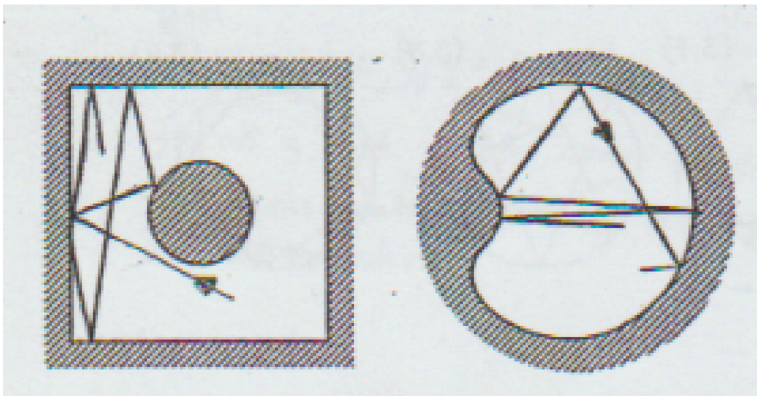
10 Chaos and Lyapunov exponents

10.1 Chaotic systems

Chaotic dynamics exhibit the following properties

- Trajectories have a finite probability to show aperiodic long-term behaviour. However, a subset of trajectories may still be asymptotically periodic or quasiperiodic in a chaotic system.
- System is deterministic, the irregular behavior is due to non-linearity of system and not due to stochastic forcing.
- Trajectories show sensitive dependence on initial condition (the 'butterfly effect'): Quantified by a positive Lyapunov exponent (this lecture).

10.1.1 Illustrative example: Convex billiards



10.1.2 More examples of chaotic systems

It is more a rule than an exception that systems exhibit chaos (often in the form of a mixture between chaotic and regular motion). Examples:

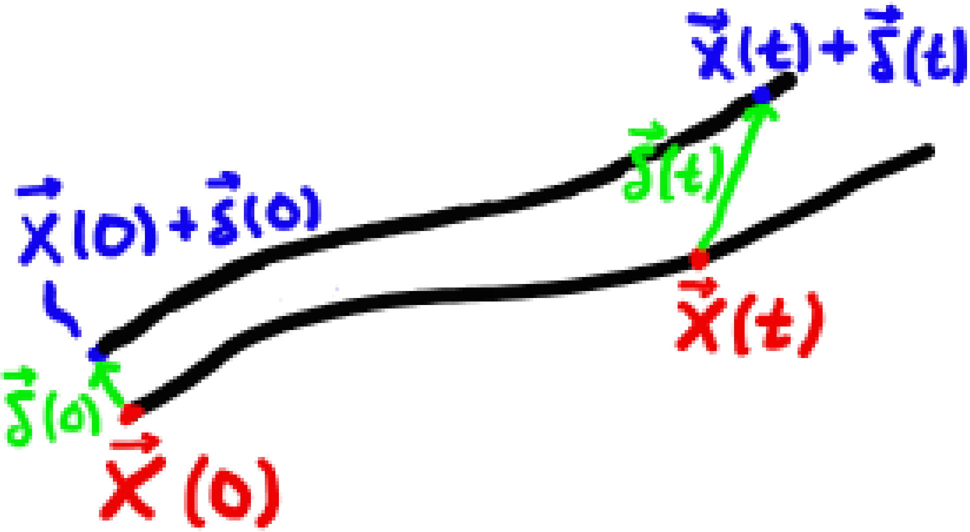
- **Biology** Population dynamics, Arrythmia (hearth), Epilepsy (brain).

- **Physics** Double pendulum, helium atom, three-body gravitational problem, celestial mechanics, mixing of fluids, meteorological systems.

- **Computer science** Pseudo-random number generators, to send secret messages (Strogatz 9.6).

10.2 The maximal Lyaponov exponent

Consider separation $\boldsymbol{\delta} \equiv \boldsymbol{x}' - \boldsymbol{x}$ between two trajectories $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$:



Assume that small distance $\delta(t) \equiv |\boldsymbol{\delta}(t)|$ changes smoothly as $\delta \to 0$ ($\dot{\delta}$ approaches zero linearly as δ approaches zero) and neglect higher-order terms in $\delta(t)$ (assume that $\delta(0)$ is small enough so that $\delta(t)$ is small for all times of consideration):

$$\dot{\delta}(t) = h(t)\delta(t) \quad \Rightarrow \quad \delta(t) = \delta(0)\exp\left[\int_0^t \mathrm{d}t' h(t')\right] .$$

Define maximal Lyapunov exponent λ_1 as the long-time average of h:

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{d}t' h(t')$$

and consider large t:

$$\delta(t) \sim e^{\lambda_1 t}\delta(0) \qquad \Rightarrow \qquad \lambda_1 \equiv \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\boldsymbol{\delta}(t)|}{|\boldsymbol{\delta}(0)|}\,.$$

Here $\delta(0)$ is made small enough so that the trajectories remain close-by at all times of interest. λ_1 describes whether a system is sensitive to small deviations in initial conditions. Depending on the sign of λ_1, a small deviation between two trajectories either decreases ($\lambda_1 < 0$) or increases ($\lambda_1 > 0$) exponentially fast for large times.

