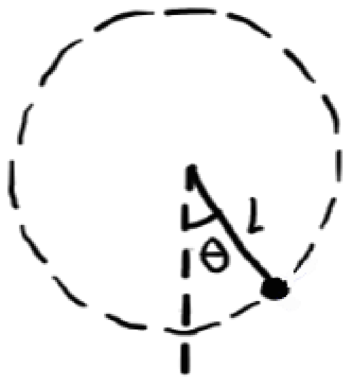
3 Linear 2D flows (Strogatz 5)

3.1 Example: Rigid pendulum in a viscous medium



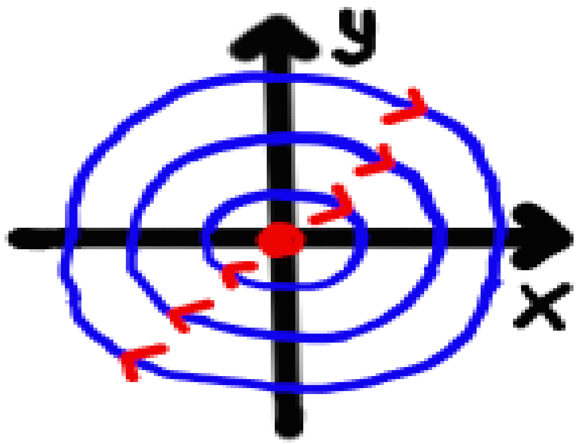
From Lecture 1:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{\gamma}{m}\dot{\theta}\,. \tag{1}$$

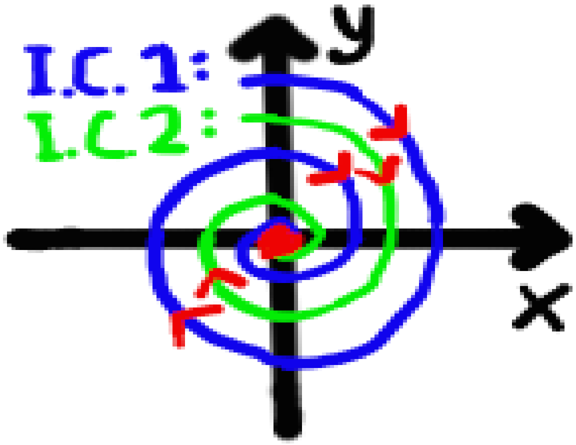
Consider small oscillations, $\sin\theta \approx \theta$ and write as a dynamical system with $x = \theta$, $y = \dot{\theta}$

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\frac{g}{l}x - \frac{\gamma}{m}y\end{aligned}$$

This is an example of a linear flow. It has a fixed point at $x^* = y^* = 0$. As for the one-dimensional systems we do a geometrical visualisation of a few representative trajectories (phase portrait) to understand the dynamics close to the fixed point. The trajectories are obtained by integration of the dynamical system starting from a suitable set of initial positions (x_0, y_0) (or by using StreamPlot[] in Mathematica):



$\gamma = 0$



$\gamma > 0$

Case $\gamma = 0$: the fixed point is surrounded by closed orbits in the form of ellipses of infinite density (which orbit is chosen depends on the initial condition). The fixed point is a center: nearby trajectories neither approach nor depart from it.
Physical interpretation: The fixed point $x^* = y^* = 0$ corresponds to the pendulum at rest, $\theta = \dot{\theta} = 0$. Non-zero initial conditions give closed orbits, corresponding to oscillations in the underlying dynamics [c.f. the ellipses formed by the explicit solution $(x, y) = (\theta, \dot{\theta}) = A_0(\cos\omega_0 t + \phi_0, -\omega_0 \sin(\omega_0 t) + \phi_0)$ with $\omega_0 = \sqrt{g/l}$].
Case $\gamma > 0$: the fixed point is a stable spiral: trajectories spiral inward towards the fixed point.
Physical interpretation: Due to the viscous damping ($\gamma > 0$) the magnitude of oscillations decreases with time.

