h'(x)h''(x) + \frac{\bar{\psi}}{2}\left(ih''(x)p + iph''(x)\right) + \bar{\psi}h''(x)h'(x) \\
&= \frac{\bar{\psi}}{2}\left(h'(x)p^2 + [p, h'(x)]p + p[p, h'(x)] - h'(x)p^2\right) + \frac{\bar{\psi}}{2}\left(ih''(x)p + iph''(x)\right) \\
&= -\frac{\bar{\psi}}{2}\left(ih''(x)p + iph''(x)\right) + \frac{\bar{\psi}}{2}\left(ih''(x)p + iph''(x)\right) \\
&= 0
\end{aligned} \tag{3.50}$$

This means that Q is a conserved quantity. Hence the notion we have of Q as a conserved charge in classical mechanics carries over into the quantum realm. In a similar calculation we find

$$[H, \bar{Q}] = 0. \tag{3.51}$$

We can also evaluate the commutator of Q with F

$$\begin{aligned}
[F, Q] &= FQ - QF \\
&= \bar{\psi}\psi\bar{\psi}(ip + h'(x)) - \bar{\psi}(ip + h'(x))\bar{\psi}\psi \\
&= \left(\{\bar{\psi}\psi\} - \psi\bar{\psi}\right)\bar{\psi}(ip + h'(x)) - \bar{\psi}\bar{\psi}\psi(ip + h'(x)) \\
&= (1 - \psi\bar{\psi})\bar{\psi}(ip + h'(x)) \\
&= \bar{\psi}(ip + h'(x)) \\
&= Q.
\end{aligned} \tag{3.52}$$

Similarly we can find that

$$[F, \bar{Q}] = -\bar{Q}. \tag{3.53}$$

Now rewrite $(-1)^F$ as

$$(-1)^F = 1 - 2F.$$

It is possible to do so since this new operator $1 - 2F$ does the same thing as $(-1)^F$. When $F = 0$ both operators equal 1, and when $F = 1$ both operators are -1. It is then easier to see that Q and $(-1)^F$ anticommutes

$$\begin{aligned}
Q(-1)^F &= Q(1 - 2F) \\
&= Q - 2Q(\bar{\psi}\psi) \\
&= Q - 2Q + 2\bar{\psi}(ip + h'(x))\psi\bar{\psi} \\
&= Q - 2Q + 2\left(\bar{\psi}\psi\right)\left(\bar{\psi}(ip + h'(x))\right) \\
&= (1 - 2 + 2FQ \\
&= (1 - 2 + 2F)Q \\
&= -(1 - 2F)Q = -(-1)^F Q.
\end{aligned} \tag{3.54}$$

Note the subtle order of operations and the use of the anti-commutator of ψ and $\bar{\psi}$. Similarly we can see that

$$\bar{Q}(-1)^F = -(-1)^F \bar{Q}. \tag{3.55}$$

This tells us that the charges Q and $\bar{Q}$ map bosonic states onto fermionic states and vice versa. Because $\psi^2 = \bar{\psi}^2 = 0$ the charges are nilpotent and we get

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0.$$

We also know that

$$Q^2 = (Q^\dagger)^2 = 0. \tag{3.56}$$

Now we compute the anti-commutator between Q and $\bar{Q}$

$$\begin{aligned}
\{Q, \bar{Q}\} &= \{\bar{\psi}\,(ip + h'(x))\,, \psi\,(-ip + h'(x))\} \\
&= \{\bar{\psi}ip, \psi(-ip)\} + \{\bar{\psi}h'(x), \psi h'(x)\} + i\{\bar{\psi}p, \psi h'(x)\} - i\{\bar{\psi}h'(x), \psi p\} \\
&= p^2 + (h'(x))^2 + i(\bar{\psi}\psi - \psi\bar{\psi})[p, h'(x)] \\
&= p^2 + h'(x)^2 + h''(x)(\bar{\psi}\psi - \psi\bar{\psi}) \\
&= 2H,
\end{aligned} \tag{3.57}$$

this means we can write the Hamiltonian as,

$$H = \frac{1}{2}(QQ^\dagger + Q^\dagger Q). \tag{3.58}$$

It is actually easier to calculate the commutators between H and Q, $\bar{Q}$, with the Hamiltonian written in the form (3.58).

We know that the operator $(-1)^F$, that denotes our $\mathbb{Z}_2$ grading, obeys the relations: $Q(-1)^F = -(-1)^F Q$ and $\bar{Q}(-1)^F = -(-1)^F \bar{Q}$. Because of that we will call the supercharges odd. The Hamiltonian however, is even for the following reason

$$\begin{aligned} H(-1)^F &= \frac{1}{2}(QQ^\dagger + Q^\dagger Q)(-1)^F \\ &= \frac{1}{2}(QQ^\dagger(-1)^F + Q^\dagger Q(-1)^F) \\ &= \frac{1}{2}(-Q(-1)^F Q^\dagger - Q^\dagger(-1)^F Q) \\ &= \frac{1}{2}((-1)^F QQ^\dagger + (-1)^F Q^\dagger Q) \\ &= (-1)^F H. \end{aligned} \tag{3.59}$$

We denote the subspace of $\mathcal{H}$ on which $(-1)^F = 1$, where F has to be an even number, by the *even* (bosonic) subspace $\mathcal{H}^B$, and the subspace of $\mathcal{H}$ on which $(-1)^F = -1$, with F an odd number, by the *odd* (fermionic) subspace $\mathcal{H}^F$.

As alluded to earlier in this section the supercharges map one subspace to the other, i.e when the charges act on one of the subspaces it takes it to the other subspace

$$\begin{aligned} Q, Q^\dagger &: \mathcal{H}^B \to \mathcal{H}^F \\ Q, Q^\dagger &: \mathcal{H}^F \to \mathcal{H}^B. \end{aligned} \tag{3.60}$$

We show this by using the basis vector for each subspace $|0\rangle$ and $\bar{\psi}|0\rangle$ and let Q and $Q^\dagger$ act on them. Note that p and $h'(x)$ have no impact on which space we are in and therefore we do not need to take them into consideration

$$Q|0\rangle \propto \bar{\psi}|0\rangle \in \mathcal{H}^F \tag{3.61}$$

and

$$\begin{aligned} \bar{Q}\bar{\psi}|0\rangle \propto \psi\bar{\psi}|0\rangle &= \left(\{\psi, \bar{\psi}\} - \bar{\psi}\psi\right)|0\rangle \\ &= 1|0\rangle - F|0\rangle \\ &= 1|0\rangle \in \mathcal{H}^B. \end{aligned} \tag{3.62}$$

This method of proving the mapping relations stated above, gives rise to some complications when we consider how $\bar{Q}$ maps a bosonic state and how Q maps a fermionic state. If we use the same method as before we get

$$\begin{aligned} \bar{Q}|0\rangle &\propto \psi|0\rangle = 0, \\ Q\bar{\psi}|0\rangle &\propto \bar{\psi}^2|0\rangle = 0. \end{aligned} \tag{3.63}$$

The zero vector is not a state, that we can define as either bosonic nor fermionic. To really prove all the relations above we will consider a more general case, in which the fermion number operator F can take values $F = n,\ n = 0, 1, 2...$ Then

$$F|\psi_n\rangle = n|\psi_n\rangle, \tag{3.64}$$

$$\psi_n = \begin{cases} \text{bosonic} & \text{if } n \text{ is even} \\ \text{fermionic} & \text{if } n \text{ is odd.} \end{cases} \tag{3.65}$$

Now, let F count on the state $Q|\psi_n\rangle$ instead

$$\begin{aligned} FQ|\psi_n\rangle &= ([F, Q] + QF)|\psi_n\rangle \\ &= (Q + QF)|\psi_n\rangle \\ &= (n+1)Q|\psi_n\rangle. \end{aligned} \tag{3.66}$$

So F counts to $n+1$ on the state $Q|\psi_n\rangle$. If n is even, $n+1$ is obviously odd and vice versa, i.e. we have made the transition from our original state to the other, by letting Q act on it.

And by using the commutation relations once again, we come to the same conclusion about the action of $\bar{Q}$ on a given state,

$$F\bar{Q}|\psi_n\rangle = (n-1)\bar{Q}|\psi_n\rangle. \tag{3.67}$$

The Hamiltonian on the other hand will map a state on itself. It is easily seen when we express the Hamiltonian as $H = \frac{1}{2}(Q\bar{Q} + Q\bar{Q})$. Now Q or $\bar{Q}$ will first take you from your original state to the other, but then Q or $\bar{Q}$ will take you back to the state you first started in. One can say that the Hamiltonian preserves the decomposition of the Hilbert space: $\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$.

The scalar product that defines the Hilbert space, implies that the norm of a vector is positive

$$\parallel Q|\alpha\rangle \parallel \geq 0. \tag{3.68}$$

Since $Q|\alpha\rangle$ is complex taking the norm of it means that we multiply it with its complex conjugate $\langle\alpha|\bar{Q}$

$$\langle\alpha|\bar{Q}Q|\alpha\rangle \geq 0. \tag{3.69}$$

In the same way we get that $\parallel \bar{Q}|\alpha\rangle \parallel \geq 0$ and

$$\langle\alpha|Q\bar{Q}|\alpha\rangle \geq 0. \tag{3.70}$$

If we add these two expressions, we can draw the conclusion that the Hamiltonian H is a positive operator,

$$\langle\alpha|\bar{Q}Q|\alpha\rangle + \langle\alpha|Q\bar{Q}|\alpha\rangle = \langle\alpha|\bar{Q}Q + Q\bar{Q}|\alpha\rangle = \langle\alpha|2H|\alpha\rangle \geq 0$$

i.e

$$H = \frac{1}{2}\{Q\bar{Q}\} \geq 0. \tag{3.71}$$

Now, assume there is a ground state with zero energy $H|\alpha\rangle = 0$. This means that $\langle\alpha|H|\alpha\rangle = 0$ and $\parallel Q|\alpha\rangle \parallel + \parallel \bar{Q}|\alpha\rangle \parallel = 0$. As we said before, the norm is always positive, and therefore

$$H|\alpha\rangle = 0 \iff Q|\alpha\rangle = \bar{Q}|\alpha\rangle = 0. \tag{3.72}$$

A state that is annihilated by the supercharges is invariant under supersymmetry and we will call such a state a supersymmetric state. What we just showed above is that the zero energy ground state is in fact a supersymmetric state. It also implies the converse: a supersymmetric state is a zero energy ground state. We will call such a state, not that surprisingly, a supersymmetric ground state.

The Hilbert space can be decomposed in terms of eigenspaces of the Hamiltonian

$$\mathcal{H} = \oplus_{n=1,2\ldots}\mathcal{H}_{(n)}, \quad \text{such that} \quad H|_{\mathcal{H}_{(n)}} = E_{(n)}. \tag{3.73}$$

We have already seen that Q, $\bar{Q}$ and $(-1)^F$ all commute with the Hamiltonian. So, if we operate on a state with one of these operators, the energy level does not change, they preserve the energy levels

$$Q, \bar{Q}, (-1)^F \ : \ \mathcal{H}_{(n)} \mapsto \mathcal{H}_{(n)}. \tag{3.74}$$

However, the decomposition of our Hilbert space does not stop here. We have earlier in this section showed that we can split up each energy level into bosonic and fermionic subspaces, or even and odd subspaces if you like (referring to the number counted by the fermion number operator F),

$$\mathcal{H}_{(n)} = \mathcal{H}^B_{(n)} + \mathcal{H}^F_{(n)}. \tag{3.75}$$

We repeat that the supercharges map one subspace to the other, but now with the new subscript

$$Q, Q^\dagger \ : \ \mathcal{H}^B_{(n)} \mapsto \mathcal{H}^F_{(n)} \ ; \ \mathcal{H}^F_{(n)} \mapsto \mathcal{H}^B_{(n)}. \tag{3.76}$$

3.3.2 Witten Index

Now it is time for the Witten index, the future gateway to the Euler characteristic [2]. Let us start by forming a new operator $Q_1 := Q + Q^\dagger$. Since Q and $Q^\dagger$ square to zero, this operator connects to the Hamiltonian in the following way

$$Q_1^2 = 2H. \tag{3.77}$$

This newly formed operator commutes with the Hamiltonian and preserves each energy level. It also maps the bosonic subspace to the fermionic subspace and vice versa. Let us look at the inverse of Q_1, such that $Q_1 Q_1^{-1} = Q_1^{-1} Q_1 = 1$. We use that Q_1^2 at the nth energy level, i.e. in the subspace $\mathcal{H}_{(n)}$, squares to $Q_1^2 = 2E_n$. The inverse is then $Q_1^{-1} = \frac{Q_1}{2E_n}$. This is easily seen by

$$Q_1 Q_1^{-1} = Q_1^{-1} Q_1 = \frac{Q_1^2}{2E_n} = 1. \tag{3.78}$$

As long as the energy $E_n > 0$ the inverse exists, and as we can see the inverse will map the two states back to their original state. So the operator Q_1 defines an *isomorphism* between the two subspaces $\mathcal{H}^B_{(n)}$ and $\mathcal{H}^F_{(n)}$

$$\mathcal{H}^B_{(n)} \cong \mathcal{H}^F_{(n)}. \tag{3.79}$$

Thus, each excited energy level comes with a paired bosonic and fermionic state. Note that we wrote each *excited* level, since we have no such restrictions on the zero energy state. Hence the supersymmetric ground states do not have to be paired.

The Witten index $\mathrm{Tr}(-1)^F$ states that the number of bosonic ground states minus the number of fermionic ground states,

$$\mathrm{Tr}(-1)^F = \{\textit{number of bosonic ground states}\} - \{\textit{number of fermionic ground states}\},$$

is invariant. To see the logic in this statement we consider a continuous deformation of the theory of energy states we have developed so far. Now the energy levels may split up into several levels. We may have newly formed energy states but there must be the same number of bosonic and fermionic states at each level, due to the isomorphism discussed above. A positive energy state may be annihilated and a zero energy state may acquire positive energy, again the positive energy state must come with both a bosonic and fermionic state. We represent this in a more mathematical way as

$$\mathrm{Tr}(-1)^F = \mathrm{dim}\mathcal{H}^B_{(0)} - \mathrm{dim}\mathcal{H}^F_{(0)}. \tag{3.80}$$

Since the operator $(-1)^F$ is -1 for a fermionic state and $+1$ for a bosonic state, when we calculate the trace, all the excited energy states will be cancelled out. The only states that survive is the supersymmetric ground states. One should not take (3.80) too literally, but rather look at it as a useful definition. Since it is an infinite summation, over all states, it is ill defined and not convergent. One can regularize $\mathrm{Tr}(-1)^F$ and get a commonly used expression for the Witten index,

$$\mathrm{Tr}(-1)^F e^{-\beta H}. \tag{3.81}$$

This expression is not dependent of β, since all non zero energy states cancels. It also gives back the first expression in the case where $\beta \to 0$.

3.4 Example: Ground States in the Supersymmetric Potential Theory

To find the supersymmetric ground states of a supersymmetric potential theory, we start by representing the supercharges in the two-dimensional ground state basis $(|0\rangle, \bar{\psi}|0\rangle)$

$$\begin{aligned} Q = \bar{\psi}(ip + h'(x)) &= \begin{pmatrix} 0 & 0 \\ d/dx + h'(x) & 0 \end{pmatrix}, \\ \bar{Q} = \psi(-ip + h'(x)) &= \begin{pmatrix} 0 & -d/dx + h'(x) \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{3.82}$$

where p is the usual operator $p = -i\frac{\partial}{\partial x}$. A supersymmetric ground state is a state annihilated by the supercharges ($\bar{Q}\psi = Q\psi = 0$). If we write a state on the form $\Psi = f_1(x)|0\rangle + f_2(x)\bar{\psi}|0\rangle$, and operate on it with Q and $\bar{Q}$ we can find the differential equations that $f_1(x)$ and $f_2(x)$ must fulfill to make Ψ a ground state

$$Q\Psi = 0 \quad \Rightarrow \quad \left(\frac{d}{dx} + h'(x)\right) f_1(x) = 0 \tag{3.83}$$

$$\bar{Q}\Psi = 0 \quad \Rightarrow \quad \left(-\frac{d}{dx} + h'(x)\right) f_2(x) = 0 \tag{3.84}$$

with the solutions

$$f_1(x) = c_1 e^{-h(x)}, \qquad f_2(x) = c_2 e^{h(x)}. \tag{3.85}$$

We want these two solutions to be square normalizable which means we have to look at how $h(x)$ behaves when $x \to \pm\infty$. We consider three different behaviours of $h(x)$

- Case 1: $h(x) \to \infty$ at both $x \to \pm\infty$. Here $e^{-h(x)}$ is normalizable but $e^{h(x)}$ is not. The supersymmetric ground state is given by

  $$\Psi = f_1(x)|0\rangle = c_2 e^{-h(x)}|0\rangle. \tag{3.86}$$

  This state belongs to $\mathcal{H}^B$ with the supersymmetric index

  $$\mathrm{Tr}(-1)^F = 1. \tag{3.87}$$

- Case 2: $h(x) \to -\infty$ at both $x \to \pm\infty$. Here $e^{h(x)}$ is normalizable but $e^{-h(x)}$ is not. We again have one supersymmetric ground state

  $$\Psi = f_2(x)\bar{\psi}|0\rangle = c_2 e^{h(x)}\bar{\psi}|0\rangle. \tag{3.88}$$

  This of course belongs to $\mathcal{H}^F$ and the Witten index is

  $$\mathrm{Tr}(-1)^F = -1. \tag{3.89}$$

- Case 3: $h(x) \to -\infty$ when $x \to -\infty$ and $h(x) \to \infty$ when $x \to \infty$ or the opposite when the sign of $h(x)$ is flipped. Here none of $e^{h(x)}$ or $e^{-h(x)}$ are normalizable, so in this case we have no supersymmetric ground state, giving us the Witten index

  $$\mathrm{Tr}(-1)^F = 0. \tag{3.90}$$

3.5 Example: Ground States and Spectrum of the Supersymmetric Harmonic Oscillator

We remember the Hamiltonian given in (3.36)

$$H = \frac{1}{2}p^2 + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)(\bar{\psi}\psi - \psi\bar{\psi}).$$

In the preceding example, the superpotential $h(x)$ was quite general. Now we consider the special case of the harmonic oscillator, with potential energy term $V(x) = \frac{1}{2}\omega^2 x^2$. Then, since in the Hamiltonian above $\frac{1}{2}\left(h'(x)\right)^2 = V(x)$, we get $(h'(x))^2 = \omega^2 x^2$, and $h(x) = \frac{1}{2}\omega x^2$. We also find that $h''(x) = \omega$. Thus the Hamiltonian becomes

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega(\bar{\psi}\psi - \psi\bar{\psi}). \tag{3.91}$$

Let us for simplicity assume that $\omega > 0$. Then $h(x) \to \infty$ when $\pm x \to \infty$, so following the analysis of the preceding example in 3.4, we find the supersymmetric ground states to be $\psi(x) = e^{-\frac{1}{2}\omega x^2}|0\rangle$.

What is then the spectrum of the supersymmetric harmonic oscillator? Let us divide the Hamiltonian H in (3.91) into two parts, one bosonic part H_B and one fermionic part

H_F. Since they commute, they will act separately and we can treat their respective spectra separately

$$H = H_B + H_F; \qquad H_B = \frac{1}{2}\left(p^2 + \omega^2 x^2\right), \qquad H_F = \frac{1}{2}\omega(\bar{\psi}\psi - \psi\bar{\psi}). \tag{3.92}$$

We found the bosonic spectrum for H_B in section 2.2.1. There we used a Hamiltonian for which $\omega = 1$. When we include ω in the calculations we get the result

$$\frac{1}{2}\omega, \frac{3}{2}\omega, \frac{5}{2}\omega, \dots.$$

One can repeat the process of using raising and lowering operators for the fermionic part of the harmonic oscillator, but we will find its spectrum in another way.

In the preceding chapters, we have used $\{|0\rangle_F; \bar{\psi}|0\rangle_F\}$ as the basis for the fermionic part of the Hilbert space. We use our matrix representations (3.45) for ψ and $\bar{\psi}$

$$\psi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \bar{\psi} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.93}$$

Then the fermionic part of the Hamiltonian may be expressed as

$$H_F = \frac{1}{2}\omega(\bar{\psi}\psi - \psi\bar{\psi}) = \frac{1}{2}\omega\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \frac{1}{2}\omega\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.94}$$

Now we want the eigenvalues of this operator

$$\frac{1}{2}\omega\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \lambda\mathbf{x}. \tag{3.95}$$

For non-trivial $\mathbf{x}$ we have

$$\begin{vmatrix} -\frac{\omega}{2} - \lambda & 0 \\ 0 & \frac{\omega}{2} - \lambda \end{vmatrix} = 0, \qquad \lambda = \pm\frac{\omega}{2}. \tag{3.96}$$

Since $-\frac{\omega}{2} < \frac{\omega}{2}$ for $\omega > 0$, we take $-\frac{\omega}{2}$ to be the fermionic ground state energy. We find the fermionic spectrum to be $-\frac{1}{2}\omega, \frac{1}{2}\omega$.

The spectrum of the total Hamiltonian is the sum of the two spectra above. We add the eigenvalues separately. We get one series of energies when the fermion number is zero

$$-\frac{\omega}{2} + \frac{\omega}{2}, -\frac{\omega}{2} + \frac{3}{2}\omega, -\frac{\omega}{2} + \frac{5}{2}\omega, \dots = 0, \omega, 2\omega, 3\omega, \dots \tag{3.97}$$

We get another series of energies when the fermion number is one

$$\frac{\omega}{2} + \frac{\omega}{2}, \frac{\omega}{2} + \frac{3}{2}\omega, \frac{\omega}{2} + \frac{5}{2}\omega, \dots = \omega, 2\omega, 3\omega, \dots \tag{3.98}$$

See figure 3.1. Note that there is no state for which $F = 1$ where the energy is zero. For

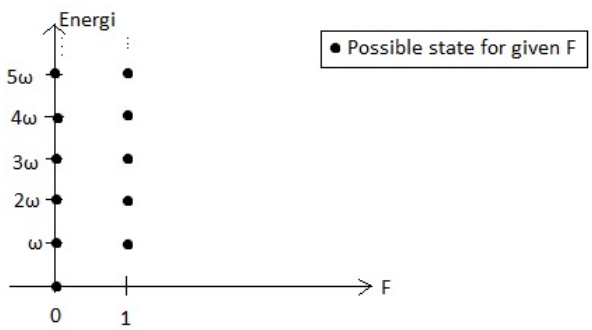


Figure 3.1: Spectrum of the supersymmetric harmonic oscillator.

all higher states there is a pairing with two states, one bosonic and one fermionic for each energy, but this pairing is broken for the ground state energy level.

Having found the spectrum of the Hamiltonian, it is not difficult to compute the partition function of the Witten index of the system. When calculating traces of operators, we must know which Hilbert space we are in. If we treat the Hamiltonian as two different parts, we must accordingly treat the Hilbert space as two different Hilbert spaces. Let $L^2 = L^2(\mathbb{R}, \mathbb{C})$ be the Hilbert space of the bosonic harmonic oscillator, and $\mathbb{C}^2 = \mathbb{C}|0\rangle \oplus \mathbb{C}\bar{\psi}|0\rangle$ be the Hilbert space of the fermionic oscillator. Then the total Hilbert space $\mathcal{H}$ is

$$\begin{aligned} \mathcal{H} &= L^2 \otimes \mathbb{C}^2 \\ &= L^2 \otimes (\mathbb{C}|0\rangle \oplus \ \mathbb{C}\bar{\psi}|0\rangle) \\ &= \left\{ L^2(\mathbb{R}, \mathbb{C}) \otimes \mathbb{C}|0\rangle \right\} \oplus \left\{ L^2(\mathbb{R}, \mathbb{C}) \otimes \mathbb{C}\bar{\psi}|0\rangle \right\}. \end{aligned} \tag{3.99}$$

We make the direct sum of two spaces which can be seen as the two columns in figure 3.1. The spaces represented by these columns are tensor products of the bosonic space and the two halves of the fermionic space. The partition function is

$$\begin{aligned} Z(\beta) &= \mathrm{Tr}_{\mathcal{H}}\ e^{-\beta H} \\ &= \mathrm{Tr}_{\mathcal{H}}\ e^{-\beta(H_B + H_F)} \\ &= \mathrm{Tr}_{\mathcal{H}} \left(e^{-\beta H_B} \cdot e^{-\beta H_F} \right) \\ &= \mathrm{Tr}_{L^2}\ e^{-\beta H_B} \cdot \mathrm{Tr}_{\mathbb{C}^2}\ e^{-\beta H_F}. \end{aligned} \tag{3.100}$$

We calculate the traces separately, first for L^2

$$\begin{aligned} \mathrm{Tr}_{L^2}\ e^{-\beta H_B} &= \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\omega} = e^{-\frac{\beta}{2}\omega} \sum_{n=0}^{\infty} \left(e^{-\beta\omega}\right)^n \\ &= \frac{e^{-\frac{\beta}{2}\omega}}{1 - e^{-\beta\omega}} = \frac{1}{e^{\frac{\beta}{2}\omega} - e^{-\frac{\beta}{2}\omega}} \\ &= \frac{1}{2\sinh\frac{\beta\omega}{2}}, \end{aligned} \tag{3.101}$$

and then for $\mathbb{C}^2$

$$\mathrm{Tr}_{\mathbb{C}^2}\ e^{-\beta H_F} = e^{-\beta\left(-\frac{\omega}{2}\right)} + e^{-\beta\frac{\omega}{2}} = 2\cosh\frac{\beta\omega}{2}. \tag{3.102}$$

In total we get

$$Z(\beta) = \frac{1}{2\sinh\frac{\beta\omega}{2}} \cdot 2\cosh\frac{\beta\omega}{2} = \coth\frac{\beta\omega}{2}. \tag{3.103}$$

We clearly see that the partition function depends only on β, the circumference in time of the circle. The Witten index becomes

$$\begin{aligned} \mathrm{Tr}(-1)^F &= \mathrm{Tr}_{\mathcal{H}} \left((-1)^F\ e^{-\beta H} \right) = \mathrm{Tr}_{\mathcal{H}} \left((-1)^F e^{-\beta H_B} e^{-\beta H_F} \right) \\ &= \mathrm{Tr}_{\mathcal{H}} \left(e^{-\beta H_B} \left[(-1)^F e^{-\beta H_F} \right] \right) \\ &= \mathrm{Tr}_{L^2}\ e^{-\beta H_B} \cdot \mathrm{Tr}_{\mathbb{C}^2} \left((-1)^F e^{-\beta H_F} \right). \end{aligned} \tag{3.104}$$

We find

$$\mathrm{Tr}_{\mathbb{C}^2} \left((-1)^F e^{-\beta H_F} \right) = (-1)^0 e^{-\beta\left(-\frac{\omega}{2}\right)} + (-1)^1 e^{-\beta\frac{\omega}{2}} = 2\sinh\frac{\beta\omega}{2}. \tag{3.105}$$

Then we get the Witten index

$$\mathrm{Tr}(-1)^F = \frac{1}{2\sinh\frac{\beta\omega}{2}}\ 2\sinh\frac{\beta\omega}{2} = 1. \tag{3.106}$$

We note that this is the same result as predicted in the preceding study of ground states for different spaces. Since $h(x) \to \infty$ for both $x \to \pm\infty$, we are in case 1, where $\mathrm{Tr}(-1)^F = 1$. We also note that $\mathrm{Tr}(-1)^F$ does not depend on β, as it must be if we want $\mathrm{Tr}(-1)^F$ to count a number of physical states. One could also study figure 3.1, take the difference of the fermionic and bosonic ground states and come to the same conclusion.

Chapter 4

Basics of Riemannian Geometry and Topology

In this chapter we will introduce a number of mathematical concepts and results that will be needed when we start working on sigma models in curved manifolds. The major part of the text will focus on elements of basic differential geometry beginning with the fundamental concept of a manifold. A manifold can be looked upon as the formalisation of the intuitive notion of a 'curved' space, we can for instance describe the sphere and the torus as manifolds. On these manifolds we can define vector space structures which will allow us to describe notions such as length, area and curvature in a mathematically rigorous manner through the concept of a differential form. These differential forms will give rise to a group structure called the de Rham cohomology, which has ties to the topological structure of the manifold it is defined on. Detailed calculations and examples will as usual be included throughout. We must stress that the purpose of this chapter is solely to introduce the necessary mathematics, so readers familiar with the theory of differential geometry may skip this chapter and move on directly to the sigma models in chapter 5.

The text is not intended to be mathematically complete in all aspects, and only covers what is needed for next chapter. For further reference, see for example [6] or [8], which will be important for all the sections of the chapter.

4.1 Manifolds

In this section we will introduce the important concept of manifolds. We will concentrate on Riemannian manifolds which are manifolds endowed with a metric. These manifolds will be useful later on as we go from flat to curved space. Let us begin with a general definition of a manifold.

Definition: *A manifold M of dimension n is a topological space that around each point $p \in M$ resembles the Euclidian space $\mathbb{R}^n$. A Riemannian manifold is a manifold M equipped with a Riemannian metric tensor g.*

We introduce a set of neighbourhoods U_i on M. The neighbourhoods are subspaces of $\mathbb{R}^n$, and patching these subspaces together gives us M. For example, one can take M to be the two-dimensional sphere in $\mathbb{R}^3$, which is patched together by the pieces U_i to build up an empty shell, i.e. it has a similar construction as a soccer ball. Between each U and $\mathbb{R}^n$ there is a coordinate generating function ϕ_i,

$$\phi_i : M \supset U_i \to \mathbb{R}^n, \tag{4.1}$$

which maps every point p in U to a point in $\mathbb{R}^n$,

$$\phi(p) = [x_p^1, x_p^2, \ldots, x_p^n] \in \mathbb{R}^n. \tag{4.2}$$

Suppose we have an intersection between two subsections on M, U_i with a function ϕ_i and U_j with a function ϕ_j, and we want to know how to relate ϕ_i and ϕ_j. Then we can define ϕ_i^{-1} as the inverse that maps back from $\mathbb{R}^n$ to M, which yields the expression for the transformation from ϕ_i to ϕ_j,

$$\phi_{ji} = \phi_j \cdot \phi_i^{-1}. \tag{4.3}$$

Important is that the map above needs to be infinitely differentiable. In other words, the map is C^∞, independent of the way we move in M.

Tangent Space and Cotangent Space

Two important spaces when working with manifolds are the tangent space $T_p(M)$ and the cotangent space $T_p^*(M)$ in M. Let us start by introducing $T_p(M)$, which is the space spanned by all the tangent vectors in a point $p \in M$. The basis of $T_p(M)$ is given by $\{\frac{\partial}{\partial x^i}\}$, and it has the same dimension as M, $(i = 1, 2, \dots, n)$. One can make a comparison to classical mechanics, where the velocity space corresponds exactly to the tangent space above.

With the basis for $T_p(M)$ given, we can define the dual space to $T_p(M)$ as the space with the basis dx^j which fulfills the inner product

$$\langle \frac{\partial}{\partial x^i}, dx^j \rangle = \delta_i^j. \tag{4.4}$$

The space with this sort of basis is called the cotangent space $T_p^*(M)$ in M, and it also has a corresponding role in classical mechanics, namely the momentum space.

Example: the n-sphere

In figure 4.1 we see an example of a simple manifold, a two-dimensional sphere in $\mathbb{R}^3$. This is a special case of the n-sphere S^n, where $n = 2$,

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = r\}. \tag{4.5}$$

In the case $n = 0$ we get a pair of points at the ends of a line segment, and in the case $n = 1$ we get a circle.

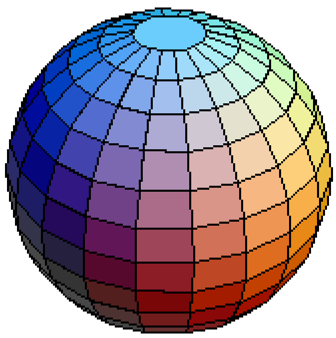


Figure 4.1: Example of manifold: a two-dimesional sphere. [18]

Orientation of M

Using the two-dimensional sphere from the example above we can define orientability of a manifold. The sphere, or the empty shell, has two sides, one inside and one outside. We can orient the sides in $\mathbb{R}^3$ by assigning normal vectors to them, $\mathbf{n} = \pm\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit vector in the radial direction.

All surfaces are not orientable, though. A Möbius strip is the two dimensional manifold obtained by taking a rectangle, twist it 180°, and then join its ends together, see figure 4.2. Pick a point p with normal vector $\mathbf{n}$. Now, follow the strip around until you come back to p. On our way around the strip we have to pass the 180° twist, which will make our normal vector switch direction to the opposite compared to before. Thus we have two normal vectors in p, and we have a non-orientable surface.

If the manifold is a more complicated one than the two-dimensional Möbius strip, is there any way to determine if it is orientable or not? Let us consider our manifold with two different bases $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$, of the n-dimensional vector space V. Between the two bases we have a transformation matrix A,

$$\mathbf{e}'_i = A\mathbf{e}_i. \tag{4.6}$$

By calculating the determinant of A we can decide whether the bases have the same orientation or not. If det $A > 0$ the bases have the same orientation, and vice versa. In our case of the Möbius strip we can take our bases to be

$$\{\mathbf{e}'_1, \mathbf{e}'_2\} = \{\mathbf{e}_1, -\mathbf{e}_2\}, \tag{4.7}$$

where the primed coordinates are the ones received after one cycle. The vector product of the two pairs of bases yields the normal vectors $\pm\mathbf{n}$.

Now, using the determinant method above, we see that our transformation matrix is given by,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.8}$$

Then det $A < 0$, and our bases do not have the same orientation, just as we expected.

The set of all bases for V can be divided into two groups, or equivalence classes, one for spaces that transform with a matrix A with a positive determinant, and one where the determinant is negative.

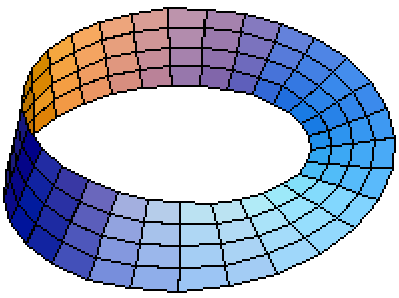


Figure 4.2: The Möbius strip. [19]

4.2 Differential Forms

We define an object called the tangent bundle as

$$T(M) = \bigcup_{p \in M} T_p(M), \tag{4.9}$$

and we can define the cotangent bundle in the same way. We can use this to define objects called differential forms which are functions on the cotangent bundle (and its tensor products). A differential $p-$form or just $p-$form is then defined as a tensor of rank p that is antisymmetric under change of any pair of indices. To begin the study of differential forms the first thing that we have to do is to define Cartan's *wedge product*, also known as the exterior product

$$dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx). \tag{4.10}$$

We stress that these forms depend on the position in M. The wedge product is an anti-symmetric tensor product of dx and dy which are differential line elements, $1-$forms. The wedge product is anti-commutative,

$$dx \wedge dy = -dy \wedge dx, \tag{4.11}$$

and it follows from the definition that the the wedge product of any pair of $1-$forms is zero, that is

$$dx \wedge dx = 0. \tag{4.12}$$

The wedge product is a way of constructing 2−forms out of 1−forms. The 2−form constructed in this way has the property of a differential area element. If we change variables to $x'(x,y)$, $y'(x,y)$ the wedge product of dx' and dy' is the given by

$$dx' \wedge dy' = \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x} \right) dx \wedge dy = \text{Jacobian}(x', y'; x, y) dx \wedge dy. \tag{4.13}$$

If we have a manifold M of dimension n, then 0−form are simply functions on M and there will exist $n+1$ different kinds of p−forms

$$\begin{array}{lll} 0- & forms & f(\vec{x}) \; functions \\ 1- & forms & a_i(\vec{x})dx^i \, covariant\ vectors \\ 2- & forms & T_{ij}(\vec{x})dx^i \wedge dx^j \, antisymmetric\ covariant\ tensors\ of\ rank\ 2 \\ & . & \\ & . & \\ & . & \\ n- & forms & f_{i_1 \ldots 1_n}(\vec{x})dx^{i_1} \wedge \ldots \wedge dx^{i_n} \, antisymmetric\ covariant\ tensors\ of\ rank\ n, \end{array} \tag{4.14}$$

where $f_{i_1 \ldots i_n}$ is a totally antisymmetric tensor. The algebra of differential forms is called the exterior algebra and is denoted by $\Lambda(V_n)$, where V_n is a vector space of dimension n and there will be $n+1$ subspaces $\Lambda^p(V_n)$. Let $\Lambda^p(x)$ be the subspace spanned by the anti-symmetric p−forms at a point x in V_n. Then this will be a vector space of dimension $\binom{n}{p} = n!/p!(n-p)!$ Then $C^\infty(\Lambda^p)$ is the space of differentiable smooth p−forms, where the p−forms are represented as

$$p - forms \qquad f_{i_1 \ldots 1_p}(\vec{x})dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{4.15}$$

The antisymmetric tensor $f_{i_1, i_2 \ldots}(\vec{x})$ will have p indices contracting with the wedge product of p differentials. There will be $n+1$ elements of $C^\infty(\Lambda^k)$ and they are explicitly given as

$$\begin{array}{lll} C^\infty(\Lambda^0) & = \{f(\vec{x})\} & dim = 1 \\ C^\infty(\Lambda^1) & = \{f(\vec{x})_i dx^i\} & dim = n \\ C^\infty(\Lambda^2) & = \{f(\vec{x})_{ij} dx^i \wedge dx^j\} & dim = n(n-1)/2! \\ C^\infty(\Lambda^3) & = \{f(\vec{x})_{ijk} dx^i \wedge dx^j \wedge dx^k\} & dim = n(n-1)(n-2)/3! \\ & . & \\ & . & \\ & . & \\ C^\infty(\Lambda^{n-1}) & = \{f(\vec{x})_{i_1 \ldots 1_{n-1}} \wedge \ldots \wedge dx^{i_{n-1}}\} & dim = n \\ C^\infty(\Lambda^n) & = \{f(\vec{x})_{i_1 \ldots 1_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n} & dim = 1. \end{array} \tag{4.16}$$

From this it is possible to observe that the two vector spaces Λ^p and Λ^{n-p} have the same dimension. If $p > n$, then $\Lambda^p = 0$ since when the p−forms are expressed in terms of local coordinates as in (4.15) then at least one pair of differentials along $dx^{i_1} \ldots dx^{i_p}$ would have to be equal and will then be annihilated.

