

The Josephson Effect

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Purpose

To describe the Josephson effect and show some consequences of it. To understand the workings of a SQUID (Superconducting QUantum Interference Device) and to demonstrate its features.

Below is given a quite detailed discussion on various effects. Do not get lost in the derivations, but try to pick out the main results.

Introduction

In 1962, B.D. Josephson analysed what happens at a junction between two closely spaced superconductors, separated by an insulating barrier. If the insulating barrier is thick, the electron pairs can not get through; but if the layer is thin enough (approximately 10 nm) there is a probability for electron pairs to tunnel. This effect later became known as 'Josephson tunnelling'. Besides displaying a broad range of interesting macroscopic quantum mechanical properties, the Josephson junction offers a vast survey of possible applications in analog and digital electronics, such as SQUID detectors, oscillators, mixers and amplifiers.

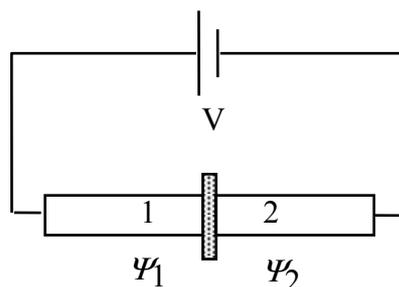


Figure 1. Two superconductors separated by a thin insulator

Theory of Josephson tunnelling

In his famous lectures Feynman made a simplified derivation of the equations describing the Josephson tunnelling. We outline this derivation here.

Suppose we have two superconductors that are connected by a thin layer of insulating material as shown in figure 1. We define Ψ_1 and Ψ_2 as the quantum mechanical wavefunction of the superconducting state in the left and the right superconductor, respectively. The dynamics of the two wavefunctions are then determined by the following coupled Schrödinger equations:

$$\begin{aligned} i\hbar \frac{\partial \Psi_1}{\partial t} &= \mu_1 \Psi_1 + K \Psi_2 \\ i\hbar \frac{\partial \Psi_2}{\partial t} &= \mu_2 \Psi_2 + K \Psi_1 \end{aligned} \tag{1}$$

where K is a constant representing the coupling across the barrier and μ_1, μ_2 are the lowest energy states on either side.

We make the ansatz that the wavefunctions can be written as

$$\begin{aligned}\psi_1 &= \sqrt{n_1} e^{i\theta_1} \\ \psi_2 &= \sqrt{n_2} e^{i\theta_2}\end{aligned}\tag{2}$$

where n_1, n_2 are the density of Cooper pairs and θ_1, θ_2 are the phases.

Substituting equation (2) into equation (1) gives

$$\hbar \frac{\partial n_1}{\partial t} = -\hbar \frac{\partial n_2}{\partial t} = 2K \sqrt{n_1 n_2} \sin(\theta_2 - \theta_1)\tag{3}$$

$$-\hbar \frac{\partial}{\partial t} (\theta_2 - \theta_1) = \mu_2 - \mu_1\tag{4}$$

These are the governing equations for the Josephson effect, but they can be written in a more compact way. The time derivative of the density of Cooper pairs describes a charge transport and thus $I = \frac{\partial n_1}{\partial t}$. If a voltage V is applied between the junctions the energy levels will shift according to $\mu_2 - \mu_1 = 2eV$. Writing $I_0 = 2K \sqrt{n_1 n_2} / \hbar$ and $\delta = \theta_1 - \theta_2$ we get

$$I = I_0 \sin \delta\tag{5}$$

$$\frac{\partial \delta}{\partial t} = \frac{2eV}{\hbar}\tag{6}$$

Equations (5) and (6) represent the main results of the general theory of the Josephson junction. The current I is called a Josephson current or a supercurrent. This strongly non-linear current-voltage behaviour is the origin for many different physical phenomena. We will study some simple cases below.

The DC Josephson effect

From the Josephson equations (5,6) it is clear that a DC current can be drawn through the junction without any voltage drop, as long as the DC current is smaller than I_0 . The phase difference will adjust according to equation (5).