3.2 Classification of linear flows

Two-dimensional flows have several additional types of fixed points compared to one-dimensional flows.
To find all possible types, consider a general linear flow (neglect constant terms, since they correspond to constant shifts in x and y):

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

On matrix form:

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x}\ ,\ \mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ . \tag{2}$$

Assume $\mathbb{A}$ is diagonalizable, $\mathbb{A} = \mathbb{P}\mathbb{D}\mathbb{P}^{-1}$ with eigenvalue matrix

$$\mathbb{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and P is a matrix spanned by the eigenvectors of $\mathbb{A}$. Then Eq. (2) can be written as

$$\begin{aligned}
&\dot{\boldsymbol{x}} = \mathbb{P}\mathbb{D}\mathbb{P}^{-1}\boldsymbol{x} \\
\Rightarrow&\frac{\mathrm{d}}{\mathrm{d}t}[\mathbb{P}^{-1}\boldsymbol{x}] = \mathbb{D}\,\underbrace{\mathbb{P}^{-1}\boldsymbol{x}}_{\boldsymbol{\xi}} \\
\Rightarrow&\dot{\boldsymbol{\xi}} = \mathbb{D}\boldsymbol{\xi} \\
\Rightarrow&\boldsymbol{\xi}(t) = (e^{\lambda_1 t}\xi_1(0), e^{\lambda_2 t}\xi_2(0))
\end{aligned}$$

The solution $\boldsymbol{\xi}(t)$ shows the prototypic behaviour of trajectories in linear systems and is quantified by λ_1 and λ_2.
In the solution of the original problem, $\boldsymbol{x}(t) = \mathbb{P}\boldsymbol{\xi}(t)$, directions are rotated and rescaled compared to $\boldsymbol{\xi}$, but the topological properties of the system are the same (structure of trajectories is rotated and stretched but the relative order between trajectories remain intact).

The eigenvalues are determined by the characteristic equation $0 = \det(\mathbb{A} - \lambda\mathbb{I})$. For an n-dimensional matrix the characteristic equation can be expressed in terms of its invariants $\mathrm{tr}\mathbb{A}$, $\mathrm{tr}(\mathbb{A}^2)$, ..., $\mathrm{tr}(\mathbb{A}^{\mathrm{n}})$ (Cayley-Hamilton).
For $n = 2$:

$$0 = \det(\mathbb{A} - \lambda\mathbb{I}) = \lambda^2 - \tau\lambda + \Delta$$

with

$$\begin{aligned}
\tau &= \mathrm{tr}\mathbb{A} \\
\Delta &= \frac{(\mathrm{tr}\mathbb{A})^2 - \mathrm{tr}(\mathbb{A}^2)}{2} = \det\mathbb{A}\,.
\end{aligned}$$

The solutions of the characteristic equation are:

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}\,, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \tag{3}$$

Example: Rigid pendulum in a viscous medium

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\gamma}{m} \end{pmatrix}$$

We have $\tau = -\frac{\gamma}{m}$, $\Delta = \frac{g}{l}$. Case $\gamma = 0$:

$$\lambda_1 = \mathrm{i}\sqrt{\frac{\mathrm{g}}{\mathrm{l}}}$$
$$\lambda_2 = -\mathrm{i}\sqrt{\frac{\mathrm{g}}{\mathrm{l}}}$$

As we saw in Section 3.1 this fixed point is a center. The eigenvalues are imaginary and the values correspond to the angular frequency $\omega_0 = \sqrt{g/l}$.

