1 Dynamical systems

Teachers
Kristian Gustafsson, kristian.gustafsson@physics.gu.se
Jan Meibohm (problem sessions)
Course page http://fy.chalmers.se/~f99krgu/dynsys/
Literature Nonlinear Dynamics and Chaos, by Steven Strogatz
Examination
Four sets of hand-in problems + Written exam on Monday, Jan 14 (All students must sign up before Dec. 20)
To pass the course a pass grade on both the problem sets and the exam are required (see home page).
Piazza
Forum for questions about homework or lectures. Questions should be asked here rather than in emails to the teachers. This allows other students to answer questions and to see answers to the questions. Anonymous posts are possible.
Lectures are interrupted by quiz questions:
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Room number: DYNSYS

1.1 What are dynamical systems?

Dynamical system = Set of quantities (system) + Rule how these change with time (dynamical)
Linear dynamical systems
Most systems encountered in introductory courses.
Often exact solutions using methods based on linear superposition.
Two examples: Small-amplitude oscillations of simple pendulum ($\theta = A\cos\omega t$) and double pendulum.
Non-linear dynamical systems
Most real-world systems are (at least to some degree) non-linear
Allows for new types of solutions (compared to linear systems).
Examples: Large-amplitude oscillations of simple pendulum and double pendulum.
Angle of single pendulum no longer well approximated by $A\cos(\omega t)$.

Motion of double pendulum becomes chaotic:

- Unpredictable (appears to be random although system is deterministic).
- Sensitive dependence on initial conditions, Two arbitrarily closeby initial conditions will show different trajectories after some time.

Non-linear systems often show chaotic behaviour.

Examples where dynamical systems are encountered -

Example	Typical variables
Classical Mechanics	Positions and momenta
Electrical circuits	Currents
Population dynamics	Number of individuals of different species
Chemical reactions	Concentrations of chemicals

Plus everywhere else you encounter ODEs or recurrence equations (such as processes in living organisms, control theory, economics, etc.)

1.1.1 Mathematical description of dynamical system

Continuous dynamical systems can be written as systems of coupled ordinary differential equations:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, \ldots, x_n)\\ \dot{x}_2 &= f_2(x_1, \ldots, x_n)\\ &\vdots\\ \dot{x}_n &= f_n(x_1, \ldots, x_n)\end{aligned}$$

Time-dependent variables x_1, x_2, $\ldots x_n$ span the phase space of dimensionality n.
$\dot{x}$ denotes total time derivative: $\dot{x} \equiv \frac{\mathrm{d}}{\mathrm{d}t}x$.
Using vector notation $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{f} = (f_1, \ldots, f_n)$ we write more compactly

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

The vector field $\boldsymbol{f}$ is called flow and the solution $\boldsymbol{x}(t)$ is called trajectory.

Concept test 1.1: Dynamical systems

Discrete dynamical systems can be written as coupled recurrence equations (on vector form):

$$\boldsymbol{x}_{i+1} = \boldsymbol{F}(\boldsymbol{x}_i)$$

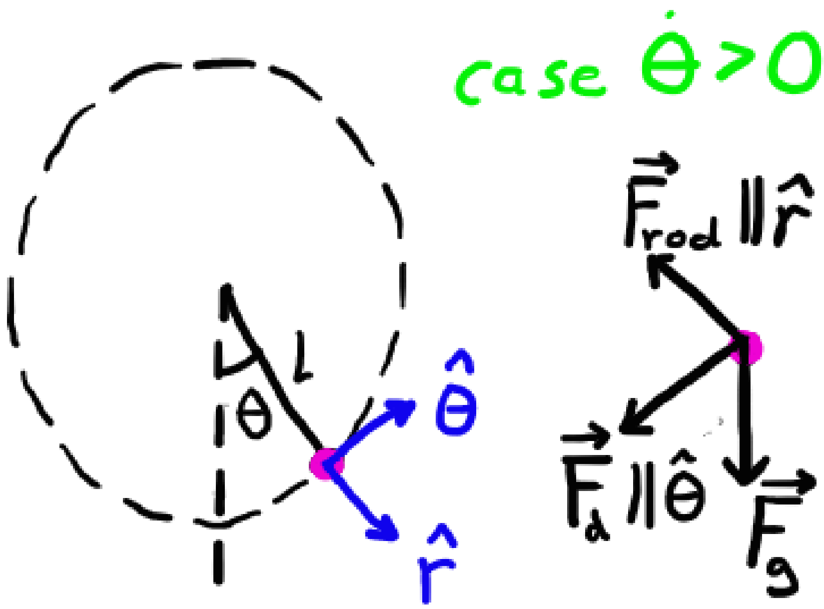
$\boldsymbol{x}_i \equiv x_{1,i}, \ldots, x_{n,i}$ denotes n phase-space variables at discrete times $i = 0, 1, \ldots$.
The functions $\boldsymbol{F} = (F_1, \ldots, F_n)$ are called a map (from $\boldsymbol{x}_i$ to $\boldsymbol{x}_{i+1}$) and the solution $\boldsymbol{x}_i$ is called orbit.