The AC Josephson effect

If we apply a DC voltage V_0 (or a DC current larger than I_0) to the junction we get from equation (5,6)

$$I = I_0 \sin \left(\frac{2eV_0}{\hbar} t \right) \quad (7)$$

Thus when a voltage is applied, the Josephson current will oscillate at a frequency

$$f = \frac{2eV_0}{h} \quad (8)$$

where $2e/h = 483.6$ GHz/mV. The junctions capacitive and resistive properties, which is modelled in the Resistively and Capacitively Shunted Junction-model, now describes its DC characteristics.

Resistively and Capacitively Shunted Junction-model

An equivalent circuit that can be used for most types of resistively junctions is shown in figure 2. The model is called the Resistively and Capacitively Shunted Junction-model (RCSJ-model). Here the Josephson junction (J) is shunted by a voltage independent resistor and a capacitor.

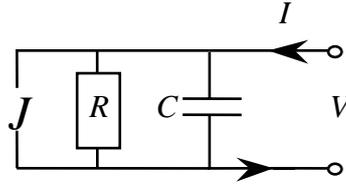


Figure 2. Equivalent circuit of a Josephson junction.

An equation is obtained from the conservation of the supplied current (I):

$$I = I_0 \sin \delta + C \frac{dV}{dt} + \frac{V}{R} \quad (9)$$

Inserting equation (6) into equation (9) yields:

$$I = I_0 \sin \delta + \frac{\hbar C}{2e} \frac{d^2 \delta}{dt^2} + \frac{\hbar}{2eR} \frac{d\delta}{dt} \quad (10)$$

which, using dimensionless variables,

$$\tau = 2\pi f_c t = \frac{2e}{\hbar} I_0 R t \quad (11)$$

$$\beta_c = 2\pi f_c R C = \frac{2e}{\hbar} I_0 R^2 C \quad (12)$$

can be rewritten as:

$$i = \sin \delta + \beta_c \frac{d^2 \delta}{d\tau^2} + \frac{d\delta}{d\tau} \quad (13)$$

The parameter β_c is called the Stewart-McCumber parameter. In general, equation (13) can not be solved analytically and computer simulations are needed. However, if the capacitance can be neglected, $\beta_c=0$, we can integrate equation (13) and taking the time average of V , find the I-V behaviour:

$$\begin{cases} V = 0 & ; \text{for } |I| < I_0 \\ V = I_0 R \left[\left(\frac{I}{I_0} \right)^2 - 1 \right]^{\frac{1}{2}} & ; \text{for } |I| > I_0 \end{cases} \quad (14)$$

The dependence is shown in figure 3. A more thorough analysis of equation (13) yields an I-V behaviour similar to that shown in figure 3 as long as $\beta_c < 1$. For $\beta_c > 1$ the junction will be hysteretic and since hysteresis is undesired in most applications a common design requirement when designing a device is $\beta_c < 1$.

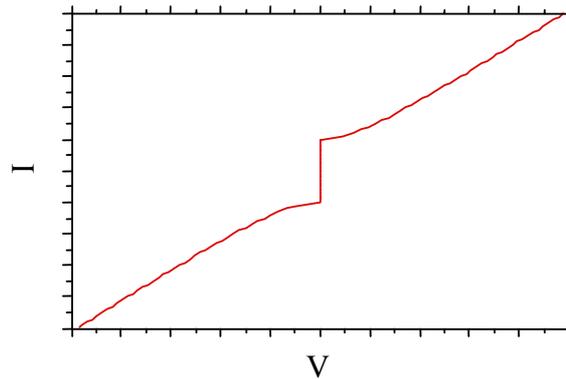


Figure 3. I-V-characteristics of a resistively shunted junction with $C = 0$.