The next thing that has to be introduced in the study of differential forms is the *exterior derivative* denoted by d, which is an operator that takes p−forms into $(p+1)$−forms and is defined as

$$\begin{array}{ll} d: C^\infty(\Lambda^0) \longrightarrow C^\infty(\Lambda^1); & d(f(x)) = \dfrac{\partial f}{\partial x^i} dx^i \\ d: C^\infty(\Lambda^1) \longrightarrow C^\infty(\Lambda^2); & d(f_j(x)dx^j) = \dfrac{\partial f_j}{\partial x^i} dx^i \wedge dx^j \\ d: C^\infty(\Lambda^2) \longrightarrow C^\infty(\Lambda^3); & d(f_{jk}(x)dx^j \wedge dx^k) = \dfrac{\partial f_{jk}}{\partial x^i} dx^i \wedge d^j \wedge dx^k \end{array} \tag{4.17}$$

and so on. An important property of the exterior derivative is that when applied twice it gives zero. That is, if for an arbitrary p-form ω_p, it follows that

$$dd\omega_p = 0. \tag{4.18}$$

Let α_p be a p−form and β_q a q−form, then

$$\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p. \tag{4.19}$$

This shows that odd forms anti-commute and the wedge product of two 1−forms will always be zero if they are identical. This is very similar to the property of the Grassmann algebra where odd Grassmann numbers anti-commute and squares to zero. The exterior derivative of $\alpha_p \wedge \beta_q$ is given by

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q \tag{4.20}$$

so the exterior derivative anti-commutes with 1-forms.

From (4.16) it was observed that $C^\infty(\Lambda^k)$ and $C^\infty(\Lambda^{n-k})$ have the same number of dimensions. Since the vector spaces have the same dimension there exists an isomorphism between them. There must exist a duality between $C^\infty(\Lambda^k)$ and $C^\infty(\Lambda^{n-k})$. We will therefore introduce the duality transformation, also know as the *Hodge dual* operator or *Hodge* $\star$. The Hodge $\star$ is defined in flat Euclidean space as

$$\star(dx^{i_1} \wedge ... \wedge dx^{i_p}) = \frac{1}{(n-p)!}\epsilon_{i_1,i_2...i_p,i_{p+1},...i_n} dx^{i_{p+1}} \wedge \ldots \wedge dx^{i_{1_n}}, \tag{4.21}$$

where $\epsilon_{ijk...}$ is the totally antisymmetric tensor in n-dimensions. The Hodge $\star$ operator thus transforms p-forms into $(n-p)-$forms. The square of the Hodge dual on an arbitrary $p-$form ω_p is given by

$$\star\star\,\omega_p = (-1)^{p(n-p)}\omega_p. \tag{4.22}$$

In the special case of $p = n$ we get

$$dx^{i_1} \wedge ... \wedge dx^{i_n} = \epsilon_{i_1,...i_n} dx^1 \wedge dx^2 \wedge ... \wedge dx^n. \tag{4.23}$$

Let us now study another concept using the Hodge dual called the *inner product*, defined as

$$(\alpha_p, \beta_p) = \int_M \alpha_p \wedge \star\beta_p, \tag{4.24}$$

where α_p and β_p are arbitrary $p-$forms. One property of the inner product that follows from the identity $\alpha_p \wedge \star\beta_p = \beta_p \wedge \star\alpha_p$ is that

$$(\alpha_p, \beta_p) = (\beta_p, \alpha_p). \tag{4.25}$$

Another important application of the Hodge $\star$ operator on a manifold is to define the adjoint $d^\dagger$ of the exterior derivative d as

$$d^\dagger = (-1)^{np+n+1} \star d \star. \tag{4.26}$$

It follows that

$$\begin{cases} d^\dagger = -\star d\star & \text{for n even, all p} \\ d^\dagger = (-1)^p \star d\star & \text{for n odd, all p.} \end{cases}$$

The adjoint $d^\dagger$ works in the opposite way of the exterior derivative d, i.e. it transforms a $p-$form into a $(p-1)-$form. Just like d, the adjoint exterior derivative squares to zero, so for a $p-$form ω_p it follows that

$$d^\dagger d^\dagger \omega_p = 0. \tag{4.27}$$

The exterior derivative and the adjoint exterior derivative are given as

$$\begin{aligned} d &: C^\infty(\Lambda^p) \longrightarrow C^\infty(\Lambda^{p+1}) \\ d^\dagger &: C^\infty(\Lambda^p) \longrightarrow C^\infty(\Lambda^{p-1}). \end{aligned} \tag{4.28}$$

The *Laplace Beltrami operator* Δ on a manifold can now be described in terms of d and $d^\dagger$ and is given as

$$\Delta = (d + d^\dagger)^2 = d^2 + dd^\dagger + d^\dagger d + (d^\dagger)^2 = dd^\dagger + d^\dagger d. \tag{4.29}$$

It takes a $p-$form back into a $p-$form, that is

$$\Delta : C^\infty(\Lambda^p) \longrightarrow C^\infty(\Lambda^p). \tag{4.30}$$

If the $p-$form ω_p obeys $\Delta\omega_p = 0$ it is *harmonic*, which happens if and only if $d\omega_p = 0$ (*closed*) and $d^\dagger\omega_p = 0$ (*co-closed*). A $p-$form is called *exact* if it can be written as

$$\omega_p = d\omega_{p-1}, \tag{4.31}$$

where ω_{p-1} is a $(p-1)-$form, and a $p-$form is said to be *co-exact* if

$$\omega_p = d^\dagger \alpha_{p+1}. \tag{4.32}$$

Let us now look at *Stokes' theorem* for differential forms which is a statement about the integration of $p-$forms in manifolds. Stokes' theorem says that the integral of a $(p-1)-$form ω_{p-1} over the boundary ∂M of some orientable manifold M is equal to the integral of the exterior derivative d of ω_{p-1} over the whole manifold M, that is

$$\int_M d\omega_{p-1} = \int_{\partial M} \omega_{p-1}. \tag{4.33}$$

For $0-$forms, i.e. functions, we get the fundamental theorem of calculus

$$\int_a^b df(x) = f(b) - f(a), \tag{4.34}$$

where M is a line segment from a to b. For a $1-$form we get

$$\int_{\text{surface}} d(\mathbf{A}\cdot dx) = \oint_{\text{line}} \mathbf{A}\cdot dx, \tag{4.35}$$

and for $3-$forms we get the familiar Gauss' law

$$\int \nabla\cdot\mathbf{E}\; d^3x = \int_{\text{volume}} d\omega = \int_{\text{surface}} \omega = \int \mathbf{E}\cdot d\mathbf{S}. \tag{4.36}$$

4.3 Curvature

We will now combine the study of manifolds and differential geometry, to form the basic structure of manifolds endowed with a metric. We will define some useful relations and identities, that will come to use later on in the paper. We end this part with a discussion about the important Riemann tensor, and how it relates to the curvature of the manifold.

4.3.1 Cartan Structure Equations and the Levi-Civita Connection

First we want to define two type of indices, Greek indices, like, α, γ, μ, and Latin indices, $a, b, c...$ The Greek letters refer to curved space, or curved manifolds, while the Latin letters refer to flat space. Note that this notation is somewhat different from the one used in the book of Nakahara [7].

Given a Riemannian manifold M, we can ascribe it a metric tensor $g_{\mu\nu}(x)$, with local coordinates x^μ. The invariant length is then written as

$$ds^2 = g_{\mu\nu}(x)dx^\mu\ dx^\nu. \tag{4.37}$$

This is the distance between two infinitesimally nearby points x^μ and $x^\mu + dx^\mu$.

There is a connection between a curved metric and a flat metric. They are connected via the so called *vielbeins*, also called solder forms, ${e^a}_\mu$. In just a moment, we will see that they transform the curved coordinate basis of the tangent space of a manifold to an orthonormal basis of the tangent space. The flat metric is denoted by η_{ab}. The classic example of a flat metric is the Euclidean space, where

$$\eta_{ab} = \delta_{ab}, \qquad a, b = 1, 2, 3, 4, \tag{4.38}$$

or the Minkowski space, where

$$\eta_{ab} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{4.39}$$

The flat metric is connected to the curved metric through the vielbeins,

$$\begin{aligned} g_{\mu\nu} &= \eta_{ab}{e^a}_\mu {e^b}_\nu \\ \eta^{ab} &= g^{\mu\nu}{e^a}_\mu {e^b}_\nu \end{aligned} \tag{4.40}$$

We define an inverse of ${e^a}_\mu$,

$$ {E_a}^\mu = \eta_{ab} g^{\mu\nu} {e^b}_\nu. \tag{4.41} $$

By the relations in (4.40), it can easily be seen that ${E_a}^\mu$ is indeed the inverse

$$ {E_a}^\mu {e^c}_\nu = \eta_{ab} \underbrace{g^{\mu\nu} {e^b}_\mu {e^c}_\nu}_{\eta^{bc}} = \eta_{ab}\eta^{bc} = \delta^c_a. \tag{4.42} $$

In the last step we used that η_{ab} and η^{ab} are each others inverses. The same thing is true for $g^{\mu\nu}$ and $g_{\mu\nu}$. Though it is not a general property of a tensor that raised or lowered indices creates an inverse, it is merely a property of the metric tensor. With the inverse of the vielbeins we can define similar relations like in (4.40),

$$ \begin{aligned} g^{\mu\nu} &= \eta^{ab} {E_a}^\mu {E_b}^\nu, \\ \eta_{ab} &= g_{\mu\nu} {E_a}^\mu {E_b}^\nu. \end{aligned} \tag{4.43} $$

Thus, we can conclude that ${e^a}_\mu$ and ${E_a}^\mu$ can be used to interconvert Greek and Latin indices.

Now, we come to an important role of the vielbeins. As we stated before, they are in fact the matrices that transforms the coordinate basis dx^μ of the dual tangent space $T^*_x(M)$ to an orthonormal basis of $T^*_x(M)$,

$$ e^a = {e^a}_\mu dx^\mu \tag{4.44} $$

Similarly, ${E_a}^\mu$ is a transformation from the basis $\partial/\partial x^\mu$ of the tangent space $T_x(M)$ to an orthonormal basis,

$$ E_a = {E_a}^\mu \partial/\partial x^\mu. \tag{4.45} $$

We will now introduce a couple of equations, called *Cartan's structure equations*. The equations themselves are not used in the rest of the text, but the first of them define what is called the *affine spin connection* ${\omega^a}_b$, which we will use later on. And once that equation is defined, we are not far away from the famous Bianchi identities. Therefore, we will continue until we have derived the Bianchi identities. One can see it as a good exercise in the rules and structure of differential forms, that we introduced in the previous section. The first of Cartan's structure equations, that defines the affine spin connection ${\omega^a}_b$ 1−form, is

$$ T^a = de^a + {\omega^a}_b \wedge e^b = \frac{1}{2} {T^a}_{bc} e^b \wedge e^c. \tag{4.46} $$

T^a is called the *torsion* 2−form of the manifold. The second of Cartan's equations defines the *curvature* 2−form ${R^a}_b$ as

$$ {R^a}_b = d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b = \frac{1}{2} {R^a}_{bcd} e^c \wedge e^d. \tag{4.47} $$

If we take the exterior derivative of (4.46), we can find a connection between (4.46) and (4.47),

$$ \begin{aligned} dT^a &= \underbrace{d(de^a)}_{=0} + d({\omega^a}_b \wedge e^b) \\ &= d{\omega^a}_b \wedge e^b - {\omega^a}_b \wedge de^b. \end{aligned} \tag{4.48} $$

The connection is then given by

$$ \begin{aligned} dT^a + {\omega^a}_b \wedge T^b &= d{\omega^a}_b \wedge e^b - {\omega^a}_b \wedge de^b + {\omega^a}_b \wedge (de^b + {\omega^b}_c \wedge e^c) \\ &= d{\omega^a}_b \wedge e^b + {\omega^a}_c \wedge {\omega^c}_b \wedge e^b \\ &= (d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b) \wedge e^b \\ &= {R^a}_b \wedge e^b. \end{aligned} \tag{4.49} $$

The Bianchi identities are found by taking the exterior derivative of the curvature 2−form of the manifold (4.47), together with two wedge products of (4.47) itself,

$$\begin{aligned} dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = & \underbrace{d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b}_{dR^a{}_b} \\ & + \underbrace{\omega^a{}_c \wedge d\omega^c{}_b + \omega^a{}_c \wedge \omega^c{}_d \wedge \omega^d{}_b}_{\omega^a{}_c \wedge R^c{}_b} \\ & - \underbrace{d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_d \wedge \omega^d{}_c \wedge \omega^c{}_b}_{R^a{}_c \wedge \omega^a{}_c} \\ = & \ 0. \end{aligned} \tag{4.50}$$

From this condition, we can define a *covariant derivative* of a general differential form $V^a_{\ b}$ of degree p,

$$DV^a_{\ b} = dV^a_{\ b} + \omega^a{}_c \wedge V^c_{\ b} - (-1)^p V^c_{\ b} \wedge \omega^a{}_c. \tag{4.51}$$

The Bianchi identities (4.50) then read,

$$DR^a{}_b = 0. \tag{4.52}$$

All the equations we have looked at so far, can of course be expressed in terms of curved coordinates. We simply multiply an expression with the vielbeins or their inverses, to make the transition from flat to curved coordinates. We can even derive an expression for the Riemann tensor from the curvature 2−form,

$$R^a{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu, \tag{4.53}$$

where the Riemann tensor then can be written as

$$R^\alpha{}_{\beta\mu\nu} = E_a{}^\alpha e^b{}_\beta R^a{}_{b\mu\nu}. \tag{4.54}$$

However, this expression gives very little insight into the meaning of the Riemann tensor. Therefore, we will soon leave what we can call the Cartan differential form approach to Riemannian geometry and continue to a more conventional formulation of the Riemann tensor. Before we do that however, we have to translate the torsion 2−form to curved coordinates as well,

$$\begin{aligned} T^a &= \frac{1}{2} T^a_{\ bc} e^b \wedge e^c = \frac{1}{2} T^a_{\ \mu\nu} dx^\mu \wedge dx^\nu \\ T^\alpha_{\ \mu\nu} &= E_a{}^\alpha T^a_{\ \mu\nu}. \end{aligned} \tag{4.55}$$

On our way to a meaningful formulation of the Riemann tensor, we first want to understand why we need a new kind of derivative, the covariant derivative, and introduce the *Christoffel symbol* $\Gamma^\lambda{}_{\mu\nu}$ of the *Levi-Civita connection*. We need a new derivative because the usual derivative of an arbitrary tensor v_μ does not transform as a tensor

$$\frac{\partial v_{\mu'}}{\partial x^{\nu'}} = \frac{\partial}{\partial x^{\nu'}} \left(v_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} \right) = \frac{\partial v_\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\mu}{\partial x^{\mu'}} + v_\mu \frac{\partial^2 x^\mu}{\partial x^{\mu'} \partial x^{\nu'}}. \tag{4.56}$$

We get an additional term. We want to avoid this with a derivative that transforms as a tensor, therefore we have the covariant derivative (in a different form than (4.51))

$$D_\mu(v_\nu) := \partial_\mu v_\nu - \Gamma^\lambda{}_{\mu\nu} v_\lambda, \tag{4.57}$$

for a covariant tensor, and

$$D_\mu(v^\nu) := \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\lambda} v^\lambda, \tag{4.58}$$

for a contravariant tensor. The Christoffel symbol takes care of the additional term from the usual derivative we got in (4.56), it is also important to note that the Christoffel symbol is not a tensor. If we want to differentiate higher ranked tensors we just add an extra Christoffel symbol for each added index. Through the covariant derivative we can get an expression for the Christoffel symbol in terms of the metric. Let the Levi-Civita connection be determined by two conditions, the covariant constancy of the metric tensor, and the absence of torsion. In tensor notation, these conditions are written as

$$\begin{aligned} \text{covariant constancy of the metric}: \quad & D_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\alpha\nu} g_{\mu\lambda} = 0 \\ \text{no torsion}: \quad & T^\mu{}_{\alpha\beta} = \frac{1}{2}(\Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha}) = 0. \end{aligned} \tag{4.59}$$

From these relations we can derive an explicit expression for the Christoffel symbol. We will see that the Christoffel symbol is written as,

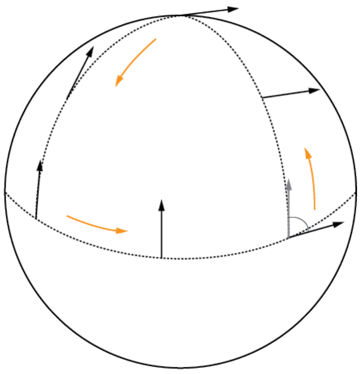
$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}). \tag{4.60}$$

By using the conditions in (4.59) we can prove this to be correct,

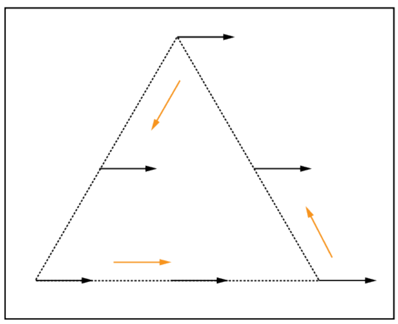
$$\begin{aligned} \Gamma^\mu{}_{\alpha\beta} &= \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}) \\ &= \frac{1}{2} g^{\mu\nu}(\underbrace{\Gamma^\lambda{}_{\alpha\nu} g_{\lambda\beta}}_{A} + \Gamma^\lambda{}_{\alpha\beta} g_{\lambda\nu} + \underbrace{\Gamma^\lambda{}_{\beta\nu} g_{\lambda\alpha}}_{B} + \Gamma^\lambda{}_{\alpha\beta} g_{\lambda\nu} - \underbrace{\Gamma^\lambda{}_{\nu\alpha} g_{\lambda\beta}}_{A} - \underbrace{\Gamma^\lambda{}_{\nu\beta} g_{\lambda\alpha}}_{B}) \\ &= \frac{1}{2} g^{\mu\nu}(2\Gamma^\lambda{}_{\alpha\beta} g_{\lambda\nu}) \\ &= \Gamma^\lambda{}_{\alpha\beta} \delta^\mu{}_\lambda \\ &= \Gamma^\mu{}_{\alpha\beta}. \end{aligned} \tag{4.61}$$

4.3.2 Riemann Tensor

Now that we have found a definition of the covariant derivative and seen how it is related to the metric of the manifold through the Christoffel symbol, we can move on to find a more intuitive form of the Riemann tensor. The Riemann tensor tells us everything we need to know about the curvature of the manifold. We can see it as a measure of how much a vector will differ from its original position when we have transported it around on the manifold. For example, consider a two dimensional spherical surface. We put a vector at a point at the equator. Then we transport the vector on a great circle to the 'north pole' without twisting its direction, then transporting it on a different great circle back to the equator and then we go back to the starting point. Now the vector will not point in the same direction as when it started, schematically shown in figure 4.3a. As a little test, think of the same procedure on a flat surface. In this case the vector will of course come back in the same condition, hence the surface is flat as in figure 4.3b. The Riemann tensor at a point on a manifold is written



(a) Vector displacement on spherical surface.



(b) Vector displacement on flat surface.

Figure 4.3: Vector displacements.

$$R^\alpha{}_{\beta\mu\nu} v_\alpha = [D_\mu, D_\nu]\, v_\beta = D_\mu D_\nu v_\beta - D_\nu D_\mu v_\beta, \tag{4.62}$$

where D_μ and D_ν are covariant derivatives. We can intuitively understand why this is a measure of the curvature, if we take the derivative in a little square on the manifold we measure the change of the vector and probe the manifold curvature. If the Riemann tensor is non-zero anywhere on the manifold, the space is curved. Writing out the derivatives explicitly we get

$$
\begin{aligned}
[D_\nu, D_\mu]\, v_\beta &= (D_\nu D_\mu - D_\mu D_\nu)\, v_\beta = D_\nu(\partial_\mu v_\beta - \Gamma^\rho{}_{\mu\beta} v_\rho) - D_\mu(\partial_\nu v_\beta - \Gamma^\rho{}_{\nu\beta} v_\rho) \\
&= \partial_\nu(\partial_\mu v_\beta - \Gamma^\rho{}_{\mu\beta} v_\rho) - \Gamma^\sigma{}_{\nu\mu}(\partial_\sigma v_\beta - \Gamma^\rho{}_{\sigma\beta} v_\rho) - \Gamma^\sigma{}_{\nu\beta}(\partial_\mu v_\sigma - \Gamma^\rho{}_{\mu\sigma} v_\rho) \\
&\quad - \partial_\mu(\partial_\nu v_\beta - \Gamma^\rho{}_{\nu\beta} v_\rho) + \Gamma^\sigma{}_{\mu\nu}(\partial_\sigma v_\beta - \Gamma^\rho{}_{\sigma\beta} v_\rho) + \Gamma^\sigma{}_{\mu\beta}(\partial_\nu v_\sigma - \Gamma^\rho{}_{\nu\sigma} v_\rho) \\
&= \partial_\mu \Gamma^\rho{}_{\nu\beta} v_\rho - \partial_\nu \Gamma^\rho{}_{\mu\beta} v_\rho + \Gamma^\sigma{}_{\nu\beta}\Gamma^\rho{}_{\mu\sigma} v_\rho - \Gamma^\sigma{}_{\mu\beta}\Gamma^\rho{}_{\nu\sigma} v_\rho \\
&= R^\rho{}_{\beta\mu\nu} v_\rho .
\end{aligned}
\tag{4.63}
$$

The last step is obtained by using the no torsion condition (4.59), $\Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}$. Omitting the test vector v_β, which we only inserted for clarity, the Riemann curvature tensor reads

$$R^\rho{}_{\beta\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\beta} - \partial_\nu \Gamma^\rho{}_{\mu\beta} + \Gamma^\sigma{}_{\nu\beta}\Gamma^\rho{}_{\mu\sigma} - \Gamma^\sigma{}_{\mu\beta}\Gamma^\rho{}_{\nu\sigma}. \tag{4.64}$$

Now we want to know how the Riemann tensor behaves, if it is symmetric in its indices or not. It is antisymmetric when interchanging its two last indices

$$R^\rho{}_{\beta\nu\mu} = \partial_\nu \Gamma^\rho{}_{\mu\beta} - \partial_\mu \Gamma^\rho{}_{\nu\beta} + \Gamma^\sigma{}_{\mu\beta}\Gamma^\rho{}_{\nu\sigma} - \Gamma^\sigma{}_{\nu\beta}\Gamma^\rho{}_{\mu\sigma} = -R^\rho{}_{\beta\mu\nu}. \tag{4.65}$$

To get the rest of the relations one will have to lower the first index. We do this with the metric tensor $g_{\alpha\rho}$

$$R_{\rho\sigma\mu\nu} = g_{\alpha\rho} R^\alpha{}_{\sigma\mu\nu} = g_{\alpha\rho}\left(\partial_\mu \Gamma^\alpha{}_{\nu\sigma} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\gamma{}_{\nu\sigma} - \partial_\nu \Gamma^\alpha{}_{\mu\sigma} - \Gamma^\alpha{}_{\nu\gamma}\Gamma^\gamma{}_{\mu\sigma}\right). \tag{4.66}$$

What is $\partial_\mu \Gamma^\alpha{}_{\nu\sigma}$ in (4.66)? It is not completely trivial, since the Christoffel symbol involves a product. In fact

$$
\begin{aligned}
\partial_\mu \Gamma^\alpha{}_{\nu\sigma} &= \partial_\mu \frac{1}{2} g^{\alpha\kappa}(g_{\kappa\nu,\sigma} + g_{\kappa\sigma,\nu} - g_{\nu\sigma,\kappa}) \\
&= \frac{1}{2}(\partial_\mu g^{\alpha\kappa})(g_{\kappa\nu,\sigma} + g_{\kappa\sigma,\nu} - g_{\nu\sigma,\kappa}) + \frac{1}{2} g^{\alpha\kappa}(g_{\kappa\nu,\sigma\mu} + g_{\kappa\sigma,\nu\mu} - g_{\nu\sigma,\kappa\mu}),
\end{aligned}
\tag{4.67}
$$

where the comma before an index μ ($,\mu$) means partial derivative with respect to the index μ. What then is $\partial_\mu g^{\alpha\kappa}$? $g^{\alpha\kappa}$ is the inverse of $g_{\gamma\lambda}$. What then is the derivative of an inverse? Generally, for a matrix M and its inverse M^{-1}, we have $MM^{-1} = \mathbf{1}$. Differentiate this equation, to get

$$0 = \delta \mathbf{1} = (\delta M)M^{-1} + M(\delta M^{-1}). \tag{4.68}$$

Multiply this equation from left by M^{-1},

$$0 = M^{-1}(\delta M)M^{-1} + (M^{-1}M)(\delta M^{-1}), \qquad \text{and} \qquad \delta M^{-1} = -M^{-1}(\delta M)M^{-1} \tag{4.69}$$

In this specific case

$$\delta g^{\alpha\kappa} = -g^{\alpha\gamma}(\delta g_{\gamma\lambda})g^{\lambda\kappa}, \tag{4.70}$$

and the corresponding partial derivative becomes

$$\partial_\mu g^{\alpha\kappa} = -g^{\alpha\gamma}(\partial_\mu g_{\gamma\lambda})g^{\lambda\kappa} = -g^{\alpha\gamma} g_{\gamma\lambda,\mu} g^{\lambda\kappa} \tag{4.71}$$

The first term in (4.67) is then

$$(\partial_\mu g^{\alpha\kappa})\frac{1}{2}(g_{\kappa\nu,\sigma} + g_{\kappa\sigma,\nu} - g_{\nu\sigma,\kappa}) = -g^{\alpha\gamma} g_{\gamma\lambda,\mu} g^{\lambda\kappa}\Gamma_{\kappa\nu\sigma}, \tag{4.72}$$

and all together

$$
\begin{aligned}
g_{\alpha\rho}(\partial_\mu \Gamma^\alpha{}_{\nu\sigma}) &= g_{\alpha\rho}\left(-g^{\alpha\gamma}g_{\gamma\lambda,\mu}g^{\lambda\kappa}\Gamma_{\kappa\nu\sigma} + \frac{1}{2}g^{\alpha\kappa}(g_{\kappa\nu,\sigma\mu} + g_{\kappa\sigma,\nu\mu} - g_{\nu\sigma,\kappa\mu})\right) \\
&= -\delta^\gamma_\rho g_{\gamma\lambda,\mu}g^{\lambda\kappa}\Gamma_{\kappa\nu\sigma} + \frac{1}{2}\delta^\kappa_\rho(g_{\kappa\nu,\sigma\mu} + g_{\kappa\sigma,\nu\mu} - g_{\nu\sigma,\kappa\mu}) \\
&= -g_{\rho\lambda,\mu}\Gamma^\lambda{}_{\nu\sigma} + \frac{1}{2}(g_{\rho\nu,\sigma\mu} + g_{\rho\sigma,\nu\mu} - g_{\nu\sigma,\rho\mu}).
\end{aligned} \tag{4.73}
$$

The first two terms in (4.66) are therefore

$$
g_{\alpha\rho}\left(\partial_\mu \Gamma^\alpha{}_{\nu\sigma} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\gamma{}_{\nu\sigma}\right) = -g_{\rho\lambda,\mu}\Gamma^\lambda_{\nu\sigma} + \frac{1}{2}(g_{\rho\nu,\sigma\mu} + g_{\rho\sigma,\nu\mu} - g_{\nu\sigma,\rho\mu}) + \Gamma_{\rho\mu\gamma}\Gamma^\gamma{}_{\nu\sigma}. \tag{4.74}
$$

The last two terms in (4.66) are just the same as the two first, except that μ has taken the place of ν and vice versa. We therefore just interchange the two indices in (4.74) to get

$$
g_{\alpha\rho}\left(-\partial_\nu \Gamma^\alpha{}_{\mu\sigma} - \Gamma^\alpha{}_{\nu\gamma}\Gamma^\gamma{}_{\mu\sigma}\right) = g_{\rho\lambda,\nu}\Gamma^\lambda{}_{\mu\sigma} - \frac{1}{2}(g_{\rho\mu,\sigma\nu} + g_{\rho\sigma,\mu\nu} - g_{\mu\sigma,\rho\nu}) - \Gamma_{\rho\nu\gamma}\Gamma^\gamma{}_{\mu\sigma}. \tag{4.75}
$$

The total Riemann tensor is thus the sum of (4.74) and (4.75)

$$
R_{\rho\sigma\mu\nu} = -g_{\rho\lambda,\mu}\Gamma^\lambda{}_{\nu\sigma} + g_{\rho\lambda,\nu}\Gamma^\lambda{}_{\mu\sigma} + \frac{1}{2}(g_{\rho\nu,\sigma\mu} - g_{\nu\sigma,\rho\mu} - g_{\rho\mu,\sigma\nu} + g_{\mu\sigma,\rho\nu}) + \Gamma_{\rho\mu\gamma}\Gamma^\gamma{}_{\nu\sigma} - \Gamma_{\rho\nu\gamma}\Gamma^\gamma{}_{\mu\sigma}. \tag{4.76}
$$

Now to see if the Riemann tensor is symmetric in the two first indices we can test it in an inertial frame where all Christoffel symbols are zero, and get

$$
R_{\rho\sigma\mu\nu} = \frac{1}{2}(g_{\rho\nu,\sigma\mu} - g_{\nu\sigma,\rho\mu} - g_{\rho\mu,\sigma\nu} + g_{\mu\sigma,\rho\nu}). \tag{4.77}
$$

Because of the commutativity of the second derivatives, $g_{\rho\nu,\sigma\mu} = g_{\rho\nu,\mu\sigma}$, we can see that

$$
R_{\rho\beta\mu\nu} = R_{\mu\nu\rho\beta} = -R_{\beta\rho\mu\nu}. \tag{4.78}
$$

We know this result holds in every frame of reference because (4.77) is a tensor equation. The Riemann tensor is therefore antisymmetric under interchange of its two first indices but symmetric under interchange of the two first indices with the two last indices. We also get the important cyclic identity

$$
R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} + R_{\rho\mu\nu\sigma} = 0. \tag{4.79}
$$

All these relations greatly reduces the work one has to do when calculating the curvature.

4.3.3 Riemann Tensor of the 2-sphere

To get more familiar with the Christoffel symbols and Riemann tensor we will calculate an example. These calculations are rather straightforward although there are many lengthy steps if you work with complicated manifolds in higher dimensions. One has to calculate all the combinations of indices in the Christoffel symbols and Riemann tensor. Therefore we will calculate the simple example of the Riemann tensor for the 2−sphere. We start by finding the metric tensor for the sphere. Distances in spherical coordinates in 3 dimensions can be written

$$
ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 = \eta_{ab}dx^a dx^b. \tag{4.80}
$$

This is our curved surface embedded in a three dimensional flat Euclidean space. We can get our curved two-dimensional metric by setting $r = R$ a constant, thus our metric will be

$$
g_{\mu\nu} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}. \tag{4.81}
$$

With the inverse

$$
g^{\mu\nu} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}. \tag{4.82}
$$

We proceed by calculating the Christoffel symbols. The no torsion condition (4.59) and the fact that only g_{22} is not constant leaves only three of the Christoffel symbols non-zero and two of them are equal

$$\begin{aligned}
\Gamma^1{}_{22} &= \frac{1}{2}g^{11}(\partial_\varphi g_{12} + \partial_\varphi g_{12} - \partial_\theta g_{22}) = \frac{1}{2R^2}(-\partial_\theta(R^2\sin^2\theta)) \\
&= -\sin\theta\cos\theta \\
\Gamma^2{}_{12} &= \frac{1}{2}g^{22}(\partial_\theta g_{22} + \partial_\varphi g_{21} - \partial_\varphi g_{12}) = \frac{1}{2R^2\sin^2\theta}\partial_\theta(R^2\sin^2\theta) \\
&= \frac{\cos\theta}{\sin\theta} = \Gamma^2{}_{21}.
\end{aligned} \tag{4.83}$$

Now we can calculate the Riemann tensor. Here we eliminate more work through the symmetry of the Riemann tensor $R_{\rho\beta\mu\nu} = -R_{\beta\rho\mu\nu} = -R_{\rho\beta\nu\mu}$ and many combinations will end up zero because of the small number of non-zero Christoffel symbols, but here is two of the non-zeros

$$\begin{aligned}
R^1{}_{212} &= \Gamma^1{}_{22,\theta} - \Gamma^1{}_{21,\varphi} + \Gamma^1{}_{11}\Gamma^1{}_{22} + \Gamma^1{}_{21}\Gamma^2{}_{22} - \Gamma^1{}_{12}\Gamma^1{}_{21} - \Gamma^1{}_{22}\Gamma^2{}_{21} \\
&= -\cos^2\theta + \sin^2\theta + \sin\theta\cos\theta\frac{\cos\theta}{\sin\theta} \\
&= \sin^2\theta, \\
R^1{}_{221} &= \Gamma^1{}_{21,\varphi} - \Gamma^1{}_{22,\theta} + \Gamma^1{}_{12}\Gamma^1{}_{21} + \Gamma^1{}_{22}\Gamma^2{}_{21} - \Gamma^1{}_{11}\Gamma^1{}_{22} - \Gamma^1{}_{21}\Gamma^2{}_{22} \\
&= \cos^2\theta - \sin^2\theta - \sin\theta\cos\theta\frac{\cos\theta}{\sin\theta} \\
&= -\sin^2\theta.
\end{aligned} \tag{4.84}$$

Already the second one is superfluous to calculate because of the antisymmetry. Here R^1_{212} is a measure of how much a vector pointing in the $\theta-$direction will swing over and point in the $\phi-$direction when parallel transported around a square on the surface. In effect the Riemann tensor tells us how a vector transported around on the surface will differ from its original direction when returned to the starting point, as mentioned earlier. In this case we can also write the Riemann tensor on a more compact form (getting rid of the Christoffel symbols)

$$R^\rho{}_{\beta\mu\nu} = \frac{1}{R^2}(\delta^\rho_\mu g_{\beta\nu} - \delta^\rho_\nu g_{\beta\mu}). \tag{4.85}$$

Here we can, in addition, see the constant value $1/R^2$ of the Gaussian curvature of the 2-sphere.

4.4 de Rham Cohomology

Two important mathematical constructs are *homology* and *cohomology*. Homology is a tool in topology to distinguish manifolds with different topological structure. Cohomology is an algebraic invariant, which distinguishes manifolds with different algebraic structure. The main issue of this section will be de Rham cohomology, but we will also encounter homology, since these concepts are closely related. In fact, considering homology and cohomology as vector spaces, they can be shown to be dual to each other. This duality builds a bridge between the topological properties of manifolds and their differentiable structure.

4.4.1 Definition of de Rham Cohomology

Let M be a manifold of dimension n and consider the differential forms in M. Differential forms were discussed in section 4.2. We will denote a general $p-$form by ω_p. Recall also the exterior derivative d from the same section, which operates on differential forms. The following terminology that we occasionally met there is crucial for the definition of de Rham cohomology:

ω_p is a *closed* form if $d\omega_p = 0$,
ω_p is an *exact form* if $\omega_p = da_{p-1}$ for some $p-1$ form a_{p-1}.

Now introduce $Z^p_{DR}(M)$ as the set of all closed forms in M. In fact, $Z^p_{DR}(M)$ is not only a set, but also a group with composition law given by addition of forms. For a brief introduction to group theory, see Appendix A.2. It is evident that $Z^p_{DR}(M)$ is a group since it fulfills the group axioms. The sum of two closed forms is always closed, $d(\omega_p + \omega'_p) = d\omega_p + d\omega'_p$, since d is linear. Associativity is immediate. The unit element is just the form 0_p ($d0_p = 0$), and the inverse of ω_p is $(-\omega_p) \in Z^p_{DR}(M)$, since $\omega_p + (-\omega_p) = 0_p$.

Also, construct the group $B^p_{DR}(M)$ as the set of all exact forms in M. The composition law is again given by addition, and it is easy to check that also $B^p_{DR}(M)$ is a group.

Remember from section 4.2 that the exterior derivative applied twice on any form always give 0, $d^2\omega_p = dd\omega_p = 0$. This means that every exact form is also closed, for if $\omega_p = da_{p-1}$ is exact, then $d\omega_p = d^2a_{p-1} = 0$, and ω_p is also closed. This means that $B^p_{DR}(M)$ is a subgroup of $Z^p_{DR}(M)$.

Next, consider the cosets of $Z^p_{DR}(M)$ with respect to its subgroup $B^p_{DR}(M)$. A general coset can be written $\omega_p + B^p_{DR}(M)$. Let da_{p-1} be an element of $B^p_{DR}(M)$, so that any element ω'_p of the coset can be written as $\omega'_p = \omega_p + da_{p-1}$. That is, $\omega'_p \sim \omega_p$ if they differ only by an exact form. Thus, the group $Z^p_{DR}(M)$ of closed forms is partitioned into a set of equivalence classes, where the elements in each equivalence class only differ by an exact form. From the set of equivalence classes we construct the quotient group

$$H^p_{DR}(M) = Z^p_{DR}(M)/B^p_{DR}(M). \tag{4.86}$$

This quotient group $H^p_{DR}(M)$ is the *de Rham cohomology* of M. It consists of the set of closed modulo exact forms. The unit element of $H^p_{DR}(M)$ is simply $B^p_{DR}(M)$, i.e. all exact forms. All the exact forms are equivalent to 0 since they all differ from 0 by themselves, i.e. they differ from 0 by exact forms. This is also clear from the fact that if $B^p_{DR}(M)$ is "multiplied" (added) to any of the other cosets of $H^p_{DR}(M)$, this operation does not change that coset. The elements in the coset will all still differ only by an exact form. A simpler analogy to this abstract group is given as an example in Appendix A.2.