10.2.1 Physical interpretation of λ_1

A positive λ_1 (and mixing) implies chaotic dynamics.
Magnitude of $1/\lambda_1$ is the Lyapunov time: when $\lambda_1 > 0$ it determines time horizon for which system is predictable. Examples:

- Motion of planets in our solar system is chaotic, but there is no problem in predicting planet motion on time scales of observation [Lyapunov time $\sim$ 50 million years for our solar system].
- Weather system: Lyapunov time (days) of same order as typical relevant time scale.
- Chaotic electric circuits (milliseconds)

Strogatz Example 9.3.1 An increase in the precision of initial condition δ_0 by factor $10^6 \Rightarrow$ system only predictable for 2.5 times longer (assuming a tolerance which is $10^4 \cdot \delta_0$) .

10.3 Deformation matrix

As before, consider a general flow $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ and a small separation $\boldsymbol{\delta} = \boldsymbol{x}' - \boldsymbol{x}$ with $|\boldsymbol{\delta}| \ll 1$ between two trajectories $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$. For the maximal Lyapunov exponent we only considered the distance $|\boldsymbol{\delta}|$, now we consider the full dynamics of $\boldsymbol{\delta}$. Linearized dynamics

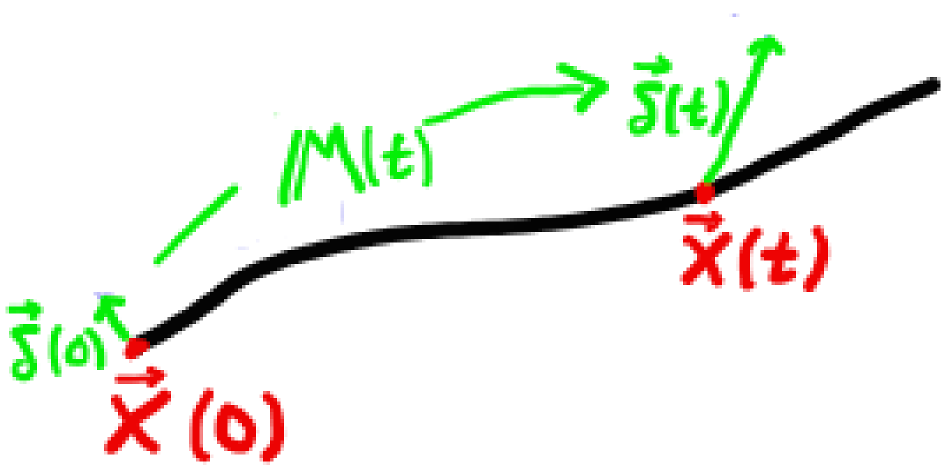
$$\begin{aligned}\dot{\boldsymbol{\delta}} &= \boldsymbol{f}(\boldsymbol{x}') - \boldsymbol{f}(\boldsymbol{x}) = [\boldsymbol{\delta} \text{ small} \Rightarrow \text{expand } \boldsymbol{f}(\boldsymbol{x}') \text{ around } \boldsymbol{x}] \\ &\approx [\boldsymbol{f}(\boldsymbol{x}) + \mathbb{J}(\boldsymbol{x})(\boldsymbol{x}' - \boldsymbol{x})] - \boldsymbol{f}(\boldsymbol{x}) = \mathbb{J}(\boldsymbol{x})\boldsymbol{\delta}\end{aligned}$$

with stability matrix $\mathbb{J}(\boldsymbol{x}) \equiv \partial \boldsymbol{f}/\partial \boldsymbol{x}$ evaluated along $\boldsymbol{x}(t)$.