Magnetic field dependence of the Josephson current

Up till now we have assumed that the phase difference δ is the same in the entire junction. This is however not true in the presence of a magnetic field. In order to derive the dependence one must consider the condition that the wave function must be single-valued. When going round one turn in the loop sketched in figure 4 the phase can only change an integer number of 2π since:

$$\psi = \sqrt{n} e^{i\theta} = \sqrt{n} e^{i(\theta + 2\pi n)} \quad (15)$$

The case of a non-zero phase change corresponds to trapped fluxons in the junction. Assuming that no flux is trapped in the junction we get by summing up phase contributions along the path in figure 4:

$$\oint \nabla \varphi \cdot d\mathbf{l} = \frac{2\pi}{\Phi_0} \oint \mathbf{A} \cdot d\mathbf{l} + \frac{m}{2e^2\rho} \oint \mathbf{J} \cdot d\mathbf{l} + \delta(x) - \delta(x+dx) = 0 \quad (16)$$

where \mathbf{A} is the magnetic vector field. The integral in the first term on the right hand side is recognised as the total magnetic flux Φ in the loop. The integral in the second term on the right hand side can be neglected since it is only non-zero in the very small barrier region. Thus we are left with:

$$\frac{\partial \delta}{\partial x} = \frac{2\pi}{\Phi_0} \Phi = \frac{2e}{\hbar} B t_{\text{eff}} \quad (17)$$

where $t_{\text{eff}} = t + 2\lambda_L$ is the effective barrier thickness, t is the barrier thickness and λ_L is the London penetration depth.

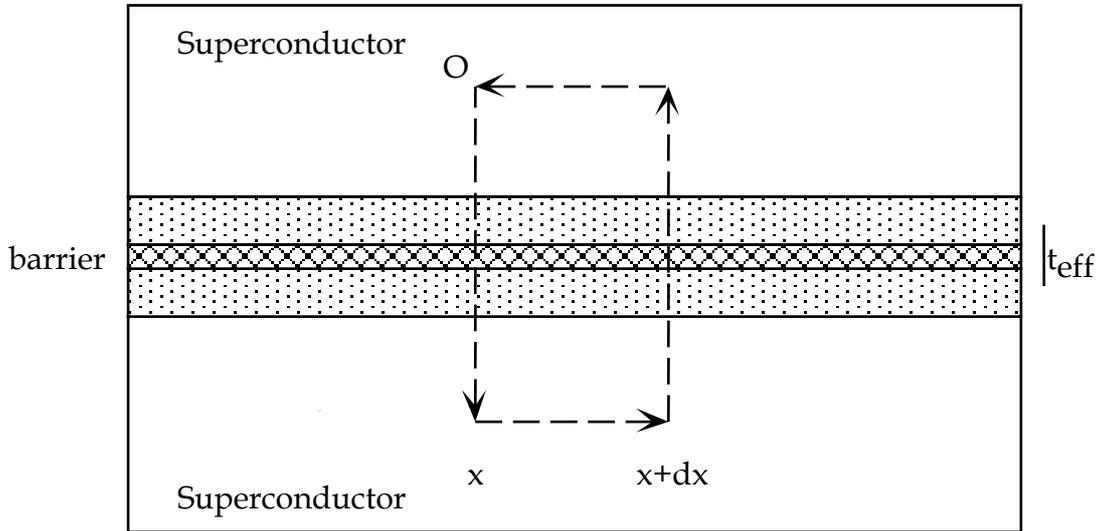


Figure 4. A Josephson tunnel junction with effective barrier thickness $t_{\text{eff}} = t + 2\lambda_L$.

Since the phase difference is not constant along the barrier, the supercurrent through the barrier will, according to equation (5), also be position dependent. The magnetic field perpendicular to the current through the junction will therefore modulate the critical current. Inserting equation (17) into equation (5) yields after integration:

$$I_0(\Phi) = I_0(0) \left| \frac{\sin\left(\pi \frac{\Phi}{\Phi_0}\right)}{\pi \frac{\Phi}{\Phi_0}} \right| \quad (18)$$

The critical current thus shows a Fraunhofer-like dependence of the magnetic field. There is an analogy in optics where light is diffracted in a single slit to give a similar modulation of the light intensity.