4.4.2 de Rham Cohomology and Harmonic Forms

In section 4.2 harmonic forms were briefly discussed. We noted that a form ω_p is harmonic if and only if it is both closed ($d\omega_p = 0$) and coclosed ($d^\dagger\omega_p = 0$). Let the set of harmonic p-forms in M be denoted by $\mathrm{Harm}^p(M, \mathbb{R})$.

Hodge decomposition theorem states that any form ω_p in a compact manifold M without boundary can be decomposed into an exact form, a coexact form and a harmonic form γ_p. In symbols

$$\omega_p = d\alpha_{p-1} + d^\dagger\beta_{p+1} + \gamma_p. \tag{4.87}$$

If now ω_p is closed so that $d\omega_p = 0$, we have

$$0 = dd\alpha_{p-1} + dd^\dagger\beta_{p+1} + d\gamma_p.$$

Since γ_p is harmonic, it is also closed. But then $dd^\dagger\beta_{p+1} = 0$ and $d^\dagger\beta_{p+1} = 0$. And this in turn implies that

$$\omega_p = d\alpha_{p-1} + \gamma_p,$$

so that γ_p only differs from ω_p by an exact form. But then ω_p and γ_p belong to the same cohomology class. This means that it is always possible to choose a harmonic representative for each cohomology class. If ω_p is harmonic, then $d^\dagger d\alpha_{p-1} = 0$, so $d\alpha_{p-1} = 0$ and $\omega_p = \gamma_p$. Hence the de Rham cohomology is isomorphic to the set of harmonic p-forms,

$$H^p_{DR}(M, \mathbb{R}) \cong \mathrm{Harm}^p(M, \mathbb{R}), \tag{4.88}$$

a fact that will be used in the end of chapter 5.

4.4.3 Closed Forms which are not exact

A question rises immediately from the definition of the de Rham cohomology. We know that exact forms are always closed, but could there be closed forms which are not exact? Let us consider a specific example which shows that it really is so.

Take the punctured plane $\mathcal{R}^2 - \{\mathbf{0}\}$ to be our manifold and consider the 1−form

$$\omega_1 = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy. \tag{4.89}$$

It is possible to choose ω_1 as a form on $\mathcal{R}^2 - \{\mathbf{0}\}$ since the origin is excluded[1], thereby avoiding division by 0. ω_1 is really closed since

$$\begin{aligned} d\omega_1 &= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) dy \wedge dx + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy \\ &= \frac{-1(x^2+y^2) - (-y)2y}{(x^2+y^2)^2} dy \wedge dx + \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} dx \wedge dy \\ &= \frac{y^2 - x^2}{(x^2+y^2)^2}(dy \wedge dx + dx \wedge dy) \\ &= 0. \end{aligned} \tag{4.90}$$

The functions $\frac{-y}{x^2+y^2}$ and $\frac{x}{x^2+y^2}$ look like partial derivatives of the arctan function. Consider the 0−form (function) $a_0(x,y) = \arctan\left(\frac{y}{x}\right)$. Then

$$\begin{aligned} da_0(x,y) &= \frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right)\right) dx + \frac{\partial}{\partial y}\left(\arctan\left(\frac{y}{x}\right)\right) dy \\ &= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) dx + \frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} dy \\ &= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy, \end{aligned} \tag{4.91}$$

which seems to be equal to ω_1. But the arctan function must be single-valued. It is defined as the inverse of the tan function with angles in $(-\pi, \pi)$. We have to delete the nonpositive real axis to guarantee single-valuedness. So the function $a_0(x,y)$ is only defined on $\mathbb{R}^2 - \mathbb{R}_{(-)}$ [2]. In this space $\omega_1 = da_0(x,y)$, but not in all of $\mathbb{R}^2 - \{\mathbf{0}\}$.

Is there any other 0−form (function), $f_0(x,y)$ say, which works better? On $\mathbb{R}^2 - \mathbb{R}_{(-)}$ we must have $\omega_1(x,y) = df_0(x,y) = da_0(x,y)$, and $d(f_0(x,y) - a_0(x,y)) = 0$. This means that $\frac{\partial}{\partial x}(f_0(x,y) - a_0(x,y)) = 0$, and

$$\frac{\partial}{\partial x} f_0(x,y) = \frac{\partial}{\partial x} a_0(x,y) \Rightarrow f_0(x,y) = a_0(x,y) + g(y),$$

where $g(y)$ is an arbitrary function of y. Also, $\frac{\partial}{\partial y}(f_0(x,y) - a_0(x,y)) = 0$ and

$$\frac{\partial}{\partial y} f_0(x,y) = \frac{\partial}{\partial y} a_0(x,y) \Rightarrow \frac{\partial a_0(x,y)}{\partial y} + g'(y) = \frac{\partial a_0(x,y)}{\partial y} \Rightarrow g'(y) = 0,$$

which means that g must be a constant. Then $f_0(x,y) = a_0(x,y) + g$ which also only is defined on $\mathbb{R}^2 - \mathbb{R}_{(-)}$, and so there are no functions on all of $\mathbb{R}^2 - \{\mathbf{0}\}$ that fulfills $\omega_1 = df$. Therefore, ω_1 is closed but not exact in M.

4.4.4 Calculation of some de Rham Cohomologies

The de Rham cohomology $H^0_{DR}(M)$ is special, since there are no (-1)−forms. That is, no closed 0−form in M can be written as $\omega_0 = da_{p-1}$. In other words, there are no exact forms except zero. Then every 0−form can only be equivalent to itself, since the only exact form by which two forms can differ is 0, and in that case the two forms are the same. Hence, the cohomology group $H^0_{DR}(M)$ is the set of equivalence classes consisting of individual elements of Z^0_{DR}, the closed forms. Then $H^p_{DR} = Z^p_{DR}$. The 0−forms are the functions f in M, and the closed 0−forms obey $df = 0$, so that f must be constant. From this

$$H^0_{DR}(M) = \{\text{space of constant functions}\}, \tag{4.92}$$

[1]The notation $\mathcal{R}^2 - \{\mathbf{0}\}$ means all real (x,y) except (0,0).

[2]The notation $\mathbb{R}^2 - \mathbb{R}_{(-)}$ means all real (x,y) except those for which $x \leq 0$ and $y = 0$.

and

$$\dim H^0_{DR}(M) = \text{number of connected pieces of the manifold.} \tag{4.93}$$

Recall from section 4.2 that there are no $p-$forms for $p > n$. This follows from the fact that the wedge product is antisymmetric. Therefore

$$H^p_{DR}(M) = 0, \qquad p > n. \tag{4.94}$$

After these general remarks about all manifolds M, we will now look specifically at manifolds which are contractible to a point. A sphere is contractible to a point, and also a circular disk, but not a circular ring. For such contractible manifolds *Poincaré's lemma* holds.

Poincaré's lemma says that if M is contractible to a point, then all closed forms on M are also exact. This means that $B^p_{DR}(M)$ coincides with $Z^p_{DR}(M)$. As stated before, B^p_{DR} is the unit element of $H^p_{DR}(M)$, and $H^p_{DR}(M)$ thus only exists of the unit element, that is $H^p_{DR} = 0$. Another way of understanding this is to say that every closed form only differs from 0 by an exact form, which is the closed form itself. Hence all forms are equivalent to 0, and they all belong to the same equivalence class. Poincaré's lemma is proved by actually writing each form ω_p as an exact form, which is a bit technical, see [8].

Directly from Poincaré's lemma we have for the space $\mathbb{R}^n$, which is contractible to a point

$$H^p_{DR}(\mathbb{R}^n) = 0, \qquad 1 \leq p \leq n. \tag{4.95}$$

From the result about $0-$forms in (4.92) and (4.93) above it is evident that

$$H^0_{DR}(\mathbb{R}^n) = \mathbb{R}, \qquad \dim H^0_{DR}(\mathbb{R}^n) = 1. \tag{4.96}$$

The content of Poincaré's lemma is probably known to the reader in the case when $M = \mathbb{R}^3$. If a $1-$form (a vector) $\mathbf{A}$ is closed, i.e. if $\nabla \times \mathbf{A} = \mathbf{0}$, then $\mathbf{A}$ is also exact, so that a $0-$form (a scalar function) ϕ exists with $\mathbf{A} = \nabla\phi$. We only get into trouble if there are singularities somewhere in the space. But as long as the space is contractible to a point, we can always find a scalar potential to $\mathbf{A}$ if the curl of $\mathbf{A}$ vanishes. In the example of section 4.4.3, the closed forms were not exact, due to the fact that the punctured plane $\mathbb{R}^2 - \{\mathbf{0}\}$ is not contractible to a point.

We also state some cohomologies when $M = S^n$, the n-sphere, which are also proven in [8]:

$$\begin{aligned} H^p_{DR}(S^n;\mathbb{R}) &= 0, \quad 1 \leq p \leq n, \\ H^p_{DR}(S^n;\mathbb{R}) &= \mathbb{R}, \quad p = n. \end{aligned} \tag{4.97}$$

The notation including $\mathbb{R}$ in $H^p_{DR}(S^n;\mathbb{R})$ will get its motivation later.

Just to see how the de Rham cohomology depends on the specific properties of the manifold M, note the following results for $p = n$

- If M is a compact, connected, orientable manifold, then

$$H^n_{DR}(M;\mathbb{R}) = \mathbb{R}. \tag{4.98}$$

- If M is a compact, connected, non-orientable manifold, then

$$H^n_{DR}(M;\mathbb{R}) = 0. \tag{4.99}$$

- If M is a non-compact, connected manifold then

$$H^n_{DR}(M;\mathbb{R}) = \mathbb{R}. \tag{4.100}$$

4.4.5 Duality between Homology and de Rham Cohomology

Homology

Homology is defined as a quotient group exactly as the de Rham cohomology, but differential forms are exchanged by *chains*. A $p-$chain is a linear combination of submanifolds of dimension p in M. For more details on chains, see Appendix A.3. Also, the exterior derivative d

is replaced by the *boundary operator* ∂. Acting with ∂ on a chain means taking its oriented boundary.

A $p-$chain c_p which is closed by ∂ so that $\partial c_p = 0$ is called a *cycle*.

A $p-$chain c_p that can be written as a boundary of a $(p+1)-$chain, $c_p = \partial b_{p+1}$ is simply called a *boundary*.

Let $Z_p(M) = \{c_p \text{ in } M | \partial c_p = 0\}$, i.e. the set of $p-$cycles in M.

Let $B_p(M) = \{c_p \text{ in } M | c_p = \partial b_{p+1}\}$, the set of $p-$boundaries in M.

These two sets can be considered as groups in the same way as Z^p_{DR} and B^p_{DR}. Then, due to the fact that $\partial^2 = 0$ (a boundary does not have a boundary), all boundaries are cycles and $B_p(M)$ is a subgroup of $Z_p(M)$. The homology of the manifold is then defined as

$$H_p(M) = Z_p(M)/B_p(M), \tag{4.101}$$

in complete analogy with the definition of de Rham cohomology. It consists of the equivalence classes of Z_p whose elements only differ by a boundary. In other words, $H_p(M)$ is the set of cycles modulo boundaries. Figure 4.4 shows three cycles a, b and c on a torus. a and b belong to the same equivalence class since they only differ by a boundary, the boundary of the gray strip. a and c do not belong to the same equivalence class, since they do not differ by a boundary.

The $\mathbb{R}$ in the notation $H_p(M;\mathbb{R})$ means that the linear combination of submanifolds of M which build a cycle has real coefficients. If one for example has complex coeffients, the homology is written $H_p(M;\mathbb{C})$.

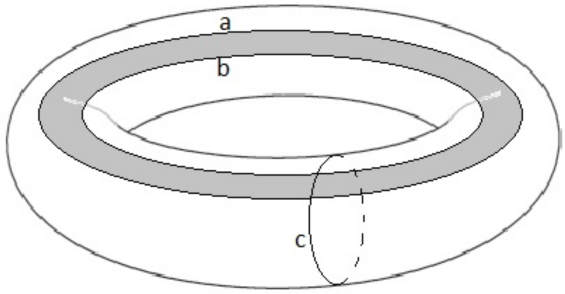


Figure 4.4: The $1-$cycles a and b are equivalent since they bound a twodimensional strip. a and c are not equivalent.

de Rham's theorem

From the similar definitions of the homology and the de Rham cohomology, one might guess that there is some relation between them. To show their duality, one starts with Stokes' theorem for differential $(n-1)-$forms in a manifold M

$$\int_M d\omega_{n-1} = \int_{\partial M} \omega_{n-1}, \tag{4.102}$$

which was introduced in section 4.2. Instead of integrating over the whole manifold, one can integrate over a cycle c in M. This defines the *inner product* $\pi(c_p, \omega_p)$ of a cycle $c_p \in Z_p$ and a closed form $\omega_p \in Z^p_{DR}$

$$\pi(c_p, \omega_p) = \int_{c_p} \omega_p. \tag{4.103}$$

The inner product is a real number and is sometimes referred to as the *period*. Now recall that the elements in H^p_{DR} and H_p are equivalence classes. The form ω_p is a representative of its equivalence class in H^p_{DR} and the cycle c_p is a representative of its equivalence class in

H_p. Is the period independent of the choice of representatives? Take another representative $w'_p \sim \omega_p$. By Stokes' theorem

$$\int_{c_p} \omega'_p = \int_{c_p} w_p + da_{p-1} = \int_{c_p} \omega_p + \int_{c_p} da_{p-1} = \int_{c_p} \omega_p + \int_{\partial c_p} a_{p-1} = \int_{c_p} \omega_p, \tag{4.104}$$

since $\partial c_p = 0$. Also try another cycle $c'_p \sim c_p$

$$\int_{c'_p} \omega_p = \int_{c_p + \partial b_{p+1}} \omega_p = \int_{c_p} \omega_p + \int_{\partial b_{p+1}} \omega_p = \int_{c_p} \omega_p + \int_{b_{p+1}} d\omega_p = \int_{c_p} \omega_p, \tag{4.105}$$

since $d\omega_p = 0$. Hence, one can choose arbitrary representatives and still get the same period. Then the inner product π defines a map

$$\pi : H_p(M;\mathbb{R}) \otimes H^p_{DR}(M;\mathbb{R}) \to \mathbb{R}.$$

In 1931 de Rham proved [13] that the homology $H_p(M;\mathbb{R})$ and the de Rham cohomology H^p_{DR} are dual with respect to π. The theorem is called *de Rham's theorem*, and applies to a compact manifold M with no boundary. We will now regard the homology and the de Rham cohomology as vector spaces. Let $\{c_i\}, i = 1, 2, \ldots, r = \dim\ H_p$ be a basis of p-cycles for $H_p(M;\mathbb{R})$. Then

- i.) one can always find a closed $p-$form ω for any set of real numbers $\nu_i, i = 1, 2, \ldots, r$ so that
$$\nu_i = \pi(c_i, \omega) = \int_{c_i} \omega, \qquad i = 1, \ldots, r.$$
Also,
- ii.) if all the ν_i are zero for a $p-$form ω,
$$0 = \pi(c_i, \omega) = \int_{c_i} \omega, \qquad i = 1, \ldots, r,$$
then ω is exact.

If ω is exact it belongs to the same equivalence class as 0, and as a vector it is 0. Thus, what ii.) says is that if ω is a basis element (which of course is not zero) of $H^p_{DR}(M;\mathbb{R})$, then the column vector $\pi(c_i, \omega)$ must be nonzero. Apply this to all basis vectors $\{\omega_j\}, j = 1, 2, \ldots, s = \dim H^p_{DR}$ of $H^p_{DR}(M;\mathbb{R})$. Then, from the s column vectors $\nu_i = (c_i, \omega_j), i = 1, \ldots, r$ we may construct the *period matrix*

$$\pi_{ij} = \pi(c_i, \omega_j) = \begin{pmatrix} \int_{c_1} \omega_1 & \int_{c_1} \omega_2 & \ldots & \int_{c_1} \omega_s \\ \int_{c_2} \omega_1 & \int_{c_2} \omega_2 & \ldots & \int_{c_2} \omega_s \\ \vdots & \vdots & \ddots & \vdots \\ \int_{c_r} \omega_1 & \int_{c_r} \omega_2 & \ldots & \int_{c_r} \omega_s \end{pmatrix}, \tag{4.106}$$

for which we know that all columns are nonzero.

Consider a linear combination with real coefficients $x_j,\ j = 1, \ldots, s$ of the s column vectors in the matrix. Then

$$x_1 \begin{pmatrix} \int_{c_1} \omega_1 \\ \vdots \\ \int_{c_r} \omega_1 \end{pmatrix} + \ldots + x_s \begin{pmatrix} \int_{c_1} \omega_s \\ \vdots \\ \int_{c_r} \omega_s \end{pmatrix} = \begin{pmatrix} \int_{c_1} x_1\omega_1 + \ldots + x_s\omega_s \\ \vdots \\ \int_{c_r} x_1\omega_1 + \ldots + x_s\omega_s \end{pmatrix}, \tag{4.107}$$

by linearity of integration. Choosing real coefficients $x_j,\ j = 1, \ldots, s$ reflects the fact that we deal with $H^p(M,\mathbb{R})$. For the cohomology $H^p(M,\mathbb{C})$ we would instead have used complex coefficients. The resulting form $x_1\omega_1 + \ldots x_s\omega_s$ is a closed $p-$form by i.). Then set (4.107) equal to zero. Applying ii.), the form $x_1\omega_1 + \ldots x_s\omega_s$ must then be equal to 0. But since $\{\omega_j\}, j = 1, \ldots, s$ is a basis for H^p_{DR}, all $x_j, j = 1, \ldots, s$ are zero. Thus, the column vectors in (4.107) are all linearly independent. The only way in which s vectors can be linearly independent in $\mathbb{R}^r$ and still span $\mathbb{R}^r$ is if $s = r$. r linearly independent column vectors in $\mathbb{R}^r$ yield an invertible matrix. We have thus found that the period matrix π_{ij} in (4.106) is invertible, and by definition $H^p_{DR}(M;\mathbb{R})$ is dual to $H_p(M,\mathbb{R})$. Often, the cohomology H^p (not de Rham) is *defined* as the dual of the homology H_p. Then, what we have shown is that the de Rham cohomology is equal to that cohomology. Let us therefore drop the subscript DR of H^p_{DR}.

Duality of H_p and H^p means that they are naturally isomorphic,

$$H^p(M;\mathbb{R}) \cong H_p(M,\mathbb{R}), \tag{4.108}$$

and having the same numbers of elements we can define the p:th *Betti number* of M as

$$b_p(M) = \dim H_p(M;\mathbb{R}) = \dim H^p(M,\mathbb{R}). \tag{4.109}$$

Then the *Euler characteristic* of M, further discussed in Appendix A.3,

$$\chi(M) = \sum_{p=0}^{n}(-1)^p b_p(M), \tag{4.110}$$

can be found from the de Rham cohomology instead of homology. Another way of expressing this is to say that the topological Euler characteristic of homology is equal to the analytic Euler characteristic of de Rham cohomology. As a bridge-builder, de Rham's theorem is clearly one of the most important theorems used in this paper.

Complexes

Now, let C_p be the set of chains in M which are infinitely differentiable, and let Ω^p denote the set of $p-$forms in M. Then the action of ∂ and d respectively, on these series of spaces, so called *complexes*, can be illustrated as

$$\begin{array}{ccccccccc} \ldots & \overset{\partial_{p-1}}{\longleftarrow} & C_{p-1} & \overset{\partial_p}{\longleftarrow} & C_p & \overset{\partial_{p+1}}{\longleftarrow} & C_{p+1} & \overset{\partial_{p+2}}{\longleftarrow} & \ldots \\ \ldots & \overset{d_{p-1}}{\longrightarrow} & \Omega_{p-1} & \overset{d_p}{\longrightarrow} & \Omega_p & \overset{d_{p+1}}{\longrightarrow} & \Omega_{p+1} & \overset{d_{p+2}}{\longrightarrow} & \ldots \\ , \end{array} \tag{4.111}$$

where the duality of H^p and H_p is reflected in the fact that ∂ and d operate in different directions. Duality also guarantees that there are as many C spaces as Ω spaces.

Regarding ∂_p as an operator $\partial_p : C_p \to C_{p-1}$, it is evident that the homology of M also may be written as

$$H_p(M,\mathbb{R}) = \text{Ker } \partial_p/\text{Im } \partial_{p+1}, \tag{4.112}$$

where Ker ∂_p is the kernel of ∂_p and Im ∂_{p+1} is the image of ∂_{p+1}. Equivalently, since d_{p+1} is the operator $d_{p+1} : \Omega^p \to \Omega^{p+1}$, the cohomology is

$$H^p(M;\mathbb{R}) = \text{Ker } d_{p+1}/\text{Im } d_p. \tag{4.113}$$

Chapter 5

Supersymmetric Quantum Mechanics - Part II: Sigma Models

In chapter 3 we analysed the supersymmetric quantum mechanics in flat space. To really see the connection between the physics of supersymmetric quantum mechanics and the mathematics introduced in chapter 4, we have to repeat the calculations we did for a flat metric, but now on a compact manifold with curvature. That is, we will find the Lagrangian, whose variation in terms of supersymmetric variations of its coordinates can be written as a total time derivative. Hence there are conserved supercharges whose commutator essentially gives the Hamiltonian of the system. These supercharges will then guide us from the physics of supersymmetric quantum mechanics to the mathematics of de Rham cohomology.

5.1 The Supersymmetric Lagrangian on a curved Manifold

We consider a particle moving in a compact Riemannian manifold M of dimension n. Since the manifold is supposed to be curved, the metric depends on the position $q^\mu, \mu = 1, 2, \dots n$ in the manifold, $g_{\mu\nu} = g_{\mu\nu}(q)$. The position vector q^μ of the particle gives the bosonic part of the theory. In the supersymmetric quantum mechanics, there must also be fermions. They will in the classical theory be represented by two variables ψ_1^μ and ψ_2^μ, which are odd Grassmann numbers.

5.1.1 The Superspace Technique

Due to the fact that the manifold is curved, the Lagrangian L now involves the Riemann tensor. To be able to write the variation of the Lagrangian δL as a total time derivative, we must differentiate the Riemann tensor. The Riemann tensor involves several Christoffel symbols of the Levi-Civita connection, which all involve derivatives of the metric. We will end up with a huge expression for δL with derivatives of the metric up to order three. To bring such a monstrous expression into a nice time derivative will become a painstaking task.

Instead, we turn to another method, the *superspace technique*. It is used in supersymmetry to solve specific problems. It provides us with a way to find what we want, a Lagrangian that is manifestly supersymmetric and supercharges whose commutator is the Hamiltonian.

For a particle moving in M the Lagrangian would just be

$$L = \frac{1}{2}v^2 = \frac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu. \tag{5.1}$$

The supersymmetric extension of this Lagrangian must also involve a fermionic part. Using the superspace technique, we have to do two important transformations. In addition to time, the system also evolves in two new coordinates θ_1 and θ_2, which are odd Grassmann numbers. Second, the position coordinate q^μ is substituted by a field, a *superfield* ϕ^μ. In short

$$\begin{aligned} t &\rightarrow t, \theta_1, \theta_2 \\ q^\mu(t) &\rightarrow \phi^\mu(t,\theta) := q^\mu(t) + i\theta_a\psi_a^\mu(t) - i\theta_1\theta_2F^\mu(t). \end{aligned} \tag{5.2}$$

In the expression for the superfield above, $a = 1, 2$. Note that a is not a tensor index, just a summation index. F is an auxiliary field that can be eliminated from the Lagrangian by using Lagrange's equations of motion, which we also will do in next section. The superspace technique also involves the introduction of two kinds of differentiation operators

$$\begin{aligned} D_a &:= \tfrac{\partial}{\partial \theta_a} - i\theta_a \tfrac{\partial}{\partial t} := \partial_a - i\theta_a \partial_t, \\ \bar{D}_a &:= \tfrac{\partial}{\partial \theta_a} + i\theta_a \tfrac{\partial}{\partial t} := \partial_a + i\theta_a \partial_t, \end{aligned} \tag{5.3}$$

which are each other's complex conjugates. To get acquainted with the algebra of these operators we investigate their commutation properties. First, it is clear that $\{\partial_a, \theta_b\} = \delta_{ab}$ since by using the Grassmann property of θ

$$\{\partial_a, \theta_b\} = \left(\frac{\partial}{\partial \theta_a}\theta_b + \theta_b \frac{\partial}{\partial \theta_a}\right) = \frac{\partial \theta_b}{\partial \theta_a} - \theta_b \frac{\partial}{\partial \theta_a} + \theta_b \frac{\partial}{\partial \theta_a} = \delta_{ab}, \tag{5.4}$$

where still $a = 1, 2;\ b = 1, 2$. The anti-commutator of $\bar{D}_a$ and $\bar{D}_b$ is then easily computed as

$$\begin{aligned} \{\bar{D}_a, \bar{D}_b\} &= \bar{D}_a \bar{D}_b + \bar{D}_b \bar{D}_a \\ &= (\partial_a + i\theta_a \partial_t)(\partial_b + i\theta_b \partial_t) + (\partial_b + i\theta_b \partial_t)(\partial_a + i\theta_a \partial_t) \\ &= \{\partial_a, \partial_b\} + i\{\partial_a, \theta_b\}\partial_t + i\{\partial_b, \theta_a\}\partial_t - \{\theta_a, \theta_b\}\partial_t \partial_t \\ &= 0 + i\delta_{ab}\partial_t + i\delta_{ba}\partial_t - 0 \\ &= 2i\delta_{ab}\partial_t. \end{aligned} \tag{5.5}$$

In the same way one can show that

$$\{D_a, D_b\} = -\{\bar{D}_a, \bar{D}_b\} = -2i\delta_{ab}\partial_t. \tag{5.6}$$

We will later need $\{D_a, \bar{D}_b\}$. Actually it is zero,

$$\begin{aligned} \{D_a, \bar{D}_b\} &= D_a \bar{D}_b + \bar{D}_b D_a \\ &= (\partial_a - i\theta_a \partial_t)(\partial_b + i\theta_b \partial_t) + (\partial_b + i\theta_b \partial_t)(\partial_a - i\theta_a \partial_t) \\ &= \{\partial_a, \partial_b\} + i\{\partial_a, \theta_b\}\partial_t - i\{\partial_b, \theta_a\}\partial_t + \{\theta_a, \theta_b\}\partial_t \partial_t \\ &= 0. \end{aligned} \tag{5.7}$$

What are the supersymmetric variations of q, ψ and F? They can be found by performing a variation of the superfield, $\phi \to \phi' = \phi + \delta\phi$. From its definition in equation (5.2), $\phi^\mu(t, \theta) = q^\mu(t) + i\theta_b \psi_b^\mu(t) - i\theta_1\theta_2 F^\mu(t)$, and the variation becomes

$$\delta\phi^\mu = \delta q^\mu + i\theta_b \delta\psi_b^\mu - i\theta_1\theta_2 \delta F^\mu. \tag{5.8}$$

The variation of ϕ can also be written as a $\bar{D}_a$ derivative acting on ϕ. Let ε_a be a small odd Grassmann number and write

$$\delta\phi^\mu = \varepsilon_a \bar{D}_a \phi^\mu = \varepsilon_a(\partial_a + i\theta_a \partial_t)\phi^\mu = \varepsilon_a \frac{\partial \phi^\mu}{\partial \theta_a} + i\varepsilon_a \theta_a \dot{\phi}^\mu. \tag{5.9}$$

For clarity, compute $\frac{\partial \phi^\mu}{\partial \theta_a}$ and $\dot{\phi}^\mu$ separately. First,

$$\begin{aligned} \frac{\partial \phi_\mu}{\partial \theta_a} &= \frac{\partial}{\partial \theta_a}(q^\mu(t) + i\theta_b \psi_b^\mu(t) - i\theta_1\theta_2 F^\mu(t)) \\ &= i\delta_{ab}\psi_b^\mu - i\varepsilon_{ab}\theta_b F^\mu \\ &= i\psi_a^\mu - i\varepsilon_{ab}\theta_b F^\mu, \end{aligned} \tag{5.10}$$

where ε_{ab} is the Levi-Civita tensor in two dimensions, $\varepsilon_{12} = 1, \varepsilon_{21} = -1, \varepsilon_{11} = \varepsilon_{22} = 0$. Hence, remembering that ε_a is an odd Grassmann number,

$$\begin{aligned} \varepsilon_a \frac{\partial \phi^\mu}{\partial \theta_a} &= i\varepsilon_a \psi_a^\mu - i\varepsilon_a \varepsilon_{ab}\theta_b F^\mu \\ &= i\varepsilon_a \psi_a^\mu + i\theta_b \varepsilon_a \varepsilon_{ab} F^\mu \\ &= i\varepsilon_a \psi_a^\mu + i\theta_a \varepsilon_b \varepsilon_{ba} F^\mu \\ &= i\varepsilon_a \psi_a^\mu - i\theta_a \varepsilon_b \varepsilon_{ab} F^\mu. \end{aligned} \tag{5.11}$$

It is evident from (5.2) that $\dot{\phi}^\mu = \dot{q}^\mu + i\theta_b \dot{\psi}^\mu_b - i\theta_1\theta_2 \dot{F}^\mu$, and

$$\begin{aligned} i\varepsilon_a \theta_a \dot{\phi}^\mu &= i\varepsilon_a \theta_a (\dot{q}^\mu + i\theta_b \dot{\psi}^\mu_b - i\theta_1\theta_2 \dot{F}^\mu) \\ &= i\varepsilon_a \theta_a \dot{q}^\mu - \varepsilon_a \theta_a \theta_b \dot{\psi}^\mu_b \\ &= -i\theta_a \varepsilon_a \dot{q}^\mu - \theta_1\theta_2 \varepsilon_a \varepsilon_{ab} \dot{\psi}^\mu_b . \end{aligned} \tag{5.12}$$

Adding the results from (5.11) and (5.12), (5.9) becomes

$$\begin{aligned} \delta\phi^\mu &= i\varepsilon_a \psi^\mu_a - i\theta_a \varepsilon_b \varepsilon_{ab} F^\mu - i\theta_a \varepsilon_a \dot{q}^\mu - \theta_1\theta_2 \varepsilon_a \varepsilon_{ab} \dot{\psi}^\mu_b \\ &= (i\varepsilon_a \psi^\mu_a) + i\theta_a(-\varepsilon_a \dot{q}^\mu - \varepsilon_b \varepsilon_{ab} F^\mu) - i\theta_1\theta_2(-i\varepsilon_a \varepsilon_{ab} \dot{\psi}^\mu_b). \end{aligned} \tag{5.13}$$

Having written $\delta\phi^\mu$ in this form, it is easy to read off the supersymmetric transformations of q, ψ and F, comparing (5.8) and (5.13). The result is

$$\begin{array}{lcl} \delta q^\mu &=& i\varepsilon_a \psi^\mu_a, \\ \delta \psi^\mu_a &=& -\varepsilon_a \dot{q}^\mu - \varepsilon_b \varepsilon_{ab} F^\mu, \\ \delta F^\mu &=& -i\varepsilon_a \varepsilon_{ab} \dot{\psi}^\mu_b . \end{array} \tag{5.14}$$

We have not shown that they yield a supersymmetric Lagrangian, yet it is clear that they mix bosonic and fermionic variables as they should. Now is the time to start the machinery of Lagrangian mechanics, where the action

$$S = \int dt L \tag{5.15}$$

should be stationary under these supersymmetric variations. The Lagrangian L is a function of $\phi = \phi(t, \theta)$ and will involve integration over the odd Grassmann variables θ_1 and θ_2. It can be written as[1]

$$L = -\frac{1}{2} \int d^2\theta (-i) g_{\mu\nu}(\phi) D_1 \phi^\mu D_2 \phi^\nu, \tag{5.17}$$

where D_1 and D_2 are the recently introduced derivatives in (5.3). We can immediately check that constructing L as in (5.17), makes its variation become a total time derivative. Remember from the first equality of (5.9) that

$$\delta\phi = \varepsilon_a \bar{D}_a \phi. \tag{5.18}$$

Then, since the metric $g_{\mu\nu}$ is a function of ϕ one gets

$$\delta g_{\mu\nu}(\phi) = \frac{\partial g_{\mu\nu}}{\partial \phi^\lambda} \delta\phi^\lambda = \frac{\partial g_{\mu\nu}}{\partial \phi^\lambda} \varepsilon_a \bar{D}_a \phi^\lambda = \varepsilon_a \frac{\partial g_{\mu\nu}}{\partial \phi^\lambda} \bar{D}_a \phi^\lambda = \varepsilon_a \bar{D}_a g_{\mu\nu}(\phi), \tag{5.19}$$

by the chain rule. Then since $\{D_1, \bar{D}_a\} = \{D_2, \bar{D}_a\} = 0$ from (5.7) we have

$$\begin{array}{lclclcl} \delta D_1 \phi^\mu &=& D_1 \delta\phi^\mu &=& D_1 \varepsilon_a \bar{D}_a \phi^\mu &=& \varepsilon_a \bar{D}_a D_1 \phi^\mu, \\ \delta D_2 \phi^\nu &=& D_2 \delta\phi^\nu &=& D_2 \varepsilon_a \bar{D}_a \phi^\nu &=& \varepsilon_a \bar{D}_a D_2 \phi^\nu. \end{array} \tag{5.20}$$

Using these results in the variation of L in (5.17) we get

[1] In the generalisation of Lagrangian mechanics to quantum field theory, one introduces the Lagrangian density $\mathcal{L}$. In this language, the action also becomes an integral over space:

$$S = \int dt L = \int dt \int dx^3 \mathcal{L}. \tag{5.16}$$

This may help us to understand why there is integration over the superspace variables θ_1 and θ_2 in (5.17).

$$
\begin{aligned}
\delta L &= \frac{i}{2}\delta\int d^2\theta\; g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu \\
&= \frac{i}{2}\int d^2\theta\,[(\delta g_{\mu\nu}(\phi))D_1\phi^\mu D_2\phi^\nu + g_{\mu\nu}(\phi)(\delta D_1\phi^\mu)D_2\phi^\nu + g_{\mu\nu}(\phi)D_1\phi^\mu(\delta D_2\phi^\nu)] \\
&= \frac{i}{2}\int d^2\theta\,\left[\varepsilon_a\bar{D}_a g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu + g_{\mu\nu}(\phi)(\varepsilon_a\bar{D}_a D_1\phi^\mu)D_2\phi^\nu + g_{\mu\nu}(\phi)D_1\phi^\mu(\varepsilon_a\bar{D}_a D_2\phi^\nu)\right] \\
&= \frac{i}{2}\int d^2\theta\;\varepsilon_a\bar{D}_a\left(g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu\right) \\
&= \frac{i}{2}\int d^2\theta\;\varepsilon_a\left(\frac{\partial}{\partial\theta_a}+i\theta_a\frac{\partial}{\partial t}\right)\left(g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu\right),
\end{aligned}
\tag{5.21}
$$

where we have used that $\bar{D}_a$ is a linear operator. From the integration rules in (3.5), the operator $\frac{\partial}{\partial\theta_a}$ in $\bar{D}_a$ will not contribute to δL in (5.21), and δL is then a total time derivative under the variations in (5.14). This will be completely obvious in section 5.2 where the supercharges are computed.

5.1.2 Calculation of the Lagrangian

The first problem we encounter in (5.17) is the fact that $g_{\mu\nu}$ depends on ϕ, which in turn depends on θ. We must somehow write $g_{\mu\nu}$ as a function only of the bosonic parameter q. This can be done by considering the fermionic part in ϕ as a little deviation from q, and then performing a Taylor expansion around q. Generally, a Taylor expansion contains an infinite number of terms. In this particular case it will only contain derivatives up to second order, thanks to the Grassmann property of θ_1 and θ_2. In mathematical terms

$$
\begin{aligned}
g_{\mu\nu}(\phi) &= g_{\mu\nu}(q^\mu + i\theta_a\psi_a^\mu - i\theta_1\theta_2F^\mu) \\
&= g_{\mu\nu}(q^\mu + (i\theta_a\psi_a^\mu - i\theta_1\theta_2F^\mu)) \\
&= g_{\mu\nu}(q^\mu) + (i\theta_a\psi_a^\rho - i\theta_1\theta_2F^\rho)\frac{\partial}{\partial q^\rho}g_{\mu\nu}(q^\mu) \\
&\quad + \frac{1}{2}(i\theta_a\psi_a^\rho - i\theta_1\theta_2F^\rho)(i\theta_b\psi_b^\sigma - i\theta_1\theta_2F^\sigma)\frac{\partial}{\partial q^\rho}\frac{\partial}{\partial q^\sigma}g_{\mu\nu} \\
&= g_{\mu\nu}(q^\mu) + (i\theta_a\psi_a^\rho - i\theta_1\theta_2F^\rho)g_{\mu\nu,\rho} - \frac{1}{2}\theta_a\psi_a^\rho\theta_b\psi_b^\sigma g_{\mu\nu,\rho\sigma},
\end{aligned}
\tag{5.22}
$$

and the θ dependence on $g_{\mu\nu}(\phi)$ has become explicit.