The deformation matrix (deformation gradient tensor, Lyapunov matrix) $\mathbb{M}$ is defined such that

$$\boldsymbol{\delta}(t) = \mathbb{M}(t)\boldsymbol{\delta}(0)$$

with small initial separation $|\boldsymbol{\delta}(0)| \ll 1$. For a given trajectory $\boldsymbol{x}(t)$, $\mathbb{M}(t)$ transforms an initial separation $\boldsymbol{\delta}(0)$ to the separation $\boldsymbol{\delta}(t)$:



To derive an equation for the evolution of $\mathbb{M}$, differentiate $\boldsymbol{\delta}$ w.r.t. t

$$\dot{\boldsymbol{\delta}}(t) = \dot{\mathbb{M}}(t)\boldsymbol{\delta}(0)$$

But we also have from the linearisation

$$\dot{\boldsymbol{\delta}}(t) = \mathbb{J}(\boldsymbol{x})\boldsymbol{\delta}(t) = \mathbb{J}(\boldsymbol{x})\mathbb{M}(t)\boldsymbol{\delta}(0)$$

and consequently

$$\dot{\mathbb{M}}(t)\boldsymbol{\delta}(0) = \mathbb{J}(\boldsymbol{x})\mathbb{M}(t)\boldsymbol{\delta}(0)\,.$$

This equation is true for any initial separation $\boldsymbol{\delta}(0) \Rightarrow$

$$\dot{\mathbb{M}}(t) = \mathbb{J}(\boldsymbol{x})\mathbb{M}(t)\,.$$

In summary, to find $\mathbb{M}(t)$ we need to integrate the joint equations

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) \\ \dot{\mathbb{M}}(t) &= \mathbb{J}(\boldsymbol{x})\mathbb{M}(t)\,, \end{aligned} \tag{1}$$

with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\mathbb{M}(0) = \mathbb{I}$ (identity matrix) for a time long enough that the initial conditions are 'forgotten'.

The eigenvalues m_i of $\mathbb{M}$ define <u>stability exponents of trajectory separations</u> $\tilde{\sigma}_i \equiv \lim_{t\to\infty} t^{-1}\ln m_i$.

Comparison to linearisation around fixed point The linearisation between closeby trajectories above, closely resembles the linearisation around a fixed point in Lecture 4:

Stability analysis of	fixed point	trajectory separation
Separation	$\boldsymbol{\delta} = \boldsymbol{x} - \boldsymbol{x}^*$	$\boldsymbol{\delta} = \boldsymbol{x}' - \boldsymbol{x}$
Dynamics $\dot{\boldsymbol{\delta}} = \mathbb{J}\boldsymbol{\delta}$	$\mathbb{J}(\boldsymbol{x}^*)$ const.	$\mathbb{J}(\boldsymbol{x}(t))$ along $\boldsymbol{x}(t)$
Solution $\boldsymbol{\delta}(t) = \mathbb{M}(t)\boldsymbol{\delta}(0)$	$\mathbb{M}(t) = \exp[\mathbb{J}(\boldsymbol{x}^*)t]$	$\mathbb{M}$ implicit from Eq. (1)

As a consequence, trajectories in the basin of attraction of a fixed-point attractor $\boldsymbol{x}^*$ have $\boldsymbol{x}(t) \to \boldsymbol{x}^*$ for large times and $\mathbb{M} \to \exp[\mathbb{J}(\boldsymbol{x}^*)t]$. Diagonalisation of $\mathbb{J}(\boldsymbol{x}^*) = \mathbb{V}\mathbb{D}\mathbb{V}^{-1}$ implies diagonalisation of $\mathbb{M}$: $\mathbb{M} = \mathbb{V}e^{\mathbb{D}t}\mathbb{V}^{-1} \Rightarrow$ the eigenvectors of $\mathbb{M}$ and $\mathbb{J}$ are the same. In this limit the the stability exponents of separations are equal to the eigenvalues $\sigma_1, \sigma_2, \ldots, \sigma_n$ of $\mathbb{J}(\boldsymbol{x}^*)$ (stability exponents), $\tilde{\sigma}_i = \sigma_i$.

In general, the eigenvalues and eigenvectors of $\mathbb{M}$ are different from the eigenvalues and eigenvectors of $\mathbb{J}$ (eigensystem of $\mathbb{J}$ requires only local knowledge of system while eigensystem of $\mathbb{M}$ is influenced by all stability matrices along a trajectory). It is in general hard to solve the equations for $\mathbb{M}$ analytically, and one needs to use a numerical method (Section 10.5 below).

10.4 Lyapunov spectrum

Using the deformation matrix $\mathbb{M}$ it is possible to generalize the maximal Lyapunov exponent in Section 10.2 describing stretching rates of small separations to stretching rates of small areas or volumes between groups of closeby trajectories.

