2 Bifurcations and catastrophes (Strogatz 3)

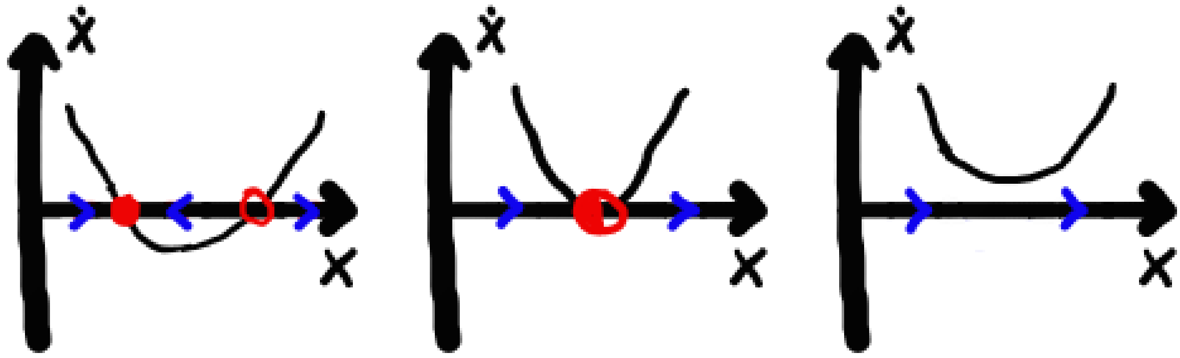
A bifurcation is a qualitative change in the dynamics (for example creation/annihilation or change in stability of fixed points) as a system parameter is varied. A bifurcation point is the value of the parameter where the bifurcation occurs.

2.1 Saddle-node bifurcation

Consider the system

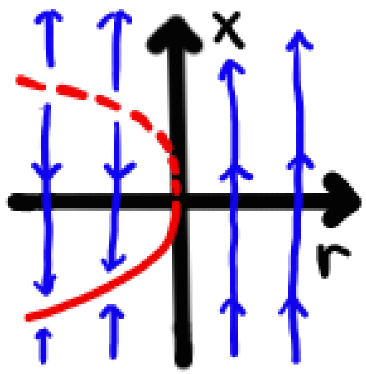
$$\dot{x} = r + x^2$$

for negative, zero, and positive values of r:



$r < 0$ $\qquad$ $r = 0$ $\qquad$ $r > 0$

As the bifurcation parameter r passes the bifurcation point r_c, two fixed points (one unstable and one stable) merge and disappear:



This is a bifurcation diagram, i.e. a plot of fixed points against the bifurcation parameter (often plotted without the blue flow). In bifurcation diagrams, solid lines denote stable fixed points and dashed lines denote unstable ones. The bifurcation at $r = 0$ is a saddle-node bifurcation (this name is explained in the lecture on bifurcations in higher dimensions). Saddle-node bifurcations is the typical mechanism for creation|annihilation of fixed points.

2.2 Analytical analysis

The geometrical approach considered so far gives the qualitative behaviour of the dynamics. To get more quantitative predictions, we consider analytical approaches.

2.2.1 Linear stability analysis

Consider general flow, $\dot{x} = f(x)$, with a fixed point $x = x^*$: $f(x^*) = 0$. A small deviation $\eta(t) = x(t) - x^*$ from the fixed point x^* evolves according to

$$\dot{\eta} = \dot{x} - \frac{\mathrm{d}}{\mathrm{d}t}x^* = \dot{x} = f(x)$$

Series expand the flow around the fixed point:

$$\begin{aligned}\dot{\eta} = f(x) &= \underbrace{f(x^*)}_{=0} + f'(x^*)\underbrace{(x - x^*)}_{=\eta} + \frac{1}{2}f''(x^*)\underbrace{(x - x^*)^2}_{=\eta^2} + \ldots \\ &\approx f'(x^*)\eta\end{aligned}$$

Solution:

$$\eta = \eta_0 e^{f'(x^*)t}$$

This is the general form of the solution close to an isolated fixed point. $\lambda = f'(x^*)$ is the stability exponent (a constant number);
$1/|\lambda|$ is the characteristic time scale of the solution close to x^* (stability time). Note that when $\lambda < 0$ the deviation from the fixed point decreases exponentially fast, but the fixed point is not reached ($\eta = 0$) in a finite time.

