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Fast Fourier Transforms and Power Spectra in LabVIEW®

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Introduction

The Fourier transform is one of the most powerful signal analysis tools, applicable to a wide variety of fields such as spectral analysis, digital filtering, applied mechanics, acoustics, medical imaging, modal analysis, numerical analysis, seismography, instrumentation, and communications.

The LabVIEW analysis virtual instruments (VIs) in the **Signal Processing** palette maximize analysis throughput in FFT-related applications. This application note discusses some FFT properties, how to interpret and display FFT results, and how to further manipulate FFT and power spectrum results to extract useful frequency information.

For more information, refer to the sections on digital signal processing, the FFT, and the power spectrum in your LabVIEW analysis VI reference manual.

FFT Properties

The Fourier transform maps time domain functions into frequency domain representations and is defined as

$$X(f) = \boldsymbol{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\ e^{-j2\pi ft}\ dt\ , \tag{1}$$

where   x(t) is the time domain signal, and
          X(f) is its Fourier Transform.

Similarly, the *discrete Fourier transform* (DFT) maps discrete-time sequences into discrete-frequency representations and is given by

$$X_k = \sum_{i=0}^{n-1} x_i\ e^{-j2\pi ik/n} \qquad \text{for } k = 0, 1, 2, \dots, n\text{-}1\ , \tag{2}$$

where   x is the input sequence,
          X is its DFT, and
          n is the number of samples in both the discrete-time and the discrete-frequency domains.

Direct implementation of the DFT, equation 2, requires approximately n^2 complex operations. However, computationally efficient algorithms can require as little as $n \log_2(n)$ operations. These algorithms are called *fast Fourier transforms* (FFTs).[4,5,6]

From the definition of the DFT, equation 2, the Fourier transform of any sequence x, whether it is real or complex, always results in a complex output sequence X of the form

$$\boldsymbol{F}\{x\} = X = X_{Re} + j\,X_{Im} = Re\{X\} + j\,Im\{X\}. \tag{3}$$

An inherent DFT property is

$$X_{n-i} = X_{-i}, \tag{4}$$

which means that the $(n-i)^{th}$ element of X contains the result of the $-i^{th}$ harmonic. Furthermore, if x is real, the ith harmonic and the $-i^{th}$ harmonic are complex conjugates

$$X_{n-i} = X_{-i} = X^*_i. \tag{5}$$

Consequently,

$$Re\{X_i\} = Re\{X_{n-i}\}, \tag{6}$$

and

$$Im\{X_i\} = -Im\{X_{n-i}\}. \tag{7}$$

These symmetrical Fourier properties of real sequences are referred to as conjugate symmetric (equation 5), symmetric or even-symmetric (equation 6), and anti-symmetric or odd-symmetric (equation 7).

FFT Output Format
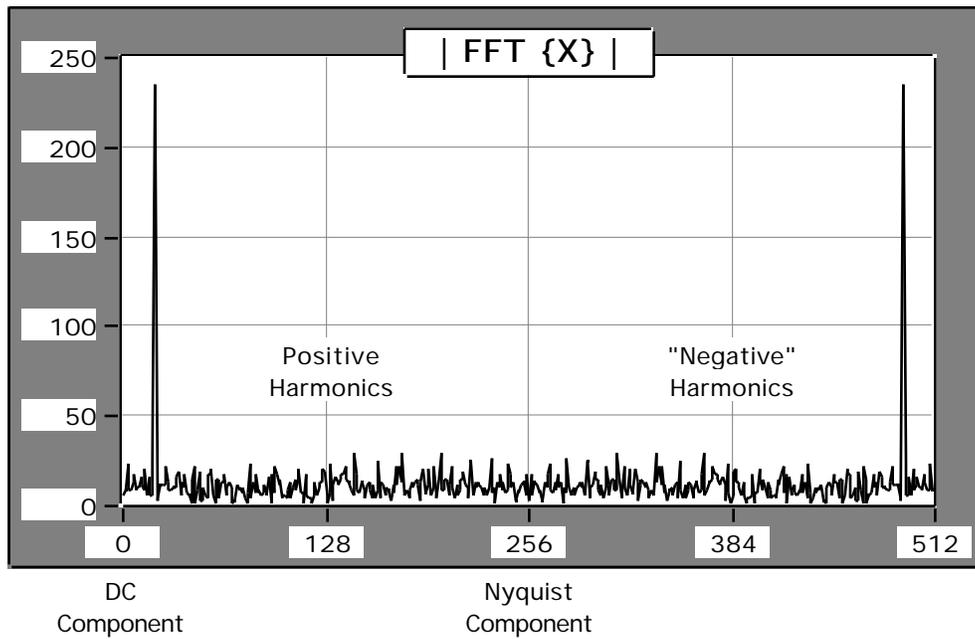
The output format of the FFT VI can now be described with the aid of equation 4. If the total number of samples n of the input sequence x is even, the format of the complex valued output sequence X and the interpretation of the results are shown in Table 1.

Table 1. Output Format for Even n, k = n ÷ 2

Array Element	Interpretation
X_0	DC component
X_1	1st harmonic or fundamental
X_2	2nd harmonic
X_3	3rd harmonic
.	.
.	.
.	.
X_{k-2}	$(k-2)^{th}$ harmonic
X_{k-1}	$(k-1)^{th}$ harmonic
$X_k = X_{-k}$	Nyquist harmonic
$X_{k+1} = X_{n-(k-1)} = X_{-(k-1)}$	$- (k-1)^{th}$ harmonic
$X_{k+2} = X_{n-(k-2)} = X_{-(k-2)}$	$- (k-2)^{th}$ harmonic
.	.
.	.
.	.
X_{n-3}	$- 3^{rd}$ harmonic
X_{n-2}	$- 2^{nd}$ harmonic
X_{n-1}	$- 1^{st}$ harmonic

Figure 1 represents the output format described in Table 1 for n = 512.

