5 Closed orbits and limit cycles

In Lecture 4 we connected the local dynamics close to fixed points in non-linear flows to the dynamics observed in linear flows. In this lecture we consider non-local properties of the flow, such as closed orbits and limit cycles.

As shown in the last lecture, non-linear terms may destroy the structurally unstable closed orbits around centers. But in systems with symmetry (conservative, volume conserving, inversion symmetry, time-reversal symmetry) closed orbits around centers often become structurally stable (insensitive to small perturbations of the flow).

5.1 Conservative systems

5.1.1 Hamiltonian systems

Newton's law

$$F = ma = m\ddot{x}$$

If the force F is conservative (no friction), it can be written as the negative gradient of a scalar potential $V(x)$:

$$F = -\frac{\partial V}{\partial x}\,.$$

Then Newton's law can be written as the following dynamical system (introduce momentum $p \equiv m\dot{x}$):

$$\begin{aligned}\dot{x} &= \frac{p}{m}\\ \dot{p} &= -\frac{\partial V}{\partial x}\,.\end{aligned} \tag{1}$$

A Hamiltonian dynamical system takes the form

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p}\\ \dot{p} &= -\frac{\partial H}{\partial x}\end{aligned}$$

for some function H. In the case of Newton's law above, choose H to be the Hamiltonian (energy function)

$$H \equiv \frac{p^2}{2m} + V(x)$$

to get

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{m}\\ \dot{p} &= -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}\,.\end{aligned}$$

In conclusion, Newton's law with a conservative force, Eq. (1), is one example of a Hamiltonian dynamical system.

In Hamiltonian systems, the energy $E = H(x,p)$ is conserved since

$$\dot{H} = \frac{\partial H}{\partial x}\underbrace{\dot{x}}_{\frac{\partial H}{\partial p}} + \frac{\partial H}{\partial p}\underbrace{\dot{p}}_{-\frac{\partial H}{\partial x}} = 0\,.$$

Systems with at least one conserved quantity (integral of motion) are called conservative systems. Remark: An integral of motion is a combination of phase space coordinates that are constant along a trajectory. A 'constant of motion' may in addition depend explicitly on time. In classical mechanics continuous symmetries imply conservation laws (Noether's theorem):

Symmetry	**Integral of motion**
Time independence	Energy
Space independence	Momentum
Rotational symmetry	Angular momentum

5.1.2 General conservative systems

More generally: Given a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ a function $E(\boldsymbol{x})$ is a conserved quantity if

$$E(\boldsymbol{x}(t)) = \text{constant} \qquad \Leftrightarrow \qquad \dot{E} = 0$$

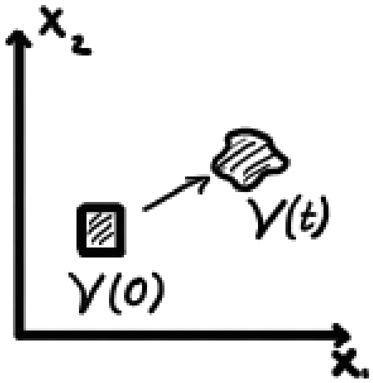
along trajectories and $E(\boldsymbol{x}(t))$ is non-constant on every open set in phase space. The latter condition implies that $E(\boldsymbol{x}(t))$ cannot be a trivial constant. For example $E(\boldsymbol{x}(t)) = x_1^2 + x_2^2$ is constant when evaluated along a circular trajectory and it is thus a conserved quantity if the system has circular trajectories, while $E(\boldsymbol{x}(t)) = 7$ (or any other constant) is constant everywhere in phase space and is therefore not a conserved quantity.

