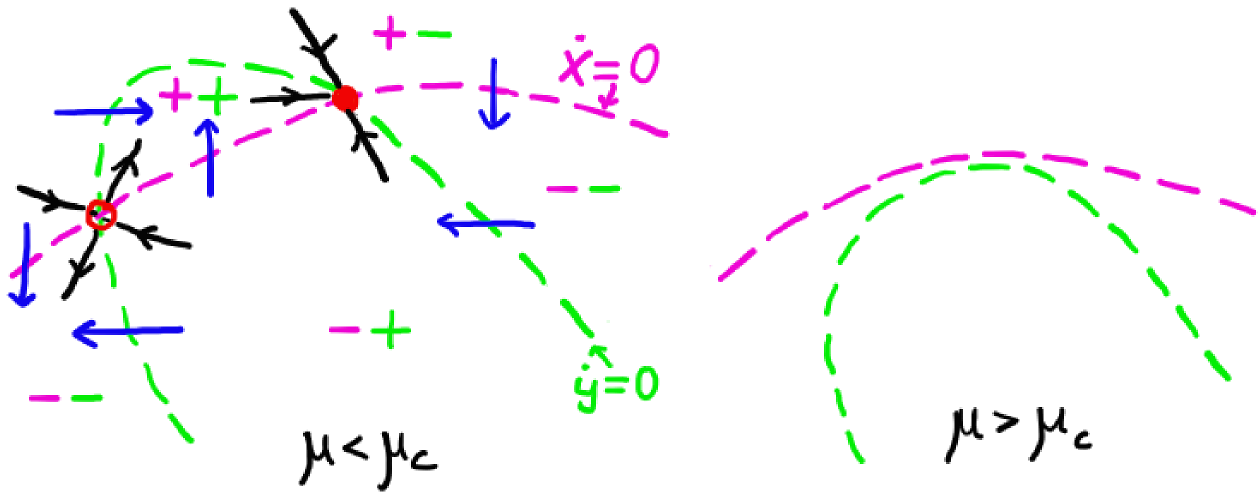
7 Two-dimensional bifurcations

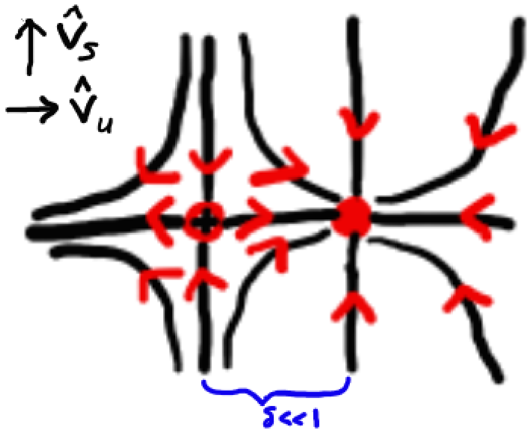
As in one-dimensional systems: fixed points may be created, destroyed, or change stability as parameters are varied (change of 'topological equivalence'). In addition closed orbits may undergo these changes.

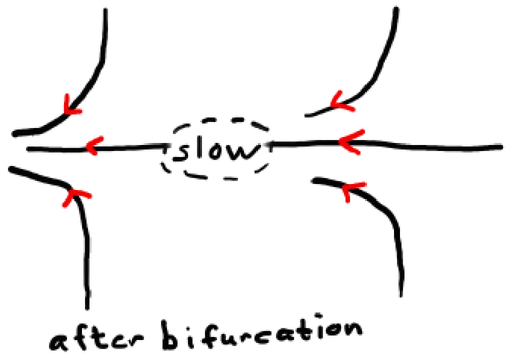
7.1 Saddle-node, transcritical, and pitchfork bifurcations

Assume that a saddle point and an attracting node collide as a parameter μ is varied. The mechanism of why the collision occurs at all (instead of the fixed points moving past each other): Fixed points are formed at intersections of nullclines. As μ is varied, the nullclines deform continuously. If they slip through each other the fixed points collide:



Change coordinates to the local eigenframe of the saddle point. Let the unstable direction of the saddle be $\hat{\boldsymbol{v}}_{\mathrm{u}} = (1, 0)$ and the stable direction $\hat{\boldsymbol{v}}_{\mathrm{s}} = (0, 1)$. When the node comes closeby, it must merge along the unstable manifold of the saddle [otherwise trajectories could not remain continuous and linear as the fixed points merge].