Next issues in (5.17) are the factors $D_1\phi^\mu$ and $D_2\phi^\nu$

$$
D_2\phi^\nu = \left(\frac{\partial}{\partial\theta_2} - i\theta_2\frac{\partial}{\partial t}\right)(q^\nu + i\theta_a\psi_a^\nu - i\theta_1\theta_2F^\nu) = i\psi_2^\nu + i\theta_1F^\nu - i\theta_2\dot{q}^\nu + \theta_2\theta_1\dot{\psi}_1^\nu, \tag{5.23}
$$

$$
D_1\phi^\mu = \left(\frac{\partial}{\partial\theta_1} - i\theta_1\frac{\partial}{\partial t}\right)(q^\mu + i\theta_a\psi_a^\mu - i\theta_1\theta_2F^\mu) = i\psi_1^\mu - i\theta_2F^\mu - i\theta_1\dot{q}^\mu + \theta_1\theta_2\dot{\psi}_2^\mu. \tag{5.24}
$$

Multiplying these factors together gives

$$
\begin{aligned}
D_1\phi^\mu D_2\phi^\nu =&(i\psi_1^\mu - i\theta_2F^\mu - i\theta_1\dot{q}^\mu + \theta_1\theta_2\dot{\psi}_2^\mu)(i\psi_2^\nu + i\theta_1F^\nu - i\theta_2\dot{q}^\nu + \theta_2\theta_1\dot{\psi}_1^\nu) \\
=& -\psi_1^\mu\psi_2^\nu + \theta_1\psi_1^\mu F^\nu - \theta_2\psi_1^\mu\dot{q}^\nu - i\theta_1\theta_2\psi_1^\mu\dot{\psi}_1^\nu + \theta_2\psi_2^\nu F^\mu - \theta_1\theta_2F^\mu F^\nu \\
&+ \theta_1\dot{q}^\mu\psi_2^\nu - \theta_1\theta_2\dot{q}^\mu\dot{q}^\nu + i\theta_1\theta_2\dot{\psi}_2^\mu\psi_2^\nu,
\end{aligned}
\tag{5.25}
$$

where we have used the important fact that both θ_a and ψ_b are odd Grassmann numbers, and hence anti-commute with each other.

The next step is to multiply $g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu$ in (5.22) and (5.25). We must as before be careful with the ordering of the factors that are odd Grassmann numbers. Some terms will disappear since they contain $\theta_a\theta_a = 0$, and we only get

$$
\begin{aligned}
g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu = &-\psi_1^\mu\psi_2^\nu g_{\mu\nu}+\theta_1\psi_1^\mu F^\nu g_{\mu\nu}-\theta_2\psi_1^\mu\dot{q}^\nu g_{\mu\nu}-i\theta_1\theta_2\psi_1^\mu\dot{\psi}_1^\nu g_{\mu\nu}+\theta_2\psi_2^\nu F^\mu g_{\mu\nu}\\
&-\theta_1\theta_2F^\mu F^\nu g_{\mu\nu}+\theta_1\dot{q}^\mu\psi_2^\nu g_{\mu\nu}-\theta_1\theta_2\dot{q}^\mu\dot{q}^\nu g_{\mu\nu}+i\theta_1\theta_2\dot{\psi}_2^\mu\psi_2^\nu g_{\mu\nu}\\
&-i\theta_a\psi_a^\rho\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho}+i\theta_1\theta_2\psi_2^\rho\psi_1^\mu F^\nu g_{\mu\nu,\rho}+i\theta_1\theta_2\psi_1^\rho\psi_1^\mu\dot{q}^\nu g_{\mu\nu,\rho}\\
&-i\theta_1\theta_2\psi_1^\rho\psi_2^\nu F^\mu g_{\mu\nu,\rho}+i\theta_1\theta_2\psi_2^\rho\psi_2^\nu\dot{q}^\mu g_{\mu\nu,\rho}+i\theta_1\theta_2F^\rho\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho}\\
&-\frac{1}{2}\theta_a\theta_b\psi_a^\rho\psi_b^\sigma\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho\sigma}.
\end{aligned}
\tag{5.26}
$$

Now we are ready to perform the integral

$$
L=-\frac{1}{2}\int d^2\theta(-i)g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu. \tag{5.27}
$$

Due to the integration rules in (3.5)

$$
\begin{aligned}
&\textstyle\int d^2\theta(-i\theta_1\theta_2)=1,\\
&\textstyle\int d^2\theta\theta_a=0,\\
&\textstyle\int d^2\theta c=0,
\end{aligned}
\tag{5.28}
$$

all terms in (5.26) that do not have two factors of θ:s will disappear when the integration in (5.27) is performed. Having already written a factor of $(-i)$ in (5.27), integration of the terms in (5.26) with $\theta_1\theta_2$, means just that one removes $\theta_1\theta_2$ from that term. The last term in (5.26) involves two terms, one with $\theta_1\theta_2$ and one with $\theta_2\theta_1$. After the integration they can still be written as one term by use of ε_{ab}. We get

$$
\begin{aligned}
L=-\frac{1}{2}\Big(&-i\psi_1^\mu\dot{\psi}_1^\nu g_{\mu\nu}-F^\mu F^\nu g_{\mu\nu}-\dot{q}^\mu\dot{q}^\nu g_{\mu\nu}+i\dot{\psi}_2^\mu\psi_2^\nu g_{\mu\nu}+i\psi_2^\rho\psi_1^\mu F^\nu g_{\mu\nu,\rho}+i\psi_1^\rho\psi_1^\mu\dot{q}^\nu g_{\mu\nu,\rho}\\
&-i\psi_1^\rho\psi_2^\nu F^\mu g_{\mu\nu,\rho}+i\psi_2^\rho\psi_2^\nu\dot{q}^\mu g_{\mu\nu,\rho}+iF^\rho\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho}-\frac{1}{2}\varepsilon_{ab}\psi_a^\rho\psi_b^\sigma\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho\sigma}\Big).
\end{aligned}
\tag{5.29}
$$

Rewriting (5.29) would make it easier to handle. The fourth term can, by first changing the order of the ψ:s and then interchanging the indices μ and ν, be written as

$$
i\dot{\psi}_2^\mu\psi_2^\nu g_{\mu\nu}=-i\psi_2^\nu\dot{\psi}_2^\mu g_{\mu\nu}=-i\psi_2^\mu\dot{\psi}_2^\nu g_{\nu\mu}=-i\psi_2^\mu\dot{\psi}_2^\nu g_{\mu\nu}, \tag{5.30}
$$

where in the last step the symmetry of the metric was used. The first and the fourth term in the parenthesis of (5.29) combine into

$$
-i\psi_1^\mu\dot{\psi}_1^\nu g_{\mu\nu}-i\psi_2^\mu\dot{\psi}_2^\nu g_{\mu\nu}=-i\psi_a^\mu\dot{\psi}_a^\nu g_{\mu\nu}. \tag{5.31}
$$

In the same way, the sixth and the eight terms in the parenthesis of (5.29) can be written as one term

$$
i\psi_1^\rho\psi_1^\mu\dot{q}^\nu g_{\mu\nu,\rho}+i\psi_2^\rho\psi_2^\nu\dot{q}^\mu g_{\mu\nu,\rho}=i\psi_a^\rho\psi_a^\nu\dot{q}^\mu g_{\mu\nu,\rho}. \tag{5.32}
$$

Further examining (5.29), there are three terms linear in F, which by changes of indices can be grouped to one single term:

$$
\begin{aligned}
&i\psi_2^\rho\psi_1^\mu F^\nu g_{\mu\nu,\rho}-i\psi_1^\rho\psi_2^\nu F^\mu g_{\mu\nu,\rho}+iF^\rho\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho}\\
&=-i\psi_1^\mu\psi_2^\rho F^\nu g_{\mu\nu,\rho}-i\psi_1^\rho\psi_2^\nu F^\mu g_{\mu\nu,\rho}+i\psi_1^\mu\psi_2^\nu F^\rho g_{\mu\nu,\rho}\\
&=-i\psi_1^\mu\psi_2^\nu F^\rho g_{\mu\rho,\nu}-i\psi_1^\mu\psi_2^\nu F^\rho g_{\rho\nu,\mu}+i\psi_1^\mu\psi_2^\nu F^\rho g_{\mu\nu,\rho}\\
&=-i\psi_1^\mu\psi_2^\nu F^\rho(g_{\mu\rho,\nu}+g_{\rho\nu,\mu}-g_{\mu\nu,\rho}).
\end{aligned}
\tag{5.33}
$$

Remember from section 4.3 that the Christoffel symbol of the Levi-Civita connection is defined in terms of three derivatives of the metric,

$$
\Gamma^\lambda{}_{\mu\nu}=\frac{1}{2}g^{\lambda\sigma}(g_{\sigma\mu,\nu}+g_{\sigma\nu,\mu}-g_{\mu\nu,\sigma}). \tag{5.34}
$$

so lowering the first index with the metric yields

$$
\begin{aligned}
\Gamma_{\rho\mu\nu} &= g_{\rho\lambda}\Gamma^{\lambda}{}_{\mu\nu} \\
&= \frac{1}{2} g_{\rho\lambda} g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}) \\
&= \frac{1}{2}\delta^{\sigma}_{\rho}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}) \\
&= \frac{1}{2}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}).
\end{aligned} \tag{5.35}
$$

The terms linear in F in (5.33) thus has a Christoffel symbol coefficient and can simply be written as

$$
-2i\psi_1^{\mu}\psi_2^{\nu}F^{\rho}\Gamma_{\rho\mu\nu}. \tag{5.36}
$$

Taking all of the above into account, and multiplying the factor $-\frac{1}{2}$ into the parenthesis of (5.29), one gets a nice expression for the Lagrangian

$$
\begin{aligned}
L = {} & \frac{1}{2}\dot{q}^{\mu}\dot{q}^{\nu}g_{\mu\nu} + \frac{1}{2}F^{\mu}F^{\nu}g_{\mu\nu} + \frac{i}{2}\psi_a^{\mu}\dot{\psi}_a^{\nu}g_{\mu\nu} - \frac{i}{2}\psi_a^{\rho}\psi_a^{\nu}\dot{q}^{\mu}g_{\mu\nu,\rho} \\
& + \frac{1}{4}\varepsilon_{ab}\psi_a^{\rho}\psi_b^{\sigma}\psi_1^{\mu}\psi_2^{\nu}g_{\mu\nu,\rho\sigma} + i\psi_1^{\mu}\psi_2^{\nu}\Gamma_{\rho\mu\nu}F^{\rho}.
\end{aligned} \tag{5.37}
$$

We immediately recognize the term $\frac{1}{2}\dot{q}^{\mu}\dot{q}^{\nu}g_{\mu\nu}$ in (5.37). Then (5.37) is, by reference to (5.21) just a supersymmetric extension of (5.1).

The auxiliary field F does not contribute to any degree of freedom, and now is the time to eliminate it from L. To do this, we use one of Lagrange's equations of motion, which reads

$$
\frac{\partial L}{\partial F} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{F}}\right) = 0, \tag{5.38}
$$

since L does not depend on $\dot{F}$. We therefore differentiate L with respect to F^{λ} and use the obtained equation to eliminate F,

$$
\begin{aligned}
0 = \frac{\partial L}{\partial F} &= \frac{1}{2}\frac{\partial}{\partial F^{\lambda}}(F^{\mu}F^{\nu})g_{\mu\nu} + i\psi_1^{\mu}\psi_2^{\nu}\Gamma_{\rho\mu\nu}\frac{\partial F^{\rho}}{\partial F^{\lambda}} \\
&= \frac{1}{2}\frac{\partial F^{\mu}}{\partial F^{\lambda}}F^{\nu}g_{\mu\nu} + \frac{1}{2}F^{\mu}\frac{\partial F^{\nu}}{\partial F^{\lambda}}g_{\mu\nu} + i\psi_1^{\mu}\psi_2^{\nu}\Gamma_{\rho\mu\nu}\delta^{\rho}_{\lambda} \\
&= \frac{1}{2}\delta^{\mu}_{\lambda}F^{\nu}g_{\mu\nu} + \frac{1}{2}F^{\mu}\delta^{\nu}_{\lambda}g_{\mu\nu} + i\psi_1^{\mu}\psi_2^{\nu}\Gamma_{\lambda\mu\nu} \\
&= \frac{1}{2}F^{\nu}g_{\lambda\nu} + \frac{1}{2}F^{\mu}g_{\mu\lambda} + i\psi_1^{\mu}\psi_2^{\nu}\Gamma_{\lambda\mu\nu} \\
&= \frac{1}{2}F^{\mu}g_{\lambda\mu} + \frac{1}{2}F^{\mu}g_{\mu\lambda} + i\psi_1^{\mu}\psi_2^{\nu}\Gamma_{\lambda\mu\nu} \\
&= F^{\mu}g_{\lambda\mu} + i\psi_1^{\mu}\psi_2^{\nu}\Gamma_{\lambda\mu\nu},
\end{aligned} \tag{5.39}
$$

which means that

$$
F^{\mu}g_{\lambda\mu} = -i\psi_1^{\mu}\psi_2^{\nu}\Gamma_{\lambda\mu\nu}, \tag{5.40}
$$

and

$$
F^{\gamma} = F^{\mu}\delta^{\gamma}_{\mu} = g^{\gamma\lambda}F^{\mu}g_{\lambda\mu} = g^{\gamma\lambda}(-i\psi_1^{\mu}\psi_2^{\nu}\Gamma_{\lambda\mu\nu}) = -i\psi_1^{\mu}\psi_2^{\nu}\Gamma^{\gamma}{}_{\mu\nu}. \tag{5.41}
$$

When inserting F from (5.41) in (5.37), we must not use the same dummy indices for different summations. The term quadratic in F becomes

$$
\frac{1}{2}F^{\mu}F^{\nu}g_{\mu\nu} = \frac{1}{2}(-i\psi_1^{\alpha}\psi_2^{\beta}\Gamma^{\mu}{}_{\alpha\beta})(-i\psi_1^{\gamma}\psi_2^{\varepsilon}\Gamma^{\nu}{}_{\gamma\varepsilon})g_{\mu\nu} = -\frac{1}{2}\psi_1^{\alpha}\psi_2^{\beta}\psi_1^{\gamma}\psi_2^{\varepsilon}\Gamma^{\mu}{}_{\alpha\beta}\Gamma_{\mu\gamma\varepsilon}. \tag{5.42}
$$

The term linear in F is

$$i\psi_1^\mu\psi_2^\nu\Gamma_{\rho\mu\nu}(-i\psi_1^\alpha\psi_2^\beta\Gamma^\rho{}_{\alpha\beta}) = \psi_1^\mu\psi_2^\nu\psi_1^\alpha\psi_2^\beta\Gamma^\rho{}_{\alpha\beta}\Gamma_{\rho\mu\nu}. \tag{5.43}$$

Thus, in total we have

$$\begin{aligned} L = &\frac{1}{2}\dot{q}^\mu\dot{q}^\nu g_{\mu\nu} + \frac{i}{2}\psi_a^\mu\dot{\psi}_a^\nu g_{\mu\nu} - \frac{i}{2}\psi_a^\rho\psi_a^\nu\dot{q}^\mu g_{\mu\nu,\rho} \\ &- \frac{1}{2}\psi_1^\alpha\psi_2^\beta\psi_1^\gamma\psi_2^\varepsilon\Gamma^\mu{}_{\alpha\beta}\Gamma_{\mu\gamma\varepsilon} + \psi_1^\mu\psi_2^\nu\psi_1^\alpha\psi_2^\beta\Gamma^\rho{}_{\alpha\beta}\Gamma_{\rho\mu\nu} + \frac{1}{4}\varepsilon_{ab}\psi_a^\rho\psi_b^\sigma\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho\sigma}. \end{aligned} \tag{5.44}$$

The last three terms in (5.44) above have the common property that they all involve four factors of ψ. Let us write them as a single term, with the ψ factors in order as $\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon$. The first of these terms becomes, after interchanging the ψ:s and then the indices

$$-\frac{1}{2}\psi_1^\alpha\psi_2^\beta\psi_1^\gamma\psi_2^\varepsilon\Gamma^\mu{}_{\alpha\beta}\Gamma_{\mu\gamma\varepsilon} = \frac{1}{2}\psi_1^\alpha\psi_1^\gamma\psi_2^\beta\psi_2^\varepsilon\Gamma^\mu{}_{\alpha\beta}\Gamma_{\mu\gamma\varepsilon} = \frac{1}{2}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon\Gamma^\mu{}_{\alpha\gamma}\Gamma_{\mu\beta\varepsilon}, \tag{5.45}$$

and the second

$$\psi_1^\mu\psi_2^\nu\psi_1^\alpha\psi_2^\beta\Gamma^\rho{}_{\alpha\beta}\Gamma_{\rho\mu\nu} = -\psi_1^\mu\psi_1^\alpha\psi_2^\nu\psi_2^\beta\Gamma^\rho{}_{\alpha\beta}\Gamma_{\rho\mu\nu} = -\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon\Gamma^\mu{}_{\beta\varepsilon}\Gamma_{\mu\alpha\gamma}. \tag{5.46}$$

The third term may be split into two according to

$$\begin{aligned} \frac{1}{4}\varepsilon_{ab}\psi_a^\rho\psi_b^\sigma\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho\sigma} &= \frac{1}{4}\psi_1^\rho\psi_2^\sigma\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho\sigma} - \frac{1}{4}\psi_2^\rho\psi_1^\sigma\psi_1^\mu\psi_2^\nu g_{\mu\nu,\rho\sigma} \\ &= -\frac{1}{4}\psi_1^\rho\psi_1^\mu\psi_2^\sigma\psi_2^\nu g_{\mu\nu,\rho\sigma} - \frac{1}{4}\psi_1^\sigma\psi_1^\mu\psi_2^\rho\psi_2^\nu g_{\mu\nu,\rho\sigma} \\ &= -\frac{1}{4}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon(g_{\beta\varepsilon,\alpha\gamma} + g_{\beta\varepsilon,\gamma\alpha}) \\ &= -\frac{1}{2}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon g_{\beta\varepsilon,\alpha\gamma}, \end{aligned} \tag{5.47}$$

where we in the last step made use of the commutativity of mixed second partial derivatives of the metric, $g_{\beta\varepsilon,\alpha\gamma} = g_{\beta\varepsilon,\gamma\alpha}$.

Collecting the results from (5.45), (5.46) and (5.47), the terms with four factors of ψ become

$$\frac{1}{2}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon\left(\Gamma^\mu{}_{\alpha\gamma}\Gamma_{\mu\beta\varepsilon} - 2\Gamma^\mu{}_{\beta\varepsilon}\Gamma_{\mu\alpha\gamma} - g_{\beta\varepsilon,\alpha\gamma}\right). \tag{5.48}$$

Note that this expression is antisymmetric under change of α and β, and also under γ and ε in the Γ-factors. Thus it is symmetric under interchange of both α with β and γ with ε. If the reader is not convinced about this, follow the manipulation of the middle term in (5.48)

$$\begin{aligned} &\frac{1}{2}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon\left(-2\Gamma^\mu{}_{\beta\varepsilon}\Gamma_{\mu\alpha\gamma}\right) = \frac{1}{2}(-\psi_1^\beta\psi_1^\alpha)(-\psi_2^\varepsilon\psi_2^\gamma)\left(-2\Gamma^\mu{}_{\beta\varepsilon}\Gamma_{\mu\alpha\gamma}\right) \\ = &\frac{1}{2}\psi_1^\beta\psi_1^\alpha\psi_2^\varepsilon\psi_2^\gamma\left(-2\Gamma^\mu{}_{\beta\varepsilon}\Gamma_{\mu\alpha\gamma}\right) = \left\{\begin{array}{c}\alpha\leftrightarrow\beta \\ \gamma\leftrightarrow\varepsilon\end{array}\right\} = \frac{1}{2}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon\left(-2\Gamma^\mu_{\alpha\gamma}\Gamma_{\mu\beta\varepsilon}\right). \end{aligned} \tag{5.49}$$

The Christoffel symbols in (5.48) add together, and (5.48) reads

$$\frac{1}{2}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon\left(-\Gamma^\mu{}_{\alpha\gamma}\Gamma_{\mu\beta\varepsilon} - g_{\beta\varepsilon,\alpha\gamma}\right) = \frac{1}{2}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon\left(\Gamma^\mu{}_{\alpha\varepsilon}\Gamma_{\mu\beta\gamma} + g_{\beta\gamma,\alpha\varepsilon}\right), \tag{5.50}$$

where in the last step we used the antisymmetry under interchange of γ and ε. The total Lagrangian is then

$$L = \frac{1}{2}\dot{q}^\mu\dot{q}^\nu g_{\mu\nu} + \frac{i}{2}\psi_a^\mu\dot{\psi}_a^\nu g_{\mu\nu} - \frac{i}{2}\dot{q}^\mu\psi_a^\rho\psi_a^\nu g_{\mu\nu,\rho} + \frac{1}{2}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon\left(\Gamma_{\mu\beta\gamma}\Gamma^\mu{}_{\alpha\varepsilon} + g_{\beta\gamma,\alpha\varepsilon}\right). \tag{5.51}$$

5.1.3 The Lagrangian in Terms of the Riemann Tensor

One could be quite content with the short expression for L that appears in (5.51) in the end of last section, but we want to express it in terms of the Riemann tensor. Let us define the time differentiation operator D_t through its action on a fermionic variable ψ^ν

$$D_t\psi^\nu = \frac{\partial}{\partial t}\psi^\nu + \Gamma^\nu{}_{\gamma\lambda}\frac{\partial q^\gamma}{\partial t}\psi^\lambda = \dot{\psi}^\nu + \dot{q}^\gamma\Gamma^\nu{}_{\gamma\lambda}\psi^\lambda. \tag{5.52}$$

Then the Lagrangian of our system is

$$L = \frac{1}{2}g_{\mu\nu}(q)(\dot{q}^\mu\dot{q}^\nu + i\psi_a^\mu D_t\psi_a^\nu) + \frac{1}{8}\psi_a^\rho\psi_a^\sigma\psi_b^\mu\psi_b^\nu R_{\rho\sigma\mu\nu}. \tag{5.53}$$

We will now show that (5.53) is equal to our old expression (5.51). First we notice that they already have the term $\frac{1}{2}\dot{q}^\mu\dot{q}^\nu g_{\mu\nu}$ in common. Next, we see that they both have two kinds of terms, terms with two factors of fermionic variables ψ, and terms with four ψ factors. It is very natural to treat and compare them separately.

The term $\frac{1}{2}g_{\mu\nu}i\psi_a^\mu D_t\psi_a^\nu$ in (5.53) can by use of the definition of D_t in (5.52) be written as

$$\frac{i}{2}g_{\mu\nu}\psi_a^\mu(\dot{\psi}_a^\nu + \dot{q}^\gamma\Gamma^\nu{}_{\gamma\lambda}\psi_a^\lambda) = \frac{i}{2}\psi_a^\mu\dot{\psi}_a^\nu g_{\mu\nu} + \frac{i}{2}\dot{q}^\gamma g_{\mu\nu}\Gamma^\nu{}_{\gamma\lambda}\psi_a^\mu\psi_a^\lambda. \tag{5.54}$$

The first of these terms is exactly what is in (5.51), so we are left with the second term. It is

$$\frac{i}{2}\dot{q}^\gamma\psi_a^\mu\psi_a^\lambda\Gamma_{\mu\gamma\lambda} = \frac{i}{2}\dot{q}^\gamma\psi_a^\mu\psi_a^\lambda\frac{1}{2}(g_{\mu\gamma,\lambda} + g_{\mu\lambda,\gamma} - g_{\gamma\lambda,\mu}). \tag{5.55}$$

The middle term in (5.55) is zero since it is symmetric in the metric and antisymmetric in the ψ factors under interchange of μ and λ. Explicitly,

$$\psi_a^\mu\psi_a^\lambda g_{\mu\lambda,\gamma} = -\psi_a^\lambda\psi_a^\mu g_{\mu\lambda,\gamma} = \{\lambda \leftrightarrow \mu\} = -\psi_a^\mu\psi_a^\lambda g_{\lambda\mu,\gamma} = -\psi_a^\mu\psi_a^\lambda g_{\mu\lambda,\gamma}, \tag{5.56}$$

which must be zero. The last term in (5.55) is just the same as the first except for sign, since it is antisymmetric under interchange of μ and λ. Hence (5.55) simplifies to

$$\frac{i}{2}\dot{q}^\gamma\psi_a^\mu\psi_a^\lambda g_{\mu\gamma,\lambda} = -\frac{i}{2}\dot{q}^\gamma\psi_a^\lambda\psi_a^\mu g_{\mu\gamma,\lambda} = \left\{\begin{array}{c}\gamma\to\mu\\ \lambda\to\rho\\ \mu\to\nu\end{array}\right\} = -\frac{i}{2}\dot{q}^\mu\psi_a^\rho\psi_a^\nu g_{\nu\mu,\rho}, \tag{5.57}$$

which is the third term in (5.51) that we wanted to find. The parts in (5.53) and (5.51) with two factors of ψ thus agree completely. What remains is the Riemann part with four factors of ψ, $\frac{1}{8}\psi_a^\rho\psi_a^\sigma\psi_b^\mu\psi_b^\nu R_{\rho\sigma\mu\nu}$. Writing it like this, one might ask if terms with $a = b$ contribute. They do not, due to the identity in (4.79)

$$R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} + R_{\rho\mu\nu\sigma} = 0. \tag{5.58}$$

Since

$$\psi^\rho\psi^\sigma\psi^\mu\psi^\nu R_{\rho\sigma\mu\nu} = \left\{\begin{array}{c}\sigma\to\nu\\ \nu\to\mu\\ \mu\to\sigma\end{array}\right\} = \psi^\rho\psi^\nu\psi^\sigma\psi^\mu R_{\rho\nu\sigma\mu} = \psi^\rho\psi^\sigma\psi^\mu\psi^\nu R_{\rho\nu\sigma\mu}, \tag{5.59}$$

and

$$\psi^\rho\psi^\sigma\psi^\mu\psi^\nu R_{\rho\sigma\mu\nu} = \left\{\begin{array}{c}\sigma\to\mu\\ \mu\to\nu\\ \nu\to\sigma\end{array}\right\} = \psi^\rho\psi^\mu\psi^\nu\psi^\sigma R_{\rho\mu\nu\sigma} = \psi^\rho\psi^\sigma\psi^\mu\psi^\nu R_{\rho\mu\nu\sigma}, \tag{5.60}$$

we have

$$3\psi^\rho\psi^\sigma\psi^\mu\psi^\nu R_{\rho\sigma\mu\nu} = \psi^\rho\psi^\sigma\psi^\mu\psi^\nu(R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} + R_{\rho\mu\nu\sigma}) = 0 \tag{5.61}$$

from (5.58). Thus $\psi^\rho\psi^\sigma\psi^\mu\psi^\nu R_{\rho\sigma\mu\nu} = 0$ and evidently

$$\frac{1}{8}\psi_a^\rho\psi_a^\sigma\psi_b^\mu\psi_b^\nu R_{\rho\sigma\mu\nu} = \frac{1}{8}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu R_{\rho\sigma\mu\nu} + \frac{1}{8}\psi_2^\rho\psi_2^\sigma\psi_1^\mu\psi_1^\nu R_{\rho\sigma\mu\nu} =$$

$$= \frac{1}{8}(\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu + \psi_1^\mu\psi_1^\nu\psi_2^\rho\psi_2^\sigma)R_{\rho\sigma\mu\nu} = \left\{\begin{array}{l}\mu\leftrightarrow\rho\\ \nu\leftrightarrow\sigma\\ \text{in last term}\end{array}\right\} = \frac{1}{8}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu(R_{\rho\sigma\mu\nu}+R_{\mu\nu\rho\sigma}) \tag{5.62}$$

From the definition of the Riemann tensor, we have

$$R^\alpha{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\alpha{}_{\nu\sigma} - \partial_\nu\Gamma^\alpha{}_{\mu\sigma} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\gamma{}_{\nu\sigma} - \Gamma^\alpha{}_{\nu\gamma}\Gamma^\gamma{}_{\mu\sigma}. \tag{5.63}$$

Lowering the first index with the metric $g_{\alpha\rho}$ yields according to (4.76)

$$R_{\rho\sigma\mu\nu} = -g_{\rho\lambda,\mu}\Gamma^\lambda{}_{\nu\sigma} + g_{\rho\lambda,\nu}\Gamma^\lambda{}_{\mu\sigma} + \frac{1}{2}(g_{\rho\nu,\sigma\mu} - g_{\nu\sigma,\rho\mu} - g_{\rho\mu,\sigma\nu} + g_{\mu\sigma,\rho\nu}) + \Gamma_{\rho\mu\gamma}\Gamma^\gamma{}_{\nu\sigma} - \Gamma_{\rho\nu\gamma}\Gamma^\gamma{}_{\mu\sigma}. \tag{5.64}$$

Just by permuting indices ($\rho\leftrightarrow\mu,\ \sigma\leftrightarrow\nu$) in (4.76) we find

$$R_{\mu\nu\rho\sigma} = -g_{\mu\lambda,\rho}\Gamma^\lambda{}_{\sigma\nu} + g_{\mu\lambda,\sigma}\Gamma^\lambda{}_{\rho\nu} + \frac{1}{2}(g_{\mu\sigma,\nu\rho} - g_{\sigma\nu,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\rho\nu,\mu\sigma}) + \Gamma_{\mu\rho\gamma}\Gamma^\gamma{}_{\sigma\nu} - \Gamma_{\mu\sigma\gamma}\Gamma^\gamma{}_{\rho\nu}. \tag{5.65}$$

Then (5.62) becomes

$$\begin{aligned}\frac{1}{8}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu(R_{\rho\sigma\mu\nu} + R_{\mu\nu\rho\sigma}) = \frac{1}{8}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu\big(&-g_{\rho\lambda,\mu}\Gamma^\lambda{}_{\nu\sigma} + g_{\rho\lambda,\nu}\Gamma^\lambda{}_{\mu\sigma} - g_{\mu\lambda,\rho}\Gamma^\lambda{}_{\sigma\nu} + g_{\mu\lambda,\sigma}\Gamma^\lambda{}_{\rho\nu}\\ &+\frac{1}{2}(g_{\rho\nu,\sigma\mu} - g_{\nu\sigma,\rho\mu} - g_{\rho\mu,\sigma\nu} + g_{\mu\sigma,\rho\nu} + g_{\mu\sigma,\nu\rho} - g_{\sigma\nu,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\rho\nu,\mu\sigma})\\ &+\Gamma_{\rho\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} + \Gamma_{\mu\rho\lambda}\Gamma^\lambda{}_{\sigma\nu} - \Gamma_{\mu\sigma\lambda}\Gamma^\lambda{}_{\rho\nu}\big).\end{aligned} \tag{5.66}$$

Such a large expression must be handled in smaller parts. Let us begin with the part involving second derivatives of the metric

$$\begin{aligned}&\frac{1}{8}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu\frac{1}{2}(g_{\rho\nu,\sigma\mu} - g_{\nu\sigma,\rho\mu} - g_{\rho\mu,\sigma\nu} + g_{\mu\sigma,\rho\nu} + g_{\mu\sigma,\nu\rho} - g_{\sigma\nu,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\rho\nu,\mu\sigma})\\ &\qquad = \frac{1}{8}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu(g_{\rho\nu,\mu\sigma} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} + g_{\mu\sigma,\rho\nu}).\end{aligned} \tag{5.67}$$

Continuing, using the fact that the above expression is antisymmetric under interchange of ρ with σ and μ with ν in the metric part, gives

$$\frac{1}{4}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu(-g_{\nu\sigma,\mu\rho} + g_{\mu\sigma,\rho\nu}) = \frac{1}{2}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu g_{\sigma\mu,\rho\nu}. \tag{5.68}$$

This antisymmetry will become even more useful when dealing with the rest of (5.66)

$$\begin{aligned}&\frac{1}{8}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu\big(-g_{\rho\lambda,\mu}\Gamma^\lambda{}_{\nu\sigma} + g_{\rho\lambda,\nu}\Gamma^\lambda{}_{\mu\sigma} - g_{\mu\lambda,\rho}\Gamma^\lambda{}_{\sigma\nu} + g_{\mu\lambda,\sigma}\Gamma^\lambda{}_{\rho\nu}\\ &\qquad +\Gamma_{\rho\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} + \Gamma_{\mu\rho\lambda}\Gamma^\lambda{}_{\sigma\nu} - \Gamma_{\mu\sigma\lambda}\Gamma^\lambda{}_{\rho\nu}\big)\\ &= \frac{1}{4}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu\big(-g_{\rho\lambda,\mu}\Gamma^\lambda{}_{\nu\sigma} + g_{\mu\lambda,\sigma}\Gamma^\lambda{}_{\rho\nu} + \Gamma_{\rho\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma_{\mu\sigma\lambda}\Gamma^\lambda{}_{\rho\nu}\big)\\ &= \frac{1}{4}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu\big(g_{\sigma\lambda,\mu}\Gamma^\lambda{}_{\nu\rho} + g_{\mu\lambda,\sigma}\Gamma^\lambda{}_{\rho\nu} - \Gamma_{\sigma\mu\lambda}\Gamma^\lambda{}_{\nu\rho} - \Gamma_{\mu\sigma\lambda}\Gamma^\lambda{}_{\rho\nu}\big)\\ &= \frac{1}{4}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu\left(g_{\sigma\lambda,\mu} + g_{\mu\lambda,\sigma} - \Gamma_{\sigma\mu\lambda} - \Gamma_{\mu\sigma\lambda}\right)\Gamma^\lambda{}_{\rho\nu}.\end{aligned} \tag{5.69}$$

The expression in the parenthesis in (5.69) above hides twice a Christoffel symbol

$$
\begin{aligned}
&g_{\sigma\lambda,\mu} + g_{\mu\lambda,\sigma} - \Gamma_{\sigma\mu\lambda} - \Gamma_{\mu\sigma\lambda} \\
&= g_{\sigma\lambda,\mu} + g_{\mu\lambda,\sigma} - \frac{1}{2}(g_{\sigma\mu,\lambda} + g_{\sigma\lambda,\mu} - g_{\mu\lambda,\sigma} + g_{\mu\sigma,\lambda} + g_{\mu\lambda,\sigma} - g_{\sigma\lambda,\mu}) \\
&= g_{\sigma\lambda,\mu} + g_{\mu\lambda,\sigma} - g_{\sigma\mu,\lambda} = 2\Gamma_{\lambda\sigma\mu}.
\end{aligned} \tag{5.70}
$$

Hence (5.69) becomes

$$
\frac{1}{2}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu\Gamma_{\lambda\sigma\mu}\Gamma^\lambda{}_{\rho\nu}, \tag{5.71}
$$

and when (5.68) and (5.71) are added together, (5.66) simply reads

$$
\frac{1}{2}\psi_1^\rho\psi_1^\sigma\psi_2^\mu\psi_2^\nu\left(g_{\sigma\mu,\rho\nu} + \Gamma_{\lambda\sigma\mu}\Gamma^\lambda{}_{\rho\nu}\right), \tag{5.72}
$$

or, after permuting indices ($\rho \to \alpha, \sigma \to \beta, \mu \to \gamma, \nu \to \varepsilon, \lambda \to \mu$)

$$
\frac{1}{2}\psi_1^\alpha\psi_1^\beta\psi_2^\gamma\psi_2^\varepsilon\left(g_{\beta\gamma,\alpha\varepsilon} + \Gamma_{\mu\beta\gamma}\Gamma^\mu{}_{\alpha\varepsilon}\right), \tag{5.73}
$$

which is exactly the part with four fermionic factors that we had in (5.51). Then, part by part we have showed the equivalence between (5.51) and (5.53), and the Lagrangian of the system may clearly be taken as

$$
L = \frac{1}{2}g_{\mu\nu}(q)(\dot{q}^\mu\dot{q}^\nu + i\psi_a^\mu D_t\psi_a^\nu) + \frac{1}{8}R_{\rho\sigma\mu\nu}\psi_a^\rho\psi_a^\sigma\psi_b^\mu\psi_b^\nu. \tag{5.74}
$$

This result is also stated in Alvarez-Gaumés paper [3], although in a bit different form. Instead of ψ_1 and ψ_2 he uses ψ and $\bar{\psi}$, which are complex conjugates of each other. While comparing one also has to be careful with the order of the indices. Applying (4.79) yields the necessary factor of 2 that makes the two Lagrangians exactly equal. A very similar Lagrangian appears in [1], but there seems to be a difference in the ratio of the coefficients of the second and the third terms. Even if the coefficients of the two terms are different in the texts, their ratio should be the same when written in comparable form. Also, in the original paper by Witten, [2], the ratio seems to differ compared to the factor in (5.74).

5.2 Calculation of the Supercharges Q_a

In this section we will deduce the expression for the supercharges Q_a. We use a similar approach to the one used in section 3.2, where we had two expressions for δL, which we could write as two total time derivatives. The difference between the two expressions contained our conserved supercharges.