Figure 1. Graph of Table 1

If the total number of samples n of the input sequence x is odd, the format of the complex valued output sequence X and the interpretation of the results are shown in Table 2.

Table 2. Output Format for Odd n, k = (n-1) ÷ 2

Array Element	Interpretation
X_0	DC component
X_1	1st harmonic or fundamental
X_2	2nd harmonic
X_3	3rd harmonic
.	.
.	.
.	.
X_{k-1}	k^{th}-1 harmonic
X_k	k^{th} harmonic
$X_{k+1} = X_{n-k} = X_{-k}$	- k^{th} harmonic
$X_{k+2} = X_{n-(k-1)} = X_{-(k-1)}$	- $(k-1)^{th}$ harmonic
.	.
.	.
.	.
X_{n-3}	- 3rd harmonic
X_{n-2}	- 2nd harmonic
X_{n-1}	- 1st harmonic

Figure 2 represents the output format described in Table 2 for n = 501.
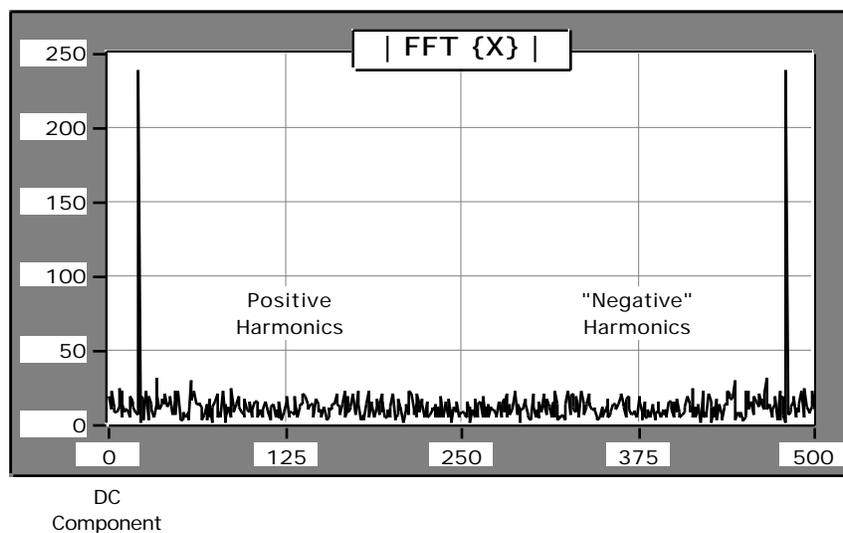
Figure 2. Graph of Table 2

Displaying FFTs of Real-Valued Sequences

In most applications, you perform Fourier and spectral analysis only on real-valued, discrete-time sequences. This section describes three common methods of displaying the FFT results of real-valued input sequences.

•   Standard Format.

•   Double-Sided Format.

•   Single-Sided Format.

The time-domain signal shown in Figure 3 demonstrates the three methods by which FFT results can be displayed graphically. This signal is used because a signal of interest is buried in the noise and because the signal of interest is easily identified in the frequency domain. Based on frequency information, digital filtering can remove the noise from the signal.
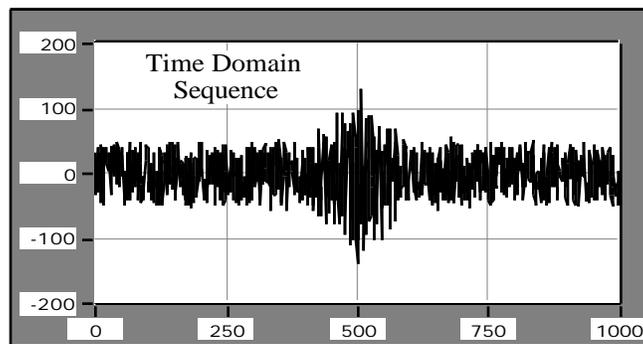
Figure 3. Time Domain Sequence for Display Examples

Standard Output

The format for even and odd sized discrete-time sequences described in tables 1 and 2 of this note is the standard output. This format is convenient because it does not require any further data manipulation.

To graphically display the results of the FFT, wire the output arrays to the Waveform Graph, as shown in Figure 4. Remember that FFT output is complex and, therefore, two graphs are required to display all the information.
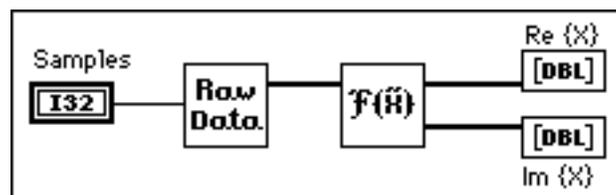