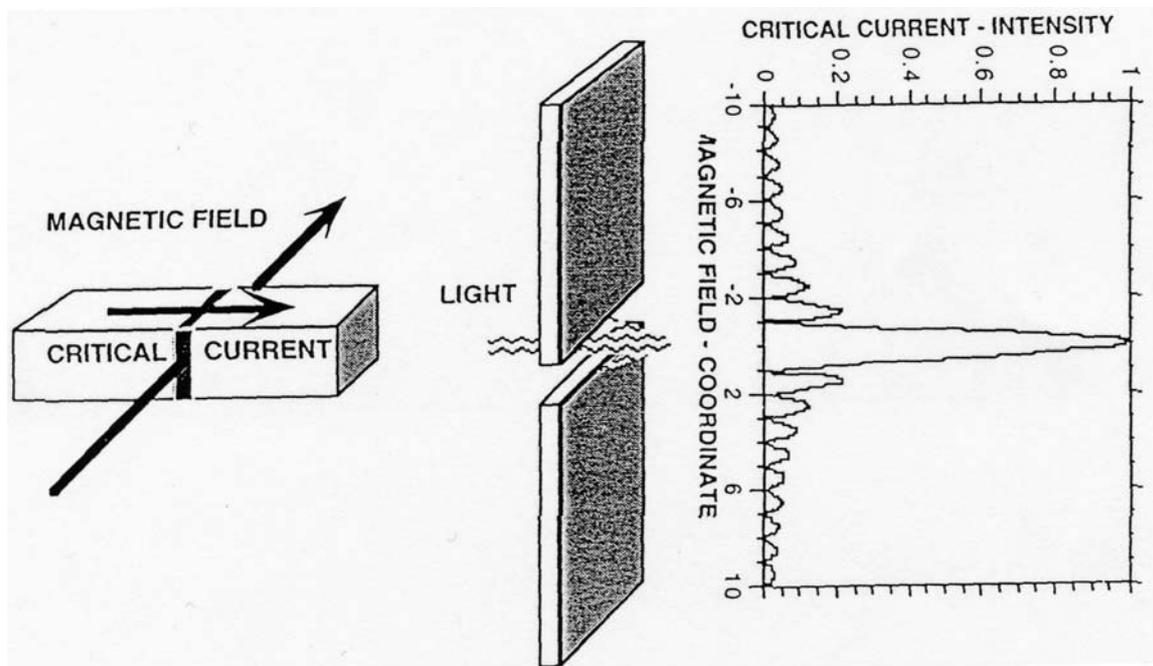


Figure 5. Analogy of a Josephson junction in a magnetic field and light diffracted in a single slit.

Superconducting quantum interference devices

A Superconducting QUantum Interference Device (SQUID) is the most sensitive magnetic flux sensor currently known. The SQUID can be seen as a flux to voltage converter and it can generally be used to sense any quantity that can be transduced into a magnetic flux, such as electrical current, voltage, position, etc. The extreme sensitivity of the SQUID is utilised in many different fields of applications, including biomagnetism, materials science, metrology, astronomy and geophysics. A SQUID is simply two Josephson junctions in parallel, see figure 6.

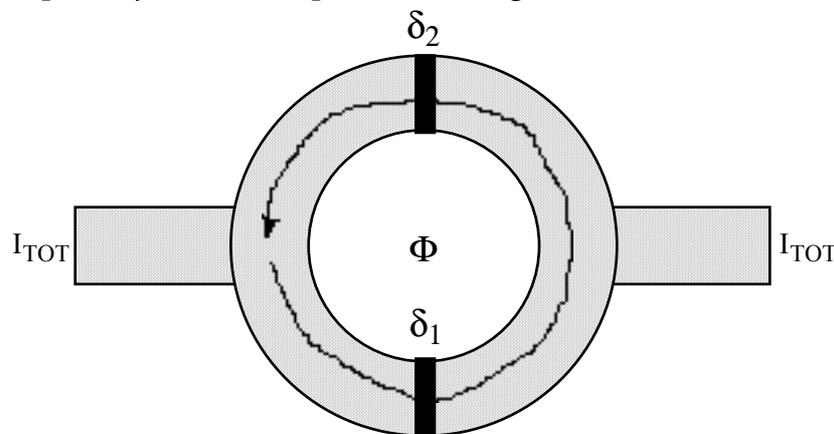


Figure 6. Two weak junctions in parallel making up a SQUID.