Consider a small spherical shell, $|\boldsymbol{\delta}(0)|^2 = \delta_0^2 = \text{const.}$, of initial separations around a test trajectory $\boldsymbol{x}(t)$. At time t, the separations have deformed to $\boldsymbol{\delta}(t) = \mathbb{M}(t)\boldsymbol{\delta}(0)$. Inverting this relation we obtain

$$1 = \frac{\boldsymbol{\delta}(0)^{\mathrm{T}}\boldsymbol{\delta}(0)}{\delta_0^2} = \boldsymbol{\delta}(t)^{\mathrm{T}} \underbrace{\frac{[\mathbb{M}(t)^{-1}]^{\mathrm{T}}\mathbb{M}(t)^{-1}}{\delta_0^2}}_{\equiv\mathbb{B}} \boldsymbol{\delta}(t)\,.$$

Here $\mathbb{B}$ is a positive definite matrix which implies that the equation

$$\boldsymbol{\delta}^{\mathrm{T}}\mathbb{B}\boldsymbol{\delta} = 1$$

forms the surface of an ellipsoid with principal axes equal to the eigenvectors of $\mathbb{B}$ and with lengths of semi-axes equal to $1/\sqrt{b_i}$, where b_i are eigenvalues of $\mathbb{B}$.

Using a singular value decomposition $\mathbb{M} = \mathbb{U}\mathbb{S}\mathbb{V}^{\mathrm{T}}$, where $\mathbb{U}$ and $\mathbb{V}$ are orthogonal matrices, $\mathbb{U}\mathbb{U}^{\mathrm{T}} = \mathbb{V}\mathbb{V}^{\mathrm{T}} = \mathbb{I}$, and $\mathbb{S}$ is diagonal with entries s_i, we find

$$\mathbb{M}^{\mathrm{T}}\mathbb{M} = \mathbb{V}\mathbb{S}^2\mathbb{V}^{\mathrm{T}}\,, \quad \mathbb{M}\mathbb{M}^{\mathrm{T}} = \mathbb{U}\mathbb{S}^2\mathbb{U}^{\mathrm{T}}\,.$$

Thus $\mathbb{M}^{\mathrm{T}}\mathbb{M}$ and $\mathbb{M}\mathbb{M}^{\mathrm{T}}$ have the same set of eigenvalues s_i^2, and

$$\mathbb{B} = \delta_0^{-2}[\mathbb{M}^{-1}]^{\mathrm{T}}\mathbb{M}^{-1} = \delta_0^{-2}[\mathbb{M}\mathbb{M}^{\mathrm{T}}]^{-1} = \delta_0^{-2}[\mathbb{U}\mathbb{S}^2\mathbb{U}^{\mathrm{T}}]^{-1} = \delta_0^{-2}\mathbb{U}\mathbb{S}^{-2}\mathbb{U}^{\mathrm{T}}$$

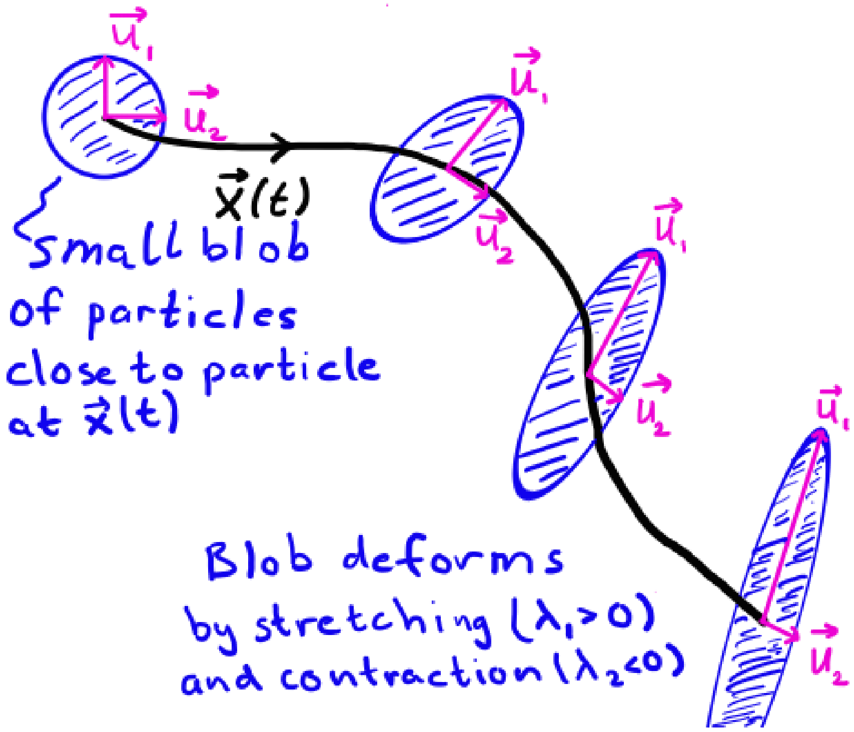
has eigenvalues $b_i = \delta_0^{-2}s_i^{-2}$. As a conclusion, $\mathbb{M}$ maps the spherical shell of radius δ_0 into an ellipsoid with lengths of semi-axes equal to $1/\sqrt{b_i} = \delta_0 s_i$.