- The bifurcation is essentially one-dimensional (in any dimension). Normal form (in unstable/stable directions of saddle):

$$\begin{aligned} \dot{x} &= \mu - x^2 \qquad\qquad \text{(same as 1D)} \\ \dot{y} &= -y \end{aligned}$$

- Along the interconnecting manifold, the eigenvalues have opposite signs $\Rightarrow$ at bifurcation (at least) one eigenvalue must vanish.

- After bifurcation a slow region remains (ghost of fixed points) [before bifurcation $t_{\text{pass}} = \infty$ (along interconnecting manifold), this time is reduced smoothly after bifurcation: $T_{\text{pass}} \sim 1/\sqrt{\mu}$] (Strogatz Sec. 4.3 and Lecture 2).

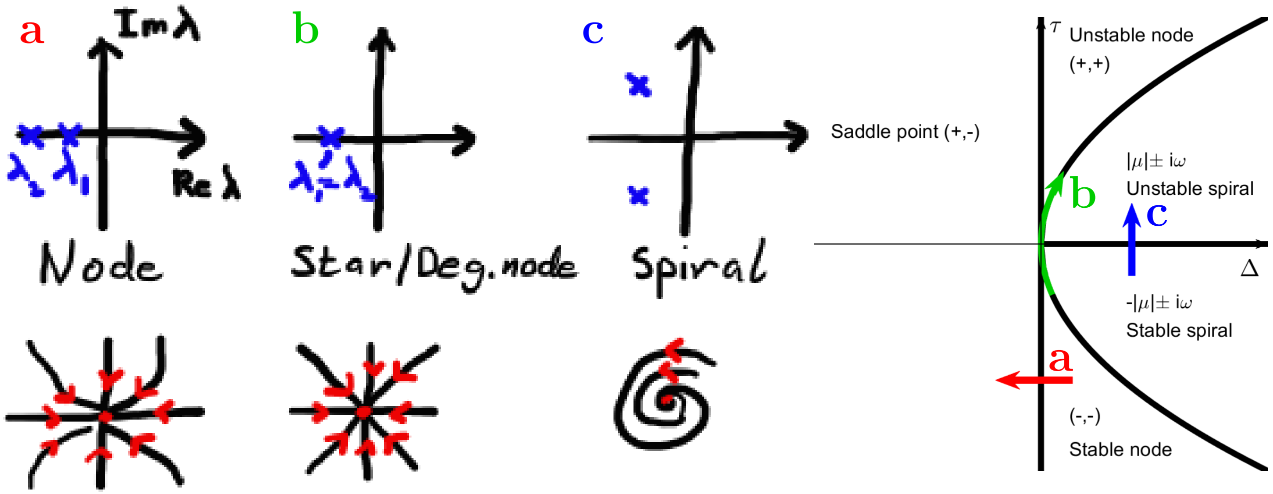
- Repelling node? $\Rightarrow$ reverse the arrows!

- The sum of all indices of the fixed points involved in a two-dimensional bifurcation in a smooth flow must be conserved (assuming that no value of the bifurcation parameter gives rise to a line of fixed points). After the saddle-node bifurcation no fixed points remain and the index must be zero. $\Rightarrow$ only fixed-points with opposite signs may annihilate. Nodes, degenerate nodes, spirals, centers, stars: $I = +1$, Saddles: $I = -1 \Rightarrow$ bifurcations where two fixed points merge and annihilate consist of one saddle and one fixed point with $I = +1$. Moreover, since one of the eigenvalues smoothly crosses zero at the bifurcation ($\Leftrightarrow \Delta$ crosses zero) the second fixed point is typically a node (unless also τ passes zero), hence the name saddle-node bifurcation.