Figure 4. Block Diagram to Display the Standard Output

Figures 5 and 6 are the plots of the real and imaginary portions, respectively, of the FFT VI results. The symmetrical properties summarized in equations 5 through 7 are clearly visible in the plots.
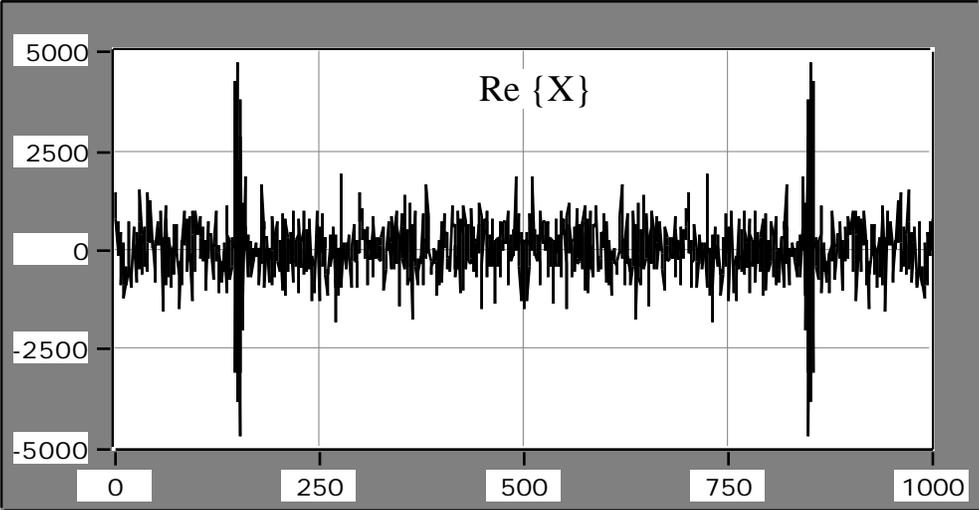
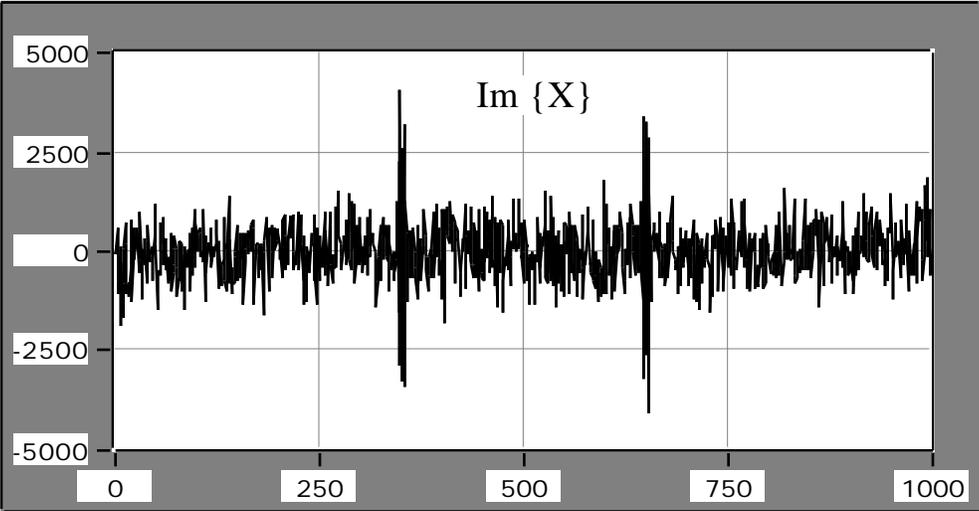Figure 5. Real Portion of the Fourier Transform

Figure 6. Imaginary Portion of the Fourier Transform

Double-Sided Output

The Fourier transform integral, equation 1, is defined in the $(-\infty,\infty)$ frequency range. Presenting Fourier transform results in the $(-\infty,\infty)$ frequency range is known as *double-sided format*, an example of which is shown in Figure 7.
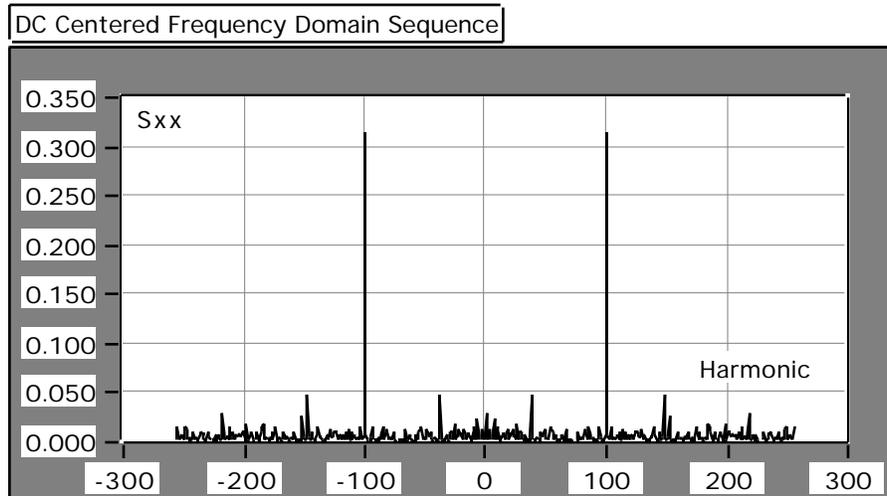
Figure 7. Double-Sided Format

You can obtain double-sided output format from the standard output by recalling the identity $X_{-i} = X_{n-i}$. To present the data in a double-sided format, you must split the arrays at their center point into two portions (corresponding to the positive and negative frequencies), and reverse the array order by appending the positive frequencies to the negative frequencies. You can do this in LabVIEW with the Split 1D Array and Build Array functions as shown in Figure 8.
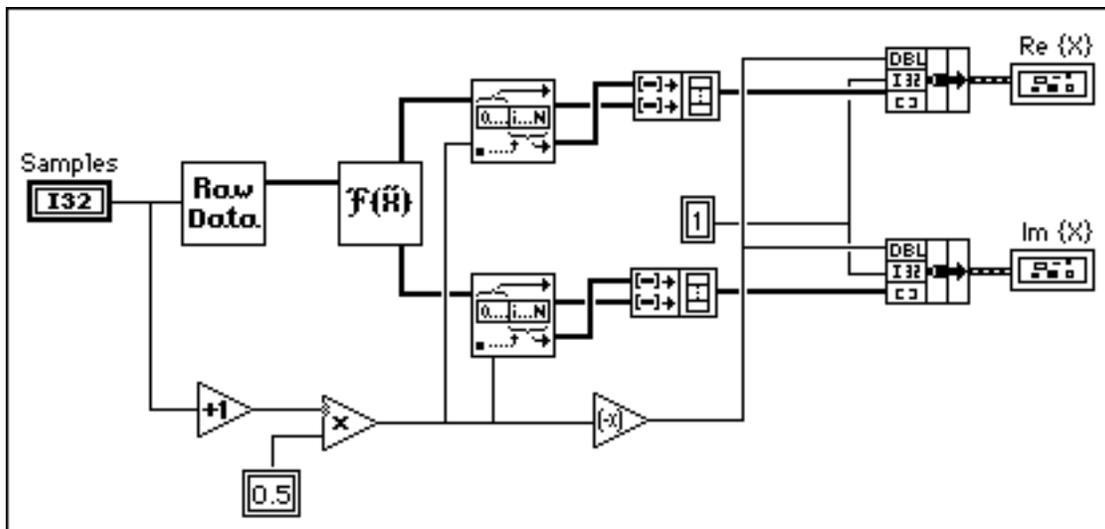

Figure 8. Block Diagram to Display Double-Sided Output Format

The lower portion of the diagram finds the index at which the arrays need to be split. This technique works for both an even and odd number of samples.