Case $\gamma > 0$ but small:

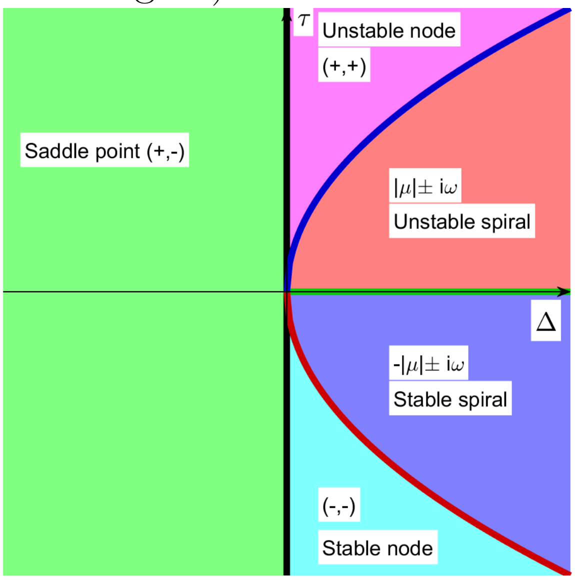
$$\lambda_1 = \frac{-\gamma/m + \mathrm{i}\sqrt{4\mathrm{g}/\mathrm{l} - (\gamma/\mathrm{m})^2}}{2} = -\frac{\gamma}{2m} + \mathrm{i}\sqrt{\frac{\mathrm{g}}{\mathrm{l}} - \frac{\gamma^2}{2\mathrm{m}^2}}$$
$$\lambda_2 = \frac{-\gamma/m - \mathrm{i}\sqrt{4\mathrm{g}/\mathrm{l} - (\gamma/\mathrm{m})^2}}{2} = -\frac{\gamma}{2m} - \mathrm{i}\sqrt{\frac{\mathrm{g}}{\mathrm{l}} - \frac{\gamma^2}{2\mathrm{m}^2}}$$

As we saw in Section 3.1 this fixed point is a stable spiral. It shows oscillating behaviour with angular frequency $\sqrt{g/l - \gamma^2/(2m^2)}$. The negative real part of the eigenvalues decreases the magnitude of the oscillations exponentially with time.

3.2.1 Different possibilities (the 'Zoo' of fixed points)

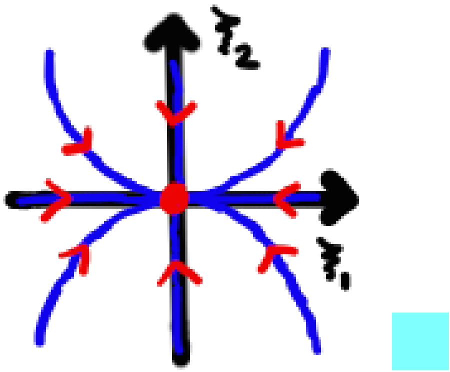
The type of fixed point depends on the relative sign of $\mathrm{Re}[\lambda_1]$ and $\mathrm{Re}[\lambda_2]$ and on whether $\mathrm{Im}[\lambda_{1,2}]$ vanishes or not.

All fixed points can be classified in five major types plus a number of boundary cases.
Parameterizing the eigenvalues by Δ and τ as in Eq. (3) we have (Fig. 5.2.8 in Strogatz):

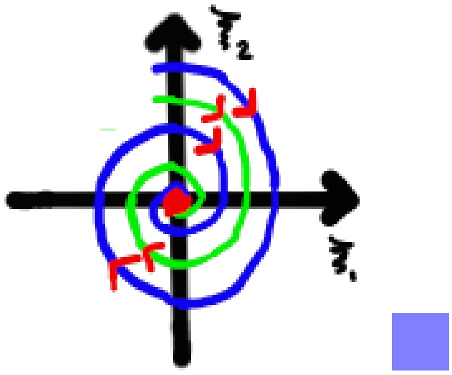


3.2.2 Major types

Stable fixed points If $\mathrm{Re}[\lambda_1] < 0$ and $\mathrm{Re}[\lambda_2] < 0$ the fixed point is stable: trajectories from all initial conditions move towards it. Moreover, if $\mathrm{Im}[\lambda] = 0$ we have a stable node, otherwise a stable spiral.