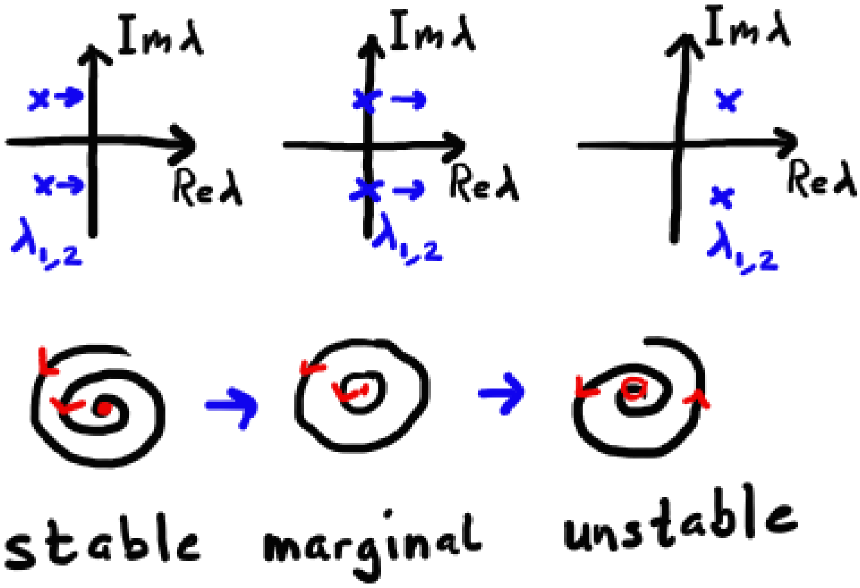
Similarly, the other bifurcations discussed in Lecture 2 (transcritical, subcritical pitchfork, supercritical pitchfork), occur in one-dimensional subspaces in higher-dimensional systems. Transversal directions are simply attracting or repelling. The bifurcations are summarized in the Table on the last page. The dynamics along the x-axis is that of 1D flows (x-component of flow plotted as black) and blue shows flow in 2D. The bifurcation diagrams show that the index is preserved.

7.2 Hopf bifurcation

A stable fixed point has $\mathcal{R}e[\lambda_{1,2}] < 0$. A bifurcation to an unstable fixed point occurs if the maximal eigenvalue crosses zero. Consider the three possible bifurcations from stable to unstable in a linear system:



Cases **a** and **b** have $\mathcal{I}m[\lambda_{1,2}] = 0$, while case **c** has $\mathcal{I}m[\lambda_{1,2}] \neq 0$. Case **a** corresponds to saddle-node, transcritical, and pitchfork bifurcations above. Case **b** is marginal and therefore not so interesting. Case **c** is a Hopf bifurcation: a new type of bifurcation that does not exist in 1D systems. Consider the transition with $\mathcal{I}m[\lambda_{1,2}] \neq 0$:



Hopf bifurcations often lead to the formation of limit cycles. The bifurcation can be either supercritical or subcritical.

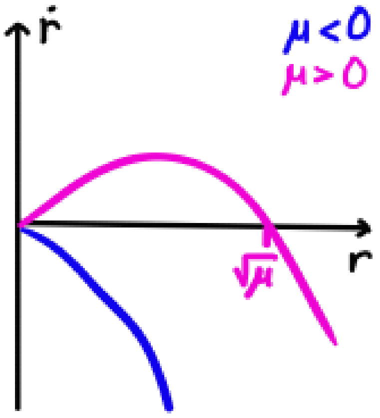
7.2.1 Supercritical Hopf bifurcation

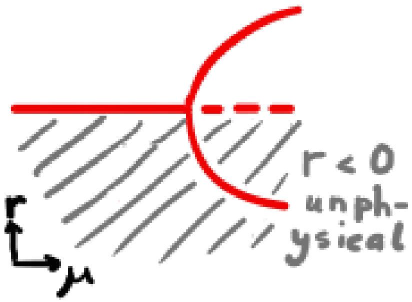
If a small-amplitude limit cycle is formed to 'catch' the unstable trajectories after the bifurcation, the bifurcation is supercritical.

Example

$$\dot{r} = r(\mu - r^2)$$
$$\dot{\theta} = \omega$$

Radial equation is on the form of a supercritical pitchfork.
When $\mu < 0$ the radial equation has a stable fixed point at $r^* = 0$ (stable spiral in x,y-space).
When $\mu > 0$ the origin becomes unstable spiral, but trajectories are caught at $r^* = \sqrt{\mu}$, i.e. limit cycle of radius $r = \sqrt{\mu}$. $\Rightarrow$ stable small-amplitude oscillations.