The results of processing the data using the block diagram shown in Figure 8 are shown in figures 9 and 10, corresponding to the real and imaginary parts, respectively. Notice that both graphs display the frequency information from - (n ÷ 2) to (n ÷ 2) and that the symmetry properties about zero are clear.
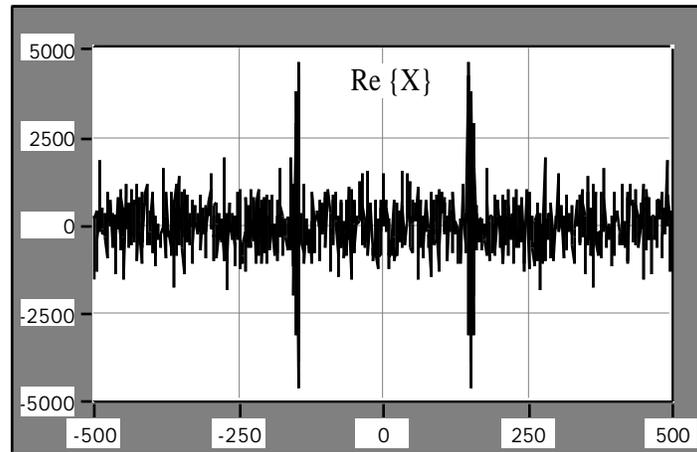

Figure 9. Real Portion of the Double-Sided Format
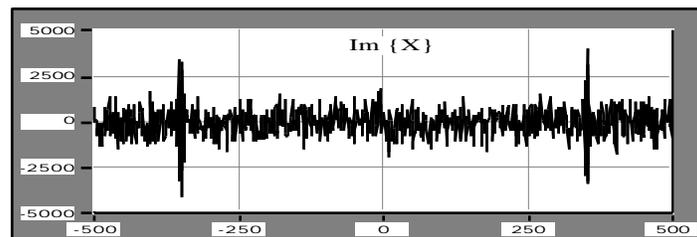
Figure 10. Imaginary Portion of the Double-Sided Format

Single-Sided Output

In the cases of standard and double-sided output formats, almost half the information is redundant. This is clear from equations 5, 6, and 7. Because the data is conjugate symmetric about the $(n÷2)^{th}$ harmonic, also known as the *Nyquist harmonic*, you can discard all the information above the Nyquist harmonic because you can reconstruct it from the frequencies below the Nyquist harmonic. Displaying only the positive frequencies gives you single-sided output.

The block diagram shown in Figure 11 uses the Array Subset function to select all the elements corresponding to the positive frequencies, including the DC component. Like the double-sided case, the lower portion of the diagram selects the total number of elements in the subset, and it also works for both an even and odd number of samples.
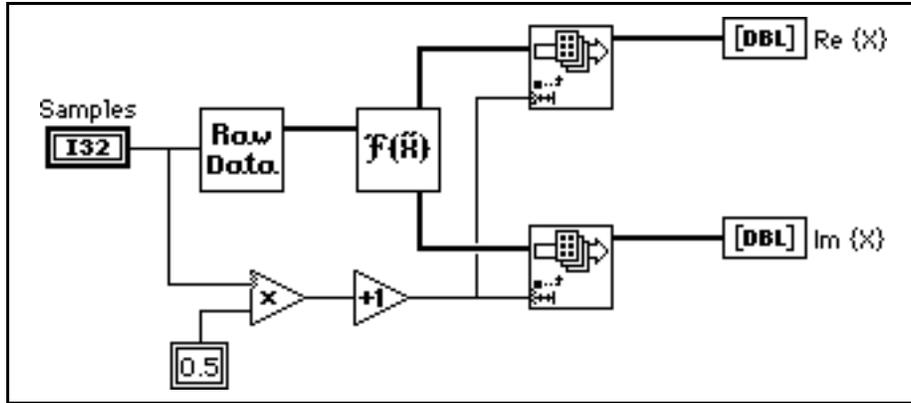
Figure 11. Block Diagram for the Single-Sided Output

The results of processing the data using the block diagram in Figure 11 are shown in figures 12 and 13, corresponding to the real and imaginary parts, respectively. Both graphs display the frequency information from 0 to (n ÷ 2), which is about half the points presented in the standard and double-sided outputs.
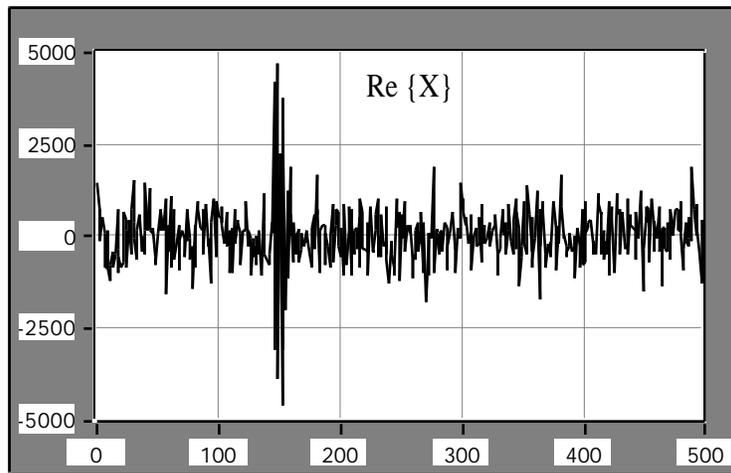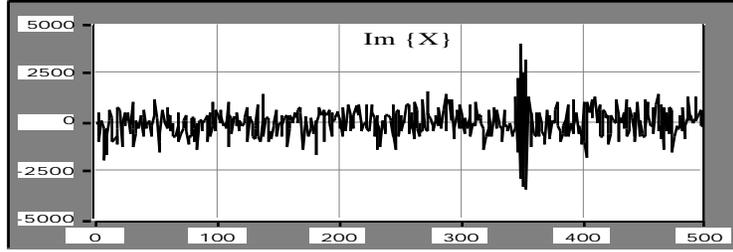