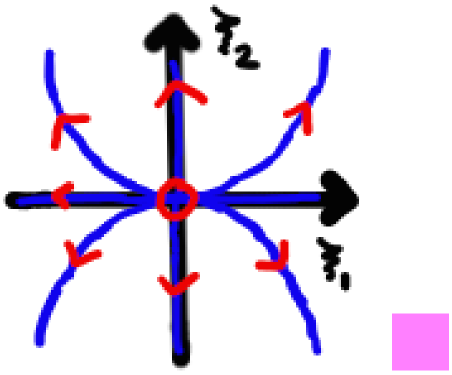


Stable node
$\tau < 0\,,\ 0 < \Delta < \tau^2/4$

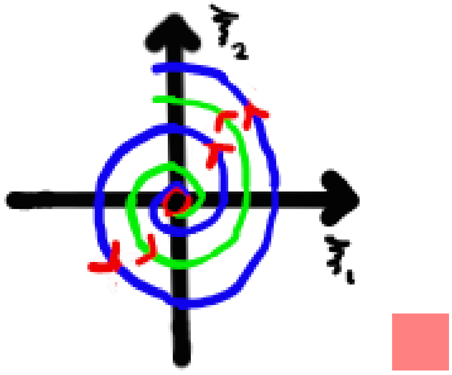


Stable spiral
$\tau < 0\,,\ \tau^2/4 < \Delta$

Unstable fixed points If $\mathrm{Re}[\lambda_1] > 0$ and $\mathrm{Re}[\lambda_2] > 0$ the fixed point is unstable: trajectories from all initial conditions move away from it.

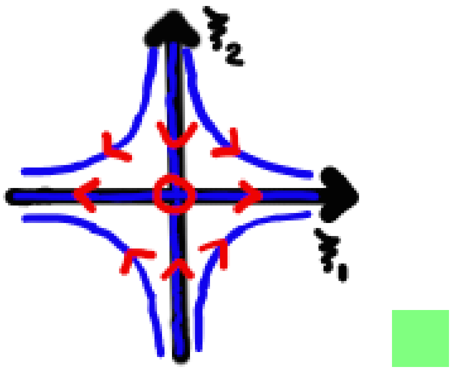


Unstable node
$\tau > 0\,,\; 0 < \Delta < \tau^2/4$



Unstable spiral
$\tau > 0\,,\; \tau^2/4 < \Delta$

Saddle points (unstable) If $\mathrm{Re}[\lambda_1] > 0$ and $\mathrm{Re}[\lambda_2] < 0$ the fixed point is a saddle point: it attracts in one direction and repels in another.



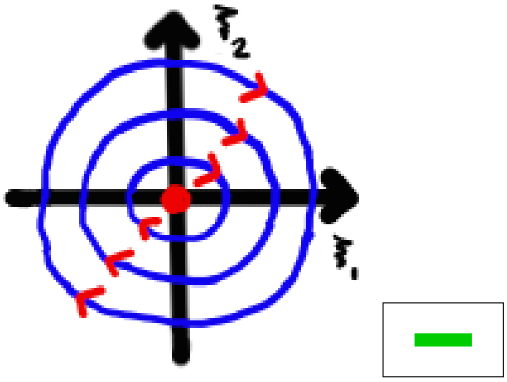
Saddle point
$\Delta < 0$

Side remark: the name of the fixed point is ‘saddle point’ or simply ‘saddle’. Not ‘saddle node’ as is common notation of the inexperienced student. Saddle-node bifurcations refer to the bifurcation between two fixed points: a saddle and a node.

Concept test 3.1: On the zoo

3.2.3 Boundary types

Centers When $\mathrm{Re}[\lambda_1] = \mathrm{Re}[\lambda_2] = 0$ and $\mathrm{Im}[\lambda \neq 0]$ we have a center (encountered for the undamped pendulum).