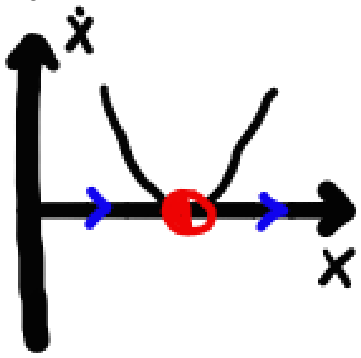
For the saddle-node bifurcation above we have $f(x) = r + x^2$ and $f'(x) = 2x$:

Parameter range	Fixed points	Stability exponents
$r < 0$	$x_1^* = -\sqrt{-r}$ $x_2^* = \sqrt{-r}$	$\lambda_1 = -2\sqrt{-r}$ (stable) $\lambda_2 = 2\sqrt{-r}$ (unstable)
$r = 0$	$x^* = 0$	$\lambda = 0$ (marginal)
$r > 0$	—	—

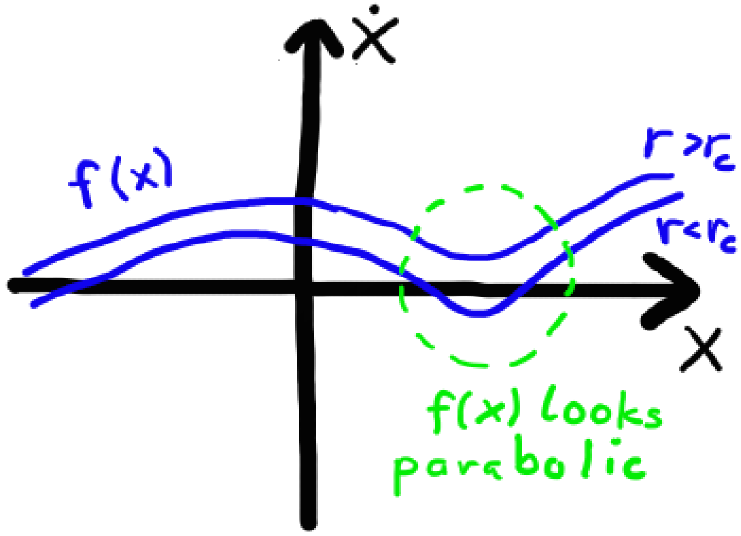
Note: The direction of a flow on the line is uniquely determined everywhere by its fixed points. Bifurcations only occur when fixed points are created, destroyed, or change stability. All these require $f'(x^*) = 0$, which is a <u>necessary condition for bifurcations</u> in flows on the line.

2.2.2 Normal form of saddle-node bifurcation

Consider a general flow with parameter r, $\dot{x}(t) = f(x(t), r)$, undergoing a saddle-node bifurcation at $x = x^*$ and $r = r_c$:



For the saddle-node bifurcation to occur, $f(x, r)$ must have two close-by roots in x:



$f(x,r)$ always looks parabolic in x close to the bifurcation (universal behavior).

At $r=r_c$, x^* is a double root: $f(x^*,r_c)=\frac{\partial f}{\partial x}(x^*,r_c)=0$. Expand f around x^* and r_c to lowest contributing orders:

$$f(x,r)=\underbrace{f(x^*,r_c)}_{=0}+\underbrace{\frac{\partial f}{\partial x}(x^*,r_c)}_{=0}(x-x^*)+\frac{\partial f}{\partial r}(x^*,r_c)(r-r_c)$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial^2 x}(x^*,r_c)(x-x^*)^2+\dots$$