We are going to continue where section 5.1.1 ends. Let us restate some useful mathematical results before we begin. The Lagrangian is given by,

$$
L = \int d^2\theta \, \underbrace{\frac{i}{2}g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu}_{\mathcal{L}}, \tag{5.75}
$$

where we, for simplicity have defined a part $\mathcal{L}$. The expressions for D_1 and D_2 are,

$$
\begin{aligned}
D_1 &= \partial_1 - i\theta_1\partial_t, \\
D_2 &= \partial_2 - i\theta_2\partial_t.
\end{aligned} \tag{5.76}
$$

For the moment, we leave the integral in (5.75) out in order to shorten our expressions. As we saw in (5.21), we can write δL accordingly,

$$
\begin{aligned}
\delta L &= \int d^2\theta \; \delta\mathcal{L} \\
&= \int d^2\theta \; \epsilon_a \tilde{D}_a \mathcal{L} \\
&= \int d^2\theta \; \epsilon_a \partial_a \mathcal{L} + i\epsilon_a\theta_a \frac{d}{dt}\mathcal{L}.
\end{aligned} \tag{5.77}
$$

As mentioned in section 5.1.1, the first part of the variation will give zero contribution in the integration. In other words, since we take the derivative of θ_a, we would have needed at least three factorns of θ_a, $\theta_a\theta_b\theta_c$ for the integration to give a nonzero contribution. This term can therefore be ignored in the further calculation of δL. We continue by focusing entirely on the second term $i\epsilon_a\theta_a\frac{d}{dt}\mathcal{L}$,

$$i\epsilon_a\theta_a\frac{d}{dt}\mathcal{L} = i\epsilon_a\theta_a\frac{d}{dt}\left(\frac{i}{2}g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu\right). \tag{5.78}$$

For simplicity the expression above is split up in parts. Let us start with the metric $g_{\mu\nu}(\phi)$, which can be rewritten as the Taylor expasion in (5.22),

$$g_{\mu\nu}(\phi) = \underbrace{g_{\mu\nu}(q^\mu) + (i\theta_a\psi_a^\lambda - i\theta_1\theta_2F^\lambda)g_{\mu\nu,\lambda}}_{A} + \underbrace{\frac{1}{2}\left(-\theta_a\psi_a^\lambda\theta_b\psi_b^\rho\right)g_{\mu\nu,\lambda\rho}}_{B}. \tag{5.79}$$

In section 5.1.2 we calculated $D_1\phi_\mu$ and $D_2\phi_\nu$. We restate the result from (5.23) and (5.24),

$$D_1\phi^\mu = i\psi_1^\mu - i\theta_2F^\mu - i\theta_1\dot{q}^\mu + \theta_1\theta_2\dot{\psi}_2^\mu, \tag{5.80}$$

$$D_2\phi^\nu = i\psi_2^\nu + i\theta_1F^\nu - i\theta_2\dot{q}^\nu + \theta_2\theta_1\dot{\psi}_1^\nu, \tag{5.81}$$

Let us now return to and calculate $i\epsilon_a\theta_a\frac{d}{dt}\left(\frac{i}{2}g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu\right)$ by multiplying the separate parts,

$$i\epsilon_a\theta_a\frac{d}{dt}\left(\frac{i}{2}g_{\mu\nu}(\phi)D_1\phi^\mu D_2\phi^\nu\right) = i\epsilon_a\theta_a\frac{d}{dt}\left(\frac{i}{2}(A+B)D_1\phi_\mu D_2\phi_\nu\right). \tag{5.82}$$

We need to change index $\epsilon_a\theta_a \to \epsilon_k\theta_k$ to avoid summation where it is not supposed to be, which yields the expression,

$$\begin{aligned} i\epsilon_k\theta_k\frac{d}{dt}\left(\frac{i}{2}(A+B)D_1\phi_\mu D_2\phi_\nu\right) = \frac{i}{2}\epsilon_k\frac{d}{dt}\,[&-i\theta_kg_{\mu\nu}\psi_1^\nu\psi_2^\nu - i\theta_1\theta_kg_{\mu\nu}\psi_1^\mu F^\nu \\ &- i\theta_k\theta_2g_{\mu\nu}\psi_1^\mu\dot{q}^\nu + i\theta_k\theta_2g_{\mu\nu}F^\mu\psi_2^\nu \\ &- i\theta_1\theta_kg_{\mu\nu}\dot{q}^\mu\psi_2^\nu \\ &+ i(-i)\theta_k\theta_a\psi_a^\lambda g_{\mu\nu,\lambda}\psi_1^\mu\psi_2^\nu\,]. \end{aligned} \tag{5.83}$$

Now we are ready to calculate δL by calculating the integral in (5.77). We apply the rules for integration in (5.28),

$$\begin{aligned} \delta L &= \int d^2\theta\ \delta\mathcal{L} \\ &= \frac{i}{2}\frac{d}{dt}\,[\epsilon_2g_{\mu\nu}\psi_1^\mu F^\nu + \epsilon_1g_{\mu\nu}\psi_1^\mu\dot{q}^\nu - \epsilon_1g_{\mu\nu}F^\mu\psi_2^\nu \\ &\qquad + \epsilon_2g_{\mu\nu}\dot{q}^\mu\psi_2^\nu + i\epsilon_k\epsilon_{ka}\psi_a^\lambda g_{\mu\nu,\lambda}\psi_1^\mu\psi_2^\nu\,] \\ &= \frac{i}{2}\frac{d}{dt}\left(\epsilon_a\epsilon_{ba}g_{\mu\nu}\psi_bF^\nu + \epsilon_ag_{\mu\nu}\psi_a^\mu\dot{q}^\nu + i\epsilon_a\epsilon_{ab}\psi_b^\lambda g_{\mu\nu,\lambda}\psi_1^\mu\psi_2^\nu\right). \end{aligned} \tag{5.84}$$

As we can see, the idea of a superfield, helped us in a very convenient way, to write the variation of the Lagrangian as a total time derivative. If we recall the technique we used when deriving the supercharges on flat space; we also wrote the variation of the Lagrangian in terms of partial derivatives. We will do the same thing now, in an exactly analogous way. Note that we could have done this part without the superfield, all we need is the Lagrangian and the supervariations. The variation of L is

$$\delta L = \delta q^\gamma\frac{\partial L}{\partial q^\gamma} + \delta\dot{q}^\gamma\frac{\partial L}{\partial\dot{q}^\gamma} + \delta\psi_a^\sigma\frac{\partial L}{\partial\psi_a^\sigma} + \delta\dot{\psi}_a^\sigma\frac{\partial L}{\partial\dot{\psi}_a^\sigma} + \delta F^\alpha\frac{\partial L}{\partial F^\alpha} + \delta\dot{F}^\alpha\frac{\partial L}{\partial\dot{F}^\alpha}. \tag{5.85}$$

As before, we can rewrite this by using Lagrange's equations, as

$$\delta L = \frac{d}{dt}\left(\delta q^\gamma\frac{\partial L}{\partial\dot{q}^\gamma} + \delta\psi_a^\sigma\frac{\partial L}{\partial\dot{\psi}_a^\sigma} + \delta F^\alpha\frac{\partial L}{\partial\dot{F}^\alpha}\right). \tag{5.86}$$

We calculated an expression for the Lagrangian earlier, we restate it for clarity

$$L = \frac{1}{2} g_{\mu\nu} \left(\dot{q}^{\mu} \dot{q}^{\nu} + i \psi_a^{\mu} D_t \psi_a^{\nu} \right) + \frac{1}{8} \cdots \tag{5.87}$$

The last part is not interesting for us in this calculation since it does not include any time derivatives, hence, it will be zero in the variation of L. By implementation of the definition of D_t in (5.52), we get

$$L = \frac{1}{2} g_{\mu\nu} \dot{q}^{\mu} \dot{q}^{\nu} + \frac{i}{2} g_{\mu\nu} \psi_a^{\mu} \left(\dot{\psi}_a^{\nu} + \dot{q}^{\lambda} \Gamma^{\nu}{}_{\lambda\alpha} \psi_a^{\alpha} \right) + \frac{1}{8} \cdots \tag{5.88}$$

We immediately see that $\partial L / \partial \dot{F}^{\alpha} = 0$. The variation of L then becomes

$$\delta L = \frac{d}{dt} \left(\delta q^{\gamma} \left(g_{\mu\gamma} \dot{q}^{\mu} + \frac{i}{2} \psi_a^{\mu} g_{\mu\nu} \Gamma^{\nu}{}_{\gamma\alpha} \psi_a^{\alpha} \right) + \delta \psi_a^{\nu} \frac{(-i)}{2} \psi_a^{\mu} g_{\mu\nu} \right) \tag{5.89}$$

The minus sign appears in the last expression because ψ^{μ} and $\dot{\psi}^{\nu}$ anticommute, and as usual, the derivative is defined so that it acts on its immediate right. We have already derived the variations δq^{γ} and $\delta \psi^{\sigma}$. It proved that they came out naturally, as a consequence of the construction of the superfield, in the calculations of the Lagrangian. We find them in (5.14), and by putting these into our expression for δL we get

$$\delta L = \frac{d}{dt} \left(i \epsilon_b \psi_b^{\gamma} g_{\mu\gamma} \dot{q}^{\mu} - \frac{1}{2} \epsilon_b \psi_b^{\gamma} \psi_a^{\mu} \Gamma_{\mu\gamma\alpha} \psi_a^{\alpha} + \frac{i}{2} \epsilon_{ab} \epsilon_b F^{\nu} \psi_a^{\mu} g_{\mu\nu} + \frac{i}{2} \epsilon_a \psi_a^{\mu} g_{\mu\nu} \dot{q}^{\nu} \right). \tag{5.90}$$

If we recall the variation of L we did earlier, (5.84), in which the variation was defined in terms of a covariant derivative, we now have all we need to construct the supercharges. Both of the two variations, (5.84) and (5.90), are written as total time derivatives. Hence, we can construct conserved supercharges. We have that

$$\delta L - \delta L = \frac{d}{dt} (\ldots) = 0 \tag{5.91}$$

Whatever is left inside the parenthesis above is the conserved supercharges, (up to the constants i and ϵ_a). They become,

$$\begin{aligned} i \epsilon_a Q_a = & \left(i \epsilon_b \psi_b^{\gamma} g_{\mu\gamma} \dot{q}^{\mu} - \frac{1}{2} \epsilon_b \psi_b^{\gamma} \psi_a^{\mu} \Gamma_{\mu\gamma\alpha} \psi_a^{\alpha} + \frac{i}{2} \epsilon_{ab} \epsilon_b F^{\nu} \psi_a^{\mu} g_{\mu\nu} + \frac{i}{2} \epsilon_a \psi_a^{\mu} g_{\mu\nu} \dot{q}^{\nu} \right) \\ & - \frac{i}{2} \left(\epsilon_a g_{\mu\nu} \psi_a^{\mu} \dot{q}^{\nu} + \epsilon_a \epsilon_{ba} \psi_b^{\mu} g_{\mu\nu} F^{\nu} + i \epsilon_a \epsilon_{ab} \psi_b^{\lambda} \psi_1^{\mu} \psi_2^{\nu} g_{\mu\nu,\lambda} \right). \end{aligned} \tag{5.92}$$

Since ψ commutes with both q and F, and q commutes with F, the ordering of these factors in each term does not matter, thus, the only terms left are

$$i \epsilon_a Q_a = i \epsilon_b \psi_b^{\gamma} g_{\mu\gamma} \dot{q}^{\mu} - \frac{1}{2} \big(\underbrace{\epsilon_b \psi_b^{\gamma} \psi_a^{\mu} \Gamma_{\mu\gamma\alpha} \psi_a^{\alpha}}_{C} - \underbrace{\epsilon_a \epsilon_{ab} \psi_b^{\lambda} \psi_1^{\mu} \psi_2^{\nu} g_{\mu\nu,\lambda}}_{D} \big). \tag{5.93}$$

As a matter of fact, C and D will cancel out each other. We just have to expand C in all its components, change some indices and use some anticommutation relations, and it will become clear that they are equal. We start off by using the explicit expression for $\Gamma_{\mu\gamma\alpha}$, C then becomes

$$\begin{aligned} C &= \epsilon_b \psi_b^{\gamma} \psi_a^{\mu} \psi_a^{\alpha} \Gamma_{\mu\gamma\alpha} \\ &= \epsilon_b \psi_b^{\gamma} \psi_a^{\mu} \psi_a^{\alpha} \frac{1}{2} (g_{\mu\gamma,\alpha} + g_{\mu\alpha,\gamma} - g_{\gamma\alpha,\mu}). \end{aligned} \tag{5.94}$$

The middle term, $\psi_a^{\mu} \psi_a^{\alpha} g_{\mu\alpha,\gamma}$ will always be zero, since $g_{\mu\alpha,\gamma}$ is symmetric in $\mu\alpha$, while $\psi_a^{\mu} \psi_a^{\alpha}$ anticommutes. Note that the 1/2 comes from the definition of $\Gamma_{\mu\gamma\alpha}$ and should not be confused with the 1/2 outside of the paranthesis in (5.93). Also, if we let $\mu \leftrightarrow \alpha$, we get that

$$-\psi_a^{\mu} \psi_a^{\alpha} g_{\gamma\alpha,\mu} = -\psi_a^{\alpha} \psi_a^{\mu} g_{\gamma\mu,\alpha} = \psi_a^{\mu} \psi_a^{\alpha} g_{\gamma\mu,\alpha}. \tag{5.95}$$

C then becomes,

$$C = \epsilon_b \psi_b^\gamma \psi_a^\mu \psi_a^\alpha g_{\mu\gamma,\alpha}. \tag{5.96}$$

Note that the terms involving $\psi_1^\gamma \psi_1^\mu \psi_1^\alpha$ and $\psi_2^\gamma \psi_2^\mu \psi_2^\alpha$ are zero, because of the antisymmetric properties of $\Gamma_{\mu\gamma\alpha}$. If we make the summation in C, we get,

$$\begin{aligned} C &= (\epsilon_1 \psi_1^\gamma \psi_2^\mu \psi_2^\alpha + \epsilon_2 \psi_2^\gamma \psi_1^\mu \psi_1^\alpha) g_{\mu\gamma,\alpha} \\ &= (\epsilon_1 \psi_2^\alpha \psi_1^\gamma \psi_2^\mu - \epsilon_2 \psi_1^\alpha \psi_1^\mu \psi_2^\gamma) g_{\mu\gamma,\alpha} \\ &= \{\text{let } \mu \leftrightarrow \gamma \text{ in the first term}\} \\ &= (\epsilon_1 \psi_2^\alpha \psi_1^\mu \psi_2^\gamma - \epsilon_2 \psi_1^\alpha \psi_1^\mu \psi_2^\gamma) g_{\mu\gamma,\alpha} \\ &= \{\alpha \to \lambda, \gamma \to \nu\} \\ &= \epsilon_a \epsilon_{ab} \psi_b^\lambda \psi_1^\mu \psi_2^\nu g_{\mu\nu,\lambda} \\ &= D. \end{aligned} \tag{5.97}$$

In rewriting C to D, we used that we are allowed to interchange the variables α and γ since $g_{\mu\gamma,\lambda}$ is symmetric in these two coordinates. So, with this result, we come the conclusion that the supercharges can be written as,

$$\begin{aligned} i\epsilon_a Q_a &= i\epsilon_a \psi_a^\gamma g_{\mu\gamma} \dot{q}^\mu, \\ Q_a &= \psi_a^\gamma g_{\mu\gamma} \dot{q}^\mu. \end{aligned} \tag{5.98}$$

These are the conserved supercharges of the system.

5.3 Poisson Bracket

Now we want to quantize the system, we go about this as we did when we had a flat manifold. First writing the Lagrangian in the Hamiltonian formalism, thereby obtaining the expression for the Poisson brackets which we can multiply by i to get the anti-commutators of quantum mechanical operators. We can still write the Lagrangian as $L = 2T - H$, where we can express $2T$ as

$$2T = \dot{q}^\mu \frac{\partial L}{\partial \dot{q}^\mu} + \dot{\psi}_a^m \frac{\partial L}{\partial \dot{\psi}_a^m}. \tag{5.99}$$

Here we take the partial derivative with respect to the curved indices in the first term but flat in the second. This may seem strange but will in the end generate a simple expression for the Poisson bracket. Therefore we have to transform the second term in the Lagrangian with curved indices to flat indices. We use the vielbeins for the transformation. The vielbeins themselves will have time derivatives because of their dependence of the coordinates q^μ, so when transforming $\dot{\psi}^\mu$ to flat indices it transforms as follows

$$\dot{\psi}^\mu = \frac{d}{dt}(E_m{}^\mu \psi^m) = \frac{\partial E_m{}^\mu}{\partial q^\lambda} \dot{q}^\lambda \psi^m + E_m{}^\mu \dot{\psi}^m. \tag{5.100}$$

The Christoffel symbol transforms into its flat counterpart $\omega^k{}_{mn}$

$$\Gamma^\nu{}_{\mu\lambda} = E_k{}^\nu (\partial_\mu e^k{}_\lambda + \omega^k{}_{m\mu} e^m{}_\lambda) = -e^k{}_\lambda (\partial_\mu E_k{}^\nu - \omega^\nu{}_{\sigma\mu} E_k{}^\sigma). \tag{5.101}$$

We can also via the vielbeins evaluate the expression $g_{\mu\nu} \psi^\mu \dot{\psi}^\nu$ more closely which is easier to handle in flat notation

$$g_{\mu\nu} \psi^\mu \dot{\psi}^\nu = e^m{}_\mu e^n{}_\nu \eta_{mn} E_k{}^\mu \psi^k E_l{}^\nu \dot{\psi}^l = \delta_k^m \delta_l^m \psi^k \dot{\psi}^l = \psi^m \dot{\psi}^m = \psi^n \dot{\psi}^n. \tag{5.102}$$

In the equation above, the index m appears twice, and both times as superscripts. So far, we have tried to be consistent, always writing summation indices as one subscript and one superscript. To do so here, we would need to write out a Kronecker delta. For convenience in the calculations to come, we skip this Kronecker delta and keep the notation with equal superscripts.

With these transformations we can now transform the second term of our Lagrangian from curved to flat indices

$$\begin{aligned}
g_{\mu\nu}\psi_a^\mu(\dot{\psi}_a^\nu + \dot{q}^\lambda \Gamma^\nu{}_{\lambda\rho}\psi_a^\rho) &= g_{\mu\nu}E_m{}^\mu \psi_a^m (\frac{d}{dt}(E_n{}^\nu \psi_a^n)) \\
&\quad + g_{\mu\nu}\psi_a^\mu \dot{q}^\lambda (e^k{}_\rho \omega^\nu{}_{\sigma\lambda} E_k{}^\sigma - e^k{}_\rho \partial_\lambda E_k{}^\nu)\psi_a^\rho \\
&= g_{\mu\nu}E_m{}^\mu \psi_a^m \psi_a^n \dot{q}^\lambda \partial_\lambda E_n{}^\nu - g_{\mu\nu}E_m{}^\mu \psi_a^m \psi_a^n \delta_n^k \dot{q}^\lambda \partial_\lambda E_k{}^\nu \\
&\quad + g_{\mu\nu}E_m{}^\mu E_n{}^\nu \psi_a^m \dot{\psi}_a^n + g_{\mu\nu}E_m{}^\mu \psi_a^m \dot{q}^\lambda \delta_n^k \omega^\nu{}_{\sigma\lambda} E_k{}^\sigma \psi_a^n \\
&= e^k{}_\mu e^l{}_\nu \eta_{kl} E_m{}^\mu E_n{}^\nu \psi_a^m \dot{\psi}_a^n + e^k{}_\mu e^l{}_\nu \eta_{kl} E_m{}^\mu \psi_a^m \dot{q}^\lambda E_n{}^\sigma \omega^\nu{}_{\sigma\lambda}\psi_a^n \\
&= \eta_{mn}\psi_a^m \dot{\psi}_a^n + \eta_{ml}\psi_a^m \dot{q}^\lambda e^l{}_\nu E^\sigma{}_n \omega^\nu{}_{\sigma\lambda}\psi_a^n \\
&= \psi_a^m(\dot{\psi}_a^m + \dot{q}^\lambda \omega_{mn\lambda}\psi_a^n).
\end{aligned} \tag{5.103}$$

Now we can calculate (5.99) using the Lagrangian with mixed curved and flat indices,

$$L = \frac{1}{2}g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu + \frac{i}{2}\psi_a^m(\dot{\psi}_a^m + \dot{q}^\lambda \omega_{mn\lambda}\psi_a^n) + \frac{1}{8}R_{\mu\nu\rho\sigma}\psi_a^\mu \psi_a^\nu \psi_b^\rho \psi_b^\sigma, \tag{5.104}$$

and find the expression

$$2T = \dot{q}^\mu \frac{\partial L}{\partial \dot{q}^\mu} + \dot{\psi}_a^m \frac{\partial L}{\partial \dot{\psi}_a^m} = p_\mu \dot{q}^\mu + \frac{i}{2}\psi_a^m \dot{\psi}_a^m. \tag{5.105}$$

The expression for p_μ is given by

$$p_\mu = g_{\mu\nu}\dot{q}^\nu + \frac{i}{2}\psi_a^m \omega_{mn\mu}\psi_a^n. \tag{5.106}$$

We proceed with the action

$$S = \int dt\ L = \int dt\ [2T - H] = \int dt \left[p_\mu \dot{q}^\mu + \frac{i}{2}\psi_a^m \dot{\psi}_a^m - H(q^\mu, p_\mu, \psi_a^m)\right]. \tag{5.107}$$

Through the variation of the action we can find the Hamiltonian equations. Keeping in mind that the ψ_a^m are odd Grassmann numbers we get

$$\begin{aligned}
\delta S &= \int dt \left[\delta(p_\mu \dot{q}^\mu + \frac{i}{2}\psi_a^m \dot{\psi}_a^m) - \delta H(q, p, \psi_a^m)\right] = \int dt\, [\delta p_\mu \dot{q}^\mu + p_\mu \delta \dot{q}^\mu \\
&\quad \left. + \frac{i}{2}\delta\psi_a^m \dot{\psi}_a^m + \frac{i}{2}\psi_a^m \delta\dot{\psi}_a^m - \frac{\partial H}{\partial q^\mu}\delta q^\mu - \frac{\partial H}{\partial p_\mu}\delta p_\mu - \delta\psi_a^m \frac{\partial H}{\partial \psi_a^m}\right] \\
&= \int dt \left[\left(\dot{q}^\mu - \frac{\partial H}{\partial p_\mu}\right)\delta p_\mu + p_\mu \frac{d}{dt}(\delta q^\mu) + \frac{i}{2}\delta\psi_a^m \dot{\psi}_a^m + \frac{i}{2}\psi_a^m \frac{d}{dt}(\delta\psi_a^m)\right. \\
&\quad \left. - \frac{\partial H}{\partial q^\mu}\delta q^\mu - \delta\psi_a^m \frac{\partial H}{\partial \psi_a^m}\right] \\
&= \int dt \left[\left(\dot{q}^\mu - \frac{\partial H}{\partial p_\mu}\right)\delta p_\mu + \frac{d}{dt}(p_\mu \delta q^\mu) - \dot{p}_\mu \delta q^\mu\right. \\
&\quad \left. + \frac{i}{2}\delta\psi_a^m \dot{\psi}_a^m + \frac{i}{2}\frac{d}{dt}(\psi_a^m \delta\psi_a^m) - \frac{i}{2}\dot{\psi}_a^m \delta\psi_a^m - \frac{\partial H}{\partial q^\mu}\delta q^\mu - \delta\psi_a^m \frac{\partial H}{\partial \psi_a^m}\right] \\
&= [p_\mu \delta q^\mu + \psi_a^m \delta\psi_a^m] + \int dt \left[\left(\dot{q}^\mu - \frac{\partial H}{\partial p_\mu}\right)\delta p_\mu - \left(\dot{p}_\mu + \frac{\partial H}{\partial q^\mu}\right)\delta q^\mu\right. \\
&\quad \left. + \left(-i\dot{\psi}_a^m + \frac{\partial H}{\partial \psi_a^m}\right)\delta\psi_a^m\right] \\
&= 0.
\end{aligned} \tag{5.108}$$

As in section 3.2.3 the first term in the last step vanishes because the variations are zero at the endpoints. Therefore each parentheses in the integral above has to equal zero, thus we get the Hamiltonian equations of motion

$$\begin{aligned}\dot{q}^\mu &= \frac{\partial H}{\partial p_\mu},\\ \dot{p}_\mu &= -\frac{\partial H}{\partial q^\mu},\\ \dot{\psi}_a^m &= -i\frac{\partial H}{\partial \psi_a^m}.\end{aligned} \tag{5.109}$$

Now we can take the time derivative of the action and use the Hamiltonian equations to obtain an expression for the Poisson bracket

$$\begin{aligned}\dot{S} = \frac{dS}{dt} &= \frac{\partial S}{\partial p_\mu}\dot{p}_\mu + \frac{\partial S}{\partial q^\mu}\dot{q}^\mu + \frac{\partial S}{\partial \psi_a^m}\dot{\psi}_a^m = -\frac{\partial S}{\partial p_\mu}\frac{\partial H}{\partial q^\mu} + \frac{\partial S}{\partial q^\mu}\frac{\partial H}{\partial p_\mu}\\ &- i\frac{\partial S}{\partial \psi_a^m}\frac{\partial H}{\partial \psi_a^m} = S\left(\frac{\overleftarrow{\partial}}{\partial q^\mu}\frac{\partial}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - i\frac{\overleftarrow{\partial}}{\partial \psi_a^m}\frac{\partial}{\partial \psi_a^m}\right)H.\end{aligned} \tag{5.110}$$

The definition of the Poisson bracket is therefore

$$\{A,B\}_P := A\left(\frac{\overleftarrow{\partial}}{\partial q^\gamma}\frac{\partial}{\partial p_\gamma} - \frac{\overleftarrow{\partial}}{\partial p_\gamma}\frac{\partial}{\partial q^\gamma} - i\frac{\overleftarrow{\partial}}{\partial \psi_a^k}\frac{\partial}{\partial \psi_a^k}\right)B. \tag{5.111}$$

Now there remains the task of computing all the relevant Poisson brackets before we can quantize the system and find the commutators and anti-commutators. The first Poisson bracket of interest is the q^μ, q^ν bracket and is given by

$$\{q^\mu, q^\nu\}_P = \frac{\partial q^\mu}{\partial q^\gamma}\frac{\partial q^\nu}{\partial p_\gamma} - \frac{\partial q^\mu}{\partial p_\gamma}\frac{\partial q^\nu}{\partial q^\gamma} - i\frac{\partial q^\mu}{\partial \psi_a^k}\frac{\partial q^\nu}{\partial \psi_a^k}. \tag{5.112}$$

The generalized coordinates $q^\mu$and q^ν do not depend on p_μ or ψ_a^m so the derivatives with respect to p_γ or ψ_a^m will be zero, this yields

$$\{q^\mu, q^\nu\}_P = 0. \tag{5.113}$$

In the same way we find that the Poisson bracket of ψ_a^μ and q^μ is

$$\{\psi_a^\mu, q^\nu\}_P = \frac{\partial \psi_a^\mu}{\partial q^\gamma}\frac{\partial q^\nu}{\partial p_\gamma} - \frac{\partial \psi_a^\mu}{\partial p_\gamma}\frac{\partial q^\nu}{\partial q^\gamma} - i\frac{\partial \psi_a^\mu}{\partial \psi_a^k}\frac{\partial q^\nu}{\partial \psi_a^k} = 0. \tag{5.114}$$

Let us now study the Poisson bracket of ψ_a^μ and ψ_b^ν, where they do not depend on p_γ. The Poisson bracket is given by

$$\begin{aligned}\{\psi_a^\mu, \psi_b^\nu\}_P &= \underbrace{\frac{\partial \psi_a^\mu}{\partial q^\gamma}\frac{\partial \psi_b^\nu}{\partial p_\gamma}}_{=0} - \underbrace{\frac{\partial \psi_a^\mu}{\partial p_\gamma}\frac{\partial \psi_b^\nu}{\partial q^\gamma}}_{=0} - i\frac{\partial \psi_a^\mu}{\partial \psi_a^k}\frac{\partial \psi_b^\nu}{\partial \psi_a^k} = -i\frac{\partial \psi_a^\mu}{\partial \psi_a^k}\frac{\partial \psi_b^\nu}{\partial \psi_a^k}\\ &= -i\delta_{ab}\frac{\partial \psi_a^\mu}{\partial \psi_a^k}\frac{\partial \psi_b^\nu}{\partial \psi_b^k} = -i\delta_{ab}\frac{\partial(E_m{}^\mu \psi_a^m)}{\partial \psi_a^k}\frac{\partial(E_n{}^\nu \psi_b^n)}{\partial \psi_b^k}\\ &= -i\delta_{ab}E_m{}^\mu \delta^{km} E_n{}^\nu \delta^{kn}\\ &= -i\delta_{ab}E_m{}^\mu E_n{}^\nu \eta^{mn}\\ &= -i\delta_{ab}g^{\mu\nu}\end{aligned} \tag{5.115}$$

where we have used the relation $\delta^{km}\delta^{kn} = \delta^{mn} = \eta^{mn}$. Now we turn to the Poisson brackets involving the supercharge and its conjugate. The supercharge is given as we saw in section 5.2 by

$$Q_a = \psi_a^\mu g_{\mu\nu}\dot{q}^\nu = \psi_a^\mu(p_\mu - \frac{i}{2}\psi_a^m \omega_{mn\mu}\psi_a^n) = \psi_a^\mu \Pi_\mu, \tag{5.116}$$

where we have introduced the variable $\Pi_\mu = p_\mu - \frac{i}{2}\psi_a^m \omega_{mn\mu}\psi_a^n$. It is then possible to calculate the Poisson bracket of Π_μ and ψ_a^ν,

$$\begin{aligned}\{\Pi_\mu, \psi_a^\nu\}_P &= \{p_\mu, \psi_a^\nu\}_P - \frac{i}{2}\{\psi_a^m \omega_{mn\mu}\psi_a^n, \psi_a^\nu\}_P \\ &= \{p_\mu, E_m{}^\nu \psi_a^m\}_P - \frac{i}{2}\{\psi_a^m \omega_{mn\mu}\psi_a^n, E_l{}^\nu \psi_a^l\}_P \\ &= \psi_a^m\{p_\mu, E_m{}^\nu\}_P - \frac{i}{2}\{\psi_a^m \omega_{mn\mu}\psi_a^n, E_l{}^\nu \psi_a^l\}_P \\ &= -\psi_a^m \frac{\partial(p_\mu)}{\partial p_\gamma}\frac{\partial(E_m{}^\nu)}{\partial q^\gamma} + \frac{1}{2}\frac{\partial(\psi_a^m \omega_{mn\mu}\psi_a^n)}{\partial \psi_a^k}\frac{\partial(E_l{}^\nu \psi_a^l)}{\partial \psi_a^k}.\end{aligned} \tag{5.117}$$

The affine spin connection $\omega_{mn\mu}$ is antisymmetric in the two first indices and commutes with ψ_a^m. This gives

$$\begin{aligned}\{\Pi_\mu, \psi_a^\nu\}_P &= -\psi_a^m E_m{}^\nu{}_{,\mu} + \omega_{kn\mu}\psi_a^n E_l{}^\nu \\ &= \psi_a^m(-E_m{}^\nu{}_{,\mu} + \omega_{km\mu}E_k{}^\nu)\end{aligned} \tag{5.118}$$

we now use the relation $\omega^a{}_{b\mu} = e^a{}_\nu(\partial_\mu E_b{}^\nu + \Gamma^\nu{}_{\mu\lambda}E_b{}^\lambda)$ given in [6]

$$\begin{aligned}&= \psi_a^m(-E_m{}^\nu{}_{,\mu} + e^k{}_\rho(E_m{}^\rho{}_{,\mu} + \Gamma^\rho{}_{\mu\lambda}E_m{}^\lambda)E_k{}^\nu) \\ &= \psi_a^m(-E_m{}^\nu{}_{,\mu} + e^k{}_\rho E_m{}^\rho{}_{,\mu}E_k{}^\nu + e^k{}_\rho \Gamma^\rho{}_{\mu\lambda}E_m{}^\lambda E_k{}^\nu) \\ &= \psi_a^m(-E_m{}^\nu{}_{,\mu} + E_m{}^k{}_{,\mu}E_k{}^\nu + \Gamma^k{}_{\mu\lambda}E_m{}^\lambda E_k{}^\nu) \\ &= \psi_a^m(-E_m{}^\nu{}_{,\mu} + E_m{}^\nu{}_{,\mu} + \Gamma^\nu{}_{\mu\lambda}E_m{}^\lambda) \\ &= \psi_a^m \Gamma^\nu{}_{\mu\lambda}E_m{}^\lambda \\ &= \Gamma^\nu{}_{\mu\lambda}\psi_a^\lambda.\end{aligned} \tag{5.119}$$

We would finally like to know the Poisson bracket of the variable Π_μ with itself. For this we first quote a definition of the Riemann tensor in the vielbein formalism from [6]

$$e^m{}_\sigma E_n{}^\lambda R^\sigma{}_{\lambda\mu\nu} = \partial_\mu \omega^m{}_{n\nu} - \partial_\nu \omega^m{}_{n\mu} + \omega^m{}_{i\mu}\omega^i{}_{n\nu} - \omega^m{}_{i\nu}\omega^i{}_{n\mu}. \tag{5.120}$$

We will use this definition in the calculation of the bracket, this gives

$$\begin{aligned}\{\Pi_\mu, \Pi_\nu\}_P &= \{p_\mu - \frac{i}{2}\psi_a^m\omega_{mn\mu}\psi_a^n, p_\nu - \frac{i}{2}\psi_a^m\omega_{mn\nu}\psi_a^n\}_P \\ &= \{p_\mu, p_\nu\}_P - \frac{i}{2}\{p_\mu, \psi_a^m\omega_{mn\nu}\psi_a^n\}_P - \frac{i}{2}\{\psi_a^m\omega_{mn\mu}\psi_a^n, p_\nu\}_P - \frac{1}{4}\{\psi_a^m\omega_{mn\mu}\psi_a^n, \psi_a^k\omega_{kl\nu}\psi_a^l\}_P \\ &= \frac{i}{2}\psi_a^m \frac{\partial(\omega_{mn\nu})}{\partial q^\mu}\psi_a^n - \frac{i}{2}\psi_a^m\frac{\partial(\omega_{mn\mu})}{\partial q^\nu}\psi_a^n - \frac{i}{4}\frac{\partial}{\partial\psi_a^i}(\psi_a^m\omega_{mn\mu}\psi_a^n)\frac{\partial}{\partial\psi_a^i}(\psi_a^k\omega_{kl\nu}\psi_a^l) \\ &= \frac{i}{2}\psi_a^m\psi_a^n\left[\frac{\partial(\omega_{mn\nu})}{\partial q^\mu} - \frac{\partial(\omega_{mn\mu})}{\partial q^\nu}\right] - i\omega_{im\mu}\psi_a^m\omega_{in\nu}\psi_a^n.\end{aligned} \tag{5.121}$$

The next few steps contain a subtle renaming of the dummy indices together with the use of the anti-commutation of the ψ_a^μ variables so pay attention here

$$\begin{aligned}
&= \frac{i}{2}\psi_a^m\psi_a^n\left[\frac{\partial\left(\omega_{mn\nu}\right)}{\partial q^\mu} - \frac{\partial\left(\omega_{mn\mu}\right)}{\partial q^\nu}\right] - \frac{i}{2}\omega_{im\mu}\psi_a^m\omega_{in\nu}\psi_a^n - \frac{i}{2}\omega_{ik\mu}\psi_a^k\omega_{il\nu}\psi_a^l \\
&= \frac{i}{2}\psi_a^m\psi_a^n\left[\frac{\partial\left(\omega_{mn\nu}\right)}{\partial q^\mu} - \frac{\partial\left(\omega_{mn\mu}\right)}{\partial q^\nu}\right] - \frac{i}{2}\omega_{im\mu}\psi_a^m\omega_{in\nu}\psi_a^n - \frac{i}{2}\omega_{il\nu}\psi_a^k\omega_{ik\mu}\psi_a^l \\
&= \frac{i}{2}\psi_a^m\psi_a^n\left[\frac{\partial\left(\omega_{mn\nu}\right)}{\partial q^\mu} - \frac{\partial\left(\omega_{mn\mu}\right)}{\partial q^\nu}\right] - \frac{i}{2}\omega_{im\mu}\psi_a^m\omega_{in\nu}\psi_a^n + \frac{i}{2}\omega_{im\nu}\psi_a^m\omega_{in\mu}\psi_a^n \\
&= \frac{i}{2}\psi_a^m\psi_a^n\left[\frac{\partial\left(\omega_{mn\nu}\right)}{\partial q^\mu} - \frac{\partial\left(\omega_{mn\mu}\right)}{\partial q^\nu}\right] + \frac{i}{2}\omega_{mi\mu}\psi_a^m\omega_{in\nu}\psi_a^n - \frac{i}{2}\omega_{mi\nu}\psi_a^m\omega_{in\mu}\psi_a^n \\
&= \frac{i}{2}\psi_a^m\psi_a^n\left[\frac{\partial\left(\omega_{mn\nu}\right)}{\partial q^\mu} - \frac{\partial\left(\omega_{mn\mu}\right)}{\partial q^\nu} + \omega_{mi\mu}\omega_{in\nu} - \omega_{mi\nu}\omega_{in\mu}\right] \\
&= \frac{i}{2}\psi_a^m\psi_a^n e^m{}_\sigma E_n{}^\lambda R^\sigma{}_{\lambda\mu\nu} \\
&= \frac{i}{2}\psi_a^m\psi_a^n R^m{}_{n\mu\nu}.
\end{aligned} \tag{5.122}$$

5.4 Quantization of the Supersymmetric Sigma Model

Before we head off to quantize the system we must remark that there is another symmetry in our Lagrangian. Namely, the system is invariant under rotation of the fermionic variables by a small real angle γ. In symbols

$$\begin{aligned}
\psi^\mu &\to e^{-i\gamma}\psi^\mu, \\
\bar{\psi}^\mu &\to e^{i\gamma}\bar{\psi}^\mu,
\end{aligned} \tag{5.123}$$

where we have defined the complex fermionic variables

$$\begin{aligned}
\psi^\mu &= \frac{1}{\sqrt{2}}(\psi_1^\mu + i\psi_2^\mu), \\
\bar{\psi}^\mu &= \frac{1}{\sqrt{2}}(\psi_1^\mu - i\psi_2^\mu).
\end{aligned} \tag{5.124}$$