To verify that the eigenvalues cross $\mathrm{Re}[\lambda_i] = 0$ with nonzero $\mathrm{Im}[\lambda_i]$ we need to convert to cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$. Some algebra gives

$$\begin{aligned}\dot{x} &= \mu x - \omega y + \text{ cubic terms}\\ \dot{y} &= \omega x + \mu y + \text{ cubic terms}\,.\end{aligned}$$

Stability matrix at fixed point in origin

$$\mathbb{J} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

Eigenvalues $\lambda = \mu \pm \mathrm{i}\omega$, become unstable as μ becomes positive.

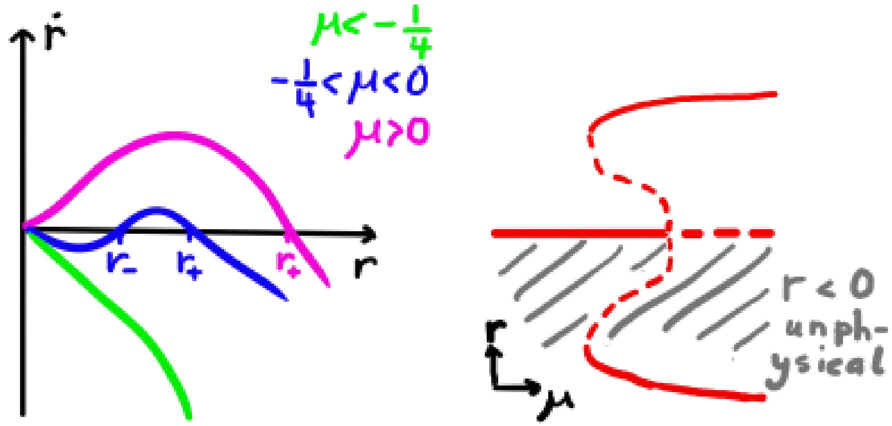
7.2.2 Subcritical Hopf bifurcation

If no stable limit cycle is formed when the fixed point becomes unstable, trajectories must run away to a distant attractor: fixed point, limit cycle, infinity (or strange attractor for $d > 2$).

Example

$$\begin{aligned}\dot{r} &= \mu r + r^3 - r^5\\ \dot{\theta} &= \omega\end{aligned} \tag{1}$$

Non-negative zeroes at $r_0^* = 0$ for all μ, $[r_\pm^*]^2 = (1 \pm \sqrt{1+4\mu})/2$ if $-1/4 \le \mu \le 0$. When μ passes 0: r_0^*, r_-^* and $-r_-^*$ merge in a subcritical pitchfork bifurcation. System settles in the distant limit cycle at $r = r_+^*$.



This jump is similar to the catastrophes in lecture 3. The system exhibits hysteresis, when μ becomes positive, the system jumps to the distant attractor $r = r_+^*$. To go back to the original stable spiral, it is not enough to reduce μ below zero, it must be reduced below the saddle-node bifurcation at $\mu = -1/4$. The bifurcation at $\mu = -1/4$ is an example of a global bifurcation, to be discussed in Section 7.3.1.

Whether one obtains a stable limit cycle after a subcritical Hopf bifurcation depends on the global properties of the flow. Before the bifurcation the system always has an unstable limit cycle.