5.1.3 Volume-conserving systems

A dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is volume conserving in phase-space if

$$\boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{x}) = 0 \tag{2}$$

everywhere. Consider a phase-space volume $\mathcal{V}(t)$:



In a volume-conserving system a phase-space element changes shape $D(t)$, but the volume $\mathcal{V}(t)$ is constant. To show this, take a small time step δt. Positions evolve according to

$$\boldsymbol{x}(\delta t) = \boldsymbol{x}(0) + \delta t\, \boldsymbol{f}(\boldsymbol{x}(0))\,.$$

The coordinate transformation $\boldsymbol{y} \equiv \boldsymbol{x}(\delta t)$ transforms all coordinates $\boldsymbol{x}_0 \in D(0)$ into the coordinates $\boldsymbol{y} \in D(\delta t)$

$$\begin{aligned}
\mathcal{V}(\delta t) &= \int_{D(\delta t)} \mathrm{d}^n y = \int_{D(0)} \mathrm{d}^n x_0 \left|\det\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}_0}\right)\right| \\
&= \int_{D(0)} \mathrm{d}^n x_0 \underbrace{\left|\det\left(1 + \delta t \mathbb{J}(\boldsymbol{x}_0)\right)\right|}_{\approx 1 + \delta t\, \mathrm{tr}\mathbb{J}(\boldsymbol{x}_0)} = \mathcal{V}(0) + \delta t \int_{D(0)} \mathrm{d}^n x_0 \;\underbrace{\mathrm{tr}\mathbb{J}(\boldsymbol{x}_0)}_{\boldsymbol{\nabla}\cdot\boldsymbol{f}(\boldsymbol{x}_0)} \\
&\Rightarrow \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = \int_{D(t)} \mathrm{d}^n x\, \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{x}) = 0\,, \text{ if } \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{x}) = 0 \text{ everywhere.}
\end{aligned}$$

A system that has $\boldsymbol{\nabla}\cdot\boldsymbol{f}(\boldsymbol{x})<0$ somewhere is called dissipative. Dissipative systems have attractors, while volume-conserving systems cannot have attractors or repellers.

Example Hamiltonian dynamical systems

$$\boldsymbol{f}=\begin{pmatrix}\frac{\partial H}{\partial p}\\ -\frac{\partial H}{\partial x}\end{pmatrix}$$

preserves phase-space volumes:

$$\nabla\cdot\boldsymbol{f}=\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial p}\end{pmatrix}\begin{pmatrix}\frac{\partial H}{\partial p}\\ -\frac{\partial H}{\partial x}\end{pmatrix}=\frac{\partial^2 H}{\partial x\partial p}-\frac{\partial^2 H}{\partial p\partial x}=0\,.$$

5.1.4 Fixed points in conservative systems

A conservative system cannot have any attracting fixed points:
If $\boldsymbol{x}^*$ was an attractive fixed point, then all points in the basin of attraction move towards the fixed point and must therefore have the same value of E (the value at the fixed point). This implies that E is constant on an open set in phase space and the flow is therefore not conservative by definition.

Find which fixed points can occur in a Hamiltonian system:

$$\begin{aligned}\dot x &= \frac{\partial H}{\partial p}\\ \dot p &= -\frac{\partial H}{\partial x}\end{aligned}\qquad\text{with}\qquad H\equiv\frac{p^2}{2m}+V(x)\,.$$

Linearize

$$\mathbb{J}=\begin{pmatrix}\frac{\partial^2 H}{\partial x\partial p} & \frac{\partial^2 H}{\partial p^2}\\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial p\partial x}\end{pmatrix}=\begin{pmatrix}0 & \frac{1}{m}\\ -\frac{\partial^2 V}{\partial x^2} & 0\end{pmatrix}\,.$$

At any fixed point we have $\tau\equiv\mathrm{tr}\mathbb{J}=0$ and $\Delta\equiv\det\mathbb{J}(\boldsymbol{x}^*)=V''(\boldsymbol{x}^*)/m$, i.e. $\lambda_{1,2}=\pm\sqrt{-\det\mathbb{J}}$:

$$\text{If } V''(\boldsymbol{x}^*)<0 \text{ then } \lambda_{1,2}=\pm\sqrt{|V''(\boldsymbol{x}^*)|/m}\qquad\text{(saddle)}$$