By Noether's theorem we get a corresponding conserved charge F (see [1]) defined as

$$F = g_{\mu\nu}\bar{\psi}^\mu\psi^\nu. \tag{5.125}$$

We can now quantize the system by applying Dirac's quantization scheme using the Poisson brackets we calculated before. This yields a set of commutators and anti-commutators of what are now operators on a Hilbert space. In symbols:

$$\begin{aligned}
[q^\mu, \Pi_\nu] &= i\delta^\mu_\nu \\
\{\psi^\mu, \bar{\psi}^\nu\} &= g^{\mu\nu} \\
\{\Pi_\mu, \bar{\psi}^\nu\} &= i\Gamma^\nu{}_{\mu\rho}\bar{\psi}^\rho \\
\{\Pi_\mu, \psi^\nu\} &= i\Gamma^\nu{}_{\mu\rho}\psi^\rho \\
\{\Pi_\mu, \Pi_\nu\} &= -\frac{1}{2}\psi_a^m\psi_a^n R^m{}_{n\mu\nu}.
\end{aligned} \tag{5.126}$$

All other (anti)- commutators vanish. The supercharges, which are now operators, are given by

$$\begin{aligned}
Q &= \bar{\psi}^\mu\Pi_\mu, \\
\bar{Q} &= \Pi_\mu\psi^\mu.
\end{aligned} \tag{5.127}$$

Notice that we avoided the problem of an ambiguity in operator ordering by enforcing a certain ordering. Why we picked this particular ordering will become clearer when we try designate a representation for the Hilbert space on which Q and $\bar{Q}$ operate. This makes the quantum mechanical Hamiltonian

$$\{Q, \bar{Q}\} = 2H. \tag{5.128}$$

This makes the quantum mechanical system manifestly supersymmetric in the way we defined it in chapter 3. We still have to calculate the relations between the charges F and Q. We will see that these relations are similar in structure to what we have seen in chapter 3. Before we calculate the commutator of F and Q, we note that F commutes with Π_μ. This is clear from the definition of Π_μ and p_μ

$$\begin{aligned}\Pi_\mu &= p_\mu - \frac{i}{2}\psi_a^m \omega_{mn\mu}\psi_a^n \\ &= \dot{q}^\nu g_{\mu\nu} + \frac{i}{2}\psi_a^m \omega_{mn\mu}\psi_a^n - \frac{i}{2}\psi_a^m \omega_{mn\mu}\psi_a^n \\ &= \dot{q}^\nu g_{\mu\nu}.\end{aligned} \tag{5.129}$$

It turns out that Π_μ is a purely bosonic operator and then it commutes with the fermionic operator F. We use this fact and find

$$\begin{aligned}[F, Q] &= FQ - QF \\ &= g_{\mu\nu}\bar{\psi}^\mu \left(\psi^\nu \bar{\psi}^\rho\right)\Pi_\rho - \bar{\psi}^\rho \Pi_\rho F \\ &= g_{\mu\nu}\bar{\psi}^\mu \left(g^{\nu\rho} - \bar{\psi}^\rho \psi^\nu\right)\Pi_\rho - \bar{\psi}^\rho F \Pi_\rho \\ &= \delta_\mu^\rho \bar{\psi}^\mu \Pi_\rho - g_{\mu\nu}\bar{\psi}^\mu \bar{\psi}^\rho \psi^\nu \Pi_\rho - \bar{\psi}^\rho g_{\mu\nu}\bar{\psi}^\mu \psi^\nu \Pi_\rho \\ &= \bar{\psi}^\rho \Pi_\rho - g_{\mu\nu}\bar{\psi}^\mu \bar{\psi}^\rho \psi^\nu \Pi_\rho + g_{\mu\nu}\bar{\psi}^\mu \bar{\psi}^\rho \psi^\nu \Pi_\rho \\ &= Q.\end{aligned} \tag{5.130}$$

A similar calculation gives

$$[F, \bar{Q}] = -\bar{Q} \tag{5.131}$$

This allows us to calculate the commutator between F and H as follows

$$\begin{aligned}[F, H] &= FH - HF \\ &= FQ\bar{Q} + F\bar{Q}Q - Q\bar{Q}F - \bar{Q}QF \\ &= FQ\bar{Q} + F\bar{Q}Q - Q([\bar{Q}, F] + F\bar{Q}) - \bar{Q}([Q, F] + FQ) \\ &= FQ\bar{Q} + F\bar{Q}Q - Q\bar{Q} - QF\bar{Q} + \bar{Q}Q - \bar{Q}FQ \\ &= FQ\bar{Q} + F\bar{Q}Q - Q\bar{Q} - ([Q, F] + FQ)\bar{Q} + \bar{Q}Q - ([\bar{Q}, F] + F\bar{Q})Q \\ &= FQ\bar{Q} + F\bar{Q}Q - Q\bar{Q} + Q\bar{Q} - FQ\bar{Q} + \bar{Q}Q - \bar{Q}Q - F\bar{Q}Q \\ &= 0.\end{aligned} \tag{5.132}$$

This means that F is also a conserved charge in the quantum realm, i.e. the F operator will respect energy levels of states. Now the only thing left to do in order to complete the quantization of the system is finding a suitable representation for the supersymmetric Hilbert space $\mathcal{H}_Q$ we just built. We will argue that a suitable Hilbert space will be

$$\mathcal{H} = \Omega(M) \otimes \mathbb{C}. \tag{5.133}$$

This is the Hilbert space of differentials forms of the manifold M, as we defined it in chapter 4, tensored with the complex plane. (This is necessary to acquire a complex Hilbert space, since the space of differential forms is a real one). We equip this Hilbert space with a Hermitian inner product defined as

$$(\omega_1, \omega_2) = \int_M \bar{\omega}_1 \wedge \star\omega_2, \tag{5.134}$$

where ω_1 and ω_2 are both $p-$forms and $\star$ is the Hodge duality operator. Like the simpler system in chapter 3, the Hilbert space can be written as the tensor product of a bosonic and a fermionic space but now the fermionic space has a higher dimensionality (it can accommodate more fermions). The bosonic space is simply the space of integrable complex functions over the manifold $L(M, \mathbb{C})$, it is infinite dimensional. In order to characterise the fermionic part of $\mathcal{H}_Q$ we go about as in the simpler, flat model and define the state $|0\rangle$ as the vector that is annihilated by all the ψ^μ's. This allows us to build up a set of states by applying the $\bar{\psi}^\mu$ operator to $|0\rangle$. We get the following set of states,

$$\begin{gathered} |0\rangle \\ \bar{\psi}^\mu|0\rangle \\ \bar{\psi}^\mu\bar{\psi}^\nu|0\rangle \\ \vdots \\ \bar{\psi}^1\dots\bar{\psi}^n|0\rangle. \end{gathered} \tag{5.135}$$

These states are also eigenstates of the operator F and they have eigenvalue p where p is the amount of different $\bar{\psi}^\mu$ operators applied. We will once again call the operator F the fermion number operator as it counts the number of 'fermions' in a certain state. Note that the dimensionality of the eigenspaces of F is $\binom{n}{p}$. This dimensionality coincides with the composition by form-degree in the space of differential forms. This means that the Hilbert space $\mathcal{H}_Q$ has the same dimensionality as the complexified (tensored with $\mathbb{C}$ space of differential forms $\mathcal{H}$. This is true because of the equivalence between the bosonic subspace and the subspace of complexified differential zero forms, which is just $L(M,\mathbb{C})$. By the theory of Hilbert spaces this means there exists an isomorphism Λ between the spaces $\mathcal{H}_Q$ and $\mathcal{H}$. One way we can always go about constructing this isomorphism is by connecting the basis elements of both spaces, i.e.

$$\begin{aligned} &\Lambda(|0\rangle) = 1 \\ &\Lambda(\bar{\psi}^\mu|0\rangle) = dx^\mu \\ &\Lambda(\bar{\psi}^\mu\bar{\psi}^\nu|0\rangle) = dx^\mu dx^\nu \\ &\vdots \\ &\Lambda(\bar{\psi}^1\dots\bar{\psi}^n|0\rangle) = dx^1\dots dx^n. \end{aligned} \tag{5.136}$$

So as a Hilbert space we now regard $\mathcal{H}_Q$ as represented by $\mathcal{H}$. Of course this knowledge will not do us much good until we also find representations for the various operators we defined on $\mathcal{H}_Q$, the most important ones being ψ^μ, $\bar{\psi}^\mu$, Q and $\bar{Q}$. We will find these representations by exploiting the properties of the function $\Lambda^* : \mathrm{Hom}(\mathcal{H}_Q,\mathcal{H}_Q) \to \mathrm{Hom}(\mathcal{H},\mathcal{H})$ associated with the function Λ. ($\mathrm{Hom}(\mathcal{H}_Q,\mathcal{H}_Q)$ is the Hilbert space of the transformations of $\mathcal{H}_Q$.) This associated function has the defining property that for a random vector $v \in \mathcal{H}_Q$ and a random operator $A \in \mathrm{Hom}(\mathcal{H}_Q,\mathcal{H}_Q)$ we get

$$\Lambda(A(v)) = \Lambda^*(A)(\Lambda(v)). \tag{5.137}$$

For the $\bar{\psi}^\mu$ operator working on a random basis element this yields the following equalities

$$\Lambda(\bar{\psi}^\mu(\bar{\psi}^\nu\dots\bar{\psi}^\sigma|0\rangle)) = \Lambda^*(\bar{\psi}^\mu)\Lambda(\bar{\psi}^\nu\dots\bar{\psi}^\sigma|0\rangle) = \Lambda^*(\bar{\psi}^\mu)dx^\nu\dots dx^\sigma, \tag{5.138}$$

but also,

$$\Lambda(\bar{\psi}^\mu(\bar{\psi}^\nu\dots\bar{\psi}^\sigma|0\rangle)) = \Lambda(\bar{\psi}^\mu\bar{\psi}^\nu\dots\bar{\psi}^\sigma|0\rangle) = dx^\mu dx^\nu\dots dx^\sigma. \tag{5.139}$$

This implies that the $\bar{\psi}^\mu$ operator must be equivalent to the operation of 'wedge multiplying' with dx^μ or

$$\Lambda^*(\bar{\psi}^\mu) = dx^\mu \wedge . \tag{5.140}$$

In order to obtain a representation for the ψ^μ operator we need to do a bit more work. Let us begin by looking at what happens when ψ^μ acts on a random basis element of fermion number p

$$\begin{aligned} \psi^\mu\left(\bar{\psi}^\nu\bar{\psi}^\sigma\dots\bar{\psi}^\lambda|0\rangle\right) &= g^{\mu\nu}\bar{\psi}^\sigma\dots\bar{\psi}^\lambda|0\rangle - \bar{\psi}^\nu\psi^\mu\bar{\psi}^\sigma\dots\bar{\psi}^\lambda|0\rangle \\ &= g^{\mu\nu}\bar{\psi}^\sigma\dots\bar{\psi}^\lambda|0\rangle - g^{\mu\sigma}\bar{\psi}^\nu\dots\bar{\psi}^\lambda|0\rangle + \bar{\psi}^\nu\bar{\psi}^\sigma\psi^\mu\dots\bar{\psi}^\lambda|0\rangle \\ &\vdots \\ &= g^{\mu\nu}\bar{\psi}^\sigma\dots\bar{\psi}^\lambda|0\rangle + (-1)^{p-1}g^{\mu\sigma}\bar{\psi}^\nu\dots\bar{\psi}^\lambda|0\rangle + (-1)^p\bar{\psi}^\nu\bar{\psi}^\sigma\psi^\mu\dots\bar{\psi}^\lambda|0\rangle \\ &= pg^{\mu\nu}\bar{\psi}^\sigma\dots\bar{\psi}^\lambda|0\rangle. \end{aligned} \tag{5.141}$$

Using the defining property of Λ^* we learn that $\Lambda^*(\psi^\mu)$ must obey the following identity in the space of differential forms

$$\Lambda^*(\psi^\mu)(dx^\nu dx^\sigma \dots dx^\lambda) = pg^{\mu\nu} dx^\sigma \dots dx^\lambda. \tag{5.142}$$

This is satisfied if

$$\Lambda^*(\psi^\mu) = g^{\mu\nu} \left(i_{\partial/\partial x^\nu}\right), \tag{5.143}$$

where i_V is the operation of contracting the differential form with the vector field V. (This operation is known as the *interior product*.) Our next operator q^μ is a purely bosonic variable, i.e. it only affects the 'function part' of an element of $\mathcal{H}_Q$. This means the q^μ operator and its differential form counterpart are the exact same thing, only in different notation. In symbols

$$q^\mu = x^\mu \times, \tag{5.144}$$

where $x^\mu\times$ is just the operation of multiplying the differential form by a factor of x^μ. The last important operator remaining is Π_μ. Since $\Pi_\mu = \dot{q}^\nu g_{\mu\nu}$, it is a purely bosonic operator. This means we can find its representation in much the same way as that of the q^μ operator. We know from quantum mechanics that the $\dot{q}^\nu$ operator acts as a partial derivative operator, which has a trivial correspondence in the space of differential forms. We get

$$\Lambda^*(\Pi_\mu) = \Lambda^*(g_{\mu\nu}\dot{q}^\nu) = g_{\mu\nu}\frac{\partial}{\partial x^\nu} = \nabla_\mu. \tag{5.145}$$

If we combine these results we can find a correspondence for the Q operator through its definition. This gives

$$\Lambda^*(Q) = \Lambda^*(\bar{\psi}^\mu \Pi_\mu) = dx^\mu \wedge \nabla_\mu = dx^\mu \wedge \frac{\partial}{\partial x_\mu} = d. \tag{5.146}$$

So Q corresponds to the exterior derivative, which was to be expected given their respective effect on the fermion number of a state and the $p-$degree of a differential form (both increment by one). We can do a similar exercise for $\bar{Q}$, and find (see [1])

$$\Lambda^*(\bar{Q}) = d^\dagger. \tag{5.147}$$

This is very reasonable since the Λ^* function preserves hermitian conjugation. We have now characterised all the operators we need to construct the rest of the theory.

5.5 Q-cohomology and the Witten Index

Now that we have found a representation of our supersymmetric Hilbert space it is time to return to a very important object we touched upon in earlier chapters, namely the Witten index. As before it is defined as the operator $\mathrm{Tr}(-1)^F$. In our earlier treatment of the Witten index we saw that it was an expression that calculates the difference between the dimensionality of ground state subspaces with an even fermion number (the eigenvalue of the F operator in that subspace) and the dimensionality of those with an odd fermion number. In symbols this gave

$$\mathrm{Tr}(-1)^F = \dim \mathcal{H}_B\big|_{E=0} - \dim \mathcal{H}_F\big|_{E=0}. \tag{5.148}$$

We now rewrites this expression as the sum

$$\mathrm{Tr}(-1)^F = \sum_{p=0}^{n} (-1)^p \dim \mathcal{H}_p\big|_{E=0}. \tag{5.149}$$

This is again true, because for any state with non-zero energy the Q operator induces an isomorphism between states of different fermion number but equal energy. The $(-1)^p$ in the equation then ensures that all contributions due to non- zero energy states cancel each other out leaving only the ground states where as before the Q-isomorphism breaks down.

There is another, more theoretically fulfilling, way to calculate this index. We can regard the total subspaces composed by fermion degree as a a cochain complex of the Hilbert space, generated by the Q operator. It can be graphically presented like

$$\mathcal{H}_0 \xrightarrow{Q} \mathcal{H}_1 \xrightarrow{Q} \dots \xrightarrow{Q} \mathcal{H}_{n-1} \xrightarrow{Q} \mathcal{H}_n. \tag{5.150}$$

For this cochain complex we can calculate the Q-cohomology groups. These are defined as

$$H^p(Q) = \frac{\text{Ker}\left(\mathcal{H}_p \xrightarrow{Q} \mathcal{H}_{p+1}\right)}{\text{Im}\left(\mathcal{H}_{p-1} \xrightarrow{Q} \mathcal{H}_p\right)}. \tag{5.151}$$

Note that H^p is also a vector space, which means we can talk about the dimension of H^p without running into trouble. This means that H^p is the quotient group of the kernel of $Q : \mathcal{H}_p \to \mathcal{H}_{p+1}$ and the image of $Q : \mathcal{H}_{p-1} \to \mathcal{H}_p$. The fact that Q is a nilpotent operator ($Q^2 = 0$) ensures that this is well defined. Note that the definition of the Q-cohomology is very similar to the de Rham cohomology we discussed in chapter 4, this will prove to be of great importance later on. Because Q commutes with the Hamiltonian, and hence preserves energy levels, we can decompose the Q-cohomology groups H^p by energy level. This gives

$$H^p(Q) = \bigoplus_i H_i^p(Q), \tag{5.152}$$

where H_i^p is defined as

$$H_i^p(Q) = \frac{\text{Ker}\left(\mathcal{H}_p\big|_{E=i} \xrightarrow{Q} \mathcal{H}_{p+1}\big|_{E=i}\right)}{\text{Im}\left(\mathcal{H}_{p-1}\big|_{E=i} \xrightarrow{Q} \mathcal{H}_p\big|_{E=i}\right)}. \tag{5.153}$$

Here i is an arbitrary element of the energy spectrum of the Hamiltonian. (The compactness of the manifold M ensures that the spectrum is discrete.) Now, for all non-zero energy levels, i.e. $i > 0$, we know that the Q operator is an isomorphism between two subspaces with adjacent fermion numbers. This means that $\text{Ker}\left(\mathcal{H}_p\big|_{E=i} \xrightarrow{Q} \mathcal{H}_{p+1}\big|_{E=i}\right)$ must be trivial (then so must $\text{Im}\left(\mathcal{H}_{p-1}\big|_{E=i} \xrightarrow{Q} \mathcal{H}_p\big|_{E=i}\right)$) and hence we have that

$$\dim\left(H^p(Q)\right) = \sum_i \dim\left(H_i^p(Q)\right) = \dim\left(H_0^p(Q)\right). \tag{5.154}$$

So the only contribution to the dimensionality of the Q- cohomology groups is due to the ground states. Now if we look at the space of ground states ($i = 0$) we see that on this space the Q operator is the zero operator. (By definition all ground states must be annihilated by Q.) This means that

$$\begin{aligned} \text{Ker}\left(\mathcal{H}_p\big|_{E=0} \xrightarrow{Q} \mathcal{H}_{p+1}\big|_{E=0}\right) &= \mathcal{H}_p\big|_{E=0} \\ \text{Im}\left(\mathcal{H}_{p-1}\big|_{E=i} \xrightarrow{Q} \mathcal{H}_p\big|_{E=i}\right) &= 0. \end{aligned} \tag{5.155}$$

Hence we have

$$\dim\left(H^p(Q)\right) = \dim\left(H_0^p(Q)\right) = \dim \mathcal{H}_p\big|_{E=0}. \tag{5.156}$$

This means that we can compute the Witten index using the cohomology of the Q operator through the formula

$$\text{Tr}(-1)^F = \sum_{p=0}^{n}(-1)^p \dim\left(\mathcal{H}_p\right) = \sum_{p=0}^{n}(-1)^p \dim\left(\mathcal{H}_p\big|_{E=0}\right) = \sum_{p=0}^{n}(-1)^p \dim\left(H^p(Q)\right). \tag{5.157}$$

Now we go back the the representation of the Hilbert space of supersymmetric quantum mechanics as the space of differential forms. Using the fact that we have identified Q with the exterior derivative d and $\bar{Q}$ with its dual operation $d^\dagger$ we can write the following correspondence for the supersymmetric Hamiltonian

$$H = \frac{1}{2}\{Q, \bar{Q}\} = \frac{1}{2}(Q\bar{Q} + \bar{Q}Q) \leftrightarrow \frac{1}{2}(dd^\dagger + d^\dagger d) = \Delta, \tag{5.158}$$

where Δ is the Laplace Beltrami operator we defined is chapter 4. This means that our supersymmetric ground states correspond to the differential forms ω for which the equation

$$\Delta\omega = 0 \tag{5.159}$$

holds. Hence we have a one-to-one correspondence between the subspace of supersymmetric ground states and the subspace of harmonic differential forms $\mathbf{H}(M,g)$ where g is the metric on M. But we saw in chapter 4 that the space of harmonic differential forms is the de Rham cohomology. This means there exists a direct correspondence between the Q-cohomology groups and the de Rham cohomology

$$H^p(Q) = H^p_{DR}, \tag{5.160}$$

where H^p_{DR} is the $p-$th de Rham cohomology group. By this correspondence we can write the Witten index in terms of the de Rham cohomology groups, yielding the following formula

$$\mathrm{Tr}(-1)^F = \sum_{p=0}^{n}(-1)^p \dim\left(H^p(Q)\right) = \sum_{p=0}^{n}(-1)^p \dim\left(H^p_{DR}\right). \tag{5.161}$$

But the term on the right hand side of this equation is none other than the Euler characteristic for the manifold M, introduced in section 4.4.5, and further discussed in appendix A.3. This is the intimate connection between supersymmetric quantum mechanics and topology that we have been working towards all this time. So in essence we have the following fundamental equation

$$\mathrm{Tr}(-1)^F = \chi(M). \tag{5.162}$$

This means that for a given manifold M we can compute the Euler characteristic by looking at the ground states of the corresponding supersymmetric Hamiltonian instead of laboriously calculating all the de Rham cohomology groups. Let us illustrate the power of this equation by calculating the Euler characteristic of a two dimensional torus.

The Witten Index for a Simple Torus

We call the angle on the big circle θ and the angle on the small circle ϕ. The metric for a torus of small radius a and big radius c is then given by

$$g_{\mu\nu} := \begin{pmatrix} (c+a\cos\phi)^2 & 0 \\ 0 & a^2 \end{pmatrix}. \tag{5.163}$$

The only non-zero connections are

$$\begin{aligned} {\Gamma^\theta}_{\theta\phi} &= {\Gamma^\theta}_{\phi\theta} = \frac{-a\sin\phi}{c+a\cos\phi} \\ {\Gamma^\phi}_{\theta\theta} &= a^{-1}\sin\phi(c+a\cos\phi). \end{aligned} \tag{5.164}$$

Our goal is to find the ground states for the Hamiltonian associated with this manifold. A state $|\alpha\rangle$ is a ground state if it is annihilated by both Q and $\bar{Q}$. Let us begin by characterising the Q operator

$$\begin{aligned} Q &= \bar{\psi}^\mu \Pi_\mu \\ &= \bar{\psi}^\mu \left(p_\mu - \frac{i}{2}\psi^m_a \omega_{mn\mu}\psi^n_a\right) \\ &= \bar{\psi}^\mu p_\mu - \frac{i}{2}\bar{\psi}^\mu \left(\psi^\rho_a {e^m}_\rho {e^m}_\nu \left(\partial_\mu {E_n}^\nu + {\Gamma^\nu}_{\mu\lambda}{E_n}^\lambda\right) {e^n}_\gamma \psi^\gamma_a\right) \\ &= \bar{\psi}^\theta p_\theta + \bar{\psi}^\phi p_\phi - \frac{i}{2}\bar{\psi}^\theta \left(\psi^\rho_a g_{\rho\nu}\left(\partial_\theta {E_n}^\nu + {\Gamma^\nu}_{\theta\lambda}{E_n}^\lambda\right) {e^n}_\gamma \psi^\gamma_a\right) \\ &\quad - \frac{i}{2}\bar{\psi}^\phi \left(\psi^\rho_a g_{\rho\nu}\left(\partial_\phi {E_n}^\nu + {\Gamma^\nu}_{\phi\lambda}{E_n}^\lambda\right) {e^n}_\gamma \psi^\gamma_a\right) \\ &= -\frac{i}{2}\bar{\psi}^\theta \left(\psi^\theta_a g_{\theta\theta}\left({\Gamma^\theta}_{\theta\phi}{E_\phi}^\phi\right) {e^\phi}_\phi \psi^\phi_a\right) - \frac{i}{2}\bar{\psi}^\theta \left(\psi^\phi_a g_{\phi\phi}\left({\Gamma^\phi}_{\theta\theta}{E_\theta}^\theta\right) {e^\theta}_\theta \psi^\theta_a\right) \\ &\quad - \frac{i}{2}\bar{\psi}^\phi \left(\psi^\theta_a g_{\theta\theta}\left(\partial_\phi {E_\theta}^\theta + {\Gamma^\theta}_{\phi\theta}{E_\theta}^\theta\right) {e^\theta}_\theta \psi^\theta_a\right) - \frac{i}{2}\bar{\psi}^\phi \left(\psi^\phi_a g_{\phi\phi}\left(\partial_\phi {E_\phi}^\phi + {\Gamma^\phi}_{\phi\phi}{E_\phi}^\phi\right) {e^\phi}_\phi \psi^\phi_a\right) \\ &\quad + \bar{\psi}^\theta p_\theta + \bar{\psi}^\phi p_\phi \\ &= \bar{\psi}^\theta p_\theta + \bar{\psi}^\phi p_\phi. \end{aligned} \tag{5.165}$$

Through a similar calculation we also obtain

$$\bar{Q} = p_\theta \psi^\theta + p_\phi \psi^\phi. \tag{5.166}$$

This means that the only states being annihilated by both Q and $\bar{Q}$ are of the form

$$\begin{aligned} |\alpha\rangle &= C|0\rangle, \\ |\alpha\rangle &= C\bar{\psi}^\theta|0\rangle, \\ |\alpha\rangle &= C\bar{\psi}^\phi|0\rangle \text{ or} \\ |\alpha\rangle &= C\bar{\psi}^\theta\bar{\psi}^\phi|0\rangle, \end{aligned} \tag{5.167}$$

where C is a normalization constant. This means that the dimensionality of the zero energy subspaces $\mathcal{H}_p\big|_{E=0}$ is given, for p going from 0 to 2, by

$$\begin{aligned} \dim \mathcal{H}_0\big|_{E=0} &= 1 \\ \dim \mathcal{H}_1\big|_{E=0} &= 2 \\ \dim \mathcal{H}_2\big|_{E=0} &= 1. \end{aligned} \tag{5.168}$$

We can now compute the Euler characteristic through the Witten index

$$\chi(M) = \mathrm{Tr}\left(-1^F\right) = \sum_{p=0}^{n} (-1)^p \dim\left(\mathcal{H}_p\big|_{E=0}\right) = 1 - 2 + 1 = 0. \tag{5.169}$$

So the the Euler index of a torus is zero, as we could have suspected from the fact that a torus is the product of two circles.

Chapter 6

Conclusions

The goal of this project was to investigate the formalism of supersymmetric quantum mechanics in general and its applications in mathematics in particular. It seems, that for a good part, we have accomplished this goal. We have investigated quantum mechanical models, both in simple one-dimensional settings like $\mathbb{R}$ and S^1, and in the more general setting of a sigma model defined on a smooth, n-dimensional manifold of arbitrary curvature. At the heart of the formalism we found a satisfying connection between the zero-energy states of the system and a modified partition function the Witten index. Later on we were able to tie these zero-energy states to a mathematical group called the Q-cohomology. We realised that the Q-cohomology was closely connected to the de Rham cohomology defined using the theory of differential forms on the manifold. Using this connection, in combination with a deep theorem connecting topology and differential geometry due to de Rham, we managed to formulate a very elegant formula for the topological Euler characteristic in terms of the quantum mechanical Witten index. This formula, though deceptively simple, summarises a deep and intricate connection between the theory of supersymmetric quantum mechanics on a manifold and the topological properties of that manifold.

Of course the connection between supersymmetric quantum mechanics and topology does not stop here. For instance one could quantize the sigma model using the path integral formalism instead of, as we did, the operator formalism. This approach would eventually lead to a characterisation of the Euler characteristic in terms of an integral over the manifold involving the Riemann curvature tensor. On the way to this characterisation one would encounter the important physical theory of localisation, a principle where integrals over functions of fermionic variables are completely determined by their value at certain 'critical points'. If one were to continue on this path of convergence between supersymmetric quantum mechanics and topology it would eventually lead to an elegant 'physics proof' of something called the Atiyah- Singer index theorem, an important theorem concerning the solvability of certain classes of differential equations in terms of the topology of the manifolds they are defined on. Readers interested in exploring this in greater detail are referred to the paper 'Supersymmetry and the Atiyah-Singer Index Theorem' [3] by Luis Alvarez-Gaumé.

Apart from the work on supersymmetric quantum mechanics we also managed to build up and present new knowlegde in basic differential geometry and topology, branches of mathematics that are very important in modern physics but are only superficially treated in a bachelor in Physics.

Appendix A

Complementary Material

A.1 Proof of the Commutator Identity

Here we present a proof for the identity

$$[p, f(x)] = -if'(x),$$

where $f(x)$ is an analytic function. We can prove this by expanding f into a power series of x

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We now prove that for every $n \in \mathbb{N}$

$$[p, x^n] = -inx^{n-1}.$$

For $n = 1$ this is just our commutation relation (2.30). (For $n = 0$ it is trivial) We now assert that the equation holds for $n = k - 1$. The equation for $n = k$ then becomes

$$\begin{aligned}[p, x^k] &= [p, x^{k-1}]x + x^{k-1}[p, x] \\ &= -i(k-1)x^{k-2}x - ix^{k-1} \\ &= -i(k-1+1)x^{k-1} \\ &= -ikx^{k-1}.\end{aligned}$$

So, by induction, the formula $[p, x^n] = -inx^{n-1}$ follows for every $n \in \mathbb{N}$. We now use this knowledge, together with the linearity of the derivative to get

$$\begin{aligned}[p, f(x)] &= \left[p, \sum_{n=0}^{\infty} c_n x^n\right] \\ &= \sum_{n=0}^{\infty} c_n [p, x^n] \\ &= \sum_{n=0}^{\infty} c_n (-i) n x^{n-1} \\ &= -if'(x).\end{aligned}$$

This proves the relation.

A.2 Brief Introduction to Group Theory

A.2.1 Groups

A *group* is a set of elements, where the elements can be combined with each other according to a specific composition law, symbolised by multiplication. This composition law can be any operation for which the following *group axioms* are fulfilled

- i.) $\forall a, b \in G : ab \in G$ (the group is closed)
- ii.) $\forall a, b, c \in G : (ab)c = a(bc)$ (associativity)
- iii.) $\exists e \in G : \forall a \in G : ae = ea = a$ (existence of unit element)
- iv.) $\forall a \in G,\ \exists a^{-1} \in G : aa^{-1} = a^{-1}a = e$ (existence of inverse).

Example 1.) The positive rational numbers, with the composition law given simply by ordinary multiplication, form a group. The product of two positive rational numbers is always a positive rational number. Associativity is immediate, and the unit element is 1. The inverse of each element is its reciprocal. 0 cannot be an element of this group since it has no inverse.

Example 2.a.) $\mathbb{Z}$. This is the group of integers with group "multiplication", i.e. the composition law, defined as addition.The sum of two integers is always a new integer, addition of integers is associative, 0 is the unit element and $-z$ is the inverse of $z \in \mathbb{Z}$.

A.2.2 Subgroups

A *subgroup* H to a group G is itself a group, whose elements are members also of G.

Example 2.b.) Let B be the group of integers that are dividable by 3, that is $B = \{\dots, -6, -3, 0, 3, 6, \dots\}$. The composition law is still addition. Since an integer b which is dividable by 3 can be written as $b = 3a$ where a is a new integer, the sum of two such integers $b_1 + b_2 = 3a_1 + 3a_2 = 3(a_1 + a_2)$ is also dividable by 3.The unit element 0 is in the group, and inverses $-b$ exist in B for each b in B. Clearly B is a subgroup of $\mathbb{Z}$.

A.2.3 Equivalence Relations and Equivalence Classes

An *equivalence relation* on a set S is a relation that connects elements in the set. Two elements $a, b \in S$ that are connected by this relation are said to be *equivalent*, written $a \sim b$. The relation must obey

- i.) Reflexivity: $a \sim a$
- ii.) Symmetry: $a \sim b \Rightarrow b \sim a$
- iii.) Transitivity: $a \sim b$ and $b \sim c \Rightarrow a \sim c$.

An equivalence relation defined on S gives rise to a partition of the set into disjoint subsets, so called *equivalence classes*, by the following procedure. Pick an element a in the set. Then find and pick out all elements in S which are equivalent to a. This is the equivalence class of the element a: $(a) = \{b | b \sim a\}$. Now if there are still elements left in the remaining set, choose a new element c from the set, and find all elements that are equivalent to c. If the remaining set is still nonempty, continue the process. In the end we are left with a set of equivalence classes, which by construction have no elements in common. See figure A.1. The concept of equivalence relations is one of the most powerful tools in all of mathematics.

A.2.4 Cosets

Let $H = \{h_1, h_2, \dots, h_r\}$ be a subgroup of G. Choose an arbitrary element $g \in G$ and construct the *coset* $gH = g\{h_1, \dots, h_r\} = \{gh_1, \dots, gh_3\}$. Generally a coset is not a group.

Now define an equivalence relation for $a, b \in G : a \sim b$ if $b \in aH$. That is, a and b are equivalent if there is an element $h \in H$ for which $b = ah$. This is really an equivalence relation, since it obeys the conditions in section A.2.3:

- i.) $a \sim a$, for $e \in H$, and $ae = a$, so that $a \in aH$.
- ii.) If $a \sim b$, then $\exists h \in H : b = ah$. Then $\exists h^{-1} \in H : bh^{-1} = a$, and $b \sim a$. So $a \sim b \Rightarrow b \sim a$.
- iii.) If $a \sim b$ and $b \sim c$, then $\exists h_i, h_j \in H : b = ah_i, c = bh_j$. By combining these one gets $c = bh_j = (ah_i)h_j = a(h_ih_j) = ah$ for some $h \in H$. Then $a \sim c$.

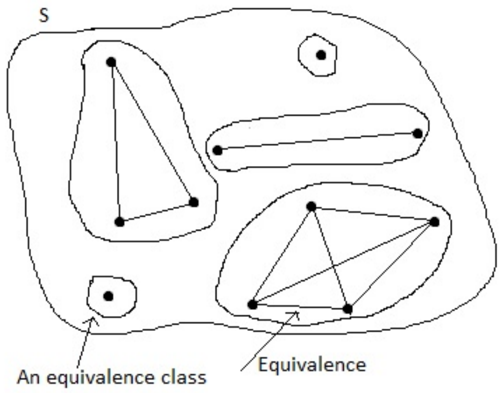


Figure A.1: The set S partioned into disjoint equivalence classes. If the elements of S (points) are equivalent, they are united by a line. The elements in an equivalence class form a subset of S.

Since the equivalence classes of this equivalence relation are cosets, we get a partition of the group G into disjoint cosets. The *representative* g for a coset gH is by no means unique.

Example 2.c) Consider the group B defined in Ex 2.b as a subgroup of $\mathbb{Z}$. Then a coset in $\mathbb{Z}$ with respect to B can be written as $z + B$ where $z \in \mathbb{Z}$, since the composition law is given by addition. This coset is the set of numbers dividable by 3 all added by an integer z. It is easy to realise that, by choosing 3 appropriate elements in $\mathbb{Z}$ as representatives for 3 cosets, one can cover $\mathbb{Z}$ completely by these cosets. For example, take 0, 1 and 2. Then the cosets

$$\begin{aligned} 0 + B &= \{\ldots, -6, -3, 0, 3, 6, \ldots\}, \\ 1 + B &= \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \\ 2 + B &= \{\ldots, -4, -1, 2, 5, 8, \ldots\}. \end{aligned}$$

are all disjoint sets, and every integer in $\mathbb{Z}$ is a member of one of these sets. The difference between two numbers in a coset is always dividable by 3, that is, they differ only by an element in B.

A.2.5 Quotient Groups

Consider the set of s cosets $\{g_{i_1}H, g_{i_2}H, \ldots g_{i_s}H\}$ which cover the group G. By defining a multiplication of such cosets, the set of cosets can be considered as a group, a *quotient group*. The product of two such cosets is given by $(g_iH)(g_jH) = g_ig_jH$. We test for the group axioms:

- i.) Since $\forall g_i, g_j \in G : g_ig_j \in G$, it is clear that g_ig_jH must be a coset.
- ii.) $\forall g_i, g_j, g_k \in G : (g_iH)[(g_jH)(g_kH)] = (g_iH)(g_jg_kH) = g_i(g_jg_k)H = (g_ig_j)(g_kH) = [(g_iH)(g_jH)](g_kH)$, and the group multiplication is associative.
- iii.) Let the unit element be $E = eH = H$. Then $\forall g \in G : gHeH = geH = gH$.
- iv.) As an inverse to gH, take $g^{-1}H$. Then $gHg^{-1}H = gg^{-1}H = eH = E$.

A problem with the definition of the composition law is that the cosets can be represented by different $g \in G$. It can be shown, see [14], that the definition works properly if the subgroup H is *normal*, that is if $\forall g \in G : gH = Hg$. So the set of cosets of a normal subgroup H is really a group, conveniently denoted by G/H. Clearly,

$$G/H = \{g_{i_1}H, g_{i_2}H, \ldots g_{i_s}H\} \tag{A.1}$$

has s elements.

Example 2.d) Consider the three cosets in Ex 2.c as a quotient group

$$\mathbb{Z}/B = \{0 + B, 1 + B, 2 + B\},$$

where the composition law is just addition of the cosets,

$$(z_i + B) + (z_j + B) = (z_i + z_j) + B.$$

Then $(z_i + z_j) + B$ is again one of the three cosets, so the group is closed. The identity element is $0 + B = B$, since

$$(z_i + B) + (0 + B) = (z_i + 0) + B = z_i + B.$$

We say that $\mathbb{Z}/B$ is the group of integers modulo 3. Each element of the group consists of integers whose differences are dividable by 3.

