The London Equations

Assume that the superfluid can be desribed by a plane wave

$$\Psi = \Psi_p e^{i\bar{k}\cdot\bar{r}} = \Psi_p \cdot e^{i\theta}$$

such that $|\Psi|^2 = \Psi_p^2 = \rho_p$

from QM we know

$$\bar{p} = -i\hbar\bar{\nabla}$$

$$\bar{p}\Psi = -i\hbar\bar{\nabla}\Psi_p e^{i\bar{k}\cdot\bar{r}} = -i\hbar i\bar{k}\,\Psi_p e^{i\bar{k}\bar{r}} = \hbar\bar{k}\Psi$$

$$-i\hbar\nabla\Psi_p e^{i\theta} = -i\hbar i\nabla\theta\,\Psi_p e^{i\theta} = \hbar\nabla\theta\Psi$$

$$\bar{J}_s = -2e\rho_p\frac{\bar{p}}{2m} = -\frac{e}{m}\hbar\nabla\theta\cdot|\Psi|^2$$

$$= -\frac{\hbar e}{m}\nabla\theta\cdot|\Psi|^2 = \frac{i\hbar e}{2m}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)$$

$$\Psi^*\nabla\Psi - \Psi\nabla\Psi^* = i\nabla\theta\,\Psi^*\Psi - (-i\nabla\theta)\Psi^*\Psi$$

$$= 2i\nabla\theta$$

$$\nabla\theta = \frac{1}{2i}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)$$

In a magnetic field

$$\bar{p} \longrightarrow \bar{p} = -i\hbar\nabla + 2e\bar{A} = \hbar\nabla\theta + eA$$

$$J_s = -2e\,\psi^*\psi\,\frac{1}{2m}(i\hbar\nabla\theta + 2eA)$$

$$\boxed{J_s = \frac{i\hbar e}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{2e^2}{m}\psi\psi^*\bar{A}}$$

insert $\psi = \sqrt{\rho_p}\,e^{i\theta}$

$$J_s = \frac{i\hbar e}{2m}\left(i\nabla\theta\,\rho_p + i\nabla\theta\,\rho_p\right) - \frac{2e^2}{m}\rho_p\bar{A}$$

$$= -\frac{\hbar e}{m}\rho_p\nabla\theta - \frac{2e^2}{m}\rho_p\bar{A}$$

The London parameter

$$\Lambda = \frac{m}{2\rho_p e^2}$$

$$\boxed{\Lambda J_s = -\left(\frac{\hbar}{2e}\nabla\theta + \bar{A}\right)}$$

$$\boxed{J_s = -\frac{1}{\Lambda}\left(\frac{\hbar}{2e}\nabla\theta + \bar{A}\right) = -2e\,\rho_p\cdot\bar{v}_s}$$

First London Equation

Accelerating the superfluid

$$\frac{\partial}{\partial t} J_s = -2e \rho_p \frac{\partial v_s}{\partial t}$$

$$\bar{F} = 2m \cdot \bar{a}_s = 2m \frac{\partial v_s}{\partial t} = -2e\bar{E}$$

$$\frac{\partial v_s}{\partial t} = -\frac{e\bar{E}}{m}$$

$$\frac{\partial}{\partial t} J_s = -2e \rho_p \left(-\frac{e}{m}\right) E = \frac{1}{\Lambda} \bar{E}$$

$$\boxed{\bar{E} = \Lambda \frac{\partial}{\partial t} J_s}$$

Second London Equation

Take the curl of J_s

$$\bar{J}_s = -\frac{1}{\Lambda}\left(\frac{\hbar}{2e}\nabla\theta + \bar{A}\right)$$

$$\nabla \times \bar{J}_s = -\frac{1}{\Lambda}\left(\frac{\hbar}{2e}\underbrace{\bar{\nabla}\times\bar{\nabla}\theta}_{0} + \underbrace{\bar{\nabla}\times\bar{A}}_{\bar{B}}\right)$$