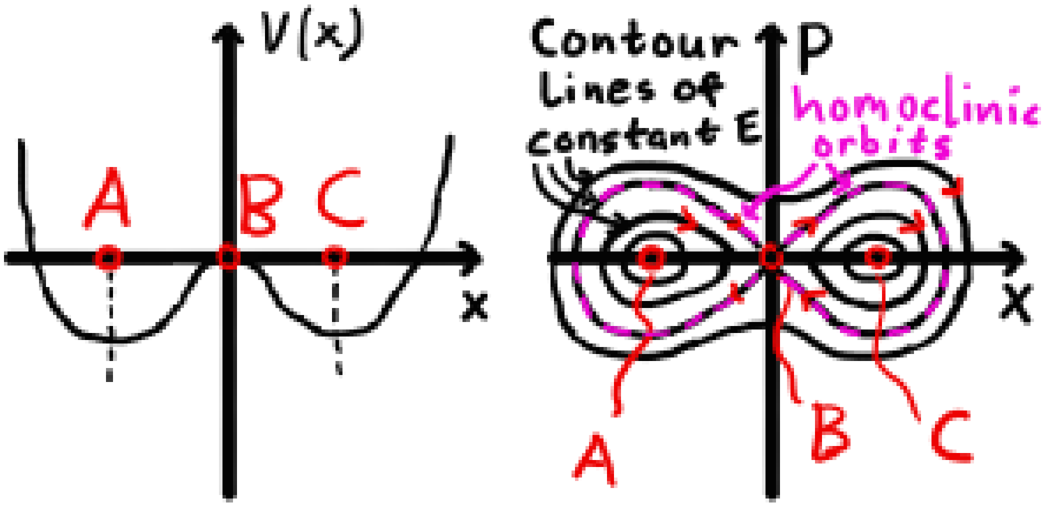
$$\text{If } V''(\boldsymbol{x}^*)>0 \text{ then } \lambda_{1,2}=\pm\mathrm{i}\sqrt{|V''(\boldsymbol{x}^*)|/m}\qquad\text{(center)}$$

Fixed points are located at $\dot{x} = p = 0$ and $\dot{p} = -V'(x) = 0 \Rightarrow$ Centers at potential minima and saddles at potential maxima.

Example: Double-well potential

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$$

The paths in the phase plane are contours of constant $H(\boldsymbol{x}) = \frac{p^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} = E$:



A and C are centers (potential minima) and B is a saddle (potential maximum). Isolated centers in Hamiltonian systems are robust to non-linear perturbations (non-linear centers). They are surrounded by a band of closed orbits despite the non-linear terms. The closed orbits around the centers correspond to small oscillations around the potential minima. The stable and unstable manifold of the saddle point connect at the fixed point and form a homoclinic orbit (this trajectory is not strictly periodic because it takes an infinite amount of time to reach the fixed point). The homoclinic orbit serves as a separatrix: outside of it, the system has high enough energy to make large-amplitude oscillations over both potential minima.

Closed orbits in conservative systems More generally, the following holds for two-dimensional conservative systems $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with smooth $\boldsymbol{f}$: If $\boldsymbol{x}^*$ is an isolated fixed point and $E(\boldsymbol{x})$ is a local minimum of E, then the paths in the vicinity of $\boldsymbol{x}^*$ are closed (Strogatz Theorem 6.5.1).

How to find conserved quantities There is no general method to find conserved quantities in a given system. As seen above, a system that can be written as $\ddot{x} = -V'(x)$ has a conserved quantity $E = y^2/2 + V(x)$ with $y = \dot{x}$.
Another method is to search for an explicit relation between y and x by dividing $\dot{y}$ by $\dot{x}$. For example

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= y^2 + x\end{aligned}$$

gives

$$\frac{\dot{y}}{\dot{x}} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 + x}{y}$$

Let $\tilde{y} = e^{-x}y$ to get

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}x} = -e^{-x}y + e^{-x}\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-x}\frac{x}{y} = e^{-2x}\frac{x}{\tilde{y}}$$

Solve by separation of variables

$$\tilde{y}\mathrm{d}\tilde{y} = e^{-2x}x\mathrm{d}x \quad \Rightarrow \quad \frac{\tilde{y}^2}{2} = -\frac{1}{4}e^{-2x}(1 + 2x) + E$$

with an integration constant E. In conclusion

$$E(x, y) = e^{-2x}\frac{y^2}{2} + \frac{1}{4}e^{-2x}(1 + 2x)$$

is a conserved quantity.