Introduce rescaled coordinates $X=a(x-x^*)$ and $R=b(r-r_c)$

$$\dot X=a\dot x=\frac{a}{b}\frac{\partial f}{\partial r}(x^*,r_c)R+\frac{1}{2a}\frac{\partial^2 f}{\partial^2 x}(x^*,r_c)X^2+\dots$$

Choose $a=\frac{1}{2}\frac{\partial^2 f}{\partial^2 x}(x^*,r_c)$, $b=a\frac{\partial f}{\partial r}(x^*,r_c)$

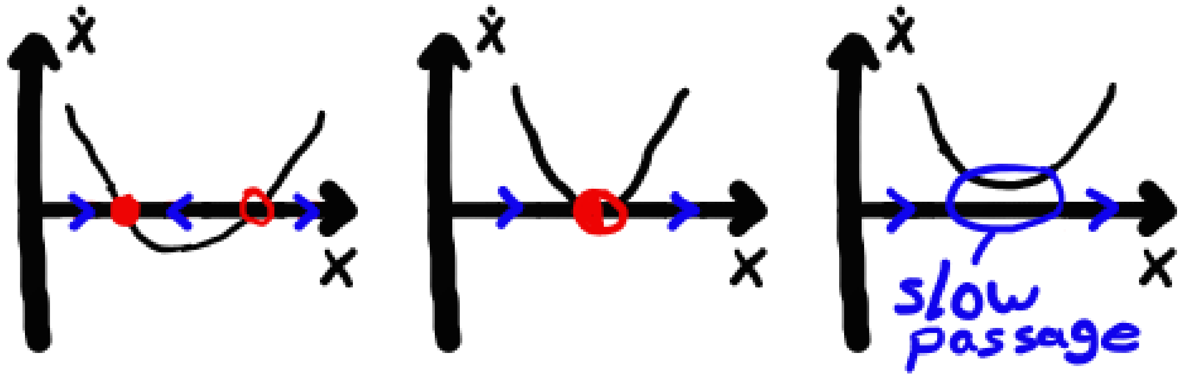
$$\dot X=R+X^2+\dots \tag{1}$$

i.e. a generic saddle-node bifurcation can be put on the form $\dot x=r+x^2$ close to the bifurcation point (provided $\frac{\partial^2 f}{\partial^2 x}(x^*,r_c)\neq 0$ and $\frac{\partial f}{\partial r}(x^*,r_c)\neq 0$).

Concept test 2.1: Normal form of saddle-node bifurcation

2.2.3 Dynamics close to a saddle-node bifurcation

The normal forms of bifurcations are very useful in order to study the dynamics close to a generic bifurcation of a given type. As an example, we can use the universal dynamics in Eq. (1), $\dot{x} = r + x^2$, to study the behaviour close to any saddle-node bifurcation, $r \approx 0$:

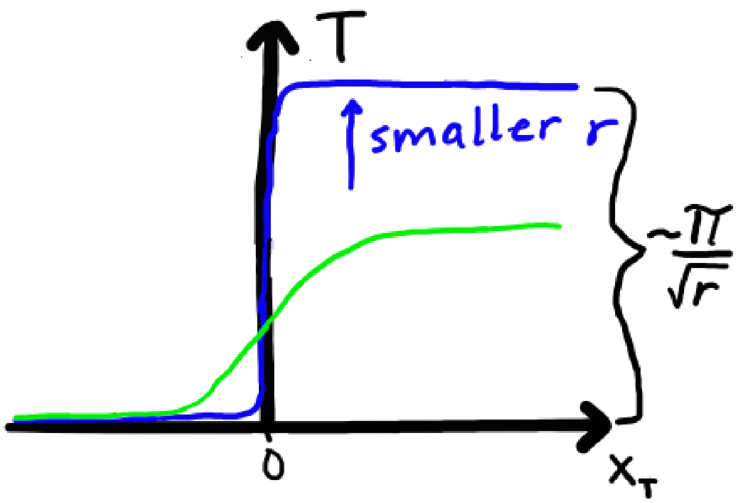


$r < 0$ $\quad$ $r = 0$ $\quad$ $r > 0$

Critical slowing down No fixed points exists after the saddle-node bifurcation ($r > 0$), but the flow velocity must be small if r is small and positive $\Rightarrow$ Passage is slow. The time T it takes to pass the bottleneck is obtained from:

$$\mathrm{d}t = \frac{\mathrm{d}x}{r + x^2} \Rightarrow T = \int_0^T \mathrm{d}t = \int_{x_0}^{x_T} \frac{\mathrm{d}x}{r + x^2} = \frac{1}{\sqrt{r}} \left[\operatorname{atan}\left(\frac{x}{\sqrt{r}}\right) \right]_{x_0}^{x_T}$$

Plotting T against x_T with large negative x_0 (to the left of the bottleneck) gives:



For small values of r the time of passage is completely dominated by

the contribution close to $x = 0$. If $x_0 < 0$ and $x_T > 0$ (on opposite sides of the bottleneck) then T denotes the time of passage. For small r we have $T \sim \frac{\pi}{2\sqrt{r}}\,[\mathrm{sign(x)}]_{x_0}^{x_T} = \frac{\pi}{\sqrt{r}}$.
In conclusion, as $r \to 0$ the time to pass the fixed point approaches infinity as $T \sim 1/\sqrt{r}$.

What about the case $r < 0$ (two fixed points)? Fixed points at $x^* = \pm\sqrt{-r}$. Stability exponents

$$\lambda = \frac{\partial}{\partial x}[r + x^2]\bigg|_{x^*} = 2x^* = \pm 2\sqrt{-r}$$

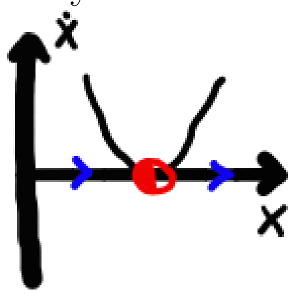
also become smaller as the bifurcation point $r = 0$ is approached, i.e. the dynamics in the vicinity of the fixed points is slow with a time scale $\sim 1/\sqrt{-r}$.

Concept test 2.2 What about $x \to \infty$?

Final case, $r = 0$

$$\dot{x} = x^2\,.$$

$x^* = 0$ is a marginal fixed point ($f'(x^*) = 0$) $\Rightarrow$ linear stability analysis is not enough to determine stability. Geometric approach:



Exact solution by separation of variables:

$$\frac{1}{x^2}\mathrm{d}x = \mathrm{d}t$$

Integrate from $t = 0$ to t (x goes from $x(0) = x_0$ to $x(t)$)

$$\frac{1}{x_0} - \frac{1}{x(t)} = t \quad \Rightarrow \quad x(t) = \frac{1}{\frac{1}{x_0} - t} = \frac{x_0}{1 - x_0 t}$$

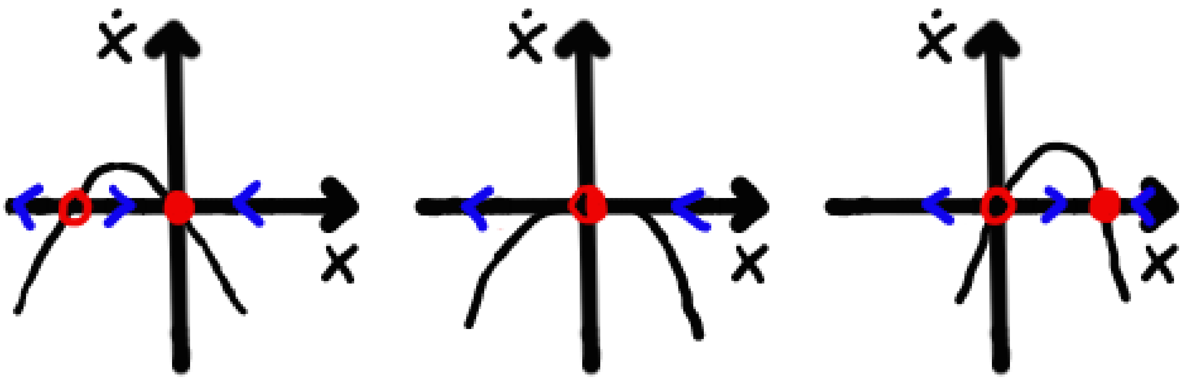
- Starting at $x_0 > 0$, it takes a finite time, $t = 1/x_0$, to reach $x = +\infty$.
- When $t > 1/x_0$ the trajectory reappears from $x = -\infty$ and eventually approaches $x = 0$ from the left.
- It takes infinite time to reach $x = 0$.