The eigenvalues s_i^2 of $\mathbb{M}^{\mathrm{T}}\mathbb{M}$ define a <u>spectrum of Lyapunov exponents</u>

$$\lambda_i \equiv \lim_{t\to\infty}\frac{1}{t}\ln|s_i|\,, \tag{2}$$

ordered such that $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$. They characterise exponential growth|decay rates in a cloud of close-by particles.

Example Consider a 2D system with $\lambda_1 > 0$, $\lambda_2 < 0$ and corresponding eigendirections $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$ from $\mathbb{U}$. A disk of initial separations around the test trajectory $\boldsymbol{x}(t)$ grows exponentially fast along $\hat{\boldsymbol{u}}_1$ with rate λ_1 and shrinks exponentially fast along $\hat{\boldsymbol{u}}_2$ with rate λ_2:



10.4.1 Physical interpretation of the Lyapunov spectrum

Consider the quantity

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{|\boldsymbol{\delta}(t)|}{|\boldsymbol{\delta}(0)|} = \lim_{t\to\infty} \frac{1}{2t} \ln\left(\hat{\boldsymbol{\delta}}(0)^{\mathrm{T}}\mathbb{M}^{\mathrm{T}}\mathbb{M}\hat{\boldsymbol{\delta}}(0)\right) .$$

Denote by $\boldsymbol{v}_i$ the eigendirections of $\mathbb{V}$ corresponding to λ_i. If $\hat{\boldsymbol{\delta}}(0)$ has a component in the direction $\boldsymbol{v}_1$, the limit above approaches the maximal Lyapunov exponent λ_1, describing the stretching rate of a typical separation in accordance with Section 10.2. For the atypical case that $\hat{\boldsymbol{\delta}}(0)$ is perpendicular to $\boldsymbol{v}_1$ but has a component along $\boldsymbol{v}_2$, the limit approaches λ_2, i.e. λ_2 describes stretching of separations in the subspace perpendicular to $\boldsymbol{v}_1$. Similarly, higher-order Lyapunov exponents describe stretching in yet lower-dimensional subspaces.

Another viewpoint is to consider partial sums of the largest Lyapunov exponents:

- λ_1 determines exponential growth rate ($\lambda_1 > 0$) or contraction rate ($\lambda_1 < 0$) of small separations between two trajectories.
- $\lambda_1 + \lambda_2$ determines exponential growth rate ($\lambda_1 + \lambda_2 > 0$) or contraction rate ($\lambda_1 + \lambda_2 < 0$) of small areas between three trajectories.
- $\lambda_1 + \lambda_2 + \lambda_3$ determines exponential growth/contraction rate of small volumes between four trajectories

and so on for sums over increasing number of Lyapunov exponents.

10.5 Numerical evaluation of Lyapunov exponents

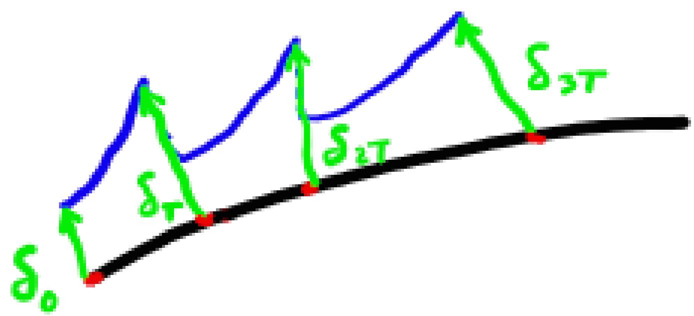
The Lyapunov exponents are hard to calculate in general and one needs to rely on numerical methods.

10.5.1 Naive numerical evaluation of λ_1

A naive approach is to solve the dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

numerically for two trajectories starting at $\boldsymbol{x}(0)$ and $\boldsymbol{x}(0) + \boldsymbol{\delta}(0)$.



At regular time intervals T, rescale separation vector to original length

$$\boldsymbol{\delta}(nT) \to \frac{1}{\alpha_n}\boldsymbol{\delta}(nT)\,, \qquad \alpha_n = \frac{|\boldsymbol{\delta}(nT)|}{|\boldsymbol{\delta}(0)|}$$

and use scaling factors α_n to evaluate

$$\lambda_1 = \frac{1}{t}\ln\frac{|\boldsymbol{\delta}(t)|}{|\boldsymbol{\delta}(0)|} = \frac{1}{NT}\sum_{n=1}^{N}\ln\alpha_n$$

with the total number of rescalings, N, large.