5.2 Reversible systems (Strogatz 6.6)

Another mechanism to form closed orbits is invariance under the simultaneous transformation $t \to -t$ and $y \to -y$ (reversible system). This transformation mirrors trajectories in the upper half-plane on the lower half-plane.
Special case: time-reversible system if reversible and $\dot{x} = y$.

Example Consider the system

$$\begin{aligned}\dot{x} &= y - y^3\\ \dot{y} &= -x - y^2\end{aligned}$$

The system has three fixed points: $(x_1^*, y_1^*) = (-1,-1)$, $(x_2^*, y_2^*) = (0,0)$, and $(x_3^*, y_3^*) = (-1,1)$.
Linear stability analysis

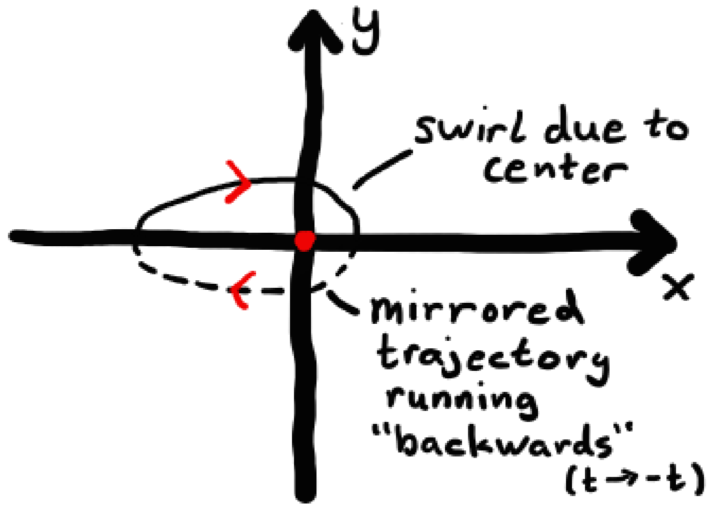
$$\mathbb{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix}$$

Classification of fixed points:

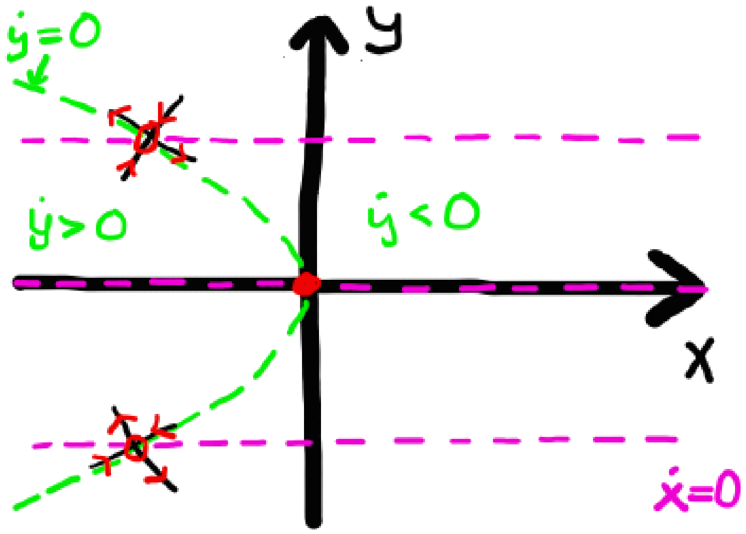
Fixed point	$(x_1^*, y_1^*) = (-1,-1)$	$(x_2^*, y_2^*) = (0,0)$	$(x_3^*, y_3^*) = (-1,1)$
Eigenvalues	$1 \pm \sqrt{3}$	$\pm\mathrm{i}$	$-1 \pm \sqrt{3}$
Type	Saddle	Center	Saddle
Stable direction	$\frac{(1+\sqrt{3},1)}{\sqrt{5+2\sqrt{3}}}$	-	$\frac{(-1+\sqrt{3},1)}{\sqrt{5-2\sqrt{3}}}$
Unstable direction	$\frac{(1-\sqrt{3},1)}{\sqrt{5-2\sqrt{3}}}$	-	$\frac{(-1-\sqrt{3},1)}{\sqrt{5+2\sqrt{3}}}$