$$\bar{\nabla}\times\bar{\nabla} f \equiv 0$$

Maxwels Equations solutions

$$\bar{E} = -\nabla\phi - \frac{\partial \bar{A}}{\partial t}$$

$$\bar{B} = \bar{\nabla}\times\bar{A}$$

$$\boxed{\bar{B} = -\Lambda\,\bar{\nabla}\times\bar{J}_s}$$

$$\boxed{\bar{E} = \Lambda\frac{\partial}{\partial t}\bar{J}_s}$$

$$\Lambda = \frac{m}{2\rho_p \cdot e^2}$$

Guage invariance

Sol to ME

$$\bar{E} = -\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t} \qquad \bar{E} = \Lambda \frac{\partial}{\partial t} \bar{J}_s$$

$$\bar{B} = \bar{\nabla} \times \bar{A} \qquad \bar{B} = -\Lambda \bar{\nabla} \times \bar{J}_s$$

Any transformation $\chi(\bar{r},t)$ = scaler function

$$\bar{A}' = \bar{A} + \nabla\chi \qquad \theta' = \theta - \frac{2e}{\hbar}\chi$$

$$\phi' = \phi - \frac{\partial \chi}{\partial t} \qquad \text{also works}$$

$$\bar{\nabla} \times \bar{A}' = \bar{\nabla} \times \bar{A} + \underbrace{\bar{\nabla} \times \bar{\nabla}\chi}_{0} = \bar{\nabla} \times \bar{A} = \bar{B}$$

$$-\bar{\nabla}\phi' - \frac{\partial}{\partial t}\bar{A}' = -\bar{\nabla}\phi + \nabla\frac{\partial}{\partial t}\chi - \frac{\partial}{\partial t}\bar{A} - \frac{\partial}{\partial t}\nabla\chi$$

$$= -\bar{\nabla}\phi - \frac{\partial}{\partial t}A = \bar{E}$$

i.e we can cloos any scaler function χ

Lorentz guage $\quad \bar{\nabla} \cdot A + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$

$$\Delta A - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A = -\mu_0 \bar{J}$$

$$\Delta \phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\frac{\rho_c}{\varepsilon}$$

London guage $\quad$ for a simply connected SC

$$\bar{\nabla} \cdot A = 0 \Rightarrow \theta \text{ const}$$

$$\bar{J}_s = -\frac{1}{\Lambda}\left(\frac{\hbar}{2e}\nabla\theta + \bar{A}\right) = -\frac{1}{\Lambda}\bar{A}$$

Londons Penetration depth

ME $\quad \overline{\nabla} \times \overline{B} = \mu_0 \overline{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\overline{E}$

Quas stat. $\Rightarrow$ Amperés law

$$\overline{\nabla} \times \overline{B} = \mu_0 (J_s + J_N)$$

curl of Amp. Law

$$\overline{\nabla} \times \overline{\nabla} \times \overline{B} = \mu_0 \overline{\nabla} \times \overline{J}_s + \mu_0 \overline{\nabla} \times J_n = \mu_0 \frac{\overline{B}}{\Lambda} + \underbrace{\mu_0 \overline{\nabla} \times J_n}_{\substack{0 \\ \text{almost always}}}$$

$$\overline{\nabla} \times (\overline{\nabla} \times \overline{B}) = \nabla \underbrace{(\nabla \cdot \overline{B})}_{ME} - \overline{\nabla} \cdot \overline{\nabla} B = \Delta \overline{B}$$

$$\Delta \overline{B} = \mu_0 \frac{B}{\Lambda}$$

1 dim $\quad \frac{\partial^2}{\partial x^2} \overline{B} = \frac{\mu_0}{\Lambda} B$

$$\overline{B} = \overline{B}_0 \cdot e^{-x/\lambda_L}$$

$$\frac{\partial^2}{\partial x^2} \overline{B} = \overline{B}_0 \cdot \left(-\frac{1}{\lambda_L}\right)^2 e^{-x/\lambda_L} = \frac{1}{\lambda_L^2} \overline{B}$$

$$\Rightarrow \quad \frac{\mu_0}{\Lambda} = \frac{1}{\lambda_L^2} \qquad \lambda_L = \sqrt{\frac{\Lambda}{\mu_0}} = \sqrt{\frac{m}{2 \rho_p \mu_0 e^2}}$$