7.3 Global bifurcations

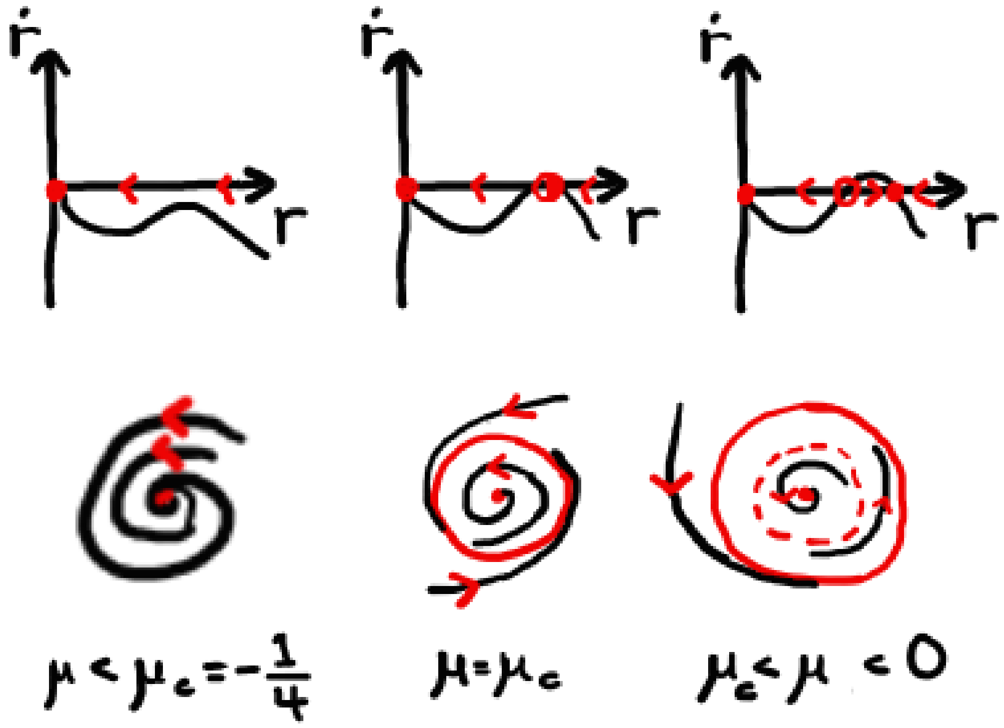
The bifurcations mentioned above are local, they happen locally as fixed points collide or change stability. It is also possible to create/destroy limit cycles in non-local regions of flow.

7.3.1 Bifurcations of cycles

Consider once again Eq. (1)

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega$$

This time, consider the bifurcation as μ passes $\mu_c = -1/4$. The one-dimensional system for r undergoes a saddle-node bifurcation at a non-zero value of $r \Rightarrow$ limit cycles in the two-dimensional system bifurcate:



7.3.2 Infinite-period bifurcation

Example: Saddle-node bifurcation on existing limit cycle.

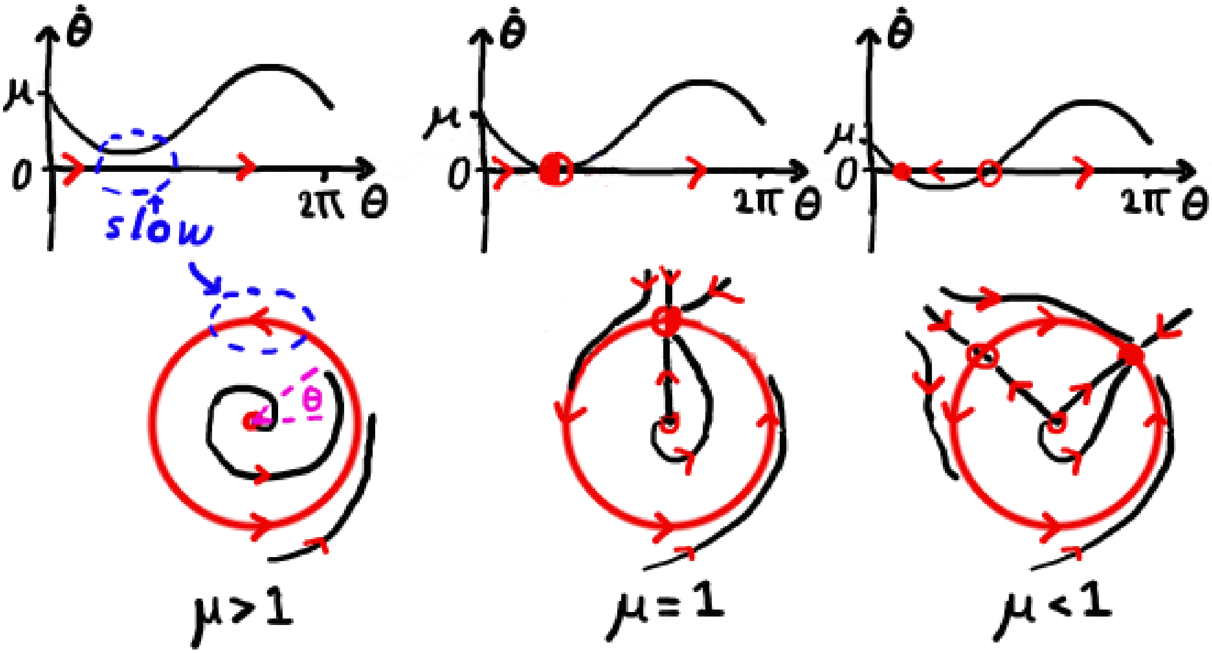
$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \mu - \sin\theta$$