Linear stability analysis shows that the origin is a center, but the non-linear terms could destabilise, there is no conservation law as for the conservative systems. However, the system is reversible: invariant under the simultaneous transform $t \to -t$ and $y \to -y$

Close to the origin, the center causes swirl for positive values of y. Time reversibility mirrors the solution in the line $y = 0$ and trajectories must close (this rules out spirals):

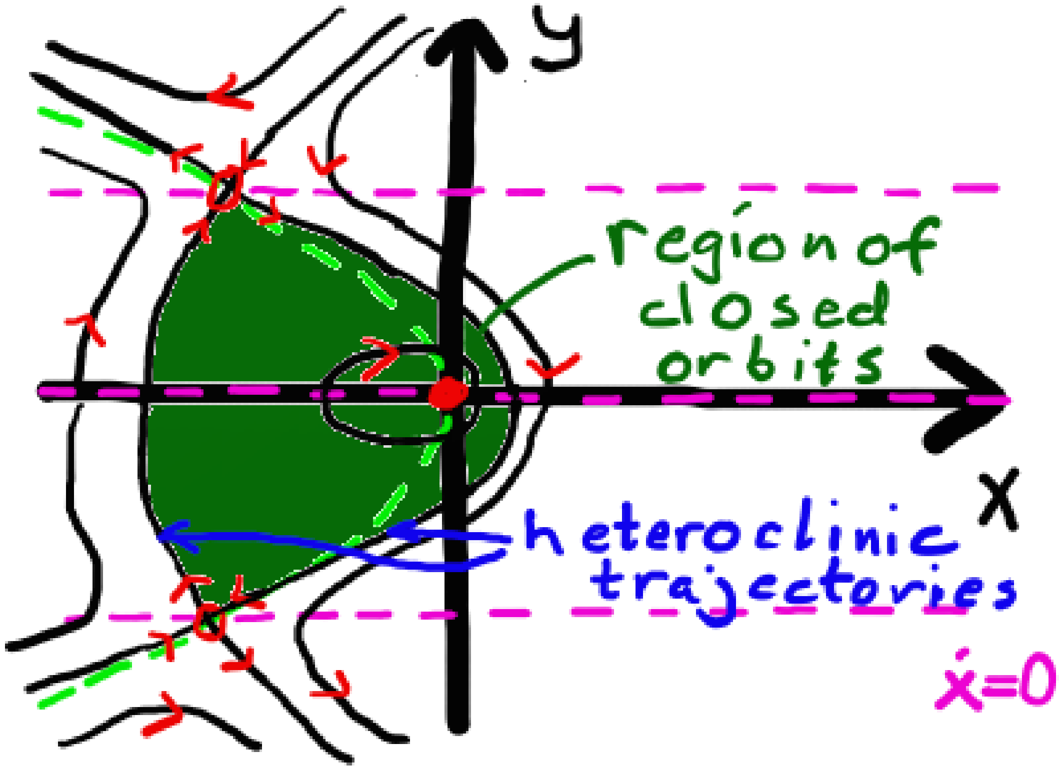


To sketch the full phase portrait, start with the nullclines and linearized dynamics around the saddle points (need to evaluate the direction of the stable and unstable manifolds of the saddle points.)



The unstable trajectory going up from the lower saddle point must continue upwards ($\dot{y} > 0$) until $y = 0$ is reached with vertical slope ($\dot{x} = 0$ at $y = 0$).
Due to the reversibility there must be a mirrored trajectory coming from the upper saddle. The joint trajectory joins two fixed points heteroclinic trajectory.
Likewise, the unstable trajectory down from the upper saddle point must move down and meet the $y = 0$ line at some $x > 0$. At the crossing point it joins with a mirrored trajectory from the lower saddle point, forming a second heteroclinic trajectory. The heteroclinic

orbits surrounds a region of closed orbits.