$B = 0$ well inside SC

Meissner effect

Current densities

LE

$$\nabla \times \overline{J}_s = -\frac{\overline{B}}{\Lambda}$$

$$\overline{\nabla} \times (\overline{\nabla} \times \overline{J}_s) = -\frac{1}{\Lambda} \nabla \times B = -\frac{\mu_0}{\Lambda} J_s$$

$$\overline{\nabla} \underbrace{(\overline{\nabla} \cdot \overline{J}_s)}_{\text{no source}} - \overline{\nabla} \cdot \overline{\nabla} \overline{J}_s = -\frac{\mu_0}{\Lambda} \overline{J}_s$$

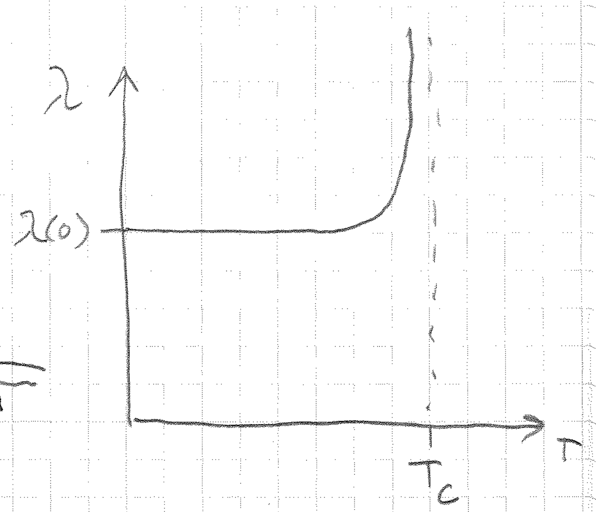
$$\nabla^2 \overline{J}_s = +\frac{\mu_0}{\Lambda} \overline{J}_s = \frac{\overline{J}_s}{\lambda_L^2}$$

$$\Rightarrow \quad \overline{J}_s = \overline{J}_{s0} e^{-x/\lambda_L}$$

For a simply connected SC we can adopt the London Guage $\Rightarrow$

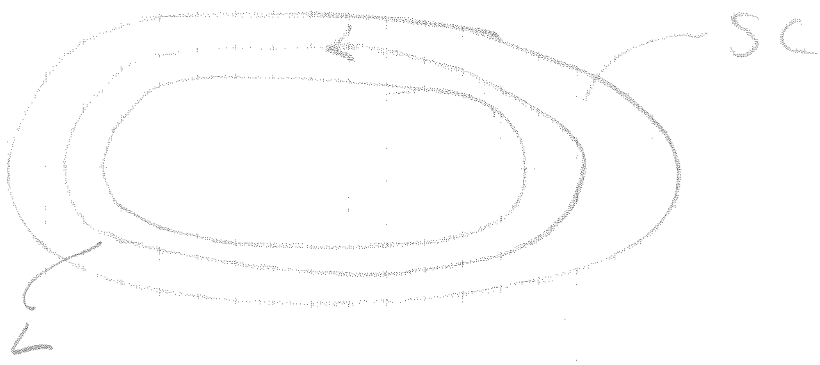
$$\nabla^2 \overline{A} = \frac{\overline{A}}{\lambda_L^2}$$

$$\lambda_L \propto \frac{1}{\sqrt{\rho_p}}$$



$$\lambda_L = \lambda(0) \frac{1}{\sqrt{1 - \left(\frac{T}{T_c}\right)^4}}$$

Flux quantization



$$\Lambda J_s = -\left(\frac{\hbar}{2e}\nabla\theta + \bar{A}\right)$$

$$\oint_L \Lambda \bar{J}_s \, dl = -\frac{\hbar}{2e}\oint_L \nabla\theta \, dl - \oint_L \bar{A}\, dl$$