The jump from $+\infty$ to $-\infty$ we could not have been predicted by the geometrical approach alone! (Note: We need to be careful in the interpretation of this result: depending on the physical system we want to model, the behaviour after $+\infty$ has been reached may be undefined.)

2.3 Transcritical bifurcation

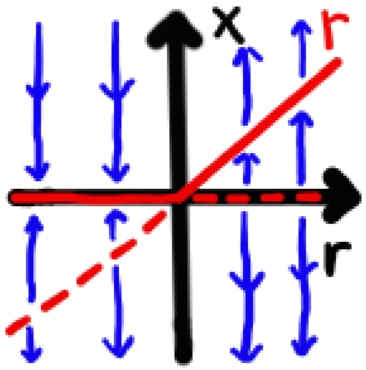
A transcritical bifurcation occurs when a fixed point exists for all values of a bifurcation parameter r surrounding r_c, but changes stability as r passes r_c. As for the saddle-node bifurcation, it is possible to derive a normal form valid close to any transcritical bifurcation:

$$\dot{x} = x(r - x) \tag{2}$$



$r < 0$ $r = 0$ $r > 0$

The normal form has a fixed point at $x^* = 0$ for all values of r, but stability changes as r passes the bifurcation point $r_c = 0$:



2.3.1 Example: Logistic growth

Let $N(t)$ be the population size of a species at time t. Assume that N changes due to births or deaths (no migration). Linear model (Malthus 1798):

$$\dot{N} = \underbrace{bN}_{b=\text{per capita birth rate }(b>0)} - \underbrace{dN}_{d=\text{per capita death rate }(d>0)}$$

Solution: $N(t) = N(0)e^{rt}$, with per capita growth rate $r \equiv b - d$. If $r > 0$ the population grows without bound. This is unrealistic, we expect population sizes to be limited due to a finite amount of resources and space. One way to model this limitation is to modify the per capita growth rate to decrease linearly with population size,

$$r \to r(1 - N/K)\,,$$

with a positive carrying capacity K. This gives a non-linear growth model

$$\dot{N} = Nr(1 - N/K)\,.$$

This is the Logistic equation (Verhulst 1836). The system has two fixed points $N_1^* = 0$ and $N_2^* = K$.
Introducing the rescaled variable $x = rN/K$ we obtain the normal form for transcritical bifurcations (2):

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}N}\dot{N} = \frac{r}{K}Nr(1 - N/K) = x(r - x)\,.$$

Following the corresponding bifurcation diagram above, we have:

- For $r < r_c = 0$ the birth rate is smaller than the death rate and the population goes extinct for any initial population size (the fixed point $x_1^* = 0$ is stable and $x_2^* = r$ is negative (unphysical)).
- For $r > r_c = 0$ the population approaches the maximal sustainable limit for any initial population size (the fixed point $x_1^* = 0$ is unstable and $x_2^* = r$ is positive and stable).