Discrete dynamical systems appear upon discretisation of continuous dynamical systems, or by themselves, for example x_i could denote the population of some species a given year i.

In this course we focus on continuous dynamical systems. Discrete dynamical systems are treated in Computational Biology A (FFR110).

1.2 Example: Derivation of a dynamical system; the rigid pendulum in a viscous medium

Consider a bead of mass m attached to a massless rod of length l that swings in a vertical plane with angle θ:



Three forces act on the bead:

1. Force from rod acting in negative $\hat{\boldsymbol{r}}$ direction, $\boldsymbol{F}_{\text{rod}}$.
2. Downward force from gravity: $\boldsymbol{F}_g = -mg\hat{\boldsymbol{y}}$.
   Component in radial direction $\hat{\boldsymbol{r}}$: $F_{g,r} = mg\cos\theta$ is balanced by force from rod, $F_{g,r}\hat{\boldsymbol{r}} = -\boldsymbol{F}_{\text{rod}}$
   Component in angular direction $\hat{\boldsymbol{\theta}}$ (tangential to pendulum motion): $F_{g,\theta} = -mg\sin\theta$
3. Drag force (due to friction with the viscous medium, e.g. air), assumed to be proportional to instantaneous velocity in the angular direction: $F_{d,\theta} = -\underbrace{\gamma}_{\text{damping coefficient}} \cdot \underbrace{l\dot{\theta}}_{\text{tangential velocity}}$

Newton's second law for angular component of forces:

$$ma = F_{\text{tot},\theta} = F_{g,\theta} + F_{d,\theta} = -mg\sin\theta - \gamma\dot{\theta}l$$

Tangential acceleration $a = l\ddot{\theta} \Rightarrow$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{\gamma}{m}\dot{\theta} \tag{1}$$

Eq. (1) and variations thereof will be studied later in the course.

We use dimensionless units to simplify the analysis (dedimensionalisation). Make time dimensionless: $t = t_0 t'$, with dimensionless time t' and a dimensional constant t_0. Using

$$\dot{\theta} = \frac{1}{t_0}\frac{\mathrm{d}\theta}{\mathrm{d}t'} \quad \text{and} \quad \ddot{\theta} = \frac{1}{t_0^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}t'^2}$$

Eq. (1) becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t'^2} = -t_0^2\frac{g}{l}\sin\theta - t_0\frac{\gamma}{m}\frac{\mathrm{d}\theta}{\mathrm{d}t'} = t_0\frac{\gamma}{m}\left[-t_0\frac{mg}{l\gamma}\sin\theta - \frac{\mathrm{d}\theta}{\mathrm{d}t'}\right]$$

Choose $t_0 = l\gamma/(mg)$ to remove one of the parameter groups

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t'^2} = \frac{l\gamma^2}{m^2 g}\left[-\sin\theta - \frac{\mathrm{d}\theta}{\mathrm{d}t'}\right]$$

To simplify the notation we replace t' with t from now on. Use dimensionless parameter $\epsilon \equiv m^2 g/(l\gamma^2)$ and multiply equation by ϵ

$$\epsilon\ddot{\theta} = -\sin\theta - \dot{\theta} \tag{2}$$

Assume large damping ($\epsilon \to 0$) to obtain a one-dimensional dynamical system (c.f. Section 2.1 in Strogatz)

$$\dot{\theta} = -\sin\theta\,. \tag{3}$$

This equation is non-linear but solvable by separation of variables:

$$\frac{1}{\sin\theta}\mathrm{d}\theta = -\mathrm{d}t$$

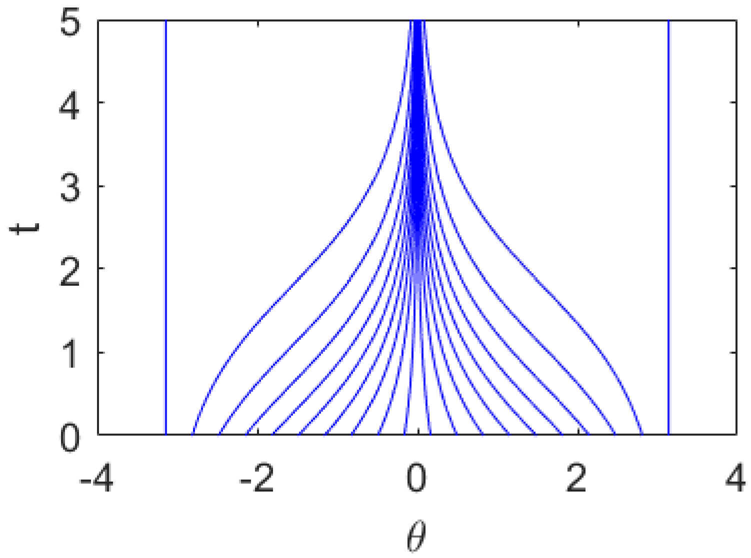
Integrate from $t=0$ to T and from $\theta(0) \equiv \theta_0$ to $\theta(T) \equiv \theta_T$

$$\begin{aligned}
I &= \int_{\theta_0}^{\theta_T} \frac{1}{\sin\theta}\mathrm{d}\theta = -\int_0^T \mathrm{d}t = -T \\
I &= \int_{\theta_0}^{\theta_T} \frac{2\mathrm{i}}{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}}\mathrm{d}\theta = \int_{\theta_0}^{\theta_T} \frac{2\mathrm{i}e^{\mathrm{i}\theta}}{e^{2\mathrm{i}\theta} - 1}\mathrm{d}\theta = \left[z = e^{\mathrm{i}\theta}\,,\ \mathrm{d}z = \mathrm{i}e^{\mathrm{i}\theta}\mathrm{d}\theta\right] \\
&= \int \frac{2}{z^2-1}\mathrm{d}z = [\text{partial fraction decomposition}] \\
&= \int \left(\frac{1}{z-1} - \frac{1}{z+1}\right)\mathrm{d}z = \left[\ln\left(\frac{e^{\mathrm{i}\theta}-1}{e^{\mathrm{i}\theta}+1}\right)\right]_{\theta_0}^{\theta_T} = \left[\ln\left(\tan(\theta/2)\right)\right]_{\theta_0}^{\theta_T}
\end{aligned}$$

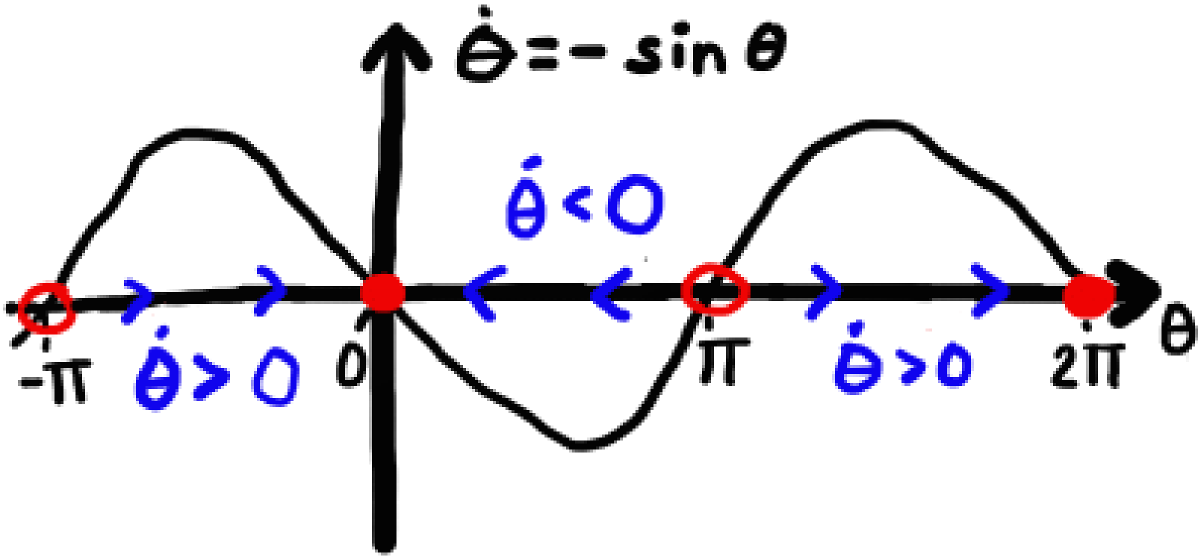
In conclusion

$$\ln\left(\frac{\tan(\theta_T/2)}{\tan(\theta_0/2)}\right) = -T \ \Rightarrow\ \boxed{\theta(t) = 2\mathrm{atan}(e^{-t}\tan(\theta_0/2))}$$