Figure 12. Real Portion of the Single-Sided Format

Figure 13. Imaginary Portion of the Single-Sided Format

Note:　　You must pay careful attention if you make measurements with single-sided data because the total energy at a particular frequency is equally divided between the positive and negative frequency (DC and Nyquist components excluded). For more information consult the *Single-Sideband Fast Fourier Transforms* example in the *LabVIEW Analysis VI Library Reference Manual.*

Fourier Analysis Using FFTs

To Fourier analyze a discrete-time signal, equation 2 must include a $1/n$ scaling factor where n is the number of samples in the sequence. Figure 14 shows a block diagram segment that scales the FFT results by the $1/n$ factor. You can apply the same scaling factor to the double-sided and single-sided formats.
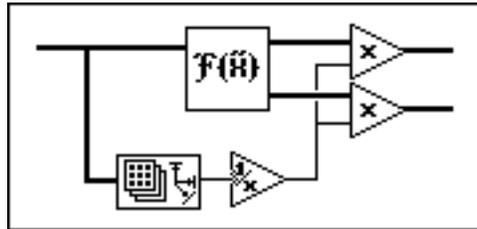

Figure 14. Scaling FFT Arrays for Fourier Analysis

For example, the Fourier Transform of the sine wave shown in Figure 15 is

$$\boldsymbol{F}\{\sin(2\pi f_0 t)\} = 0.5 \times [\delta(f - f_0) + \delta(f + f_0)] \tag{8}$$

where $\qquad \delta(f) = \begin{cases} 1 & \text{if } f = 0 \\ 0 & \text{elsewhere} \end{cases}$ $\qquad\qquad$ (9)
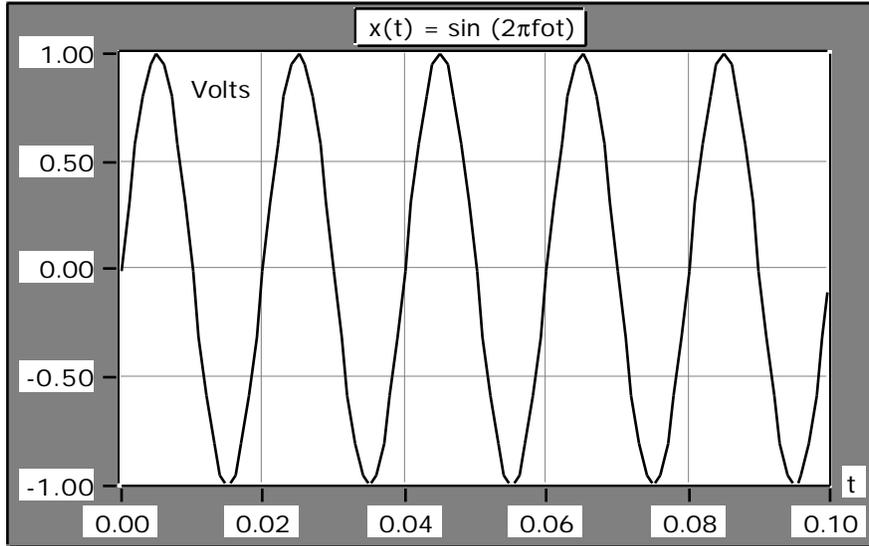

Figure 15. Sine Wave To Be Fourier Analyzed

The double-sided Fourier analysis of the sinusoidal waveform is shown in Figure 16.
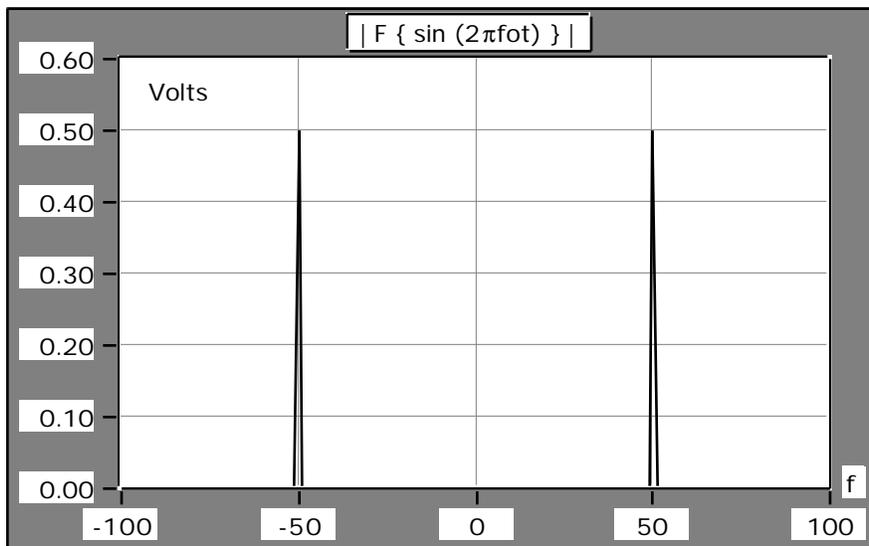