A.3 The Euler Characteristic

The Euler characteristic is a topological invariant defined for a topological space. This means that it is unchanged under *homeomorphisms* (continuous deformations) of that space. It is a number, conventionally denoted by the Greek letter χ and intuitively it gives a rough measure of the difference between the number of 'even dimensional parts' and 'odd dimensional parts' of the space. Although it is defined for general topological spaces we will normally use it in the context of smooth, boundary-less manifolds. Topological invariants are useful to determine whether or not two topological spaces are topologically equivalent, because if two spaces have a different value of some topological invariant there can be no homeomorphism between the two spaces. For instance a sphere and a torus have different Euler characteristics (2 and 0 respectively, we will prove this later on) whereas a torus and a coffee mug have the same Euler number (there is no difference between them, topologically speaking). Originally the characteristic was defined by the eponymous mathematician Leonhard Euler. His original definition only encompassed polyhedra and it was defined as

$$\chi = (\text{NUMBER OF VERTICES}) - (\text{NUMBER OF EDGES}) + (\text{NUMBER OF FACES}).$$

For instance the Euler characteristic of a cube would be 2 since it has 8 vertices, 12 edges and 6 faces. In modern topology the Euler characteristic is defined in a more sophisticated way, through a concept known as homology, which we will describe very briefly. For a more in depth discussion of homology we refer to (reference here)

A.3.1 Homology of a Smooth Manifold

In order to define a version of the Euler characteristic we must first go through a process called triangulation of the manifold. Intuitively this means we will try to build an 'approximation' of the manifold using generalisations of basic triangles called n-simplexes. An n-simplex is formally defined as a collection of vertices in n-dimensional affine space. A two-simplex, which is just a triangle in this formalism looks like

$$\sigma_2 = \{(1,0), (0,1), (0,0)\}. \tag{A.2}$$

Keep in mind that this is really a two dimensional object, so we are including the face of the triangle. A zero-simplex would be a point, a one-simplex a line and so on. Now imagine glueing simplexes of the same dimension together. If we do this in a nice enough way we get something called a simplicial chain. (The definition of a simplicial complex is rather technical, but it basically boils down to having no loose ends.) So a simplicial chain is a set of simplexes, glued together in a proper manner. A set of simplicial chains of different dimension is called a simplicial complex. On this simplicial complex we can define something called a boundary operator ∂. This operator takes a simplicial chain of dimension n to another chain of dimension $n - 1$. So a two-simplex would be taken to a triangle with the face cut out of it. A very important property of this operator is that $\partial^2 = 0$, namely a boundary has no boundary of itself. We now focus our attention on two types of simplicial chains. The first is defined as having no boundary, or

$$\partial\omega = 0. \tag{A.3}$$

Where ω is some simplicial chain. We call these particular chains $n-$cycles. And the second set is defined as being the boundary of some other simplicial chain of higher dimension (+1). We call these chains n-boundaries. It is possible to define a group structure on these objects and the group of $n-$cycles is called Z^n whereas the group of $n-$boundaries is called B^n. Because of the property $\partial^2 = 0$ we know that

$$B^n \subseteq Z^m. \tag{A.4}$$

This allows us to define the quotient group

$$H^n = \frac{Z^n}{B^n}. \tag{A.5}$$

This quotient group we call the $n-$th homology group of the simplicial complex. In a sense, the rank of the $n-$th homology group measures the size of the $n-$dimensional part of the simplicial complex. In this spirit we define the Euler characteristic for a simplicial complex of dimension k as

$$\chi = \sum_{n=0}^{k} (-1)^n \text{rank}(H^n). \tag{A.6}$$

If necessary one could endow these homology groups with a vector space structure, then the equation for the Euler index becomes

$$\chi = \sum_{n=0}^{k} (-1)^n \dim(H^n). \tag{A.7}$$

This way it can be connected to the de Rham cohomology, as done in chapter 4. Now if we want to find the Euler characteristic of a manifold, we try to find a simplicial complex that is homeomorphic to the manifold (we triangulate it) and then compute the homology groups. Because the Euler characteristic is invariant under homeomorphisms, it carries over to the manifold. Needless to say this is a rather laborious way of going about, and far more efficient ways have been found to calculate the Euler characteristic of a manifold, as we hope to show in the body of this paper.

Appendix B

Some Analytic Function Theory

B.1 The Infinite Product Representation of the sinh Function

We want to prove that

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 n^2}\right). \tag{B.1}$$

Mainly we will follow the route outlined in [15]. Consider the function

$$C(z) = \pi \cot \pi z = \frac{\pi \cos \pi z}{\sin \pi z}$$

with simple poles at all integers z. It is analytic everywhere else. Especially, it is analytic on the square γ_N in the complex plane with corners in

$$\begin{aligned} z_1 &= N + \tfrac{1}{2} - i(N + \tfrac{1}{2}); & z_2 &= N + \tfrac{1}{2} + i(N + \tfrac{1}{2}); \\ z_3 &= -(N + \tfrac{1}{2}) + i(N + \tfrac{1}{2}); & z_4 &= -(N + \tfrac{1}{2}) - i(N + \tfrac{1}{2}); \end{aligned}$$

where N is a large positive integer. It does not cross the real axis at integers. We may also note that $C(z)$ is bounded on γ_N. Write $C(z)$ as

$$C(z) = i\pi \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = i\pi \frac{e^{i2\pi z} + 1}{e^{i2\pi z} - 1}.$$

For example, the line from z_1 to z_2 may be parametrized by
$z = N + \frac{1}{2} + it,\ t\epsilon \left[-(N + \frac{1}{2}), N + \frac{1}{2}\right]$. On this line

$$|C(z)| = \pi \left| \frac{e^{i2\pi(N+\frac{1}{2}+it)} + 1}{e^{i2\pi(N+\frac{1}{2}+it)} - 1} \right| = \pi \left| \frac{1(-1)e^{-2\pi t} + 1}{1(-1)e^{-2\pi t} - 1} \right| = \pi \left| \frac{e^{\pi t} + e^{-\pi t}}{e^{\pi t} + e^{-\pi t}} \right| = \pi \left|\tanh(\pi t)\right| \leq \pi \leq 4.$$

On the line from z_2 to z_3, we have: $z = t + i(N + \frac{1}{2}),\ t\epsilon \left[(N + \frac{1}{2}), -(N + \frac{1}{2})\right]$. Here

$$|C(z)| = \pi \left| \frac{e^{i2\pi(t+i(N+\frac{1}{2})} + 1}{e^{i2\pi(t+i(N+\frac{1}{2})} - 1} \right| = \pi \frac{|e^{-2\pi(N+\frac{1}{2})} + 1|}{|e^{-2\pi(N+\frac{1}{2})} - 1|} \leq \pi \frac{e^{-3\pi} + 1}{|e^{-3\pi} - 1|} = \pi \frac{e^{3\pi} + 1}{e^{3\pi} - 1} \leq 4.$$

On the two other sides of the square, we find the same result, $C(z) \leq 4$.
Now, look at the function

$$g(z) = \frac{C(z)}{z(z - w)} = \frac{\pi \cos \pi z}{z(z - w) \sin \pi z},$$

where w is inside γ_N and not an integer. We see immediately that g has simple poles in $\pm 1, \pm 2, \pm 3 \ldots$, a simple pole in w, and a pole of order 2 in 0. $g(z)$ is analytic on γ_N. The residue theorem gives for the poles z_j to $g(z)$ which are inside γ_N

$$\frac{1}{2\pi i}\oint_{\gamma_N} g(z)dz = \sum_{\{z_j \text{ inside } \gamma_N\}} \text{res}(g, z_j). \tag{B.2}$$

First carry out the LHS

$$\left|\frac{1}{2\pi i}\oint_{\gamma_N} g(z)dz\right| \leq \frac{1}{2\pi}\{\text{lenght of curve}\}\cdot \max_{\{z\ \epsilon\ \gamma_N\}} |g(z)|$$

$$\leq \frac{1}{2\pi}4(2N+1)\cdot\frac{4}{\min|z(z-w)|} \leq \frac{8}{\pi}\frac{2N+1}{(N+\frac{1}{2})(N+\frac{1}{2}-|w|)} \to 0 \text{ when } N\to\infty.$$

We will first find the residues at $k = \pm 1, \pm 2, \pm 3\ldots$ We write

$$g(z) = \frac{\frac{\pi cos\pi z}{z(z-w)}}{sin\pi z} =: \frac{F(z)}{G(z)}.$$

Since $z = k$ are all simple zeros to $G(z)$, and $G'(k) \neq 0$ we have

$$\text{res}(g(z), k) = \frac{F(k)}{G'(k)} = \frac{\pi\cos\pi k}{k(k-w)\pi\cos\pi k} = \frac{1}{k(k-w)}.$$

In a similar way we get the residue of $g(z)$ in $z = w$ by choosing $G(z) = z - w$, to find

$$\text{res}(g(z), w) = \frac{\pi\cot\pi w}{w}.$$

The residue at the double pole $z = 0$ is probably most easily find by just reading off the z^{-1} coefficient of the Laurent expansion of $g(z)$

$$\begin{aligned}
g(z) &= \frac{\pi\cos\pi z}{z(z-w)\sin\pi z} \\
&= \frac{\pi(1-\frac{\pi^2 z^2}{2}+\ldots)}{z(-w)(1-z/w)(\pi z - \frac{\pi^3 z^3}{6}+\ldots)} \\
&= \frac{\pi(1-\frac{\pi^2 z^2}{2}+...)(1+\frac{z}{w}+\frac{z^2}{w^2}+...)}{(-w)\pi z^2(1-(\frac{\pi^2 z^2}{6}-...))} \\
&= \frac{(1-\frac{\pi^2 z^2}{2}+...)(1+\frac{z}{w}+...)(1+(\frac{\pi^2 z^2}{6}-...)+...)}{-wz^2}.
\end{aligned}$$

The z coefficient in the nominator is $1/w$, so the z^{-1} coefficient for $g(z)$ is simply $-\frac{1}{w^2}$. Then (B.2) gives when $N \to \infty$

$$0 = -\frac{1}{w^2} + \frac{\pi\cot\pi w}{w} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k(k-w)}.$$

The last term in this expression becomes

$$\begin{aligned}
\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k(k-w)} &= \sum_{l=-\infty}^{-1}\frac{1}{l(l-w)} + \sum_{k=1}^{\infty}\frac{1}{k(k-w)} \\
&= \sum_{k=1}^{\infty}\left[\frac{1}{(-k)((-k)-w)} + \frac{1}{k(k-w)}\right] \\
&= \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{k+w}+\frac{1}{k-w}\right) \\
&= \sum_{k=1}^{\infty}\frac{2}{k^2-w^2}.
\end{aligned}$$

After this simplification, we get

$$\pi \cot \pi w - \frac{1}{w} = \sum_{k=1}^{\infty} \frac{2}{w^2 - k^2}. \tag{B.3}$$

Now integrate this equation from $w = 0$ to $w = z$. Here we need to use a branch for the logarithmic function. Delete the non-positive real axis from the complex plane. Let us define $w = \text{Log}\, z$ as the number for which $e^w = z$ and where $\text{Log}\, z = \ln|z| + i \arg z,\ \arg z\, \epsilon(-\pi, \pi)$. Then, integrating the LHS and the RHS separately, we get

$$\begin{aligned}
\text{LHS} &= \int_0^z \left(\pi \cot \pi w - \frac{1}{w}\right) dw \\
&= [\text{Log}(\sin \pi w) - \text{Log} w]_0^z \\
&= \left[\text{Log}\frac{\sin \pi w}{w}\ (+in2\pi)\right]_0^z \\
&= \text{Log}\frac{\sin \pi z}{z} - \lim_{|w|\to 0} \text{Log}\frac{\sin \pi w}{\pi w}\pi \\
&= \text{Log}\left(\frac{\sin \pi z}{z}\right) - \ln \pi,
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} &= \int_0^z \sum_{k=1}^{\infty} \frac{2w}{w^2 - k^2} dw \\
&= \sum_{k=1}^{\infty} \int_0^z \frac{2w}{w^2 - k^2} dw \\
&= \sum_{k=1}^{\infty} \left[\text{Log}(w^2 - k^2)\right]_o^z \\
&= \sum_{k=1}^{\infty} \left(\text{Log}(z^2 - k^2) - \text{Log}(-k^2)\right) \\
&= \sum_{k=1}^{\infty} \left(\text{Log}\frac{k^2 - z^2}{k^2}\ + in2\pi\right).
\end{aligned}$$

Then, $e^{\text{LHS}} = e^{\text{RHS}}$ and we find

$$\frac{\sin \pi z}{z}\frac{1}{\pi} = e^{\sum_{k=1}^{\infty} \text{Log}\left(1 - \frac{z^2}{k^2}\right)}, \qquad \sin \pi z = \pi z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right). \tag{B.4}$$

Now we have found the infinite product representation of the sine function. From the definition it is easy to find the corresponding product for sinh,

$$\sinh \pi z = \frac{e^{\pi z} - e^{-\pi z}}{2} = i\frac{e^{-i(i\pi z)} - e^{i(i\pi z)}}{2i} = -i \sin i\pi z = (-i)\pi(iz) \prod_{k=1}^{\infty}\left(1 - \frac{(iz)^2}{k^2}\right)$$

and (B.1) is finally proven.

B.2 Computation of $\zeta(0)$

One way of doing the analytic continuation of Riemann's ζ-function is by using the Γ-function. We start by some preliminaries concerning this function.

B.2.1 The Γ-function

We may state three definitions of the Γ-function, that are all equal. First, we have the *Euler limit at infinity*

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{z(z+1)(z+2)\cdot\ldots(z+n)} n^z, \qquad z \neq 0, \pm 1, \pm 2, \ldots \tag{B.5}$$

Then, $\Gamma(z+1)$ becomes:

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{(z+1)(z+2)\cdot\ldots(z+1+n)} n^{z+1} = \lim_{n\to\infty} \frac{nz}{z+1+n}\Gamma(z) = z\Gamma(z).$$

We see that the Γ-function has the very important difference property

$$\Gamma(z+1) = z\Gamma(z). \tag{B.6}$$

Since

$$\Gamma(1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{1\cdot 2\cdot 3\cdot \ldots \cdot n(1+n)} n^1 = 1,$$

by using (B.6), we have $\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1$, and $\Gamma(3) = 2\Gamma(2) = 2$. Continuing in this way, we see that for all positive integers

$$\Gamma(n) = (n-1)! \tag{B.7}$$

Second, we have the *definite integral form*, also due to Euler.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \Re(z) > 0. \tag{B.8}$$

By just handling out a change of variable, $t = x^2$, we get

$$\Gamma(z) = \int_0^\infty e^{-x^2}(x^2)^{z-1} 2x\, dx, \qquad \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\, dx = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}. \tag{B.9}$$

Now we want to show that these two definitions, (B.5) and (B.8) are equivalent. Define a function $F(z,n)$ as

$$F(z,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt, \qquad \Re(z) > 0,\ n \in \mathbb{Z}^+.$$

The definition of e gives that $e^{-t} = \lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n$. Then we see that

$$\lim_{n\to\infty} F(z,n) \equiv F(z,\infty) = \int_0^\infty e^{-t} t^{z-1}\, dt = \Gamma(z),$$

the definite integral form of the Γ-function. Now carry out a change of variables, $u = \frac{t}{n}$.

$$F(z,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt = \int_0^1 (1-u)^n n^{z-1} u^{z-1} n\, du.$$

Finally, we do partial integration until we reach the first form (B.5) of the Γ function

$$\begin{aligned}
n^z \int_0^1 (1-u)^n u^{z-1}\, du &= n^z \left(\left[(1-u)^n \frac{u^z}{z} \right]_0^1 + n \int_o^1 (1-u)^{n-1} \frac{u^z}{z}\, du \right) \\
&= n^z \frac{n}{z} \int_0^1 (1-u)^{n-1} u^z\, du \\
&= n^z \frac{n}{z} \left(\left[(1-u)^{n-1} \frac{u^{z+1}}{z+1} \right]_0^1 + \frac{n-1}{z+1} \int_0^1 (1-u)^{n-2} u^{z+1}\, du \right) \\
&= n^z \frac{n(n-1)}{z(z+1)} \int_0^1 (1-u)^{n-2} u^{z+1}\, du \\
&= \ldots = \frac{n(n-1)\cdot\ldots\cdot 1}{z(z+1)\cdot\ldots\cdot(z+n-1)} n^z \int_0^1 u^{z+n-1}\, du \\
&= \frac{n(n-1)\cdot\ldots\cdot 1}{z(z+1)\cdot\ldots\cdot(z+n-1)} n^z \left[\frac{u^{z+n}}{z+n} \right]_0^1 \\
&= \frac{n(n-1)\cdot\ldots\cdot 1}{z(z+1)\cdot\ldots\cdot(z+n-1)(z+n)} n^z.
\end{aligned}$$

Take the limit $n \to \infty$ and we get the Γ-function as (B.5).

The third definition of the Γ-function is an infinite product, introduced by Weierstrass

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}. \tag{B.10}$$

Here, γ is the famous Euler constant

$$\gamma = \lim_{n\to\infty} \left(\sum_{m=1}^{n} \frac{1}{m} - \ln n \right) = 0.5772156619\ldots$$

Now we show that this definition is equivalent to (B.5), and therefore also to (B.8). We had

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{z(z+1)(z+2)\cdot \ldots (z+n)} n^z = \lim_{n\to\infty} \frac{1}{z} \prod_{m=1}^{n} \frac{m}{m+z} n^z = \lim_{n\to\infty} \frac{1}{z} \prod_{m=1}^{n} \frac{1}{1+\frac{z}{m}} n^z.$$

Just take the reciprocal of both sides of the equation

$$\frac{1}{\Gamma(z)} = z \lim_{n\to\infty} \prod_{m=1}^{n} \left(1+\frac{z}{m}\right) n^{-z} = z \lim_{n\to\infty} e^{(-\ln n)z} \prod_{m=1}^{n} \left(1+\frac{z}{m}\right).$$

Then note that

$$e^{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)z} = \prod_{m=1}^{n} e^{\frac{z}{m}}.$$

Use this in the equation for $\frac{1}{\Gamma(z)}$. By multiplying and dividing we don't change its value

$$\frac{1}{\Gamma(z)} = z \lim_{n\to\infty} e^{(-\ln n)z} \prod_{m=1}^{n} \left(1+\frac{z}{m}\right) \frac{e^{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)z}}{\prod_{m=1}^{n} e^{\frac{z}{m}}},$$

$$\frac{1}{\Gamma(z)} = z \lim_{n\to\infty} e^{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n\right)z} \lim_{n\to\infty} \prod_{m=1}^{n} \left(1+\frac{z}{m}\right) e^{-\frac{z}{m}} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}},$$

which is Weierstrass' definition of Γ. This gives

$$\Gamma(z) = \left(ze^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \right)^{-1}, \quad \Gamma(-z) = \left(-ze^{-\gamma z} \prod_{n=1}^{\infty} \left(1-\frac{z}{n}\right) e^{\frac{z}{n}} \right)^{-1}$$

$$\Gamma(z)\Gamma(-z) = \left(-z^2 \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right) \right)^{-1}.$$

But we know from (B.6) that $\Gamma(1-z) = -z\Gamma(-z)$.

We also know from the preceding section in the appendix that $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right)$. This gives

$$\Gamma(z)\frac{\Gamma(1-z)}{-z} = -\frac{1}{z^2} \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right)^{-1} = -\frac{1}{z^2} \frac{\pi z}{\sin \pi z}$$

and we have finally established the important formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{B.11}$$

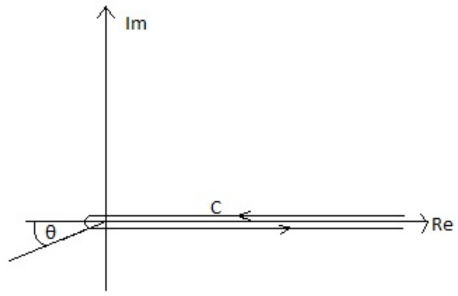


Figure B.1: The curve C

B.2.2 Analytic continuation of the Γ-function

Let C be the curve which goes arbitrarily close to and above the positive real axis from $+\infty$ to 0, then circles around the origin and then goes back to $+\infty$ arbitrarily close to and below the positive real axis. See figure B.1. Then consider the integral

$$\int_C \frac{d\zeta}{\zeta}(-\zeta)^s e^{-\zeta}. \tag{B.12}$$

The function $f(\zeta) = \frac{1}{\zeta}(-\zeta)^s e^{-\zeta}$ is analytic along C. The exponential factor $e^{-\zeta}$ makes the function disappear at infinity.

Make a change of variables so that $-\zeta = \rho e^{i\theta}$, where θ is measured counterclockwise from the negative real axis. Then integrate the two parts of C (above and below the positive real axis). We begin with the integration below the axis, where $\theta = \pi$

$$\int_0^\infty \frac{d\zeta}{\zeta}(-\zeta)^s e^{-\zeta} = \int_0^\infty \frac{-d\rho\ e^{i\pi}}{-\rho e^{i\pi}}\rho^s e^{i\pi s} e^{\rho e^{i\pi}} = e^{i\pi s}\int_0^\infty \frac{d\rho}{\rho}\rho^s e^{-\rho}.$$

Above the axis, $\theta = -\pi$.

$$\int_\infty^0 \frac{d\zeta}{\zeta}(-\zeta)^s e^{-\zeta} = \int_\infty^0 \frac{-d\rho\ e^{-i\pi}}{-\rho e^{-i\pi}}\rho^s e^{-i\pi s} e^{\rho e^{-i\pi}} = -e^{-i\pi s}\int_0^\infty \frac{d\rho}{\rho}\rho^s e^{-\rho}.$$

In total we get

$$\int_C \frac{d\zeta}{\zeta}(-\zeta)^s e^{-\zeta} = \left(e^{i\pi s} - e^{-i\pi s}\right)\int_0^\infty \frac{d\rho}{\rho}\rho^s e^{-\rho} = 2i\sin\pi s\ \Gamma(s).$$

We have found an analytic continuation of the Γ-function

$$\Gamma(s) = \frac{1}{2i\sin\pi s}\int_C \frac{d\zeta}{\zeta}(-\zeta)^s e^{-\zeta}, \tag{B.13}$$

or even more elegant, using (B.11)

$$\frac{1}{\Gamma(1-s)} = \frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta}(-\zeta)^s e^{-\zeta}. \tag{B.14}$$

B.2.3 Analytic continuation of the ζ-function

Normally, we define Riemann's ζ-function as

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \qquad s > 1.$$

We instead introduce a more general function of two parameters: $\zeta(s,a)$

$$\zeta(s,a) = \sum_{n=0}^\infty \frac{1}{(n+a)^s}. \tag{B.15}$$

We see that $a = 1$ leads us back to the original definition. By performing a simple change of variables, we may be able to express this generalised ζ-function in terms of the Γ-function

$$\Gamma(s) = \int_0^\infty \frac{d\rho}{\rho}\rho^s e^{-\rho} = \quad \{\rho = x(n+a)\} \quad = (n+a)^s \int_0^\infty \frac{dx}{x} x^s e^{-(n+a)x}.$$

Divide and multiply (B.15) with the expression above, so that

$$\zeta(s,a) = \sum_{n=0}^\infty \frac{1}{(n+a)^s} = \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \int_0^\infty \frac{dx}{x} x^s e^{-(n+a)x} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dx}{x} x^s e^{-ax} \left(\sum_{n=0}^\infty (e^{-x})^n \right),$$

and we simply get

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dx}{x} x^s \frac{e^{-ax}}{1-e^{-x}}. \tag{B.16}$$

Now consider the curve integral

$$\int_C \frac{d\xi}{\xi} (-\xi)^s \frac{e^{-a\xi}}{1-e^{-\xi}}, \tag{B.17}$$

where C is the same curve (Figure B.1) as before. We see that the integrand is analytic on C. Repeat the calculation procedure we did before, but with this new integral (B.17). Perform the same change of variables, $-\xi = \rho e^{i\theta}$ and divide the integration into two parts. Integration below the real axis gives

$$\int_0^\infty \frac{d\xi}{\xi} (-\xi)^s \frac{e^{-a\xi}}{1-e^{-\xi}} = e^{i\pi s} \int_0^\infty \frac{d\rho}{\rho} \rho^s \frac{e^{-a\rho}}{1-e^{-\rho}}.$$

Above the real axis we get

$$\int_\infty^0 \frac{d\xi}{\xi} (-\xi)^s \frac{e^{-a\xi}}{1-e^{-\xi}} = -e^{-i\pi s} \int_0^\infty \frac{d\rho}{\rho} \rho^s \frac{e^{-a\rho}}{1-e^{-\rho}}.$$

In total the curve integral (B.17) becomes

$$\int_C \frac{d\xi}{\xi} (-\xi)^s \frac{e^{-a\xi}}{1-e^{-\xi}} = \left(e^{i\pi s} - e^{-i\pi s}\right) \int_0^\infty \frac{d\rho}{\rho} \rho^s \frac{e^{-a\rho}}{1-e^{-\rho}} = 2i \sin \pi s \ \Gamma(s)\zeta(s,a),$$

where in the last step we have used (B.16), with the real variable $\rho = x$. Again using (B.11), we can write the ζ-function as

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \frac{1}{2i \sin \pi s} \int_C \frac{d\xi}{\xi} (-\xi)^s \frac{e^{-a\xi}}{1-e^{-\xi}} = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{d\xi}{\xi} (-\xi)^s \frac{e^{-a\xi}}{1-e^{-\xi}}. \tag{B.18}$$

This is our analytic continuation of Riemann-s ζ-function. We want to calculate $\zeta(0)$. To do this, we may turn the curve C to a closed curve D, by connecting the two ends at infinity. We can do this since the integrand goes to zero when $\Re(\xi) \to \infty, \ a > 0$

$$\zeta(s,a) = \frac{\Gamma(1-s)}{2\pi i} \oint_D \frac{d\xi}{\xi} (-\xi)^s \frac{e^{-a\xi}}{1-e^{-\xi}}. \tag{B.19}$$

The only residue of $g(\xi) = \frac{(-\xi)^s}{\xi} \frac{e^{-a\xi}}{1-e^{-\xi}}$ inside D will be at 0. Then the theorem of residues gives for the integral

$$\oint_D \frac{d\xi}{\xi} (-\xi)^s \frac{e^{-a\xi}}{1-e^{-\xi}} = 2\pi i \ \mathrm{res}(g,0). \tag{B.20}$$

Expand $g(\xi)$ about $\xi = 0$

$$g(\xi) = \frac{(-\xi)^s}{\xi} \frac{e^{-a\xi}}{1-e^{-\xi}} = \frac{(-\xi)^s}{\xi} \frac{1 - a\xi + \frac{a^2\xi^2}{2} - \dots}{\xi - \frac{\xi^2}{2} + \frac{\xi^3}{6} + \dots} = \{s=0\} = \frac{1}{\xi^2} \frac{1 - a\xi + \frac{a^2\xi^2}{2} - \dots}{1 - \left(\frac{\xi}{2} - \frac{\xi^2}{6} + \dots\right)}$$

$$= \frac{1}{\xi^2}\left(1 - a\xi + \frac{a^2\xi^2}{2} - \dots\right)\left(1 + \frac{\xi}{2} - \dots\right) = \frac{1 + \left(-a + \frac{1}{2}\right)\xi + \dots}{\xi^2}.$$

We easily read off the ξ^{-1} coefficient to be $-a + \frac{1}{2} = -\frac{1}{2}$, since $a = 1$ in our case. Then we use (B.19) and (B.20)

$$\zeta(0) = \zeta(0,1) = \frac{\Gamma(1-0)}{2\pi i} 2\pi i \ \mathrm{res}(g,0) = 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}. \tag{B.21}$$

B.3 Computation of $\zeta'(0)$

We have in the preceding section found $\zeta(0)$. To be able to calculate $\zeta'(0)$, we have to use a heavier mathematical artillery. We begin with some preliminary calculations which at first sight do not seem to have anything to do with Riemann's ζ function, but later will become quite useful. Throughout the text, $\log z$ will be the natural logarithm with a suitable branch for a complex number, whereas $\ln z$ will be used when z is a real number. Our main reference is the excellent Modern Analysis [16], by Whittaker and Watson.

B.3.1 Preparatory Calculations

An integral for $\log z$

We start by proving the following expression for the logarithm

$$\log z = \int_0^\infty \frac{e^{-t} - e^{-tz}}{t} dt, \tag{B.22}$$

where $\Re z > 0$. The RHS becomes

$$\text{RHS} = \int_0^\infty \frac{e^{-t} - e^{-tz}}{t} dt = \lim_{\delta \to 0, \rho \to \infty} \left\{ \int_\delta^\rho \frac{e^{-t}}{t} dt - \int_\delta^\rho \frac{e^{-tz}}{t} dt \right\}.$$

Performing a change of variables in the second integral, $u = tz$, with $\Re z > 0$, gives

$$\text{RHS} = \lim_{\delta \to 0, \rho \to \infty} \left\{ \int_\delta^\rho \frac{e^{-t}}{t} dt - \int_{\delta z}^{\rho z} \frac{e^{-u}}{u} du \right\}.$$

δ and ρ are two points on the positive real axis. Together with δz and ρz, these points are corners of a quadrilateral γ, inside which the function e^{-t}/t will be analytic. Then, by Cauchy's theorem, the closed curve integral along γ will be zero,

$$0 = \oint_\gamma \frac{e^{-t}}{t} dt = \int_\delta^\rho \frac{e^{-t}}{t} dt + \int_\rho^{\rho z} \frac{e^{-t}}{t} dt + \int_{\rho z}^{\delta z} \frac{e^{-t}}{t} dt + \int_{\delta z}^{\delta} \frac{e^{-t}}{t} dt,$$

and therefore

$$\int_\delta^\rho \frac{e^{-t}}{t} dt - \int_{\delta z}^{\rho z} \frac{e^{-t}}{t} dt = \int_\delta^{\delta z} \frac{e^{-t}}{t} dt - \int_\rho^{\rho z} \frac{e^{-t}}{t} dt.$$

We use this result in the calculation above, so that

$$\text{RHS} = \lim_{\delta \to 0, \rho \to \infty} \left\{ \int_\delta^{\delta z} \frac{e^{-t}}{t} dt - \int_\rho^{\rho z} \frac{e^{-t}}{t} dt \right\}.$$

As $\Re(z) > 0$, the last integral $\to 0$ when $\rho \to \infty$. The first integral becomes

$$\begin{aligned}
\int_\delta^{\delta z} \frac{e^{-t}}{t} dt &= \int_\delta^{\delta z} \frac{1 + e^{-t} - 1}{t} dt \\
&= \int_\delta^{\delta z} \frac{1}{t} dt + \int_\delta^{\delta z} \frac{e^{-t} - 1}{t} dt \\
&= \log \delta z - \log \delta - \int_\delta^{\delta z} \frac{1 - e^{-t}}{t} dt \\
&= \log z - \int_\delta^{\delta z} \frac{e^t - 1}{t} e^{-t} dt.
\end{aligned}$$

Since $\frac{e^t - 1}{t} \to 1$ when $t \to 0$, the integral vanishes when $\delta \to 0$. Hence we arrive at the result (B.22).

The harmonic series and Euler's constant

Next, we will prove that the first n terms of the harmonic series can be written as

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}=\int_0^1 \frac{1-(1-t)^n}{t}dt. \tag{B.23}$$

The result follows directly if we note that

$$\sum_{k=0}^{n-1}(1-t)^k=\frac{1-(1-t)^n}{1-(1-t)}=\frac{1-(1-t)^n}{t},$$

which is true when $|1-t|<0$, as is the case in the integral. Then,

$$\int_0^1 \frac{1-(1-t)^n}{t}dt=\int_0^1\sum_{k=0}^{n-1}(1-t)^k dt=\sum_{k=0}^{n-1}\left[-\frac{(1-t)^{k+1}}{k+1}\right]_0^1=\sum_{k=0}^{n-1}\frac{1}{k+1}=\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}.$$

In fact, Euler's constant γ can be written as

$$\gamma=\lim_{n\to\infty}\left\{\int_0^1\frac{1-\left(1-\frac{t}{n}\right)^n}{t}dt-\int_1^n\frac{\left(1-\frac{t}{n}\right)^n}{t}dt\right\}. \tag{B.24}$$

To see this, we write the integrals within {} brackets as

$$\begin{aligned}\int_0^1\frac{1-\left(1-\frac{t}{n}\right)^n}{t}dt+\int_1^n\frac{1-\left(1-\frac{t}{n}\right)^n}{t}dt-\int_1^n\frac{1}{t}dt &= \int_0^n\frac{1-\left(1-\frac{t}{n}\right)^n}{t}dt-\ln n\\ &=\left\{u=\frac{t}{n}\right\}\\ &=\int_0^1\frac{1-(1-u)^n}{u}du-\ln n.\end{aligned}$$

Using (B.23), the result (B.24) follows directly.