This often works! But it is unreliable: what is a good value for $|\boldsymbol{\delta}(0)|$ and the regularisation time T? Also, it does not give the stretching rate in directions other than the maximal, needed to calculate $\lambda_2,\ldots,\lambda_n$. In order to calculate these, one would need to follow $n+1$ trajectories and rescale and reorthonormalize the volume spanned between the trajectories. This is quite complicated.

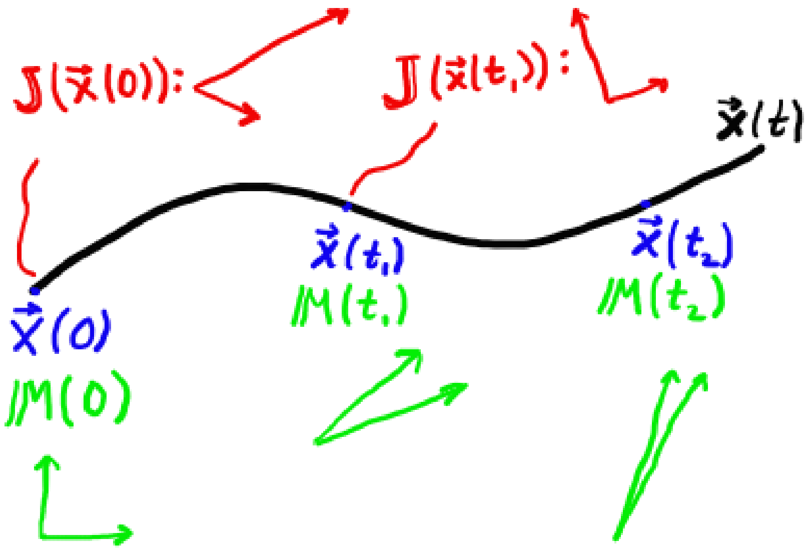
10.5.2 Evaluation using the deformation matrix

In principle, the Lyapunov exponents can be obtained from the eigenvalues of the matrix $\mathbb{M}^{\mathrm{T}}\mathbb{M}$ following Eq. (1). However, direct evaluation of Eq. (1) is in general numerically problematic (the elements in $\mathbb{M}$ blow up exponentially with increasing t). As a workaround, discretize time $t \to t_n \equiv n\delta t$ (n integer and δt small time step):

$$\begin{aligned}
&\frac{\mathbb{M}(t_n)-\mathbb{M}(t_{n-1})}{\delta t} = \mathbb{J}(\boldsymbol{x}(t_{n-1}))\mathbb{M}(t_{n-1})\\
\Rightarrow \mathbb{M}(t_n) &= [\mathbb{I}+\mathbb{J}(\boldsymbol{x}(t_{n-1}))\delta t]\mathbb{M}(t_{n-1})\\
&= [\mathbb{I}+\mathbb{J}(\boldsymbol{x}(t_{n-1}))\delta t][\mathbb{I}+\mathbb{J}(\boldsymbol{x}(t_{n-2}))\delta t]\mathbb{M}(t_{n-2})\\
&= [\mathbb{I}+\mathbb{J}(\boldsymbol{x}(t_{n-1}))\delta t][\mathbb{I}+\mathbb{J}(\boldsymbol{x}(t_{n-2}))\delta t]\ldots[\mathbb{I}+\mathbb{J}(\boldsymbol{x}(t_0))\delta t]\underbrace{\mathbb{M}(t_0)}_{\mathbb{I}}
\end{aligned}$$

i.e. $\mathbb{M}(t_n)$ consists of product of n matrices $\mathbb{M}(t_n) = \mathbb{M}^{(n-1)}\mathbb{M}^{(n-2)}\cdots\mathbb{M}^{(0)}$ where $\mathbb{M}^{(i)} \equiv \mathbb{I}+\mathbb{J}(\boldsymbol{x}(t_i))\delta t$.

The time evolution of the deformation matrix $\mathbb{M}$ driven by stability matrices $\mathbb{J}(\boldsymbol{x}(t_n))$ along a trajectory $\boldsymbol{x}(t)$:



Arrows show eigensystems of $\mathbb{M}$ (green) and $\mathbb{J}$ (red). At each time step, the eigendirections of $\mathbb{M}$ strives against the maximal direction of $\mathbb{J}$ and becomes longer if maximal eigenvalue of $\mathbb{J}$ is positive. $\Rightarrow$ eigenvectors of $\mathbb{M}$ (and $\mathbb{M}^{\mathrm{T}}\mathbb{M}$) tend to become very long and almost aligned. $\Rightarrow$ hard numerics

QR-trick Use QR-decomposition to evaluate the eigenvalues of the product $\mathbb{M}(t_n) = \mathbb{M}^{(n-1)} \cdots \mathbb{M}^{(0)}$ without numerical overflow.
OBS: QRDecomposition[M] in Mathematica gives matrices named Q and R, but $\mathbb{M} = \mathbb{Q}^{\mathrm{T}}\mathbb{R}$, i.e. one must transpose $\mathbb{Q}$.