Figure 16. Fourier Analysis of a Sinusoidal Waveform

Obtaining Magnitude and Phase Information

In many applications it is not convenient to think in terms of complex data. You can instead present complex data as magnitude and phase data. To illustrate this point, figures 17 and 18 show the frequency response of a filter in terms of complex data. The same frequency response is shown in figures 19 and 20 as magnitude and phase data.
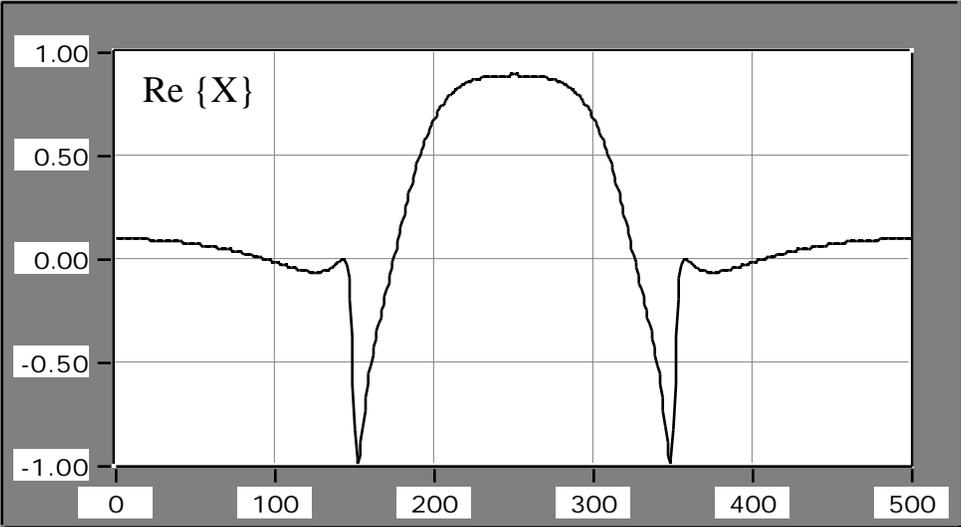

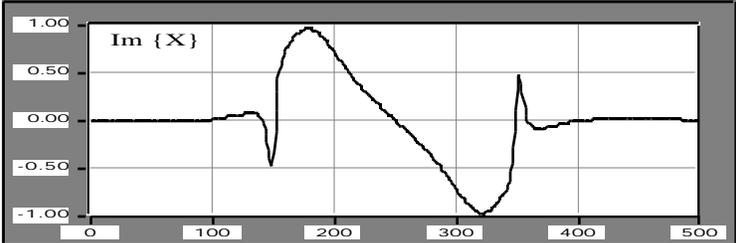Figure 17. Real Portion of Filter Response

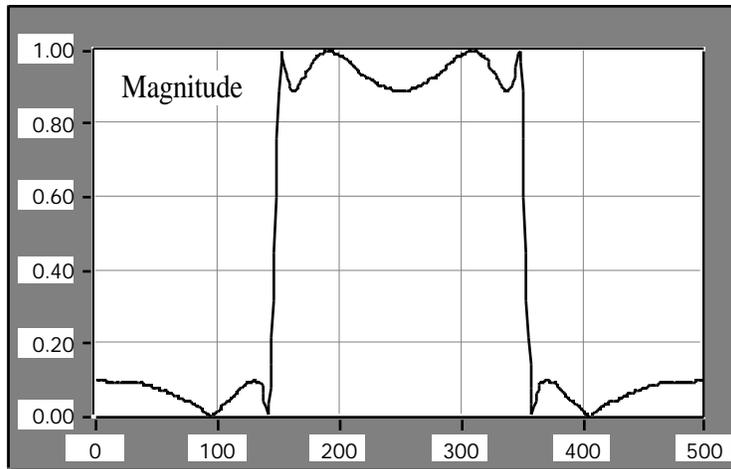Figure 18. Imaginary Portion of Filter Response
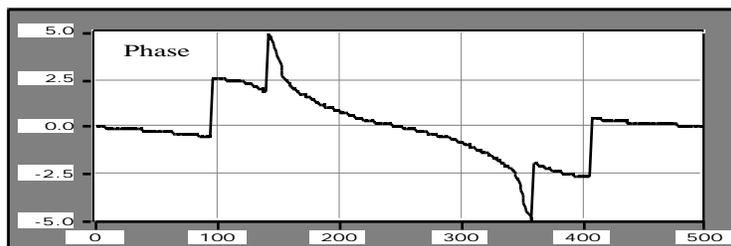
Figure 19. Magnitude Filter Response

Figure 20. Phase Filter Response

Although the phase information may not appear to carry useful information, the filter's magnitude frequency response is clearly that of a bandpass filter.

You obtain magnitude and phase information from complex data using the 1D Rectangular To Polar VI. Because the phase information is computed using an arc tangent function and the arc tangent function results are in the $-\pi$ to π range, you can use the Unwrap Phase VI to smooth out some of the discontinuities that may arise when complex data is converted to magnitude and phase data.