If the two links have the phase difference δ_1 and δ_2 respectively the total current through the SQUID will be:

$$I_{\text{TOT}} = I_1 + I_2 = I_0 \sin \delta_1 + I_0 \sin \delta_2 \quad (19)$$

The phase differences δ_1 and δ_2 are dependent of the total flux Φ threading the ring. By summing up phase contributions from the flux and the junctions we get:

$$\oint \nabla\varphi \cdot d\mathbf{l} = 2\pi n = \frac{2\pi}{\Phi_0} \oint \mathbf{A} \cdot d\mathbf{l} + \frac{m}{2e^2 n} \oint \mathbf{I} \cdot d\mathbf{l} + \delta_1 - \delta_2 \quad (20)$$

By choosing an integration path well inside the two superconductors the integral in the first term on the right hand side is recognised as the total magnetic flux Φ . Since the current inside the superconductor is zero, there will not be any contribution from the second term and the expression can be simplified to:

$$2\pi n = \frac{2\pi}{\Phi_0} \Phi + \delta_1 - \delta_2 \quad (21)$$

Combining equation (19) and equation (21) gives us the expression for the total current through the SQUID:

$$I_{\text{TOT}} = 2I_0 \left| \cos \left(\frac{\pi\Phi}{\Phi_0} \right) \right| \quad (22)$$

The current is modulated with a period of Φ_0 but since it is the total area of the loop and not only the junctions that are picking up flux, the SQUID is much more sensitive to a magnetic flux than a single junction is. The flux that is picked up in the junctions is according to equation (18) modulating the critical current so in total there will be a slow critical current modulation as an envelope to the faster SQUID-modulation. Even in this case there is an analogy with optics. It is the Young's double-slit experiment where monochromatic light coming in from two slits is interfering constructively and destructively and thus giving rise to intensity maxima and minima. As an envelope to this intensity modulation there is the diffraction patterns from the two slits.