Trajectories starting with $-\pi < \theta_0 < \pi$ converge to $\theta = 0$ as $t \to \infty$.
Trajectories starting at $\theta_0 = \pm\pi$ remain at $\pm\pi$.



Solution using a dynamical systems approach It is easier to solve the system geometrically. Plot $\dot{\theta} = f(\theta)$ against θ:



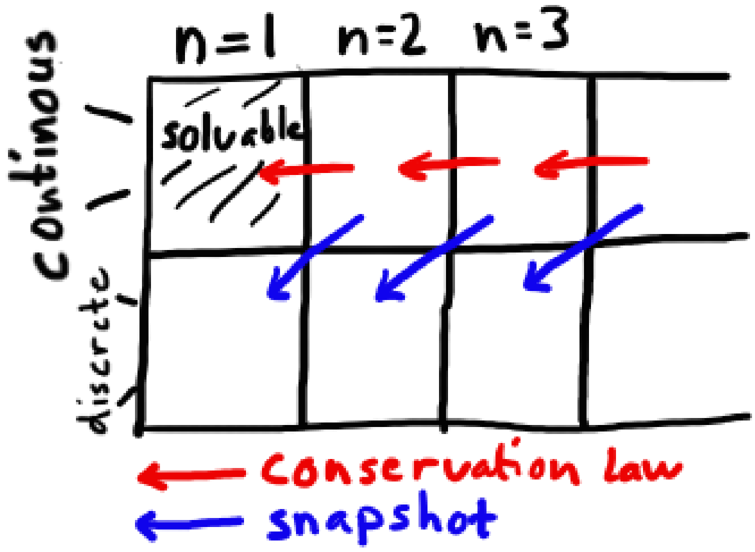
Arrows denote the directions of trajectories along the line (c.f. exact trajectories in previous figure).
Points with no flow ($\dot{\theta} = 0$): fixed points (also called: equilibrium points or steady states) correspond to constant solutions of the ODE.
● Stable fixed point (attractor/sink). Surrounding flow is directed towards the fixed point $\Rightarrow$ dynamics is stable to small perturbations.
○ Unstable fixed point (repeller/source). Surrounding flow is directed away from the fixed point $\Rightarrow$ small deviations from the fixed point grow with time, the fixed point is unstable to small perturbations.

The geometric solution gives the qualitative picture: all trajectories end up at $\theta = 0$ (or multiples of 2π), unless they start exactly at an unstable fixed point. Some details are missing but often it is enough to have qualitative information about the solution. More examples in Strogatz 2.2 and 2.3

Remark on validity of Eq. (3) Eq. (3) was derived as the limit $\epsilon \to 0$ in Eq. (2). But since Eq. (2) depends on two initial conditions θ_0 and $\dot{\theta}_0$ and Eq. (3) only depend on one initial condition θ_0, the solutions are only identical if $\dot{\theta}_0 = -\sin\theta_0$. For the case $\dot{\theta}_0 \neq -\sin\theta_0$ trajectories are quickly damped (on the time scale ϵ) to the condition $\dot{\theta} = -\sin\theta$, and the solution to Eq. (3) is a good approximation to the actual solution for times larger than ϵ (c.f. Strogatz 3.5).

1.3 Reduction of dimensionality

In Section 1.2 we analyzed a system of dimensionality $n = 1$, Eq. (3). The possible behaviors a dynamical system can show depends on its dimensionality. Low-dimensional systems are often easier to analyze than high-dimensional systems. The only type of dynamical systems that can be solved in general is flows on the line (continuous dynamical systems with $n = 1$), by separation of variables: $\frac{\mathrm{d}x}{f(x)} = \mathrm{d}t$ (c.f. example in Section 1.2)



Some methods to reduce the dimensionality of a dynamical system:

- Taking snapshots of a continuous dynamical system when its trajectory intersects a chosen lower-dimensional subspace (Poincaré map) result in a discrete system of lower dimensionality.