Figure 21 shows how to use the 1D Rectangular To Polar and the Unwrap Phase VIs to convert complex data into magnitude and phase format. Figure 21 shows single-sided output format, but the same technique can be applied to the standard output and double-sided output as well.
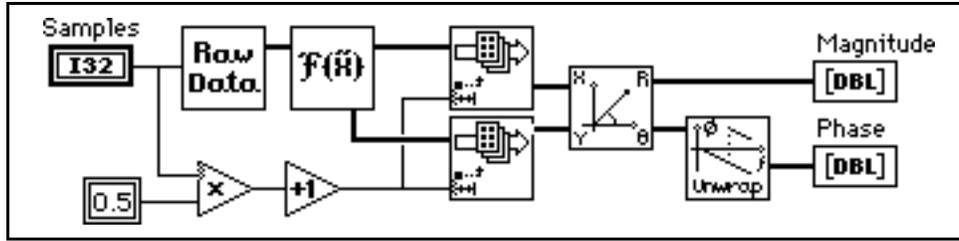
Figure 21. Converting Complex Data to Magnitude and Phase Format

Power Spectrum

You use the power spectrum function, which is closely related to the Fourier Transform, to calculate the harmonic power in a signal. The power spectrum $S_{XX}(f)$ of a time domain signal x(t) is defined as

$$S_{XX}(f) \;=\; X(f)\,X^{*}(f) \;=\; |\,X(f)\,|^{\,2} \tag{10}$$

where $X(f) = F\{x(t)\}$ and

$X^{*}(f)$ is the complex conjugate of X(f).

Because the power spectrum format is identical to the real part of the FFT, the discussion of standard, double-sided, and single-sided formats apply to the Power Spectrum VI. Furthermore, the single-sided format uses less memory, because this format eliminates redundancy while retaining the complete power spectrum information.

The Power Spectrum VI calculates the harmonic power in discrete-time, real-valued sequences. Figure 22 shows the single-sided output of the power spectrum of two 1-V peak sinusoids–one with 150 cycles and one with 250 cycles. The total power for each one of these sinusoids is 2×0.25 W.
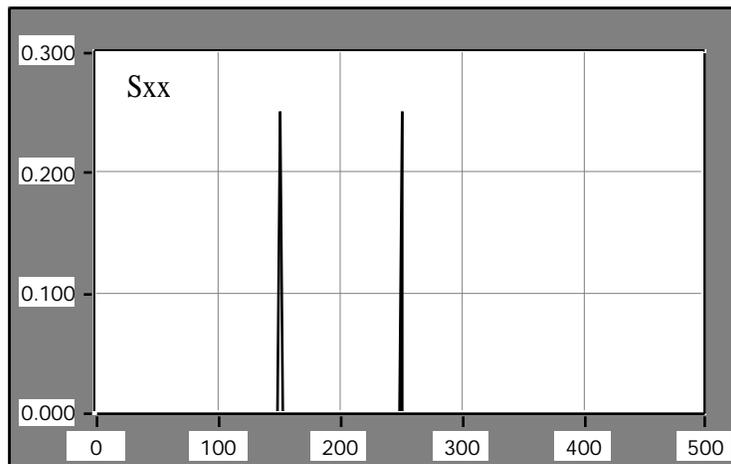

Figure 22. Power Spectrum of Two Sinusoids

The discrete implementation of the power spectrum S_{XX} is given by

$$S_{XX} = \frac{|F\{x\}|^2}{n^2} \; ,$$

(11)

where   x is the discrete-time, real-valued sequence and
n is the number of elements in x.

Unlike the FFT, power spectrum results are always real. The Power Spectrum VI executes faster than the FFT VI because the computation is done in place and there is no need to allocate memory to accommodate complex results. On the other hand, phase information is lost and cannot be reconstructed from the power spectrum's output sequence. You must use the FFT if phase information is important.

Converting Magnitude Information to Decibels

You can use the 10 Log X[i] and 20 Log X[i] VIs to convert 1D numeric arrays into decibels (dB), which is a common unit for representing power ratios.

The 10 Log X[i] VI can convert magnitude squared or power values, such as acoustic pressure waves, to decibels and the 20 Log X[i] VI can convert magnitude or amplitude values, such as voltages or currents, to decibels.

The block diagram in Figure 23 shows an example that converts the result of the power spectrum to decibel notation. The results are shown in Figure 24.
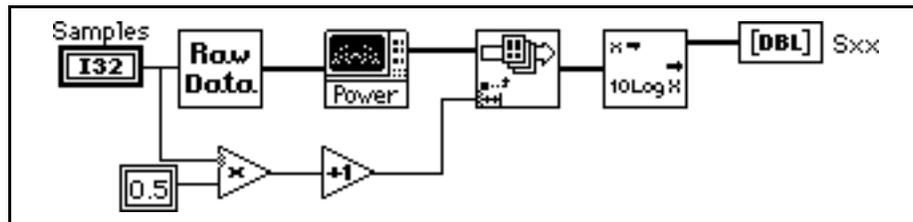

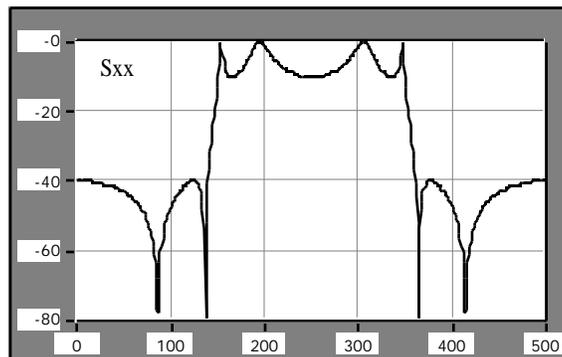Figure 23. Converting Magnitude Squared to Decibels Block Diagram

Figure 24. Converting Magnitude Squared to Decibels Results

Obtaining Frequency Information from Transforms

As discussed in the FFT VI description in the *LabVIEW Analysis VI Reference Manual*, the discrete implementation of the Fourier transform maps a digital signal into its Fourier series coefficients or harmonics. The arrays involved in FFT related operations do not contain discrete-time or discrete-frequency information, and modern acquisition systems, whether they use add-on boards or instruments to capture the data, allow you to control or specify the sampling interval Δt and obtain frequency scaling information from Δt.