Uncoupled equations.
r-equation 'usual' equation for attracting limit cycle at $r = 1$.

- When $\mu > 1$ we have a stable limit cycle with a bottleneck (slow velocity) at $\theta = \pi/2$.

- When $\mu = 1$ a half-stable fixed point appears at $(r, \theta) = (1, \pi/2)$ $\Rightarrow$ it takes an infinite time to pass $\theta = \pi/2$ along the homoclinic orbit (former limit cycle). At $\theta = \pi/2$ the flow must be vertical towards $r = 1$.

- When $\mu < 1$ a saddle-node pair is formed, joined by heteroclinic trajectories.



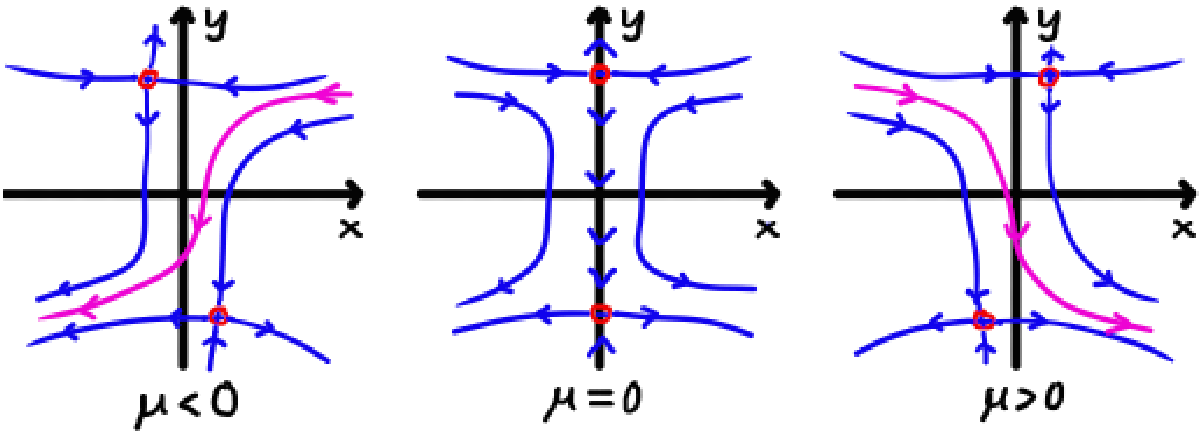
As shown in lecture 2 the dynamics is slow close to the saddle-node bifurcation (the time scale along the limit cycles scales as $1/\sqrt{\mu-1}$ for both sides of the bifurcation).
The scaling of the period with the control parameter μ is important in order to investigate oscillating systems in numerical or real-life experiments. Observing amplitude and period time as μ is varied allows to identify or rule out what kind of system we have.
Another infinite-time bifurcation (with another scaling in period time, $T \sim \ln \mu$, see Problem set 2) is the Homoclinic bifurcation.