- Using limiting behaviours, one example being the limit of large damping in Section 1.2 that reduced the dimensionality of the system from $n = 2$, Eq. (1), to $n = 1$, Eq. (3).

- By finding conservation laws. A conservation law implies that some combination of phase-space variables is independent of time (a conserved quantity) $\Rightarrow$ One of the constituting variables can be eliminated, reducing the problem dimensionality.

- Symmetries can also be used to decouple variables we are not interested in. For example, spherical symmetry in a three-dimensional system allows us to write the time evolution of the radial coordinate independently from the angular coordinates $\Rightarrow$ 3 to 1 dimension.

Motivated by the above-mentioned methods, we start the general analysis of dynamical systems with the simplest case $n = 1$.

1.4 Flows on the line

Dynamical systems of phase-space dimensionality $n = 1$

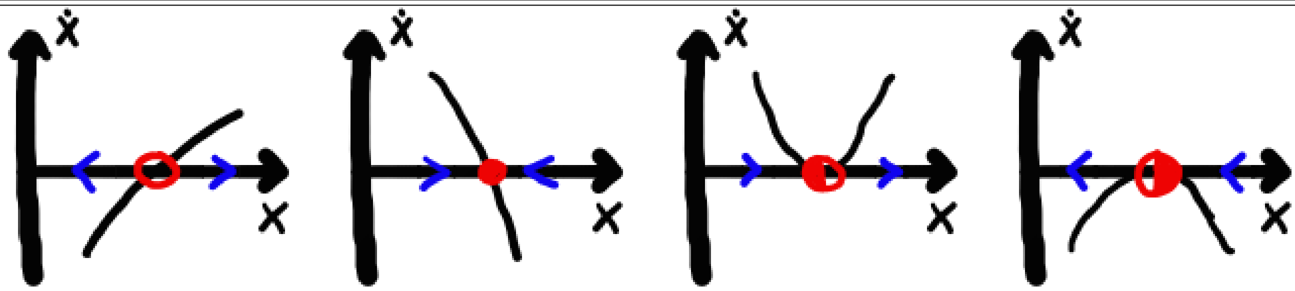
$$\dot{x} = f(x)$$

f is smooth and real-valued. x takes any real value. No explicit time dependence in f. One example is given by the overdamped pendulum [Eq. (3) in Section 1.2].

Concept test: 1.2 Flow on the line

1.4.1 Types of fixed points

Assume x^* is an isolated fixed point on the line, $f(x^*) = 0$. The possible types are summarized as follows:



Type	Unstable ○	Stable ●	Half stable/semi stable ◐, ◑	
Slope	$f'(x^*) > 0$	$f'(x^*) < 0$	$f'(x^*) = 0$	$f'(x^*) = 0$
Example	$f(x) = x$	$f(x) = -x$	$f(x) = x^2$	$f(x) = -x^2$

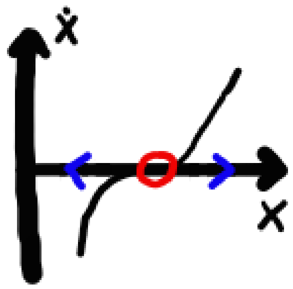
Half-stable fixed points:

◐ Dynamics attracted to the left of fixed point, repelled to the right.

◑ Repelled to the left, attracted to the right.

The case $f'(x^*) = 0$ is called marginal.

Note that $f'(x^*) = 0$ is not a sufficient condition for a fixed point to be half-stable, for example $f(x) = x^3$ is unstable:



1.4.2 Potential problems: (Strogatz 2.5)

- If the flow is not smooth, solutions from a given initial condition are not necessarily unique.

- If the solution reaches infinity in a finite time and no solution exists for later times (blow-up).

Most flows encountered in this course are smooth, implying that there exist unique solutions starting from any initial condition.

1.4.3 No periodic solutions

The trajectories resulting from a smooth flow on a line can not cross (c.f. the resulting trajectories in the overdamped limit of the example in Section 1.2). The reason is that for each point in space the solution moves in a unique direction determined by the flow. As a result:

- Trajectories move monotonically towards fixed points or towards plus|minus infinity

- No oscillatory motion is possible.
  Note however: by introducing periodic boundaries (for example at $\theta = 0$ and $\theta = 2\pi$) to a one-dimensional dynamical system, we obtain a flow on the circle (Strogatz 4). Such flow behaves as the flow on the line, with the additional property that it allows for periodic solutions, one example being $\dot{\theta} =$ const. with periodicity 2π.