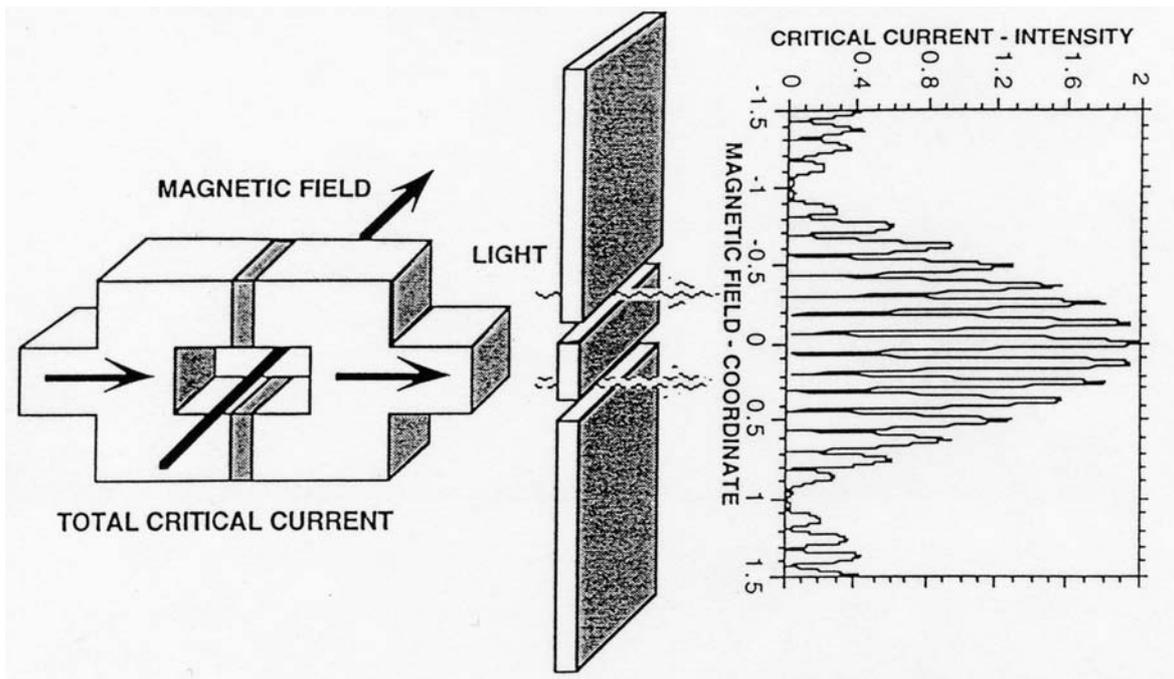


Figure 7. Analogy of a SQUID in a magnetic field and light interfering through two slits.

For a zero flux the SQUID's I-V-characteristic looks just like the one for a Josephson junction, with the critical current $I_{TOT,max} = 2I_0$ and the normal resistance $R_N/2$, see figure 8. For an increasing flux the total current will decrease down to a minimal value $I_{TOT,min}$ when the flux is $\Phi_0/2$. According to the theory given above $I_{TOT,min}$ should be equal to zero, but a more detailed analysis taking into account the SQUID-inductance gives a non-zero value for $I_{TOT,min}$. When the flux is increased even more, the total current is increased back to its maximum value at the flux Φ_0 .

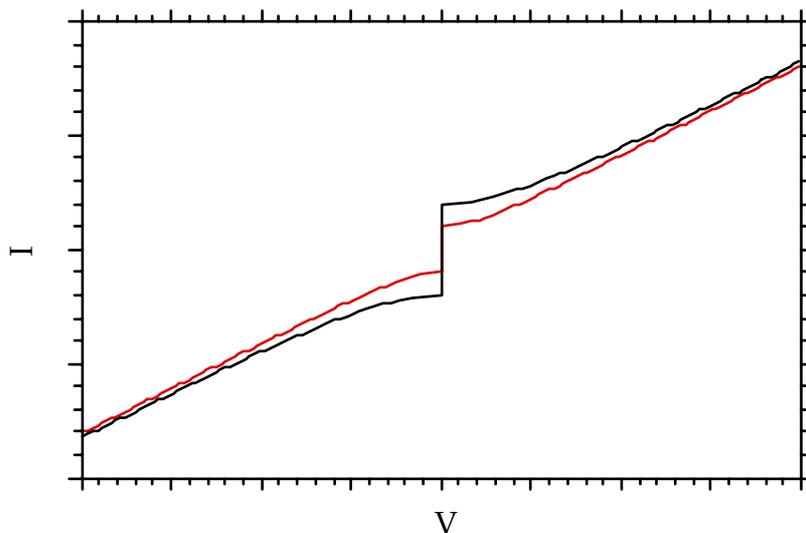


Figure 8. I-V-modulation of a SQUID in a magnetic field.

If the SQUID is current biased above $2I_0$ ($= I_{TOT,max}$), the voltage over the SQUID will be modulated with a modulation depth ΔV (typically $10 \mu\text{V}$ for a SQUID at 77 K) by the magnetic field. We thus have a flux to voltage converter. It is, in principle, possible to use the SQUID as a flux sensor in this operational mode, but it is more common to use feedback electronics to lock the flux in the SQUID to a certain value. This is done in order to optimise the SQUID's sensitivity by locking the flux to a value where the SQUID has its maximal gain, which is where $\left. \frac{\partial V}{\partial \Phi} \right|_{I_{bias}}$ is at a maximum.