7.3.3 Bifurcation of heteroclinic trajectory

Consider the dynamical system

$$\dot{x} = \mu + x^2 - xy$$
$$\dot{y} = y^2 - x^2 - 1$$

This system has two saddle points ($\det \mathbb{J} = -2(x-y)^2 < 0$) at: $(x^*_\pm, y^*_\pm) = \pm(\mu, 1-\mu)/\sqrt{1-2\mu}$. When $\mu = 0$, they lie on the y-axis, $(x^*_\pm, y^*_\pm) = (0, \pm 1)$, and since $\dot{x} = 0$ along the y-axis, they must be connected by a heteroclinic trajectory. When $|\mu|$ is small but non-zero, the fixed points move to either side of the y-axis, $(x^*_\pm, y^*_\pm) \approx \pm(\mu, 1)$. Since $\dot{x} = \mu$ along the y-axis, the connection between the stable and unstable manifolds must be broken as seen in the following phase portrait:



Purple trajectories shows that for $\mu < 0$ there ar solutions in x from $+\infty$ to $-\infty$, while after the bifurcation the system has solutions in x from $-\infty$ to $+\infty$.

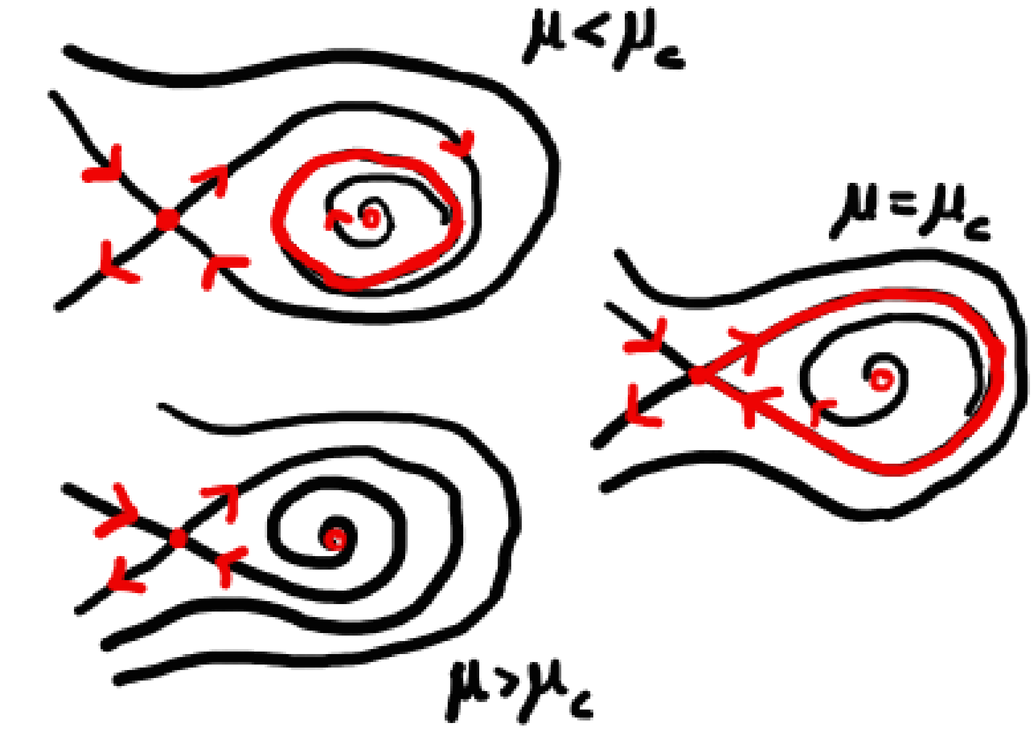
7.3.4 Bifurcation of homoclinic orbit

Similar to the heteroclinic trajectory above, one may obtain a bifurcation for the homoclinic orbit of a single saddle point. For example, a symmetric system (as considered in Lecture 6) with a homoclinic orbit at the bifurcation point μ_c. A small deviation from μ_c may break the symmetry and the homoclinic orbit breaks differently depending on the sign of the deviation.

Another example is the collision between a limit cycle and a saddle point to form a homoclinic orbit.