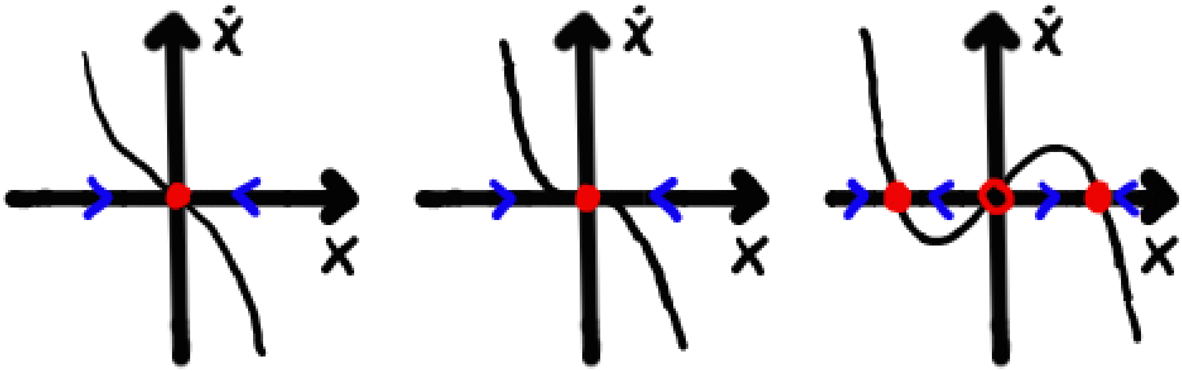
2.4 Pitchfork bifurcation

In a pitchfork bifurcation one fixed point splits into three.
The pitchfork bifurcation can be either supercritical or subcritical.

2.4.1 Supercritical pitchfork bifurcation

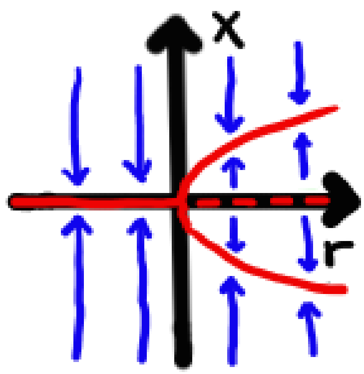
Normal form of supercritical pitchfork bifurcations:

$$\dot{x} = x(r - x^2)$$



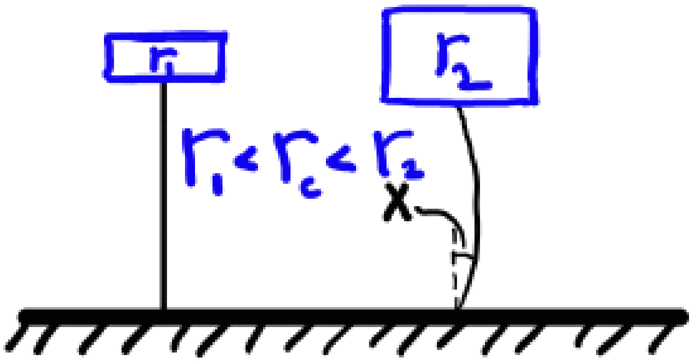
$r < 0$ $r = 0$ $r > 0$

Bifurcation diagram:



Example: Buckling of elastic ruler It may seem unlikely that three fixed points join at one point, but this often happens in systems with mirror symmetry (equations invariant under $x \to -x$).

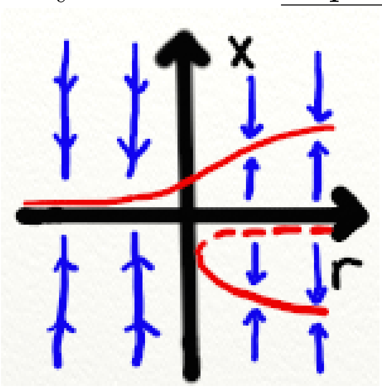
As an example, consider an up-standing perfectly mirror symmetric elastic ruler with a weight applied from above. Let r be the the mass of the weight and let x be the 'buckling angle':



The ruler can sustain a small weight r without deformation. If r is increased above a threshold (the bifurcation point r_c), the slightest asymmetry in the applied mass causes the ruler to buckle in the direction determined by the asymmetry. When the mass is lightened, the ruler moves back to its original state ($x^* = 0$).