well inside SC $J_s = 0$

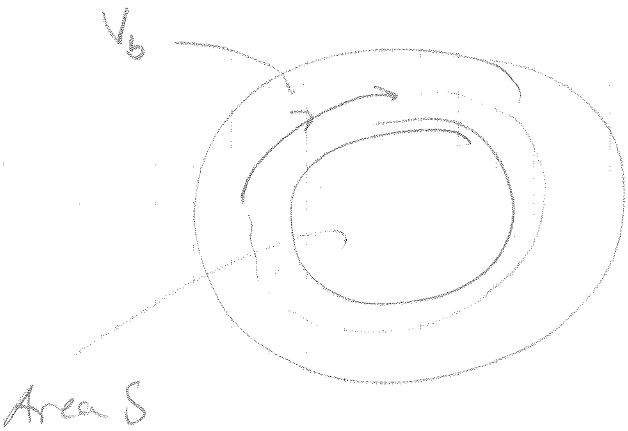
$$0 = -\frac{\hbar}{2e} n \cdot 2\pi - \Phi$$

$$\Phi = \frac{h}{2e}(+n) = \Phi_0 \cdot n$$

flux quantum $\frac{h}{2e} = 2{,}07 \cdot 10^{-15}$ Vs

$0{,}002T = 20{,}7$ G in $1 \mu m^2$

A rotating SC



rotating with th angular frequency Ω

$$\bar{\nabla}\times\bar{B} = \mu\bar{J} = -\rho_p 2e\left(\bar{V}_s-\bar{V}_b\right)$$

$$\bar{\nabla}\times\bar{\nabla}\times\bar{B} = \mu_0\bar{\nabla}\times\bar{J} = -\mu_0\rho_p 2e\left(\bar{\nabla}\times\bar{V}_s-\bar{\nabla}\times\bar{V}_b\right)$$

$$\nabla B = \frac{1}{\lambda_L^2}\bar{B} - \mu_0\rho_p 2e\underbrace{\bar{\nabla}\times\bar{V}_b}_{2\bar{\Omega}} \equiv \frac{1}{\lambda_L^2}\left(\bar{B}-\bar{B}_L\right)$$

The London Field

$$\bar{B}_L = \frac{2m_0}{e}\bar{\Omega}$$

the field in the SC falls off to B_L not to zero

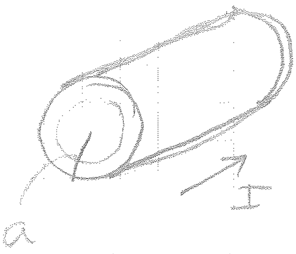
$$\oint_L \Lambda J dl = \int_L \Im\Lambda J_s dl + \oint_L \Lambda\frac{\rho_p 2e}{\mu_0}\bar{V}_b\, dl$$

$$0 = -n\Phi_0 - \Phi + \underbrace{\int_L \frac{m}{2e^2\rho_p}\frac{2\rho_p e^2}{\mu_0}\bar{V}_b\cdot dl}_{\frac{2m_0}{e}\bar{\Omega}\cdot\bar{S}}$$

$$\bar{\Phi} = n\Phi_0 + \frac{2m}{e}\bar{\Omega}\cdot\bar{S}$$

Kinnetic inductanse

$$\bar{J}_s = -2e\rho_p \bar{v}_s$$



$$E_k = \rho_p \frac{1}{2} 2m \cdot \bar{v}_s^2$$

Combining eqns

$$v_s = -\frac{\bar{J}_s}{2e\rho_p}$$

$$E_k = \rho_p \cdot m \frac{\bar{J}_s^2}{4e^2\rho_p^2} = \frac{m}{4e^2\rho_p}\bar{J}_s^2$$

$$= \frac{\Lambda}{2} J_s^2 = \frac{\mu_0\lambda_L^2}{2}\bar{J}_s^2$$

$$E_{tot} = E_{mag} + E_K$$

$$\frac{LI^2}{2} = \overbrace{\int_V dV \frac{B^2}{2\mu_0}}^{\propto I^2} + \overbrace{\int_V dV \frac{\mu_0\lambda_L^2}{2}\bar{J}_s^2}^{\propto I^2}$$

$$L = L_{mag} + L_{k.in.} = L_{or} + L_{oo} + L_K$$

L_{kin} dep strongly on jeometry

Grows quickly as d becomes smaller than λ

d