A simple mathematical model of a dripping tap

Thomas Schmidt† and Marko Marhl‡§

† 2 Obergymnasium Steglitz, Gritznerstrasse 57, D-12163 Berlin, Germany
‡ University of Maribor, Faculty of Education, Department of Physics, Koroška c. 160, SI-2000 Maribor, Slovenia

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Abstract. A mathematical model of the well known chaotic behaviour of a dripping tap is presented. The purpose of the model is to give a good insight into the dynamics of the system, whereas the usual experimental measurements can only give us its discrete states. Such a combined approach should be of special importance for the introduction of chaotic systems in school. For this purpose we constructed such a model which is simple enough to be understandable by pupils in secondary school, whereby all the main processes of the real system are included. This is confirmed by good agreement of the model's predictions with the experimental measurements.

Zusammenfassung. Im folgenden soll ein mathematisches Modell des bekannten chaotischen Verhaltens eines tropfenden Wasserhahnes präsentiert werden. Das Modell soll einen guten Einblick in das dynamische Verhalten des Systems ermöglichen, da experimentelle Messungen lediglich die diskreten Zustände beschreiben. Solch ein kombinierter Ansatz ist von besonderer Bedeutung für die Einführung chaotischer Systeme in der Schule. Es wurde ein Modell konstruiert, das einfach genug ist, von Schülern der Oberstufe verstanden zu werden und dabei alle grundlegenden Prozesse des realen Systems beinhaltet. Dies wird durch die große Übereinstimmung der Ergebnisse des Modells mit den experimentellen Messungen bestätigt.

1. Introduction

Recently, many experiments have been carried out on the chaotic behaviour of nonlinear systems. A well known example of such a system is the dripping water tap studied by many authors. The pioneering work in the field were the experiments carried out by Shaw [1] which were analysed in detail by Sternemann [2]. Subsequently the system was used by many other authors to consider its role in school [3]. These studies have shown that the dripping tap, considered to be a simple discrete chaotic system, can function as a very suitable introduction to nonlinear physics.

Usually, nonlinear physics deals with dynamical systems; based on positive experiences with the dripping tap experiment we show that it can be used as an introduction to dynamic chaotic systems as well. Indeed, a simple mathematical model has been found which represents a dynamical description of the dripping water. It enables us to observe the complete time course of the dripping water, as well as to make a comparison with the discrete system characteristics obtained by the experiments.

In the present paper, the development of the mathematical model compared with the known experimental data is presented. To show how well the experiment is reproducible and how easily it can be used in school, the experimental treatment in our laboratory is discussed briefly in section 2. The mathematical modelling of the system follows in section 3. The model predictions, presented and compared with the experimental data in section 4, are discussed in section 5.

2. Real experiment

These experiments are based on suggestions of Sternemann [2] and other works [3, 4]. The simple construction and straightforward measurements with good reproducibility make the experimental work appropriate for secondary school students. The experiment consists simply of a tap connected to a water container from which water flows out continuously (figure 1). In particular, we use an open plastic bottle as a water container. The tap is made of glass and has a drain pipe of length 40 mm and a diameter of 8 mm. The end of the pipe is cut straight.

In the experiment, the dripping behaviour of the tap is studied. This behaviour is evaluated by measuring the dependence of time intervals between two successive drips T_i on the water afflux from the container. Note that the afflux changes continuously with time t over which the experiment was carried out. The functional

§ E-mail address: marko.marhl@uni-mb.si

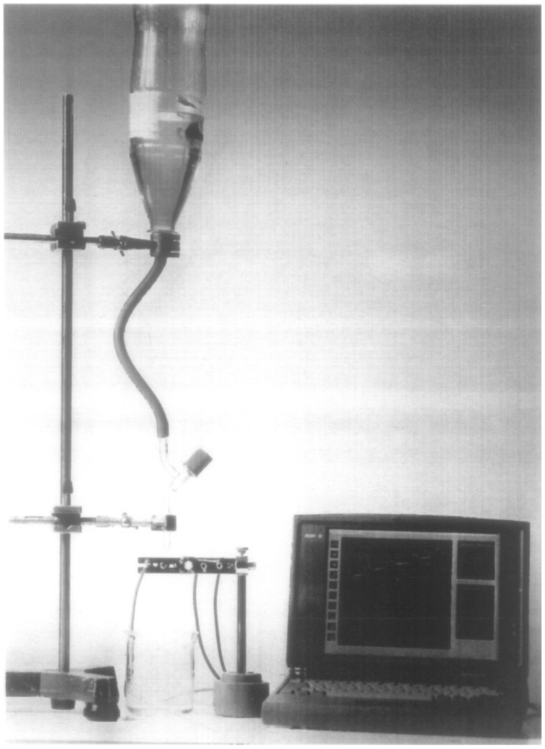

Figure 1. Experiment used in our laboratory.