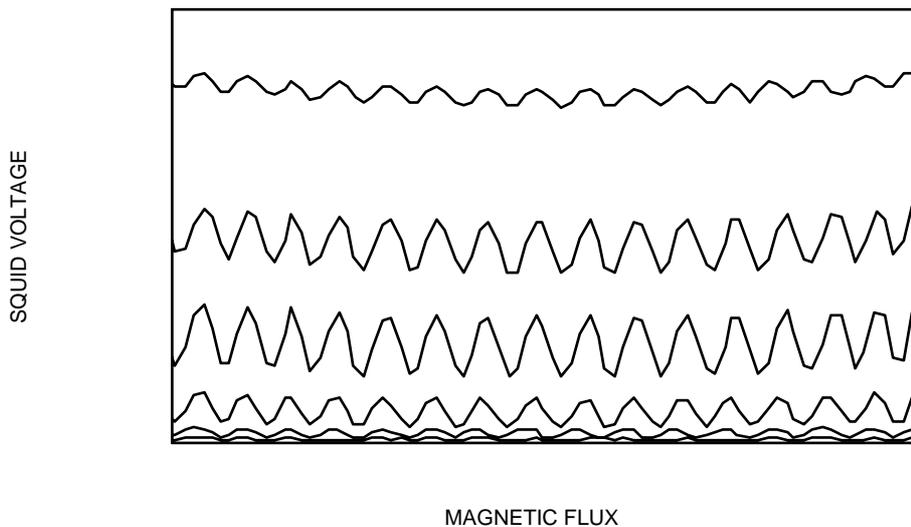


Figure 9. Schematic $V - \Phi$ curve of a SQUID at different bias currents.

The sensitivity of the SQUID is limited by its intrinsic noise. The flux noise density of a high quality high- T_c SQUID is typically $10 \mu\Phi_0 / \text{Hz}^{1/2}$. The flux noise is found by multiplying the flux noise density with the square root of the bandwidth of the measurement setup. Thus with a bandwidth of, for instance, 100 Hz it is possible to measure a flux of $100 \mu\Phi_0$.

The area of a SQUID is very small, typically 0.01 mm^2 , which means that a magnetic field of about $2 \text{ pT}/\text{Hz}^{1/2}$ can be measured with a bare high- T_c SQUID. With various flux focusing techniques it is possible to increase the effective area of the SQUID up to about 1 mm^2 , which means that a magnetic field of about $20 \text{ fT}/\text{Hz}^{1/2}$ can be measured with a high- T_c SQUID with a flux antenna. The low- T_c SQUIDs are approximately 10 times more sensitive. Figure 10 shows typical magnetic fields for various sources and disturbances.