2.4.2 Imperfect bifurcation and catastrophes

If the symmetry of the ruler in the example above is not perfect, we may obtain an imperfect bifurcation.



Here small initial buckling angles in either direction makes the ruler buckle towards positive x. However, a large enough negative initial buckling angle makes the ruler buckle in the opposite direction (lower branch on the saddle-node bifurcation). Note that if the mass is slowly decreased from this state, the ruler makes a sudden switch to positive x as r becomes smaller than the saddle-node bifurcation point. This jump in the state of the system is a catastrophe (sudden change in state). If r is once again increased, the ruler does not flip back to negative x (hysteresis).

Cusp catastrophe Imperfect bifurcations are often described by addition of an imperfection parameter h to the normal form. For the supercritical pitchfork bifurcation we obtain:

$$\dot{x} = x(r - x^2) + h\,.$$

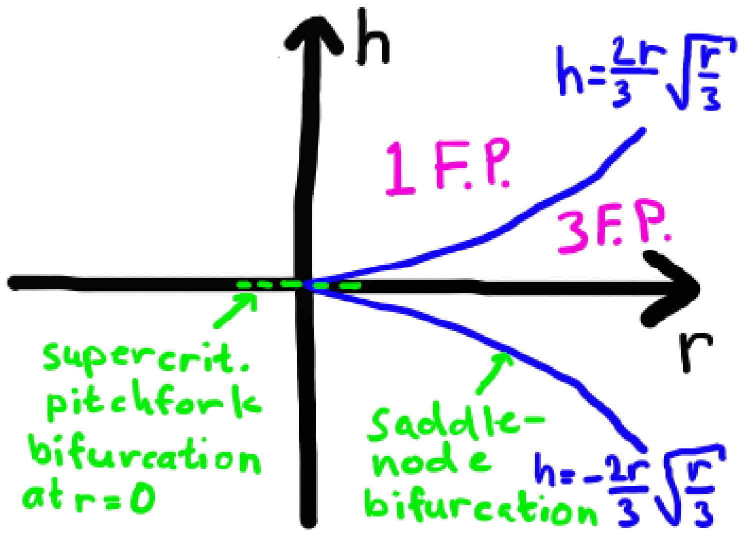
This is a two-parameter problem. When the perturbation h is zero, the normal form is reobtained. As discussed earlier, a necessary condition for bifurcations of fixed points is that both $f(x^*) = 0$ and $f'(x^*) = 0$. The condition $f'(x^*) = 0$ gives

$$0 = \frac{\partial}{\partial x}[x(r - x^2) + h]|_{x=x^*} = r - 3(x^*)^2$$

Inserting the solution $x^* = \pm\sqrt{r/3}$ into the condition $f(x^*) = 0$ gives

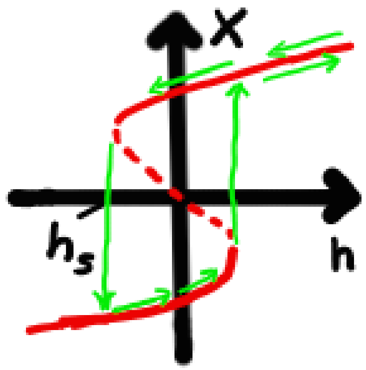
$$0 = \pm\sqrt{\frac{r}{3}}\left(r - \left[\pm\sqrt{\frac{r}{3}}\right]^2\right) + h \quad \Rightarrow \quad h = \mp\frac{2}{3}r\sqrt{\frac{r}{3}}$$

Thus, bifurcations involving at least two fixed points occur at curves $h = \mp\frac{2}{3}r\sqrt{\frac{r}{3}}$:



These curves separates regions with one fixed point from regions with three fixed points. For the bifurcation to involve three fixed points we must have a triple root, i.e. $0 = f''(x^*) = -6x^*$. This condition is only satisfied when $r = h = 0$. We can therefore conclude that the bifurcations occurring along $h = \mp\frac{2}{3}r\sqrt{\frac{r}{3}}$ with $h \neq 0$ involves two fixed points that are created out of the blue (saddle-node bifurcations), just as in the figure illustrating an imperfect bifurcation in example with the ruler above.