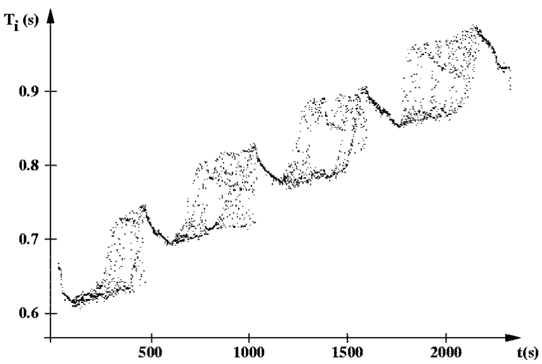

Figure 2. Typical measurement obtained in our laboratory.

dependence between the afflux and the time is given by an exponential function (see the appendix). It enables us to measure the real time simply. So, our results represent measurements of time intervals between two successive drips T_i against real time t. A typical measurement is shown in figure 2. It shows the characteristic behaviour of water dripping, where areas with stable dripping behaviour alternate with areas of chaotic behaviour. It should be pointed out that results are well reproducible and structurally independent of the equipment.

Table 1. The model parameters for which all results are calculated unless otherwise stated.

Parameter	Value (arbitrary units)
Geometrical parameters	
k	1.00×10^{-2}
β_1	3.00×10^{-4}
β_2	1.00×10^{-1}
α	2.86×10^{-2}
x_0	3.00×10^{-2}
Φ	3.50×10^{-5}
First initial conditions	
$x\|_{\tau_0}$	2.00×10^{-2}
$m\|_{\tau_0}$	1.00×10^{-6}

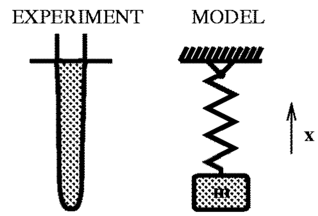

Figure 3. Schematic presentation of the model system.

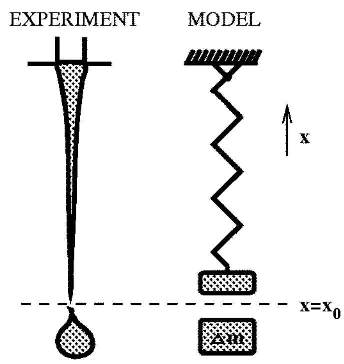

Figure 4. Schematic presentation of the water drop disconnection.

3. Mathematical model

In the following, the experimental observations and results are be described by a mathematical formalism. Note that the experimental observations are strictly discrete and now the idea is to describe the whole dynamical behaviour of the system. It requires the construction of a model system on the basis of which an appropriate mathematical model can be constructed. The purpose of this work is to develop such a model which is simple enough to be understandable by secondary school pupils, and at the same time the model has to be a good description of all the main processes in the real system.