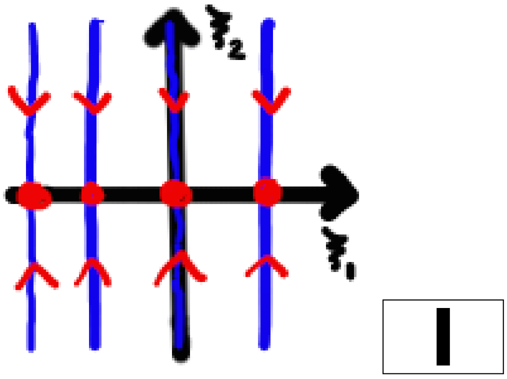


Center
$\tau = 0\,,\ \Delta > 0$

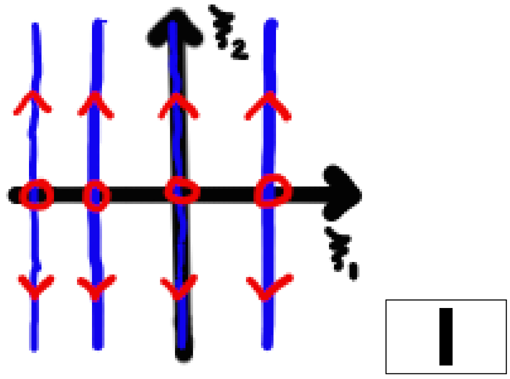
Degenerate case 1 When one eigenvalue, say λ_1, is zero the system

$$\dot{\xi}_1 = 0$$
$$\dot{\xi}_2 = \lambda_2 \xi_2$$

has a line of fixed points at $\xi_2 = 0$.



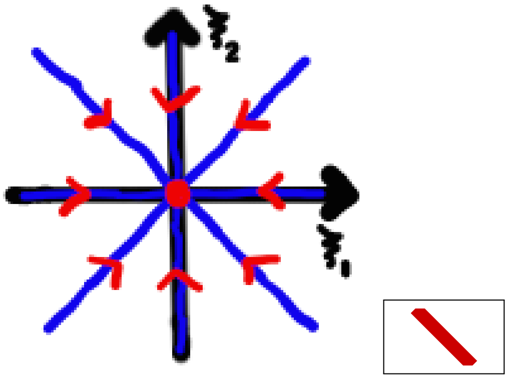
Line of stable fixed points
$\Delta = 0\,,\ \tau < 0$



Line of unstable fixed points
$\Delta = 0\,,\ \tau > 0$

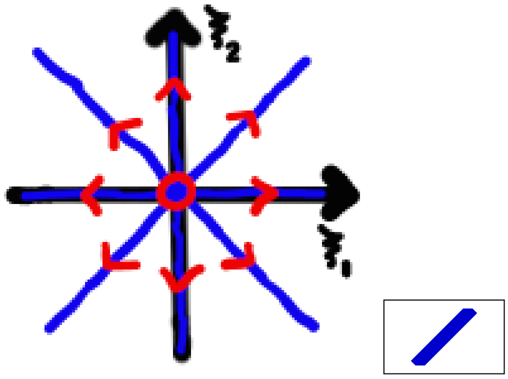
Degenerate case 2 When $\lambda_1 = \lambda_2$ (i.e. when $\tau^2 = 4\Delta$) there are two possibilities:

1. $\mathbb{A}$ is a multiple of the unit matrix, two (arbitrary) independent eigenvectors exists. The fixed point is then a star



Stable star

$$\mathbb{A} = \begin{pmatrix} -|\mu| & 0 \\ 0 & -|\mu| \end{pmatrix}$$



Unstable star

$$\mathbb{A} = \begin{pmatrix} |\mu| & 0 \\ 0 & |\mu| \end{pmatrix}$$

2. $\mathbb{A}$ is not diagonalizable (there is no transformation $\mathbb{P}$ such that $\mathbb{D} = \mathbb{P}^{-1}\mathbb{A}\mathbb{P}$ is diagonal). It is always possible to find a transformation to Jordan normal form:

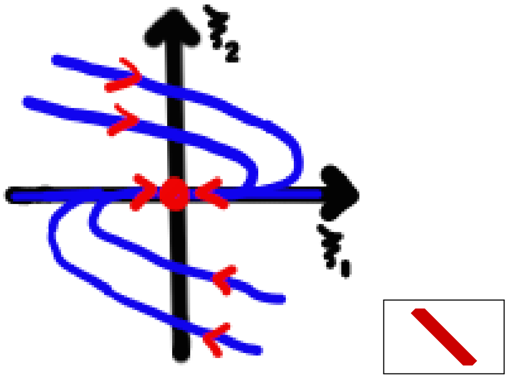
$$\mathbb{P}^{-1}\mathbb{A}\mathbb{P} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Eigenvalues $\lambda_{1,2} = \frac{2\lambda \pm \sqrt{(2\lambda)^2 - 4\lambda^2}}{2} = \lambda$. The matrix has only one eigenvector:

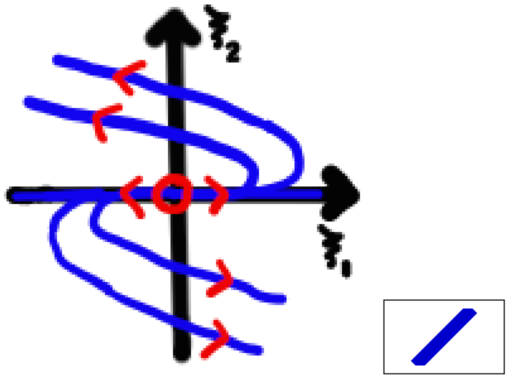
$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The fixed point is then a degenerate node (borderline between spiral and node)



Stable degenerate node
$\tau < 0,\ \tau^2 = 4\Delta$



Unstable degenerate node
$\tau > 0,\ \tau^2 = 4\Delta$

The non-diagonalizable case has a special solution:

$$\dot{\boldsymbol{\xi}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \boldsymbol{\xi}$$

$$\Rightarrow \boldsymbol{\xi}(t) = \exp\left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t\right] \boldsymbol{\xi}(0) = \sum_{i=0}^{\infty} \frac{1}{i!} \left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t\right]^i \boldsymbol{\xi}(0) = \ldots$$

where the sum can be evaluated by brute force. A more elegant solution is to write

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \mathbb{B} + \mathbb{C}\,, \quad \text{with} \quad \mathbb{B} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \mathbb{C} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and use that $\mathbb{B}$ and $\mathbb{C}$ commute because $\mathbb{B}$ is a multiple of the unit matrix:

$$[\mathbb{B}, \mathbb{C}] = \mathbb{B}\mathbb{C} - \mathbb{C}\mathbb{B} = \lambda \mathbb{I}\mathbb{C} - \lambda \mathbb{C}\mathbb{I} = \lambda(\mathbb{C} - \mathbb{C}) = 0\,.$$

For commuting matrices we have $e^{\mathbb{B}} e^{\mathbb{C}} = e^{\mathbb{B}+\mathbb{C}} \Rightarrow$

$$\begin{aligned} \boldsymbol{\xi}(t) &= \exp\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} t + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t\right] \boldsymbol{\xi}(0) \\ &= \exp\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} t\right] \exp\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t\right] \boldsymbol{\xi}(0) \\ &= e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi}(0) \end{aligned}$$

where in the last step we used that

$$\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}^i = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i = 0 \\ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} & \text{if } i = 1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } i = 2, 3, \ldots \end{cases}$$

In conclusion, the solution of diagonalizable matrices can be written on the form

$$\boldsymbol{\xi}(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \boldsymbol{\xi}(0)$$

and the solution for non-diagonaliozable matrices can be written on the form

$$\boldsymbol{\xi}(t) = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi}(0)$$

under a suitable choice of basis.