Euler's constant in terms of e

We will now rewrite (B.24) in another way. To do this, we need to recall the definition of the exponential e,

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Taking both sides to the power of t, and performing a change of variables $-nt\to t$ gives

$$e^{-t}=\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n.$$

The careful reader might get nervous about this last operation on such a crucial definition, and also wonder how fast the convergence is. We will show that it is in fact always true for $n=1,2,\ldots$ that

$$0\le e^{-t}-\left(1-\frac{t}{n}\right)^n\le\frac{t^2e^{-t}}{n}. \tag{B.25}$$

From the series of e^y and $1/(1-y)$,

$$e^y=1+y+\frac{y^2}{2}+\ldots;\qquad \frac{1}{1-y}=1+y+y^2+\ldots,$$

it follows that $1+y\le e^y\le 1/(1-y)$, and letting $y=t/n$, we see that

$$1+\frac{t}{n}\le e^{\frac{t}{n}}\le\left(1-\frac{t}{n}\right)^{-1};\qquad \left(1+\frac{t}{n}\right)^{-n}\ge e^{-t}\ge\left(1-\frac{t}{n}\right)^n,$$

and we notice immediately that $e^{-t} - \left(1-\frac{t}{n}\right)^n \geq 0$. Further, we have that

$$\begin{aligned}
e^{-t} - \left(1-\frac{t}{n}\right)^n &= e^{-t}\left(1 - e^t\left(1-\frac{t}{n}\right)^n\right) \\
&= e^{-t}\left(1 - \left(e^{\frac{t}{n}}\left(1-\frac{t}{n}\right)\right)^n\right) \\
&= e^{-t}\left(1 - \left(1+\frac{t}{n}+\frac{t^2}{2n^2}+\dots\right)^n\left(1-\frac{t}{n}\right)^n\right) \\
&\leq e^{-t}\left(1 - \left[\left(1+\frac{t}{n}\right)\left(1-\frac{t}{n}\right)\right]^n\right) \\
&= e^{-t}\left(1 - \left(1-\frac{t^2}{n^2}\right)^n\right).
\end{aligned}$$

If $0 \leq a \leq 1$, we have the inequality $(1-a)^n \geq 1-na$. It is obviously true when $n = 1$. Now, suppose it is true for $n = p \geq 1$. Then $(1-a)^p \geq 1-pa$. Is it then true for $n = p+1$? The RHS becomes $1-(p+1)a = 1-pa-a$. The LHS becomes $(1-a)^{p+1} = (1-a)(1-a)^p \geq (1-a)(1-pa) = 1-a-pa+a^2p \geq 1-pa-a =$RHS. From the principle of induction, it is true that $(1-a)^n \geq 1-na$ for all $n \geq 1$. Setting $a = t^2/n^2$, we get

$$\left(1-\frac{t^2}{n^2}\right)^n \geq 1-\frac{t^2}{n}; \qquad 1-\left(1-\frac{t^2}{n^2}\right)^n \leq \frac{t^2}{n},$$

and (B.25) follows. Then, we may instead write (B.24) as

$$\gamma = \int_0^1 \frac{1-e^{-t}}{t}dt - \int_1^\infty \frac{e^{-t}}{t}dt. \tag{B.26}$$

Another way of writing this is

$$\gamma = \lim_{\delta\to 0}\left(\int_\delta^1 \frac{dt}{t} - \int_\delta^\infty \frac{e^{-t}}{t}dt\right).$$

Now perform a change of variables in the first integral, $\Delta = 1-e^{-\delta}$. We notice that

$$\int_\Delta^\delta \frac{dt}{t} = \log\frac{\delta}{\Delta} = \ln\frac{\delta}{1-e^{-\delta}} = \ln\left(e^\delta \frac{\delta}{e^\delta - 1}\right) \to 0,$$

when $\delta \to 0$. We may therefore include this tiny part in the expression for γ, so that

$$\gamma = \lim_{\delta\to 0}\left(\int_\Delta^1 \frac{dt}{t} - \int_\delta^\infty \frac{e^{-t}}{t}dt\right).$$

Perform another change of variables in the first integral, $t = 1-e^{-u},\ u = -\ln(1-t)$,

$$\gamma = \lim_{\delta\to 0}\int_\delta^\infty \frac{e^{-t}}{1-e^{-t}}dt - \int_\delta^\infty \frac{e^{-t}}{t}dt = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-t}dt. \tag{B.27}$$

Derivatives of $\log\Gamma(z)$

We conclude this section with a short calculation, starting from Weierstrass expression (B.10) for the Γ function, and the logarithm of it,

$$\Gamma(z+1) = e^{-\gamma z}\prod_{n=1}^\infty \frac{e^{\frac{z}{n}}}{1+\frac{z}{n}}; \qquad \log\Gamma(z+1) = -\gamma z + \sum_{n=1}^\infty\left(\frac{z}{n} - \log\left(1+\frac{z}{n}\right)\right).$$

Now differentiate $\log\Gamma(z+1)$,

$$\frac{d}{dz}\log\Gamma(z+1) = -\gamma + \sum_{n=1}^\infty\left(\frac{1}{n} - \frac{1}{1+\frac{z}{n}}\frac{1}{n}\right) = -\gamma + \sum_{n=1}^\infty \frac{z}{n(z+n)}. \tag{B.28}$$

Using the property (B.6), we have that

$$\log\Gamma(z+1) = \log z + \log\Gamma(z); \qquad \frac{d}{dz}\log\Gamma(z+1) = \frac{1}{z} + \frac{d}{dz}\log\Gamma(z),$$

and therefore

$$\frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\log\Gamma(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(z+n)}. \tag{B.29}$$

Differentiating (B.29) gives

$$\begin{aligned}
\frac{d^2}{dz^2}\log\Gamma(z) &= \frac{1}{z^2} + \frac{d}{dz}\sum_{n=1}^{\infty}\frac{z}{n(z+n)} \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{d}{dz}\frac{z}{n(z+n)} \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{1}{n}\frac{1(z+n) - z\cdot 1}{(z+n)^2} \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{1}{(z+n)^2} \\
&= \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}.
\end{aligned} \tag{B.30}$$

B.3.2 Gauss' Expression for $\frac{\Gamma'(z)}{\Gamma(z)}$

We will in this section try to find a formula for $\frac{\Gamma'(z)}{\Gamma(z)}$, applicable whenever $\Re z > 0$. Remember the expression (B.29) for the logarithmic derivative of $\Gamma(z)$,

$$\frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\log\Gamma(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(z+n)} = -\gamma - \frac{1}{z} + \lim_{n\to\infty}\sum_{m=1}^{n}\left(\frac{1}{m} - \frac{1}{z+m}\right).$$

Note that $\frac{1}{z+m} = \int_0^\infty e^{-t(z+m)}dt$, $m = 0, 1, 2, \ldots,$ and $\Re z > 0$. Hence we have

$$\begin{aligned}
\frac{\Gamma'(z)}{\Gamma(z)} &= -\gamma - \int_0^\infty e^{-tz}dt + \lim_{n\to\infty}\int_o^\infty \sum_{m=1}^{n}\left(e^{-mt} - e^{-(m+z)t}\right)dt \\
&= -\gamma + \lim_{n\to\infty}\int_0^\infty -e^{-tz} + e^{-t}\frac{1-e^{-tn}}{1-e^{-t}}(1-e^{-zt})dt,
\end{aligned}$$

where we have used the formula for the geometric sum. We continue by writing the integrand as a ratio

$$\begin{aligned}
\frac{\Gamma'(z)}{\Gamma(z)} &= -\gamma + \lim_{n\to\infty}\int_0^\infty \frac{-e^{-tz}(1-e^{-t}) + (e^{-t} - e^{-(n+1)t})(1-e^{-zt})}{1-e^{-t}}dt \\
&= -\gamma + \lim_{n\to\infty}\int_0^\infty \frac{e^{-t} - e^{-zt} - e^{-(n+1)t} + e^{-(n+z+1)t}}{1-e^{-t}}dt.
\end{aligned}$$

Rewrite γ as an integral with (B.27), to get

$$\begin{aligned}
\frac{\Gamma'(z)}{\Gamma(z)} &= \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-t}}{1-e^{-t}}dt + \lim_{n\to\infty}\int_0^\infty \frac{e^{-t} - e^{-zt} - e^{-(n+1)t} + e^{-(n+z+1)t}}{1-e^{-t}}dt \\
&= \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}dt - \lim_{n\to\infty}\int_0^\infty \frac{1-e^{-zt}}{1-e^{-t}}e^{-(n+1)t}dt.
\end{aligned}$$

The second integral is in fact zero. First look at the case $0 < t \leq 1$. The factor $\left|\frac{1-e^{-zt}}{1-e^{-t}}\right|$ has a finite limit when $t \to 0$, namely $|z|$, and is therefore bounded, so when $n \to \infty$, the part of the integral from $t = 0$ to $t = 1$ goes to zero. Next, look at the case when $t \geq 1$. Then, since $\Re z > 0$,

$$\left|\frac{1 - e^{-zt}}{1 - e^{-t}}\right| < \frac{1 + |e^{-zt}|}{1 - e^{-1}} < \frac{2}{1 - e^{-1}},$$

and when $n \to \infty$ we get the integral from 1 to ∞ also equal to zero. Then we have the interesting result named after Gauss,

$$\frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}}\, dt. \tag{B.31}$$

B.3.3 Binet's First Expression for $\log \Gamma(z)$

By letting $z \to z + 1$ in (B.31), we get

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-(z+1)t}}{1 - e^{-t}} dt = \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-zt}}{e^t - 1} dt.$$

We remember from (B.22) that

$$\log z = \int_0^\infty \frac{e^{-t} - e^{-tz}}{t} dt; \qquad \int_0^\infty \frac{e^{-t}}{t} dt = \log z + \int_0^\infty \frac{e^{-tz}}{t} dt.$$

Use this, together with adding and subtracting the identity $\int_0^\infty \frac{1}{2} e^{-tz} dt = \frac{1}{2z}$ in the expression for $\frac{\Gamma'(z+1)}{\Gamma(z+1)}$, we find

$$\begin{aligned}
\frac{d}{dz} \log \Gamma(z+1) &= \log z + \int_0^\infty \frac{e^{-tz}}{t} - \frac{e^{-tz}}{e^t - 1} dt + \left(\frac{1}{2z} - \int_0^\infty \frac{1}{2} e^{-tz} dt\right) \\
&= \frac{1}{2z} + \log z - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) e^{-tz} dt.
\end{aligned}$$

The integrand is continuous when $t \to 0$, and since $\Re z > 0$ it really converges uniformly. We are therefore allowed to integrate with respect to z under the integral sign

$$\int_1^z \frac{d}{dz'} \left(\log \Gamma(z'+1)\right) dz' = \int_1^z \left(\frac{1}{2z'} + \log z' - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) e^{-tz'} dt\right) dz',$$

and

$$\begin{aligned}
\log \Gamma(z+1) - \log \Gamma(2) &= \frac{1}{2} \log z + [z' \log z' - z']_1^z - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \left[-\frac{e^{-tz'}}{t}\right]_1^z dt \\
&= \frac{1}{2} \log z + z \log z - z + 1 + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tz} - e^{-t}}{t} dt.
\end{aligned}$$

Now, from (B.6), $\log \Gamma(z+1) = \log\left(z\Gamma(z)\right) = \log z + \log \Gamma(z)$, and

$$\begin{aligned}
\log \Gamma(z) = {}& \left(z - \frac{1}{2}\right) \log z - z + 1 \\
&+ \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tz}}{t} dt - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-t}}{t} dt.
\end{aligned} \tag{B.32}$$

We put our attention on the last integral, which we call I. Also define an integral J,

$$I = \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-t}}{t} dt; \qquad J = \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-\frac{t}{2}}}{t} dt.$$

Using these definitions, we see that

$$\log \Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2} - \frac{1}{2}\right) \ln \frac{1}{2} - \frac{1}{2} + 1 + J - I = \frac{1}{2} + J - I. \tag{B.33}$$

Let us make a change of variables $t \to \frac{t}{2}$ in the expression for I. We get

$$I = \int_0^\infty \left(\frac{1}{2} - \frac{2}{t} + \frac{1}{e^{t/2} - 1}\right) \frac{e^{-t/2}}{t} dt.$$

The difference between J and I becomes

$$\begin{aligned}
J - I &= \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-\frac{t}{2}}}{t} dt - \int_0^\infty \left(\frac{1}{2} - \frac{2}{t} + \frac{1}{e^{t/2} - 1}\right) \frac{e^{-t/2}}{t} dt \\
&= \int_0^\infty \left(\frac{1}{t} + \left(\frac{1}{e^t - 1} - \frac{1}{e^{t/2} - 1}\right)\right) \frac{e^{-\frac{t}{2}}}{t} dt \\
&= \int_0^\infty \left(\frac{1}{t} + \left(\frac{1}{e^t - 1} - \frac{e^{t/2} + 1}{(e^{t/2} - 1)(e^{t/2} + 1)}\right)\right) \frac{e^{-\frac{t}{2}}}{t} dt \\
&= \int_0^\infty \left(\frac{1}{t} - \frac{e^{t/2}}{e^t - 1}\right) \frac{e^{-t/2}}{t} dt \\
&= \int_0^\infty \left(\frac{e^{-t/2}}{t} - \frac{1}{e^t - 1}\right) \frac{dt}{t}.
\end{aligned}$$

Then $J = (J - I) + I$, so

$$J = \int_0^\infty \left(\frac{e^{-t/2}}{t} - \frac{1}{e^t - 1} + \frac{1}{2} e^{-t} - \frac{e^{-t}}{t} + \frac{e^{-t}}{e^t - 1}\right) \frac{dt}{t}.$$

Since $\frac{1}{2} e^{-t} - \frac{1}{e^t - 1} + \frac{e^{-t}}{e^t - 1} = \frac{1}{2} e^{-t} + e^{-t} \frac{1 - e^t}{e^t - 1} = -\frac{1}{2} e^{-t}$, we get

$$\begin{aligned}
J &= \int_0^\infty \left(e^{-t/2} - e^{-t}\right) \frac{dt}{t^2} - \int_0^\infty \frac{1}{2} \frac{e^{-t}}{t} dt \\
&= \left[-\frac{e^{-t/2} - e^{-t}}{t}\right]_0^\infty + \int_0^\infty \frac{-\frac{1}{2} e^{-t/2} + e^{-t}}{t} dt - \int_0^\infty \frac{1}{2} \frac{e^{-t}}{t} dt \\
&= \lim_{t \to 0} \frac{e^{-t/2} - e^{-t}}{t} + \frac{1}{2} \int_0^\infty \frac{e^{-t} - e^{-t/2}}{t} dt.
\end{aligned}$$

The integral is just $\ln \frac{1}{2}$ by (B.22). The limit is

$$\lim_{t \to 0} \frac{e^{-t/2} - e^{-t}}{t} = \lim_{t \to 0} e^{-t} \frac{e^{t/2} - 1}{2 \cdot \frac{t}{2}} = 1 \cdot \frac{1}{2} = \frac{1}{2},$$

so that $J = \frac{1}{2} + \frac{1}{2} \ln \frac{1}{2}$. From (B.33), using the result $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ in (B.9), we have

$$I = \frac{1}{2} + J - \log \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \ln \frac{1}{2} - \ln \sqrt{\pi} = 1 - \frac{1}{2} \ln 2\pi.$$

Hence, (B.32) becomes

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \ln 2\pi + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tz}}{t} dt,$$

which is called Binet's first expression for $\log \Gamma(z)$. The factor $\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{1}{t}$ is continuous as $t \to 0$, which may be shown in the study of Bernoulli polynomials. For large t it is clearly bounded, and we may set $\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{1}{t} < K$, where K is a constant independent of t. From this, with $z = x + iy$, we have

$$\left|\log \Gamma(z) - \left(z - \frac{1}{2}\right) \log z + z - \frac{1}{2} \ln 2\pi\right| < K \int_0^\infty e^{-tx} dt = \frac{K}{x} \to 0, \ x \to \infty.$$

Therefore, we have the approximate formula for $\log\Gamma(z)$ when $\Re z \gg 0$,

$$\log\Gamma(z) \approx \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\ln 2\pi. \tag{B.34}$$

B.3.4 Plana's Formula

Suppose that we have an analytic function $\phi(z)$, which is bounded whenever $x_1 \le \Re z \le x_2$, where x_1 and x_2 are two integers. Then we will show that

$$\frac{1}{2}\phi(x_1)+\phi(x_1+1)+\phi(x_1+2)+\ldots+\phi(x_2-1)+\frac{1}{2}\phi(x_2)$$

$$=\int_{x_1}^{x_2}\phi(z)dz+\frac{1}{i}\int_0^{\infty}\frac{\phi(x_2+iy)-\phi(x_1+iy)-\phi(x_2-iy)+\phi(x_1-iy)}{e^{2\pi y}-1}dy, \tag{B.35}$$

a relation that is called Plana's formula, and which we will use twice in the sections to come. We prove it by adding the equations for two curve integrals. The first curve integral is $\oint_{C_1}\frac{\phi(z)dz}{e^{-i2\pi z}-1}$, where C_1 is the rectangle with corners (given in order) $x_1, x_2, x_2+i\infty$ and $x_1+i\infty$. The second is $\oint_{C_2}\frac{\phi(z)dz}{e^{i2\pi z}-1}$, where C_2 is the rectangle with corners (given in order) $x_1, x_2, x_2-i\infty$ and $x_1-i\infty$. See figure B.2.

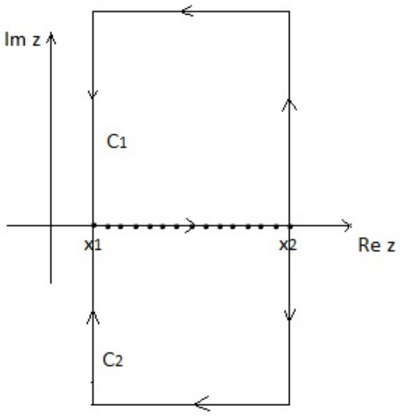


Figure B.2: The closed contours C_1 and C_2.

Let $z=x+iy$. The first integral becomes

$$\oint_{C_1}\frac{\phi(z)dz}{e^{-i2\pi z}-1}$$

$$=\int_{x_1}^{x_2}\frac{\phi(x)dx}{e^{-i2\pi x}-1}+i\int_0^{\infty}\frac{\phi(x_2+iy)dy}{e^{-i2\pi(x_2+iy)}-1}+\lim_{r\to\infty}\int_{x_2+ir}^{x_1+ir}\frac{\phi(z)dz}{e^{-i2\pi z}-1}+i\int_{\infty}^{0}\frac{\phi(x_1+iy)dy}{e^{-i2\pi(x_1+iy)}-1}.$$

Using the fact that $\phi(z)$ is bounded whenever $x_1 \le \Re z \le x_2$, the integral involving r vanishes, and also using the fact that x_1 and x_2 are integers, we get

$$\oint_{C_1}\frac{\phi(z)dz}{e^{-i2\pi z}-1}=\int_{x_1}^{x_2}\frac{\phi(x)dx}{e^{-i2\pi x}-1}+i\int_0^{\infty}\frac{\phi(x_2+iy)-\phi(x_1+iy)}{e^{2\pi y}-1}dy. \tag{B.36}$$

The second integral becomes

$$\oint_{C_2}\frac{\phi(z)dz}{e^{i2\pi z}-1}$$

$$=\int_{x_1}^{x_2}\frac{\phi(x)dx}{e^{i2\pi x}-1}+i\int_0^{-\infty}\frac{\phi(x_2+iy)dy}{e^{i2\pi(x_2+iy)}-1}+\lim_{r\to\infty}\int_{x_2-ir}^{x_1-ir}\frac{\phi(z)dz}{e^{i2\pi z}-1}+i\int_{-\infty}^{0}\frac{\phi(x_1+iy)dy}{e^{i2\pi(x_1+iy)}-1}.$$

Here, the third integral on the RHS also vanishes as $r \to \infty$. In the second and the fourth integrals, we perform a change of variable $y \to -y$, to get

$$\oint_{C_2} \frac{\phi(z)dz}{e^{i2\pi z} - 1} = \int_{x_1}^{x_2} \frac{\phi(x)dx}{e^{i2\pi x} - 1} + i\int_0^\infty \frac{\phi(x_1 - iy) - \phi(x_2 - iy)}{e^{2\pi y} - 1} dy. \tag{B.37}$$

We want to use the theorem of residues for the integrals over C_1 and C_2. The poles of $\frac{\phi(z)}{e^{\mp i2\pi z}-1}$ are simple (remember that $\phi(z)$ is analytic), they are just the integers, lying on the real axis. See figure B.2. The curves C_1 and C_2 crosses the integer poles from x_1 to x_2. How do we treat poles which are not inside the contour, but instead on the boarder? Look at a specific example with the curve C in figure B.3.

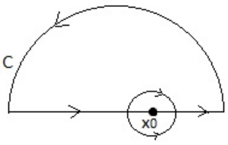


Figure B.3: Example of a contour C that crosses a pole x_0. One may include or exclude the pole by taking different paths around it.

We may avoid the pole in x_0 by taking a trip around it along a semicircle, clockwise or counterclockwise. The relevant term of the Laurent expansion around x_0 is $\frac{a_{-1}}{z-x_0}$. The integration of this term along the semicircles becomes

$$\int \frac{a_{-1}}{z - x_0} dz = \left\{ \begin{array}{l} z - x_0 = \delta e^{i\theta}, \\ dz = i\delta e^{i\theta} d\theta \end{array} \right\} = \left\{ \begin{array}{ll} a_{-1}\int_\pi^0 i\; d\theta = -a_{-1}i\pi, & \text{if clockwise} \\ a_{-1}\int_{-\pi}^0 i\; d\theta = a_{-1}i\pi, & \text{if counterclockwise.} \end{array} \right\}$$

If taking the semicircle clockwise, the pole is excluded. In the residue theorem,

$$\oint_C f(z)dz = 2\pi i \sum (\text{residues inside C}), \tag{B.38}$$

a factor $-a_{-1}i\pi$ will appear on the LHS, due to x_0. It becomes $+a_{-1}i\pi$ on the RHS. If we instead take the semicircle counterclockwise, the pole is included. A term $+a_{-1}i\pi$ will appear on the LHS of (B.38) due to x_0, and a term $a_{-1}i2\pi$ will appear on the RHS, since x_0 is included. The net result is a term $a_{-1}i\pi$ on the RHS of (B.38), which is the same result as obtained in the clockwise semicircle case. Hence, it is very natural to interpret "a crossed residue" as "half a residue". In the same way, if there is a right-angled corner of the contour at a residue, it will contribute with a fourth of the value that an included residue would give.

Having discussed crossed poles for a while, we now return to the proof of Plana's formula. The residue for $f(z) = \frac{\phi(z)}{e^{\mp i2\pi z}-1}$ at an (included) pole $z_0 \in \mathbb{Z}$, such that $x_1 \leq z_0 \leq x_2$ is easy to find. Let $f(z) = \frac{\phi(z)}{e^{\mp i2\pi z}-1} \equiv \frac{F(z)}{G(z)}$. We see that $G'(z) \neq 0$, so

$$\text{res} f(z)|_{z=z_0} = \frac{F(z_0)}{G'(z_0)} = \frac{\phi(z_0)}{\mp 2\pi i e^{\mp i2\pi z_0}} = \mp \frac{\phi(z_0)}{2\pi i}.$$

Using these residues, counting the poles x_1 and x_2 as quarter residues and the integers in between as half residues, we complete (B.36)

$$\int_{x_1}^{x_2} \frac{\phi(x)dx}{e^{-i2\pi x} - 1} + i\int_0^\infty \frac{\phi(x_2 + iy) - \phi(x_1 + iy)}{e^{2\pi y} - 1} dy = \oint_{C_1} \frac{\phi(z)dz}{e^{-i2\pi z} - 1}$$

$$= 2\pi i \left(-\frac{1}{4}\frac{\phi(x_1)}{2\pi i} - \frac{1}{2}\frac{\phi(x_1 + 1)}{2\pi i} - \frac{1}{2}\frac{\phi(x_1 + 2)}{2\pi i} - \ldots - \frac{1}{2}\frac{\phi(x_2 - 1)}{2\pi i} - \frac{1}{4}\frac{\phi(x_2)}{2\pi i} \right),$$

and by just shuffling minus signs we find

$$- \int_{x_1}^{x_2} \frac{\phi(x)dx}{e^{-i2\pi x} - 1} + \frac{1}{i}\int_0^\infty \frac{\phi(x_2 + iy) - \phi(x_1 + iy)}{e^{2\pi y} - 1} dy$$
$$= \frac{1}{4}\phi(x_1) + \frac{1}{2}\phi(x_1 + 1) + \frac{1}{2}\phi(x_1 + 2) + \ldots + \frac{1}{2}\phi(x_2 - 1) + \frac{1}{4}\phi(x_2). \tag{B.39}$$

Paying attention to the fact that C_2 is negatively oriented, we complete (B.37) and get

$$\int_{x_1}^{x_2} \frac{\phi(x)dx}{e^{i2\pi x}-1} + i\int_0^\infty \frac{\phi(x_1-iy)-\phi(x_2-iy)}{e^{2\pi y}-1}dy = \oint_{C_2} \frac{\phi(z)dz}{e^{i2\pi z}-1}$$

$$= -2\pi i\left(\frac{1}{4}\frac{\phi(x_1)}{2\pi i} + \frac{1}{2}\frac{\phi(x_1+1)}{2\pi i} + \frac{1}{2}\frac{\phi(x_1+2)}{2\pi i} + \ldots + \frac{1}{2}\frac{\phi(x_2-1)}{2\pi i} + \frac{1}{4}\frac{\phi(x_2)}{2\pi i}\right),$$

and by again just shuffling minus signs we find

$$-\int_{x_1}^{x_2} \frac{\phi(x)dx}{e^{i2\pi x}-1} + \frac{1}{i}\int_0^\infty \frac{\phi(x_1-iy)-\phi(x_2-iy)}{e^{2\pi y}-1}dy$$
$$= \frac{1}{4}\phi(x_1) + \frac{1}{2}\phi(x_1+1) + \frac{1}{2}\phi(x_1+2) + \ldots + \frac{1}{2}\phi(x_2-1) + \frac{1}{4}\phi(x_2). \quad \text{(B.40)}$$

Next, add the equations (B.39) and (B.40) side by side

$$-\int_{x_1}^{x_2} \phi(x)\left(\frac{1}{e^{-i2\pi x}-1} + \frac{1}{e^{i2\pi x}-1}\right)dx$$
$$+\frac{1}{i}\int_0^\infty \frac{\phi(x_2+iy)-\phi(x_1+iy)+\phi(x_1-iy)-\phi(x_2-iy)}{e^{2\pi y}-1}dy$$
$$= \frac{1}{2}\phi(x_1) + \phi(x_1+1) + \phi(x_1+2) + \ldots + \phi(x_2-1) + \frac{1}{2}\phi(x_2).$$

We find that $\frac{1}{e^{-i2\pi x}-1} + \frac{1}{e^{i2\pi x}-1} = \frac{e^{i2\pi x}-1+e^{-i2\pi x}-1}{(e^{-i2\pi x}-1)(e^{i2\pi x}-1)} = -1$, and by just writing z instead of x, we receive the wanted result (B.35).

B.3.5 Binet's Second Expression for $\log\Gamma(z)$

We will now use the result (B.30)

$$\frac{d^2}{dz^2}\log\Gamma(z) = \sum_{n=0}^\infty \frac{1}{(z+n)^2},$$

to find another expression for $\log\Gamma(z)$. Let $\frac{1}{(z+\zeta)^2}$ be the function $\phi(\zeta)$ in Plana's formula (B.35), assuming that $\Re z > 0$. We see that

$$\frac{d^2}{dz^2}\log\Gamma(z) = \frac{1}{z^2} + \frac{1}{(z+1)^2} + \ldots = \phi(0) + \phi(1) + \phi(2) + \ldots$$

Since $\Re z > 0$, $\phi(\zeta)$ is analytic and bounded when $\Re\zeta \geq 0$, and we can apply Plana's theorem where $x_1 = 0$ and $x_2 \to \infty$. Since $\phi(\zeta) \to 0$ when $\Re\zeta \to \infty$, the factor $\frac{1}{2}$ in front of $\phi(\infty)$ does not matter, and we write

$$\frac{1}{2}\phi(0) + \phi(1) + \phi(2) + \ldots$$

$$= \int_0^\infty \frac{1}{(z+\xi)^2}d\xi + \lim_{x_2\to\infty}\frac{1}{i}\int_0^\infty \frac{\phi(x_2+it)-\phi(0+it)-\phi(x_2-it)+\phi(0-it)}{e^{2\pi t}-1}dt. \quad \text{(B.41)}$$

The two terms in the right integral which depend on x_2 will disappear when $x_2 \to \infty$. Now add $\frac{1}{2}\phi(0) = \frac{1}{2z^2}$ to both sides of (B.41), and perform the integration over the real variable ξ

$$\begin{aligned}\frac{d^2}{dz^2}\log\Gamma(z) &= \frac{1}{2z^2} + \left[-\frac{1}{z+\xi}\right]_0^\infty + \frac{1}{i}\int_0^\infty \left(\frac{1}{(z-it)^2} - \frac{1}{(z+it)^2}\right)\frac{1}{e^{2\pi t}-1}dt \\ &= \frac{1}{2z^2} + \frac{1}{z} + \frac{1}{i}\int_0^\infty \frac{(z+it)^2-(z-it)^2}{((z-it)(z+it))^2}\frac{dt}{e^{2\pi t}-1} \\ &= \frac{1}{2z^2} + \frac{1}{z} + \int_0^\infty \frac{4tz}{(z^2+t^2)^2(e^{2\pi t}-1)}dt.\end{aligned} \quad \text{(B.42)}$$

The integral converges uniformly. Hence, we may integrate under the integral sign in (B.42) from 1 to z,

$$\left[\frac{d}{dz'}\log\Gamma(z')\right]_1^z = \left[-\frac{1}{2z'}\right]_1^z + [\log z']_1^z + \int_0^\infty -2t\left[\frac{1}{z'^2+t^2}\right]_1^z \frac{1}{e^{2\pi t}-1}dt.$$

We find

$$\frac{d}{dz}\log\Gamma(z) = -\frac{1}{2z} + \frac{1}{2} + \log z + A - 2\int_0^\infty \frac{t\ dt}{(z^2+t^2)(e^{2\pi t}-1)} + 2\int_0^\infty \frac{t\ dt}{(1+t^2)(e^{2\pi t}-1)}.$$

The last integral is clearly convergent, and will be treated just as a constant. Then, collecting all the constants into a constant C, we get

$$\frac{d}{dz}\log\Gamma(z) = -\frac{1}{2z} + \log z + C - 2\int_0^\infty \frac{t\ dt}{(z^2+t^2)(e^{2\pi t}-1)}. \tag{B.43}$$

Integrate again from 1 to z. There is still no problem to integrate under the integral sign, so

$$[\log\Gamma(z')]_1^z = -\frac{1}{2}[\log z']_1^z + [z'\log z' - z']_1^z + [Cz']_1^z + 2\int_0^\infty \left[\arctan\frac{t}{z'}\right]_1^z \frac{dt}{e^{2\pi t}-1}.$$

Continuing,

$$\log\Gamma(z) = -\frac{1}{2}\log z + z\log z - z + 1 + Cz - C + B + 2\int_0^\infty \frac{\arctan(t/z)}{e^{2\pi t}-1}dt - 2\int_0^\infty \frac{\arctan t}{e^{2\pi t}-1}dt,$$

and again the last integral is convergent and has just a constant value. Collect all the different constants into one constant C' and find that

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z + (C-1)z + C' + 2\int_0^\infty \frac{\arctan(t/z)}{e^{2\pi t}-1}dt. \tag{B.44}$$

We now want to determine the constants C and C' in (B.44). Since $\arctan\xi = \int_0^\xi \frac{dt}{1+t^2} \leq \int_0^\xi \frac{dt}{1} = \xi$, we know that

$$\left|\log\Gamma(z) - \left(z-\frac{1}{2}\right)\log z - (C-1)z - C'\right| < \frac{2}{z}\int_0^\infty \frac{t}{e^{2\pi t}-1}dt \to 0,$$

when $\Re z \to \infty$. Also, we found in (B.34) that for large $\Re z$

$$\left|\log\Gamma(z) - \left(z-\frac{1}{2}\right)\log z + z - \frac{1}{2}\ln 2\pi\right| \to 0.$$

Comparing these two results, (which only apply for large $\Re z$), we determine the constants C and C' to be

$$C = 0; \qquad C' = \frac{1}{2}\ln 2\pi.$$

Then, (B.44) becomes

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\ln 2\pi + 2\int_0^\infty \frac{\arctan(t/z)}{e^{2\pi t}-1}dt, \tag{B.45}$$

which is Binet's second expression for $\log\Gamma(z)$.

B.3.6 Hermite's Formula for $\zeta(s,a)$

Having for long only dealt with the Γ function, we now turn our attention to Riemann's ζ function. This becomes obvious immediately when we apply Plana's formula (B.35) to the function $\phi(z) = \frac{1}{(a+z)^s}$. We apparently have

$$\frac{1}{2}\phi(0) + \phi(1) + \phi(2) + \ldots = \frac{1}{2}\frac{1}{(a+0)^s} + \frac{1}{(a+1)^s} + \frac{1}{(a+2)^s} + \ldots,$$

so, using Plana's formula with $x_1 = 0$ and $x_2 \to \infty$,

$$\begin{aligned}
\zeta(s,a) =& \phi(0) + \phi(1) + \ldots = \frac{1}{2}\phi(0) + \left(\frac{1}{2}\phi(0) + \phi(1) + \phi(2) + \ldots\right) \\
=& \frac{1}{2a^s} + \int_0^\infty \phi(z)dz + \lim_{x_2\to\infty} \frac{1}{i}\int_0^\infty \frac{\phi(x_2+iy) - \phi(0+iy) - \phi(x_2-iy) + \phi(0-iy)}{e^{2\pi y}-1}dy \\
=& \frac{1}{2a^s} + \int_0^\infty \phi(z)dz - \lim_{x_2\to\infty} \frac{1}{i}\int_0^\infty \left(\frac{1}{(a+x_2-iy)^s} - \frac{1}{(a+x_2+iy)^s}\right)\frac{dy}{e^{2\pi y}-1} \\
&+ \frac{1}{i}\int_0^\infty \left(\frac{1}{(a+0-iy)^s} - \frac{1}{(a+0+iy)^s}\right)\frac{dy}{e^{2\pi y}-1}.
\end{aligned} \tag{B.46}$$

We have not yet motivated that it was possible to set $x_1 = 0$ and let $x_2 \to \infty$ in Plana's formula. We will do that now, by noting that the obtained integrals are convergent. Introduce

$$q(x,y) = \frac{1}{2i}\left(\frac{1}{(a+x-iy)^s} - \frac{1}{(a+x+iy)^s}\right) = \frac{1}{2i}\left(\frac{(a+x+iy)^s - (a+x-iy)^s}{((a+x)^2+y^2)^s}\right).$$

The two complex numbers $z = (a+x+iy)^s$ and $\bar{z} = (a+x-iy)^s$ are each other's complex conjugates. Their difference is $z - \bar{z} = 2i\Im z$. We write

$$a + x + iy = \sqrt{(a+x)^2+y^2}(\cos\theta + i\sin\theta),$$

where $\theta = \arctan\left(\frac{y}{a+x}\right)$, and by de Moivre's formula,

$$z = (a+x+iy)^s = \left((a+x)^2+y^2\right)^{\frac{s}{2}}(\cos s\theta + i\sin s\theta).$$

Then

$$\Im z = \left((a+x)^2+y^2\right)^{\frac{s}{2}}\sin s\theta = \left((a+x)^2+y^2\right)^{\frac{s}{2}}\sin\left(s\arctan\frac{y}{a+x}\right),$$

and

$$q(x,y) = \frac{1}{2i}\frac{2i\left((a+x)^2+y^2\right)^{\frac{s}{2}}\sin\left(s\arctan\frac{y}{a+x}\right)}{\left((a+x)^2+y^2\right)^s} = \frac{\sin\left(s\arctan\frac{y}{a+x}\right)}{\left((a+x)^2+y^2\right)^{\frac{s}{2}}}. \tag{B.47}$$

It is always true that $\arctan\left|\frac{y}{a+x}\right| \le \frac{\pi}{2}$, and we have earlier found that $\arctan\left|\frac{y}{a+x}\right| \le \left|\frac{y}{a+x}\right|$. Using the first inequality, writing σ for $\Re s$, we find

$$\begin{aligned}
|q(x,y)| &\le \frac{1}{((a+x)^2+y^2)^{\sigma/2}}\frac{\left((a+x)^2+y^2\right)^{1/2}}{|y|}\left|\sin s\frac{\pi}{2}\right| \\
&\le \frac{1}{((a+x)^2+y^2)^{\frac{\sigma}{2}-\frac{1}{2}}}\frac{1}{|y|}\sinh\left(\frac{\pi}{2}|s|\right).
\end{aligned} \tag{B.48}$$

To confirm the last inequality, study the function $f(x) = \sinh x - \sin x$. Differentiate, to get $f'(x) = \cosh x - \cos x$, which is always a non-negative function. Since $f(0) = 0, |\sinh x| \ge |\sin x|$.

From $\arctan\left|\frac{y}{a+x}\right| \le \left|\frac{y}{a+x}\right|$ it follows that

$$|q(x,y)| \le \frac{1}{((a+x)^2+y^2)^{\sigma/2}} \sinh\left(\frac{y|s|}{a+x}\right). \quad \text{(B.49)}$$

Now look at the case $y > a$ and $\sigma > 0$, and use (B.48) to see that the integral $\int_a^\infty \frac{q(x,y)}{e^{2\pi y}-1}dy$ is convergent when $x \ge 0$ and tends to zero when $x \to \infty$.
For the case $y < a$ (and still $\sigma > 0$), we instead use (B.49) to see that the integral $\int_0^a \frac{q(x,y)}{e^{2\pi y}-1}dy$ is convergent when $x \ge 0$ and tends to zero when $x \to \infty$.

These results together give that $\int_0^\infty \frac{q(x,y)}{e^{2\pi y}-1}dy$ is convergent when $x \ge 0$ and tends to zero when $x \to \infty$.

The last two integrals in (B.46) involve exactly the integral $\int_0^\infty \frac{q(x,y)}{e^{2\pi y}-1}dy$, when $x \to \infty$ and when $x = 0$, respectively. The integral involving $x_2 \to \infty$ will disappear, so we are left with $2\int_0^\infty \frac{q(0,y)}{e^{2\pi y}-1}dy$, which is convergent when $\sigma > 0$. The other integral in (B.46), $\int_0^\infty \phi(z)dz = \int_0^\infty \frac{1}{(a+x)^s}dx$, is convergent if $\sigma > 1$.These convergences guarantee that we are allowed to use Plana's formula in the interval from 0 to ∞. Using (B.46), (B.47) and what we have discussed above, we can write

$$\begin{aligned}\zeta(s,a) &= \frac{1}{2a^s} + \int_0^\infty \frac{1}{(a+x)^s}dx + 2\int_0^\infty \frac{\sin(s\arctan(y/a))}{(a^2+y^2)^{s/2}(e^{2\pi y}-1)}dy \\ &= \frac{1}{2a^s} + \left[\frac{(a+x)^{1-s}}{1-s}\right]_0^\infty + 2\int_0^\infty \frac{\sin(s\arctan(y/a))}{(a^2+y^2)^{s/2}(e^{2\pi y}-1)}dy.\end{aligned}$$

If still $\sigma = \Re s > 1$, we see that the integrated term $\to 0$ when $x \to \infty$, and

$$\zeta(s,a) = \frac{1}{2a^s} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty \frac{\sin(s\arctan(y/a))}{(a^2+y^2)^{s/2}(e^{2\pi y}-1)}dy, \quad \text{(B.50)}$$

which is Hermite's formula.

We have thus far assumed that $\sigma = \Re s > 1$. We want Hermite's formula to be valid when $s = 0$. Then we need to do an analytic continuation of this function in the complex plane, as we did with another function in the section where we computed $\zeta(0)$. We will not repeat the process which we did there. We just note that the integrand in (B.50) is an analytic function in the entire complex plane (both $e^{2\pi y} - 1$ and $\sin s \arctan y/a$ become zero when $y = 0$, so the pole at $y = 0$ is "cancelled".) We can therefore generalise (B.50) to be valid in the whole complex plane, except for the point $s = 1$. Certainly, more mathematical theory is needed to be completely sure of this. However, let us compute

$$\zeta(0,a) = \frac{1}{2} - a + 2\int_0^\infty \frac{\sin(0)dy}{(a^2+y^2)^0(e^{2\pi y}-1)} = \frac{1}{2} - a; \qquad \zeta(0) = \zeta(0,1) = -\frac{1}{2},$$

which we already know. To obtain $\zeta'(0)$, we differentiate (B.50) with respect to s,

$$\frac{d}{ds}\zeta(s,a) = -\frac{1}{2}a^{-s}\ln a + \frac{(-1a^{1-s}\ln a)(s-1) - a^{1-s}\cdot 1}{(s-1)^2}$$

$$+2\int_0^\infty \frac{\cos(s\arctan\frac{y}{a})\arctan\frac{y}{a}(a^2+y^2)^{\frac{s}{2}} - \sin(s\arctan\frac{y}{a})\frac{1}{2}(a^2+y^2)^{\frac{s}{2}}\ln(a^2+y^2)}{(a^2+y^2)^s}\frac{dy}{e^{2\pi y}-1},$$

where the integral can be shown to be convergent for all values of s, and consequently we were allowed to differentiate. Find the limit $s \to 0$ as

$$\zeta'(0,a) = \left\{\frac{d}{ds}\zeta(s,a)\right\}_{s=0} = \lim_{s\to 0}\left\{-\frac{\ln a}{2a^s} - \frac{a^{1-s}\ln a}{s-1} - \frac{a^{1-s}}{(s-1)^2}\right.$$

$$\left. +2\int_0^\infty \frac{\cos(s\arctan y/a)\arctan y/a - \frac{1}{2}\sin(s\arctan y/a)\ln(a^2+y^2)}{(a^2+y^2)^{s/2}(e^{2\pi y}-1)}dy\right\}$$

$$= -\frac{\ln a}{2} + a\ln a - a + 2\int_0^\infty \frac{\arctan y/a}{e^{2\pi y}-1}dy = \left(a - \frac{1}{2}\right)\ln a - a + 2\int_0^\infty \frac{\arctan y/a}{e^{2\pi y}-1}dy. \quad \text{(B.51)}$$

Now recall the result (B.45), where we let $z = a$ and $t = y$, and find that

$$\log\Gamma(a) = \left(a - \frac{1}{2}\right)\ln a - a + \frac{1}{2}\ln 2\pi + 2\int_0^\infty \frac{\arctan y/a}{e^{2\pi y} - 1}dy,$$

and so

$$\left(a - \frac{1}{2}\right)\ln a - a + 2\int_0^\infty \frac{\arctan y/a}{e^{2\pi y} - 1}dy = \log\Gamma(a) - \frac{1}{2}\ln 2\pi.$$

Using this in (B.51), we finally arrive at

$$\zeta'(0, a) = \log\Gamma(a) - \frac{1}{2}\ln 2\pi; \qquad \zeta'(0) = \zeta'(0, 1) = -\frac{1}{2}\ln 2\pi.$$

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