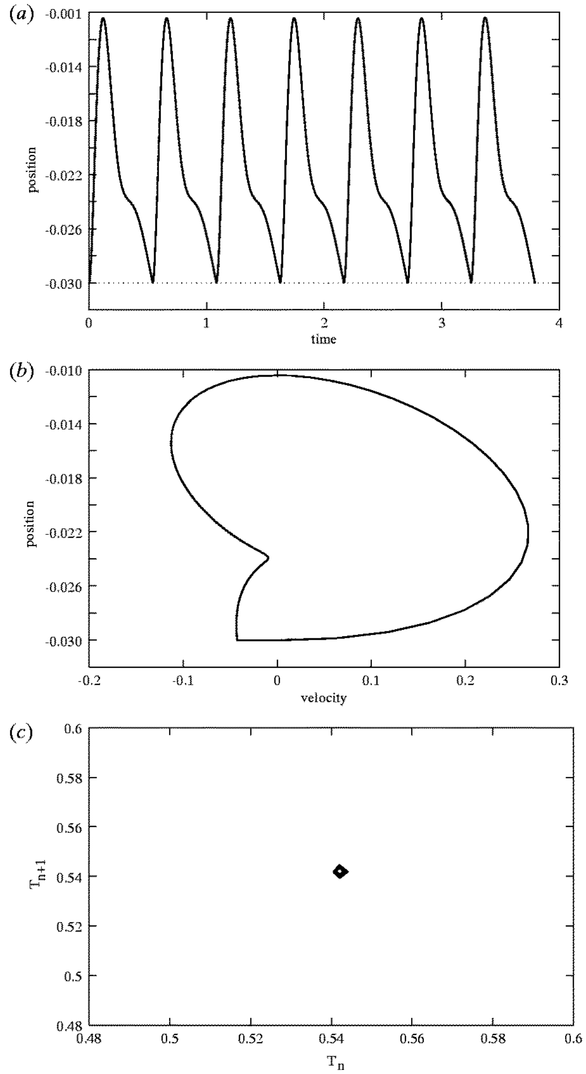

Figure 5. Water dripping with a well defined frequency at a water afflux of $\Phi = 3.5 \times 10^{-5}$ (for other parameters see table 1). Plots represent (*a*) the drop position versus time, (*b*) the phase plane drop position versus its velocity and (*c*) the return map of time intervals between successive drips.

An idea for a model system has been given by Shaw [1], but to our knowledge it has never been developed in the sense of a mathematical model which would be able to reproduce the experimental results. However, following this idea the water dripping system can be seen as a simple mechanical model considered as an increasing mass attached to a spring. This increasing mass is related to the growing drop on the tap, whereas the surface tension of the drop is modelled by the spring (figure 3).

The mathematical model describing the model system presented above is quite simple and straightforward. It consists of the following two equations:

$$m\ddot{x} = mg - kx - \beta\dot{x}, \quad \text{where} \quad \beta = \beta_1 + \beta_2\Phi, \tag{1}$$

$$\dot{m} = \Phi, \tag{2}$$

where x is the displacement of the oscillator, m is the mass of the oscillating water, k is the spring constant, Φ is the influx of water and β is the damping constant. Note that the damping of the oscillation is caused by the movement of the water in the drop (β_1) as well as by the influx of water (β_2).

Additionally, the model has two constraints. They determine the time τ_i at which the drop falls and also the mass of this drop. The following two conditions are taken in our model:

$$x = x_0 = \text{constant}, \tag{3}$$

$$\Delta m = \frac{\dot{x}}{\alpha + \dot{x}} m, \tag{4}$$

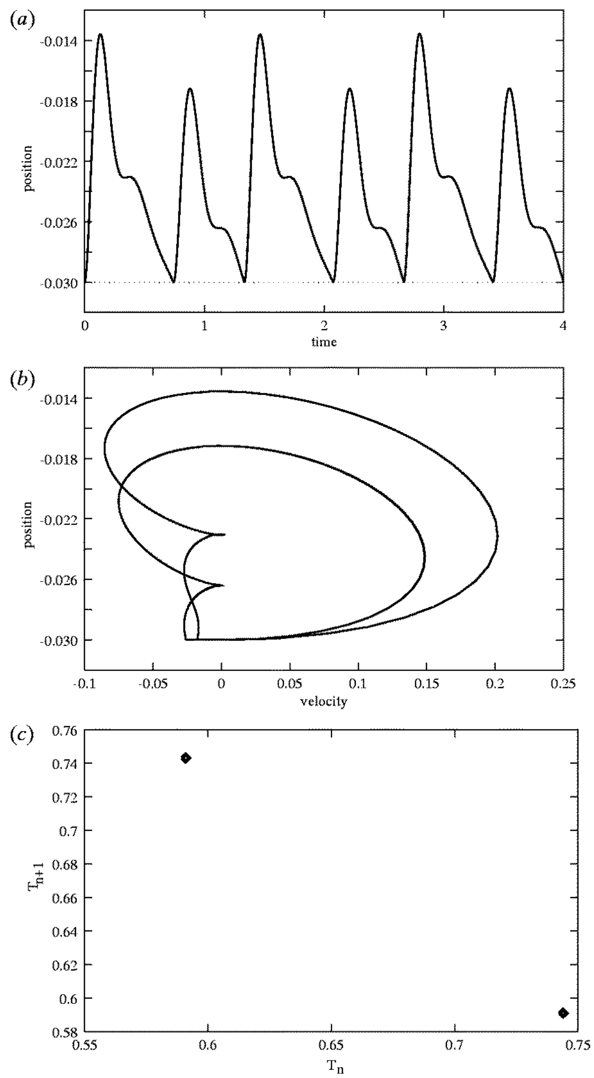

Figure 6. Water dripping with two frequencies at a water afflux of $\Phi = 2.0 \times 10^{-5}$ (for other parameters see table 1). Plots represent (*a*) the drop position versus time, (*b*) the phase plane drop position versus its velocity and (*c*) the return map of time intervals between successive drips.

where x_0 is the displacement at which the surface tension of the drop can no longer hold the form of the drop; it represents the displacement at which the drop falls (figure 4). The mass of the falling drop is denoted by Δm. It depends on the drop velocity $\dot{x}$ at position $x = x_0$. This determination of Δm with respect to $\dot{x}$ takes into account the inertia of the water in the drop. Namely, for small velocities small values for Δm are expected, whereas for larger $\dot{x}$ the values for Δm become closer to its limiting value m.