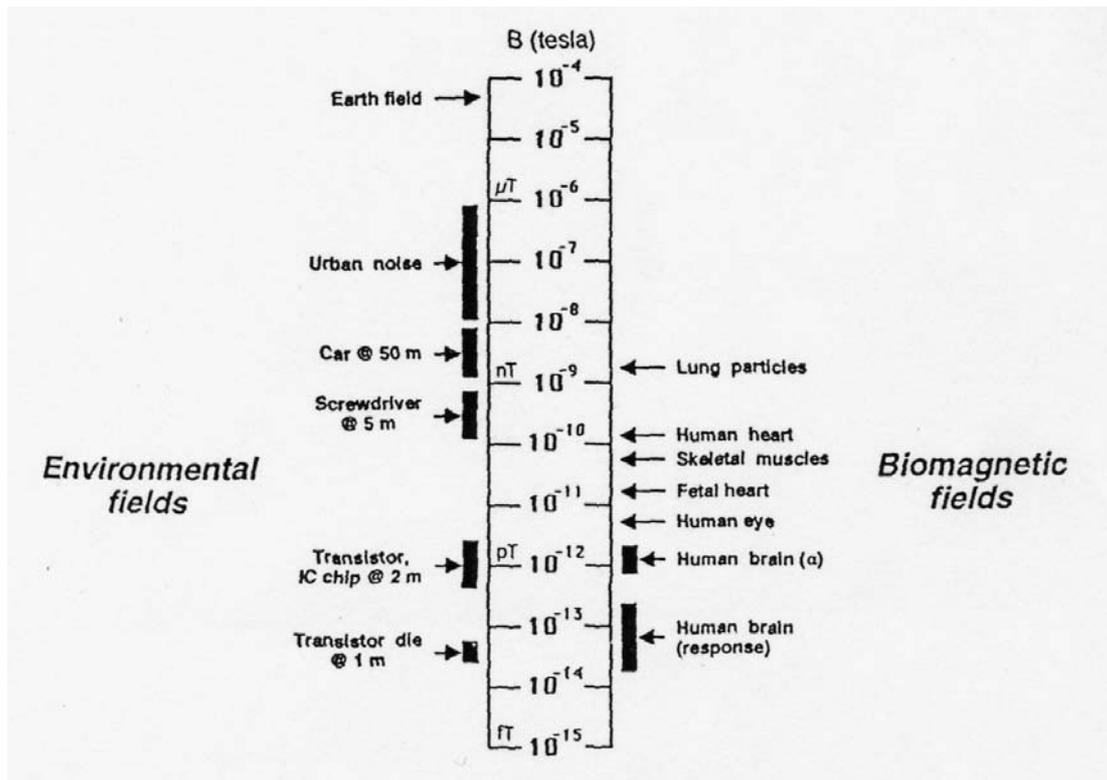


Figure 10. Comparison of selected biomagnetic fields and environmental disturbances.

References

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Van Duzer, T. and Turner, C.W., Principles of Superconductive Devices and Circuits, 1981, Elsevier North Holland Inc.

Laboratory exercise.

In this exercise we will examine a SQUID made from the high temperature superconductor $\text{YBa}_2\text{Cu}_3\text{O}_{7-x}$, which becomes superconducting at approximately 90 K. We will look at its properties both at room temperature and at 77 K (liquid nitrogen).

1) To get familiar with the electronics and the oscilloscope, first look at the IV curve of a $10\ \Omega$ resistor. The current through the resistor and the voltage over it are both amplified 10000 times by the electronics box. Verify that the resistance really is $10\ \Omega$.

2) Short the input of the electronics box and connect the SQUID, the 4-pole cable (I and V) to contact 9 to 12 and the 2-pole cable (input coil current) to contact 1 to 2. Open the short and measure the resistance of the SQUID at room temperature.

3) Display the IV-curve on the oscilloscope. Put the stick slowly into the liquid nitrogen and watch the IV-curve as the temperature drops.

4) Turn up the current AMPLitude to display the IV-curve. Look at the effect of changing the current through the input coil, which will change the flux in the SQUID-loop, by varying $\emptyset\text{BIAS}$. Maximize and minimize the critical current. Use Lab-VIEW to copy some traces displayed on the digital oscilloscope to the computer. Measure the critical current I_0 and the normal resistance R_n .

5) Turn down the sweep and move to the 'knee' on the IV curve. Turn the $\text{IV}/V\emptyset$ switch to $V\emptyset$. In this mode the current through the input coil is displayed on the I-output, the gain is still 10000 times. Maximize the voltage modulation by adjusting the IBIAS knob. Use Lab-VIEW to copy some traces displayed on the digital oscilloscope to the computer. Measure the modulation depth ΔV and the input coil current I_Φ needed to induce one flux quantum in the SQUID-loop.

6) *This exercise does not require any further measurements.* The effective area of the SQUID, defined as $A_{\text{eff}} = \frac{\Phi_{\text{SQUID}}}{B_{\text{SQUID}}}$,

generated by the input coil with the field in the center of a one turn circular coil, $B = \frac{\mu_0}{4\pi} \frac{2IA}{r^3}$, where A is the area enclosed by the turn and r is radius of the coil, see for instance Physics Handbook. The effective area can also be estimated from the SQUID's geometry by using the formula $A_{\text{eff}} = ad$, where a and d are given in figure on next page. The outer input coil is used in this exercise to induce the flux. Compare the results of both estimations.

Washer-type SQUID with 2 μm junctions.