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \mu y + x - x^2 + xy\end{aligned}$$

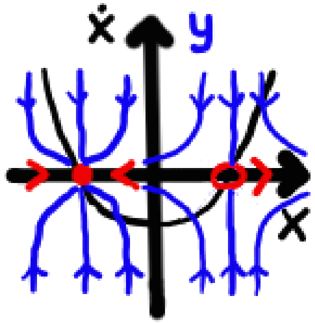
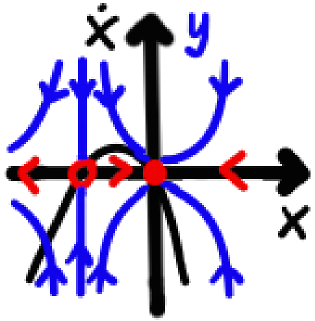
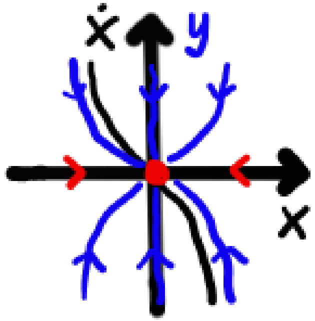
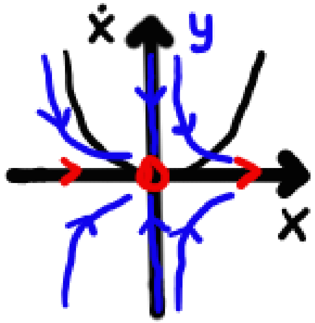
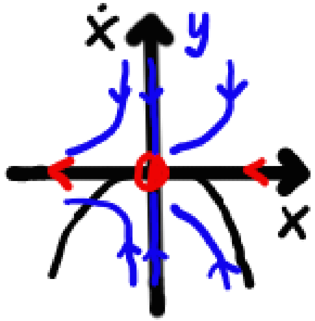
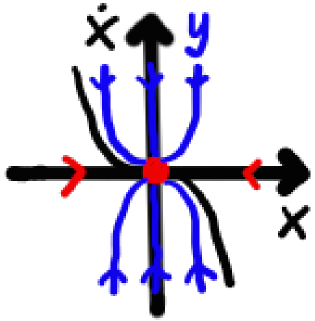
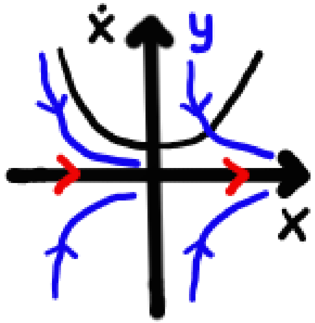
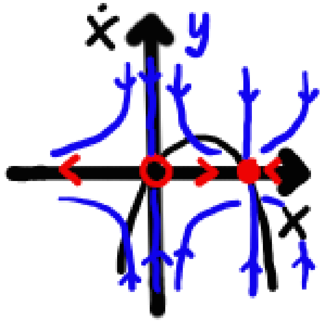
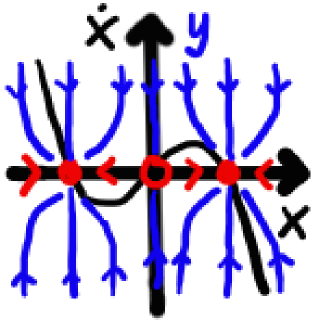
Non-local bifurcation $\Rightarrow$ need to use computer! The result for some values of μ:



For $\mu < \mu_{rmc} \approx -0.8645$ a saddle point and a limit cycle are isolated. As μ is increased, the limit cycle expands until it eventually collides with the saddle point at $\mu = \mu_{\mathrm{c}}$, forming a homoclinic orbit.
When $\mu > \mu_{rmc}$ the homoclinic orbit breaks.

An example is a Josephson junction which is equivalent to a forced pendulum with friction (Lecture 9).

The homoclinic bifurcation is another example of an infinite-period bifurcation (the homoclinic orbit has an infinite period time). The upper right unstable manifold lies inside of stable manifold when $\mu < \mu_{\mathrm{c}}$ and lies outside when $\mu > \mu_{\mathrm{c}}$.

Type Normal form	saddle-node $\dot{x} = \mu + x^2$ $\dot{y} = -y$	transcritical $\dot{x} = \mu x - x^2$ $\dot{y} = -y$	supercrit. pitchfork $\dot{x} = \mu x - x^3$ $\dot{y} = -y$	subcrit. pitchfork $\dot{x} = \mu x + x^3$ $\dot{y} = -y$
$\mu < 0$				Similar to super crit.
$\mu = 0$				
$\mu > 0$				
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