The initial conditions of the model system equations at every time τ_i are determined by the following set of equations:

$$x|_{\tau_i} = x_0 \tag{5}$$

$$\dot{x}|_{\tau_i} = 0 \tag{6}$$

$$m|_{\tau_i} = m - \Delta m. \tag{7}$$

The model equations with their constraints and for initially determined conditions are integrated numerically by a Runge–Kutta method. The results of some calculations are presented in the next section.

4. Results

With the mathematical model a mathematical description of the real system is given. To show the time-dependent behaviour of the system the model equations

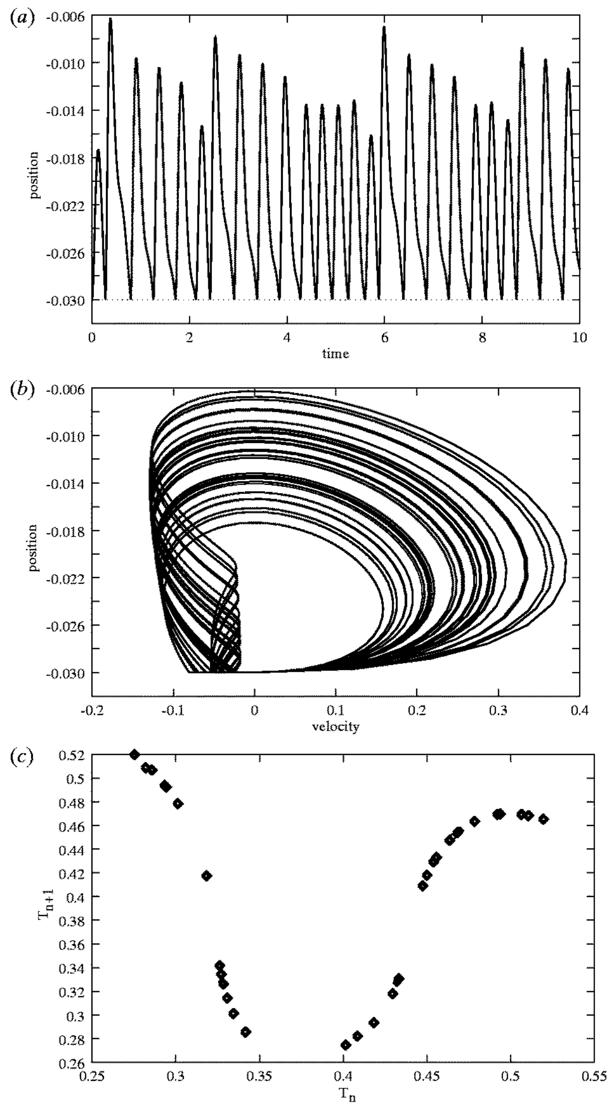

Figure 7. Chaotic behaviour of the system at a water afflux of $\Phi = 4.6 \times 10^{-5}$ (for other parameters see table 1). Plots represent (*a*) the drop position versus time, (*b*) the phase plane drop position versus its velocity and (*c*) the return map of time intervals between successive drips.

should be integrated. In our case, all results are obtained by a numerical integration of equations (1) and (2) for a given set of model parameters. Note that all results are calculated for parameters listed in table 1 unless otherwise stated.

In the first case, a model prediction for a rather large water afflux $\Phi = 3.5 \times 10^{-5}$ is given. With this afflux only one water dripping frequency is found (figure 5(*a*)). It can be seen more clearly in the phase plane (figure 5(*b*)) or in a return map (figure 5(*c*)). In our context, the return map represents a simplified analysis of a dynamical system in the sense of its discrete states. For our model it is of particular interest to show how T_n predicts T_{n+1}. This idea of mapping was introduced by Lorentz (1963) and the map is also known as the 'Lorentz map' (cf [5]).

Let us note that water dripping with only one well defined frequency can also be observed in the real experiment. This has been proved in our laboratory using a microphone and in a more precise way using a stroboscope.