A QR-decomposition of a general matrix $\mathbb{P}$ factorizes $\mathbb{P} = \mathbb{Q}\mathbb{R}$, where $\mathbb{Q}\mathbb{Q}^{\mathrm{T}} = \mathbb{I}$ and $\mathbb{R}$ is upper triangular.

General principle (not identical to what you should implement):

- Before first time step, QR-decompose $\mathbb{M}^{(0)} = \mathbb{Q}^{(0)}\mathbb{R}^{(0)}$, i.e. $\mathbb{Q}^{(0)} = \mathbb{R}^{(0)} = \mathbb{I}$ because $\mathbb{M}^{(0)} = \mathbb{I}$.
- After first time step, rewrite $\mathbb{M}^{(1)}\mathbb{M}^{(0)} = \underbrace{\mathbb{M}^{(1)}\mathbb{Q}^{(0)}}_{\mathbb{Q}^{(1)}\mathbb{R}^{(1)}}\mathbb{R}^{(0)} = \mathbb{Q}^{(1)}\mathbb{R}^{(1)}\mathbb{R}^{(0)}$
- After second time step, rewrite $\mathbb{M}^{(2)}\mathbb{M}^{(1)}\mathbb{M}^{(0)} = \underbrace{\mathbb{M}^{(2)}\mathbb{Q}^{(1)}}_{\mathbb{Q}^{(2)}\mathbb{R}^{(2)}}\mathbb{R}^{(1)}\mathbb{R}^{(0)} =$

$\mathbb{Q}^{(2)}\mathbb{R}^{(2)}\mathbb{R}^{(1)}\mathbb{R}^{(0)}$

- Repeat for each time step: $\mathbb{M}^{(n-1)} = \mathbb{Q}^{(n-1)}\mathbb{R}^{(n-1)}\cdots\mathbb{R}^{(1)}\mathbb{R}^{(0)}$

The Lyapunov spectrum can be evaluated for large N, corresponding to a final time $t_N = N\delta t$, using the limit in Section 10.4.1:

$$\lambda_i = \lim_{N\to\infty}\frac{1}{2N\delta t}\ln\left(\hat{\boldsymbol{\delta}}_i^{\mathrm{T}}[\mathbb{M}^{(N-1)}]^{\mathrm{T}}\mathbb{M}^{(N-1)}\hat{\boldsymbol{\delta}}_i\right) = \lim_{N\to\infty}\frac{1}{N\delta t}\ln\left|\mathbb{R}\hat{\boldsymbol{\delta}}_i\right| .$$

Here $\mathbb{Q}^{\mathrm{T}}\mathbb{Q} = \mathbb{I}$ was used, $\mathbb{R} = \mathbb{R}^{(n-1)}\cdots\mathbb{R}^{(1)}\mathbb{R}^{(0)}$ is an upper diagonal matrix and $\hat{\boldsymbol{\delta}}_i$ with $i = 1, 2, \ldots, d$ denotes an orthonormal set of vectors such that $\hat{\boldsymbol{\delta}}_i$ lies in the subspace excluding the directions corresponding to $\lambda_1, \ldots, \lambda_{i-1}$ (Oseledets theorem ensures this limit exist for almost all initial conditions).

It is possible to show that the elements of $\mathbb{R}$ typically order such that different Lyapunov exponents are given by different diagonal entries of $\mathbb{R}$:

$$\lambda_i = \lim_{N\to\infty}\frac{1}{N\delta t}\ln[R_{ii}] = \lim_{N\to\infty}\frac{1}{N\delta t}\sum_{n=0}^{N-1}\ln|R_{ii}^{(n)}| . \tag{3}$$

The $\ln|R_{ii}^{(n)}|$ can be added one at a time to avoid overflow.

What you should implement:

1. Solve the equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ for some time to end up close to the fractal attractor (Lorenz system in Problem 3.2).

2. Start with matrix $\mathbb{Q} = \mathbb{I}$ and zero-valued variables λ_i for the sums in Eq. (3)

3. At each time step you get a new matrix $M^{(n)} = \mathbb{I} + \mathbb{J}(\boldsymbol{x}(t_n))\delta t$ where $\boldsymbol{x}(t_n)$ is taken from solution of the $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ equation.