The bifurcation curve above is an example of a cusp catastrophe (named so because the two branches of saddle-node bifurcations meet tangentially in a cusp (peak) at the origin). The bifurcation diagram along constant $r > 0$ in the figure above is:



Assume that the system starts at the top fixed point with a large value of h. When h is decreased, the system eventually moves over the left saddle-node bifurcation point, $h_{\rm s}$, and makes a big jump to a fixed point far away (a catastrophe). After the jump the system does not revert back to the original fixed point by a small increase in h (hysteresis). To move back to the original fixed point (remaining at constant r) we must increase h beyond the right saddle point, where a new jump occurs (forming a hysteresis loop).

Some examples on catastrophes:

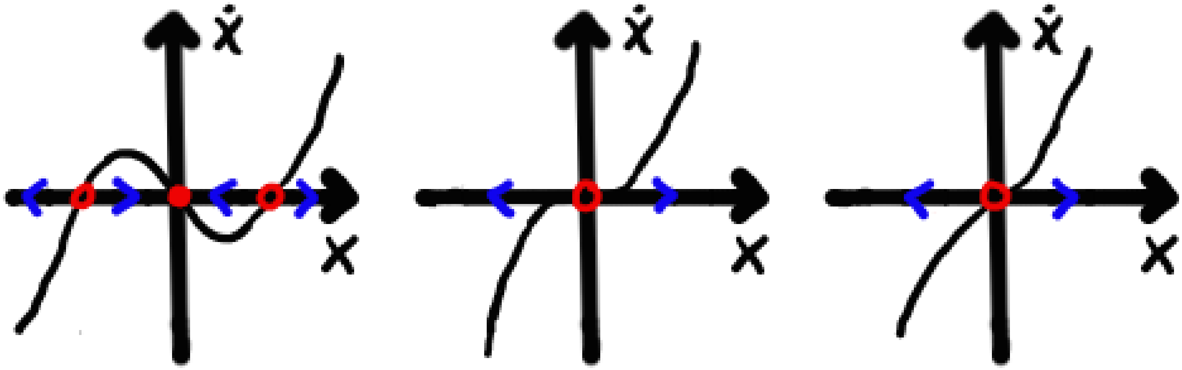
- A sudden change in equilibrium could be catastrophic for buildings and other constructions.
- The problem of hysteresis could be catastrophic for ecological systems: if the system makes a big jump to a new equilibrium (for example due to human influence), it may be very hard to restore the system to its original state due to hysteresis.

- Models in behavioural sciences [Scientific American article by Zeeman (1976)]

2.4.3 Subcritical pitchfork bifurcation

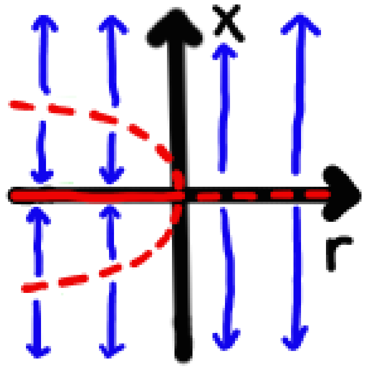
Normal form of subcritical pitchfork bifurcations:

$$\dot{x} = x(r + x^2)$$



$r < 0$ $r = 0$ $r > 0$

Bifurcation diagram:



As for the supercritical case, we have a stable fixed point at $x^* = 0$ for $r < r_c$. When r passes r_c there are no stable fixed points and a small deviation from $x = 0$ grows to infinity in a finite time (blow-up due to the cubic dynamics). Most physical systems have higher-order non-linear corrections that counteract the blow-up (the pitchfork bifurcation happens locally at small x and the system may have other fixed points at larger values of $|x|$). However, the system must make a jump to the new fixed points making subcritical pitchfork bifurcations potentially dangerous, similar to the catastrophes discussed in Section 2.4.2.