In figure 6 the system behaviour for a smaller water afflux of $\Phi = 2.0 \times 10^{-5}$ is presented, where two different frequencies of water dripping are obtained. Such a behaviour of the system can also be found in the real experiment. Also in this case the frequency

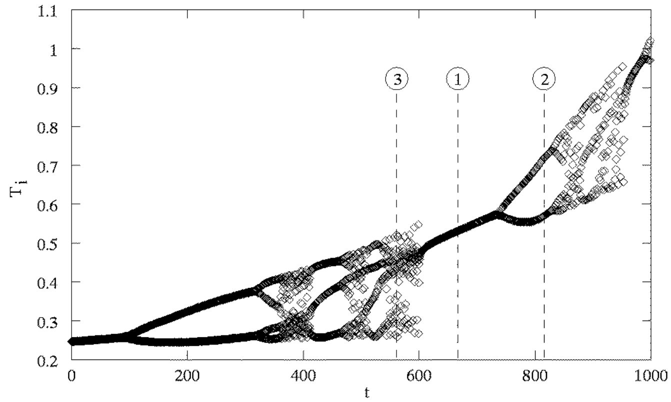

Figure 8. Feigenbaum diagram obtained by varying the water afflux and thus the time from 0 to 1000. Three cases are labelled: water dripping with one frequency (1), with two frequencies (2) and chaotic behaviour (3).

determination has been done by the use of a microphone and a stroboscope.

In general, different frequencies of water dripping can be obtained at different affluxes. In particular, we have parameter ranges where the system shows its chaotic behaviour (see figure 7).

Here for a better presentation of the whole behaviour of the system the Feigenbaum diagram is used. Generally, it represents all possible discrete states observed in a given system versus one of its parameters. The intervals between two drops T_i versus the afflux Φ and thereby the total time (see the appendix) are presented by the Feigenbaum diagram in figure 8. For a better understanding of the system the three sections of the above described solutions are labelled 1 for system behaviour represented in figure 5, 2 for system behaviour represented in figure 6 and 3 for system behaviour represented in figure 7.

In this way, figure 8 enables us to make a qualitative comparison with the experimentally obtained Feigenbaum diagram (see figure 2). The main point of this comparison is the structure of both diagrams, which clearly confirms the good agreement of the model predictions with the experimental results.

5. Conclusion

In this paper we have established a mathematical model of the well known dripping water tap experiments. In this way this system together with the experiment and its mathematical description can be used as a complete introductory example of chaotic systems in school. It enables us to make not only a discrete observation and description of the system but of its whole dynamics as well.

Note that the dripping water tap is well suited as an introductory experiment to the theory of discrete chaotic systems. The experimental arrangements can easily be carried out in the class and the results provide sufficient material for discussion. In addition, our mathematical model enables us to use this system as an introduction to dynamic chaotic systems. The necessary knowledge about typical methods and commonly used diagrams in nonlinear physics can be gained with the help of this model.

Although our model represents only a very simple simulation of the real dripping behaviour, typical situations such as stable or chaotic dropping can be described with the help of it. The structure of the model's Feigenbaum diagram is in good agreement with the experimental data. Comparing the results of our work with those of Shaw [1], a great correspondence can be seen especially with the comparison of the phase plane diagrams. Furthermore, the return maps show typical structures for such systems.

With regard to the suitability in schools, our model has been designed to be very simple. The model needs extension if the experiment is to be described in a more specific way. To start with, the vibration of the water-chute above the drop should also be examined. Also the disconnection of the water drop could be described more precisely. In our opinion though, such an extension would not lead to any important new findings, but rather render more difficulties for the use in school.

Acknowledgments

We want to thank Mr W Sternemann for quickly providing much needed literature and Professor Hampbrecht of the Freie Universität Berlin for checking our work.

Appendix

Here the relation between the afflux from the container and the real time is presented in greater detail. In our case, in which a cylindrical water container is assumed, the following equations are valid:

$$\Phi = \frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\rho A\, \mathrm{d}h}{\mathrm{d}t}, \tag{A1}$$

and

$$\Phi = c\Delta p = c\rho g h, \tag{A2}$$

where A is the cross sectional area of the water container, h the height of the water over the drain pipe, ρ is the density of the water and c is a constant of proportionality between the afflux and the appropriate water pressure. Equations (A1) and (A2) give a simple differential equation system and by separation of variables and integration one gets

$$h = h_0 \mathrm{e}^{-\lambda t}, \tag{A3}$$

where λ is the geometric parameter of the system. After inserting into equation (A2) the result is

$$\Phi = \Phi_0 \mathrm{e}^{-\lambda t}. \tag{A4}$$

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