Reversible or conservative systems tend to have trajectories connecting one fixed point with itself (homoclinic orbits, c.f. the double-well potential in Section 5.1.4) or with another fixed point (heteroclinic trajectory).
Note that the homoclinic orbit is not a periodic orbit: it takes an infinite amount of time to reach the fixed point.
Homoclinic orbits and heteroclinic cycles (several heteroclinic trajectories forming a loop) can act as attractors (or repellers) for other trajectories, or as divisors between regions of separated dynamics (separatrices): In our case between bands of closed orbits and other trajectories (as in the example above).
It is also possible that isolated closed orbits, not containing fixed points, act as attractors|repellers. Such orbits are limit cycles.

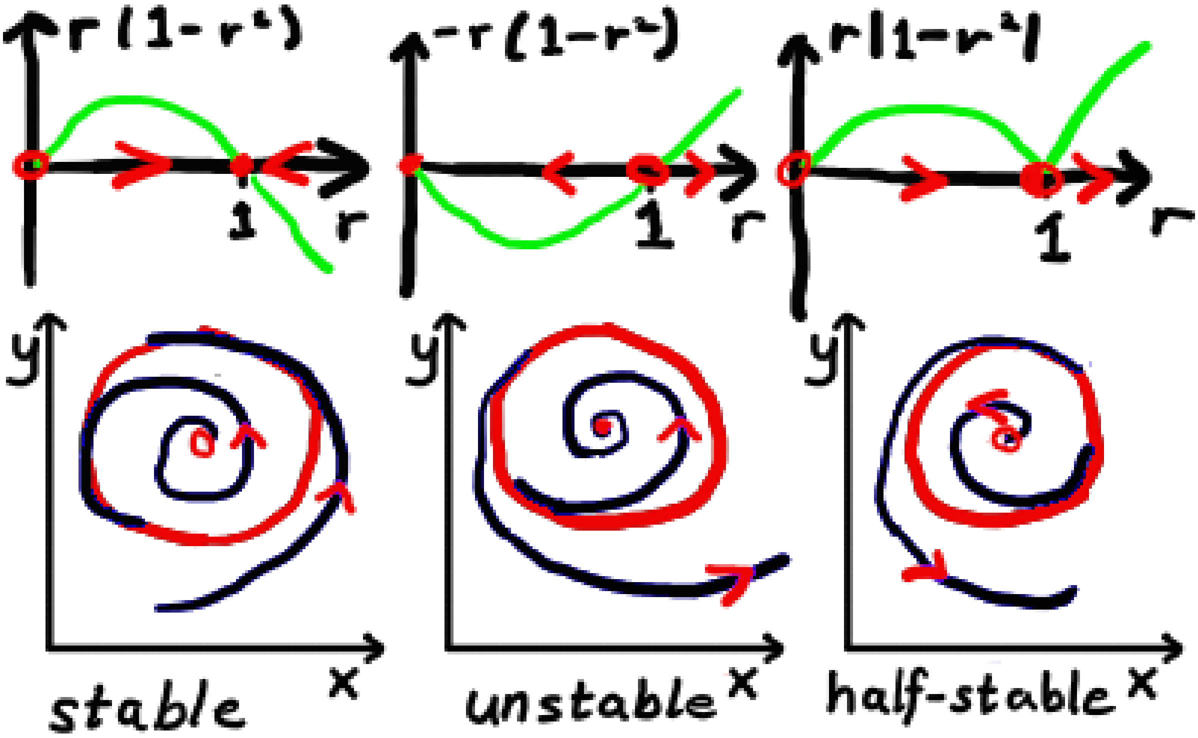
5.3 Limit cycles (Strogatz 7)

Systems with limit cycles are useful in order to model self-sustained oscillations (oscillations without external periodic forcing), such as the firing of a pacemaker, cycles in the body , oscillating chemical reactions, unwanted or dangerous self-excitations in mechanical systems.