Because an acquired array of samples commonly represents real-valued signals acquired at equally spaced time intervals, you can determine the value corresponding to the observed frequency in hertz. To associate a frequency axis to operations that map time signals into frequency domain representations, such as the FFT, power spectrum and Hartley transforms, the sampling frequency f_s must first be determined from Δt

$$f_s \ = \ \frac{1}{\Delta t} \ . \tag{12}$$

The frequency interval is given by

$$\Delta f \ = \ \frac{f_s}{n} \ = \ \frac{1}{n\Delta t} \ , \tag{13}$$

where n is the number of samples in the sequence.

Given the sampling interval 1.000E-3, the block diagram shown in Figure 25 demonstrates how to display a graph with the correct frequency scale.
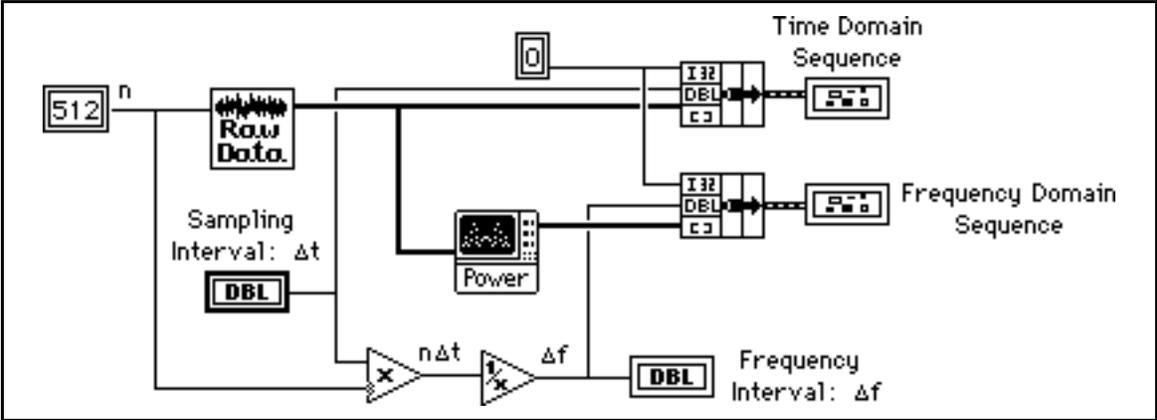

Figure 25. Obtaining Frequency Information from the Sampling Interval

Thus, for the signal x(t) represented in Figure 26, the resulting single-sided power spectrum graph with the correct frequency axis is presented in Figure 27, and the resulting frequency interval is 1.953E0.
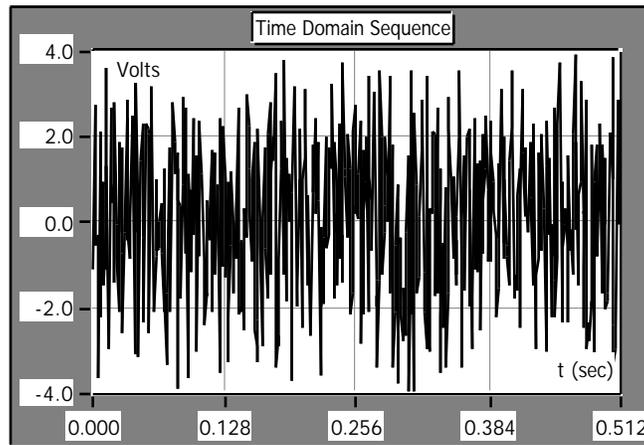

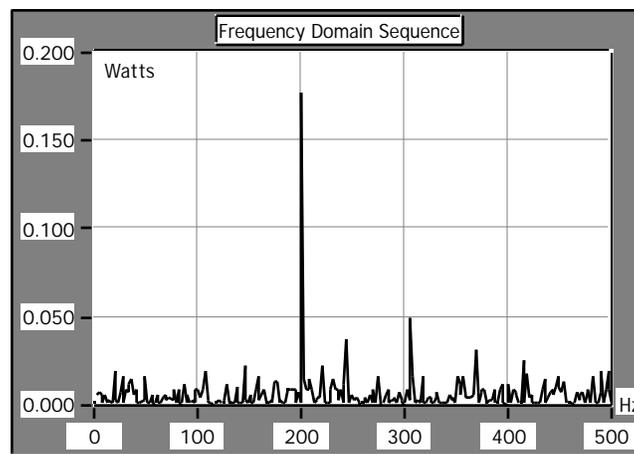Figure 26. Discrete-Time Sequence with Correct Time Axis

Figure 27. Discrete-Frequency Sequence with Correct Frequency Axis

The sampling interval is the smallest frequency that the system can resolve through FFT or related routines. A simple way to increase the resolution is to increase the number of samples or increase the sampling interval.

Summary

LabVIEW and its analysis VI library provide a complete set of tools to perform Fourier and Spectral Analysis. The FFT and Power Spectrum VIs have been optimized and their outputs adhere to the standard DSP format. This application note has demonstrated how to further manipulate the output data so that you can obtain double-sided and single-sided representations. It also demonstrated how to convert complex data into magnitude and phase information, how to convert results into decibel notation, and how to obtain the correct frequency information from the sampling interval.

References

1. Brigham, E.O. *The Fast Fourier Transform and its Applications*. Englewood Cliffs, New Jersey: Prentice Hall. 1988.

2. Elliot, D.F. *Handbook of Digital Signal Processing Engineering Applications*. San Diego: Academic Press. 1987.

3. Oppenheim, A.V. & Schafer, R.W. *Discrete-Time signal Processing*. Englewood Cliffs, New Jersey: Prentice Hall. 1989.

4. Rabiner, L.R. & Gold, B. *Theory and Application of Digital Signal Processing*. Englewood Cliffs, New Jersey: Prentice Hall. 1975.

5. Sorensen, H.V. et al. "On Computing the Split-Radix FFT." IEEE Transactions on ASSP. ASSP-34 (1986):1 (February).

6. Sorensen, H.V. et al. "Real-Valued Fast Fourier Transform Algorithms." IEEE Transactions on ASSP. ASSP-35 (1987): 6 (June).