4. At each time step QR-decompose $\mathbb{M}^{(n)}\mathbb{Q}_{\text{old}} = \mathbb{Q}_{\text{new}}\mathbb{R}_{\text{new}}$

5. At each time step add the diagonal elements of $\mathbb{R}_{\text{new}}$ to λ_i in Eq. (3)

6. Repeat from step 3 with $\mathbb{Q} = \mathbb{Q}_{\text{new}}$ (total of N iterations)

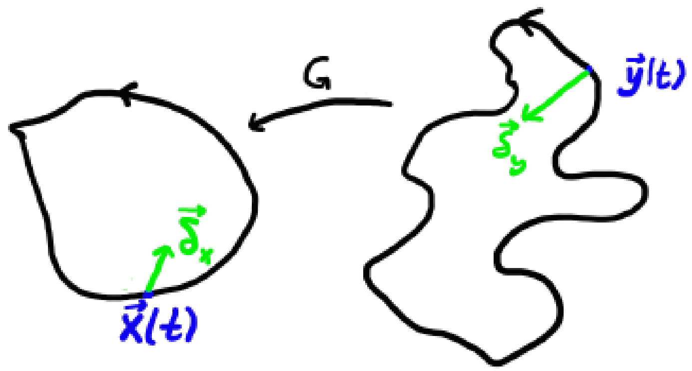
10.6 Coordinate transform of the deformation matrix for closed orbits

At multiples of the period time of a closed orbit, the deformation matrix $\mathbb{M}$ is (similarity-) invariant under general coordinate transformations. This property can sometimes be useful for analytical calculations of eigenvalues of the deformation matrix (Problem set 3.1).

Start from the equation defining $\mathbb{M}$ (subscripts denote original coordinates $\boldsymbol{x}$)

$$\boldsymbol{\delta_x}(t) = \mathbb{M}_{\boldsymbol{x}}(t)\boldsymbol{\delta_x}(0)$$

Make a coordinate transform $\boldsymbol{x} = \boldsymbol{G}(\boldsymbol{y})$



For small separations $\boldsymbol{\delta_x}$ and $\boldsymbol{\delta_y}$ we have

$$\boldsymbol{\delta_x}(t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}}\boldsymbol{\delta_y}(t) \equiv \mathbb{J}_G(\boldsymbol{y}(t))\boldsymbol{\delta_y}(t)$$

and

$$\boldsymbol{\delta_x}(0) = \mathbb{J}_G(\boldsymbol{y}(0))\boldsymbol{\delta_y}(0)$$

where $\mathbb{J}_G$ is the gradient matrix of the transformation $\boldsymbol{G}$. Consequently

$$\begin{aligned}\boldsymbol{\delta_y}(t) &= \mathbb{J}_G^{-1}(\boldsymbol{y}(t)) \underbrace{\boldsymbol{\delta_x}(t)}_{\mathbb{M}_{\boldsymbol{x}}(t)\boldsymbol{\delta_x}(0)} \\ &= \mathbb{J}_G^{-1}(\boldsymbol{y}(t))\mathbb{M}_{\boldsymbol{x}}(t)\mathbb{J}_G(\boldsymbol{y}(0))\boldsymbol{\delta_y}(0)\,.\end{aligned}$$

But from the definition of the deformation matrix $\mathbb{M}_{\boldsymbol{y}}(t)$ in the y-system we also have $\boldsymbol{\delta}_{\boldsymbol{y}}(t) = \mathbb{M}_{\boldsymbol{y}}(t)\boldsymbol{\delta}_{\boldsymbol{y}}(0)$.

$$\Rightarrow \mathbb{M}_{\boldsymbol{y}}(t) = \mathbb{J}_G^{-1}(\boldsymbol{y}(t))\mathbb{M}_{\boldsymbol{x}}(t)\mathbb{J}_G(\boldsymbol{y}(0))$$

For a closed orbit at multiples of the period time (so that $\boldsymbol{y}(t) = \boldsymbol{y}(0)$), eigenvalues of $\mathbb{M}_{\boldsymbol{y}}$ = eigenvalues of $\mathbb{M}_{\boldsymbol{x}}$. This can be seen by diagonalisation $\mathbb{M}_{\boldsymbol{x}}(t) = \mathbb{P}^{-1}\mathbb{D}\mathbb{P} \Rightarrow \mathbb{M}_{\boldsymbol{y}}(t) = [\mathbb{P}\mathbb{J}_G^{-1}(\boldsymbol{y}(0))]^{-1}\mathbb{D}[\mathbb{P}\mathbb{J}_G(\boldsymbol{y}(0))]$, i.e. also $\mathbb{M}_{\boldsymbol{y}}(t)$ is diagonalized with the same diagonal matrix $\mathbb{D}$.