Simplest construction using uncoupled polar coordinates

$$\dot{r} = f(r)$$
$$\dot{\theta} = \omega$$

where $f(r)$ has zeroes for $r > 0$. r describes a one-dimensional system whose fixed points determines the stability of the limit cycles. Examples of stable, unstable, and half-stable limit cycles:



Note that the system above is given in terms of polar coordinates and that the limit cycle appears after changing to Cartesian coordinates.

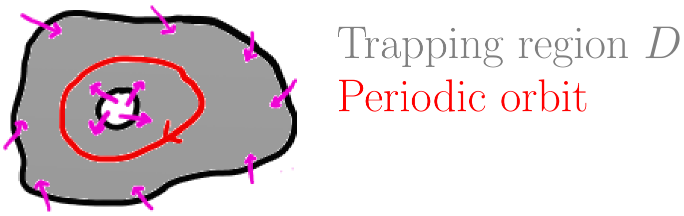
Limit cycles (in the Cartesian coordinates) can not be found using linear-stability analysis ⇒ non-local effect, non-linear terms are necessary. They can be stable, unstable and half-stable (it is also possible to have a limit cycle from outside|inside, while band of closed orbits on the inside|outside).

More non-trivial examples of limit cycles will be discussed later in the course (van der Pol oscillators).

5.4 To prove that closed orbits exist

Poincaré-Bendixson theorem Assume a smooth flow in a bounded domain D of the plane. Assume further that D does not contain any fixed point and that there exists a trajectory that is confined in D for all times. Then at least one periodic orbit exists in D. This is a consequence of the fact that trajectories for smooth flows cannot intersect in two dimensions.

To satisfy the condition that a confined trajectory exists, one can construct a trapping region, i.e. choose D such that the flow points inward everywhere. If it is possible to construct a trapping region, then the Poincaré-Bendixson theorem ensures that at least one closed orbit exists in D.



As a consequence in two dimensions: possible attractors are fixed points, periodic orbits, or union of fixed points and homo|hetero-clinic orbits. In higher dimensions: infinite wandering of trajectories is possible (trajectories never repeats).

5.5 To rule out closed orbits or limit cycles

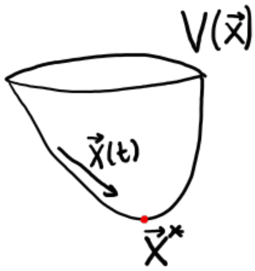
- If system is a gradient system: $\dot{\boldsymbol{x}} = -\boldsymbol{\nabla} V(\boldsymbol{x})$ with potential function V. During one revolution of a supposed periodic orbit with period time T the potential changes by $\Delta V \equiv V(\boldsymbol{x}_T) - V(\boldsymbol{x}_0)$

$$\Delta V = \int_0^T \mathrm{d}t\,\dot{V} = \int_0^T \mathrm{d}t\,\dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla} V = -\int_0^T \mathrm{d}t\,|\dot{\boldsymbol{x}}|^2 < 0$$

  But ΔV bust be 0 because $\boldsymbol{x}(T) = \boldsymbol{x}(0)$ for the periodic orbit. Hence, our assumed periodic orbit cannot exist.

This criterion rules out limit cycles in one-dimensional systems, ($f(x)$ can be written as ∇V). When $d > 1$ gradient systems are atypical.

- Construct a function $V(\boldsymbol{x})$ (Lyapunov function) that
  - is positive everywhere except at a fixed point $\boldsymbol{x}^*$ where it is zero
  - decreases along any trajectory ($\dot{V} \equiv \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla} V < 0$).



If a Lyapunov function can be constructed, the fixed point $\boldsymbol{x}^*$ is globally attracting and no closed orbits can exist.

- Dulac's criterion: consider a differentiable function $g(\boldsymbol{x})$ such that $\boldsymbol{\nabla} \cdot (g\dot{\boldsymbol{x}})$ does not change sign in some domain. If such function exists, there are no closed orbits in the domain (Green's theorem).

Concept test 5.1: Closed orbits