9 Another example and chaotic billiards

9.1 Example 2: Driven damped pendulum (Strogatz 8.5)

Add a constant torque τ to the damped pendulum from Lecture 1:

$$\ddot{\theta} = -\frac{\gamma}{m}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{\tau}{I_0}$$

with I_0 the moment of inertia.

This equation can be written as a dimensionless dynamical system by a suitable rescaling of t and $\dot{\theta}$ (Problem set 2) . In dimensionless units we have:

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t'} &= y \\ \frac{\mathrm{d}y}{\mathrm{d}t'} &= -\alpha y - \sin\theta + \underbrace{\frac{\tau l}{I_0 g}}_{I}\end{aligned} \tag{1}$$

where α = dimensionless damping, I = dimensionless torque.

This system is identical to that of the phase difference in a Josephson junction (superconducting device): Strogatz 4.6. It can also be used as a lowest-order approximative model for a large number of problems.

We can view the system Eq. (1) as a flow on a cylinder with θ periodic $-\pi < \theta \leq \pi$ and keep aperiodic angular velocity y.

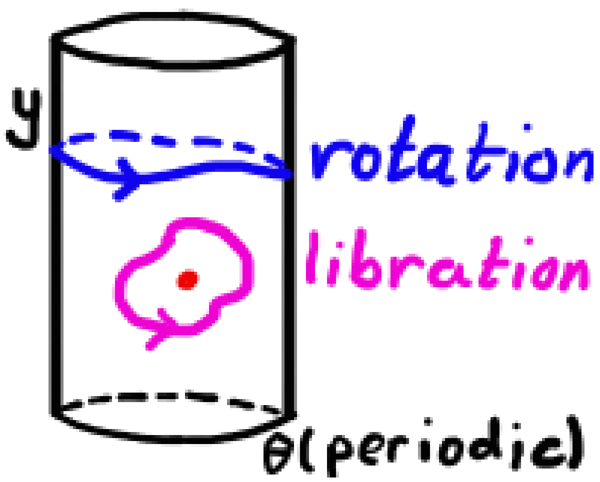
9.1.1 Analysis for small α and I

Simple pendulum (Strogatz 6.7) When $\alpha = I = 0$ Eq. (1) simplifies to a simple (undamped, undriven) pendulum:

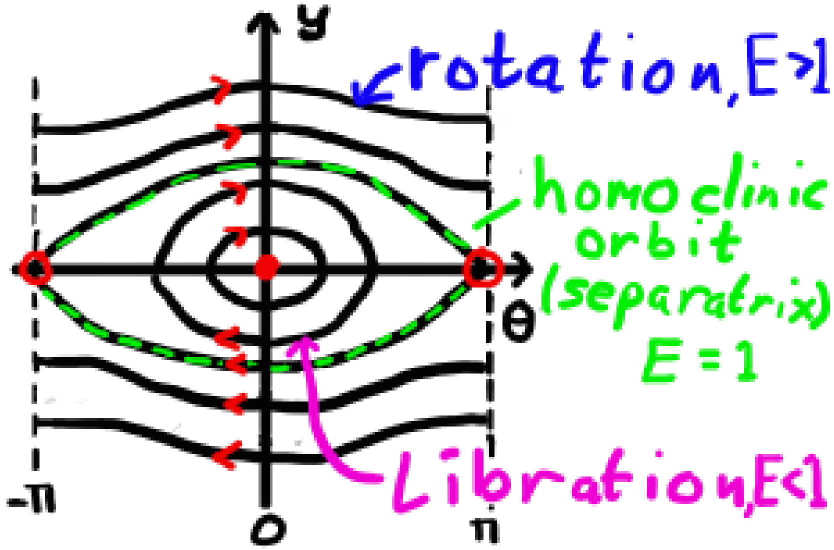
$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t'} &= y \\ \frac{\mathrm{d}y}{\mathrm{d}t'} &= -\sin\theta\,.\end{aligned}$$

This is a Hamiltonian system with $V'(\theta) = \sin\theta \Rightarrow$ potential energy $V(\theta) = -\cos\theta$. On the cylinder there are two distinct fixed points given by the potential minimum at $(\theta^*, y^*) = (0, 0)$ (nonlinear center) and the potential maximum at $(\theta^*, y^*) = (\pi, 0)$ (saddle).

On the cylinder periodic orbits comes in two types: librations and rotations

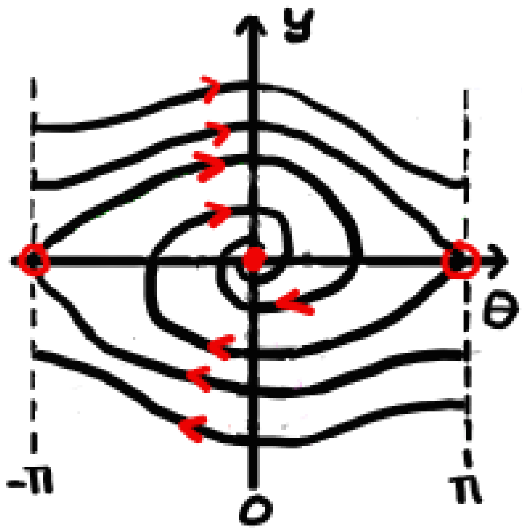


Note that due to index theory librations always encircle a fixed point, while rotations instead encircle the cylinder.
For the simple pendulum two homoclinic orbits [homoclinic because $(\theta^*, y^*) = (-\pi, 0)$ and $(\theta^*, y^*) = (\pi, 0)$ are the same point on the cylinder] separate regions with closed orbits in the forms of librations and rotations:



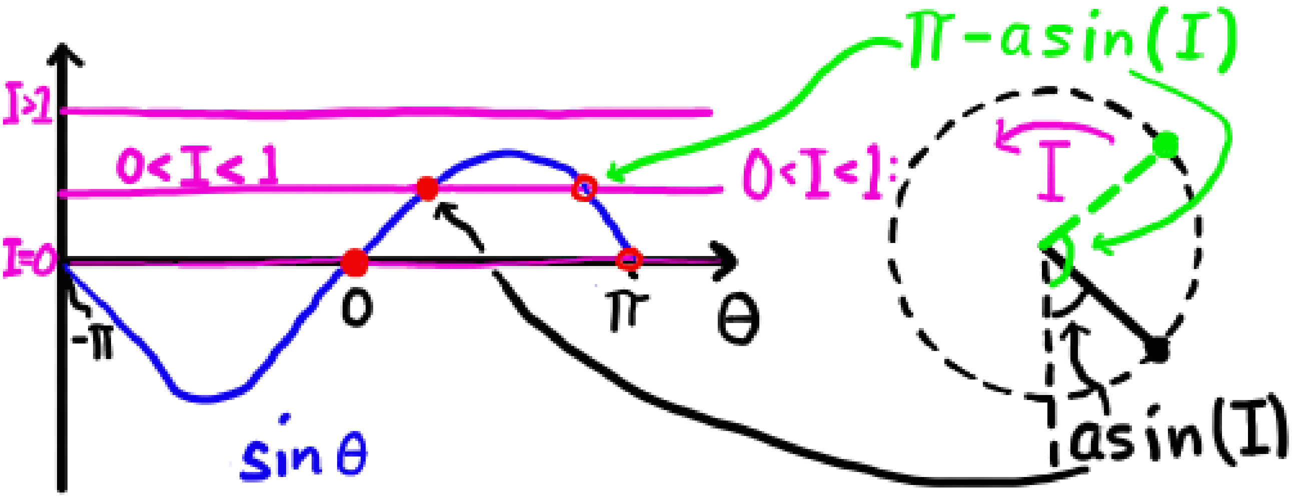
Which orbit is obtained depends on the energy E of the initial condition.
Along any trajectory we have the dimensionless energy $E = y^2/2 - \cos\theta = \text{const}$ (conservation of energy in Hamiltonian systems). At the saddle point $(\theta^*, y^*) = (-\pi, 0)$ we have $E = 1$, the energy for the homoclinic orbits. Librations have $E < 1$ and rotations $E > 1$. Due to energy conservation, $E = y^2/2 - \cos\theta = 1$, the two homoclinic orbits can be parameterized as $y_{\rm h}^{\pm} = \pm\sqrt{2 + 2\cos\theta}$.

Weakly damped pendulum When $0 < \alpha \ll 1$ and $I = 0$ we have a weakly damped pendulum: the homoclinic orbits are broken and the fixed point $(\theta^*, y^*) = (0, 0)$ becomes a stable spiral ($\lambda_{1,2} = -\alpha/2 \pm {\rm i}\sqrt{1 - \alpha^2/4}$).



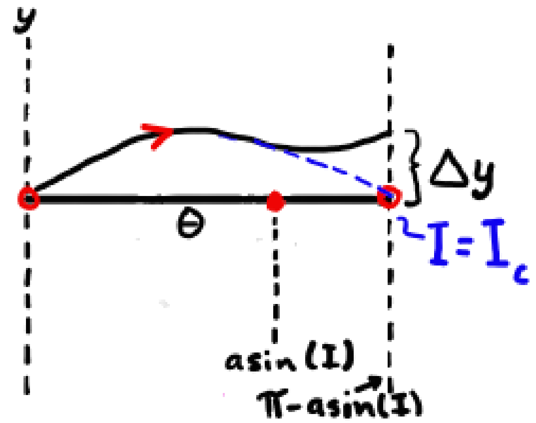
Letting $\alpha > 0$ destroys the homoclinic orbit due to the dissipative damping term. Is it possible to recover the homoclinic orbit by input of energy from the driving term, $I > 0$? Yes! by Melnikov's method.

Weakly damped and weakly driven pendulum For general values of α and I the system (1) has fixed points where $y^* = 0$ and $\sin\theta^* = I$. Depending on the value of I we have 0 ($I > 1$), 1 ($I = 1$), or 2 ($0 \le I < 1$) fixed points:



When the driving is weak ($0 < I \ll 1$) the position of the saddle point is shifted to $(\theta^*, y^*) = (\pi - \text{asin}(I), 0)$ (the constant driving shifts the unstable equilibrium from up-side down to an angle).

Evaluate the energy function from the unperturbed system, $H = y^2/2 - \cos\theta$, along a trajectory originating from the saddle. For most values of α and I we end up at a point different from the saddle after one lap on the cylinder. Such trajectories have non-zero $\Delta H = H_{\text{start}} - H_{\text{end}}$ (H is assumed to vary smoothly with the small parameters α and I):



Aim: Find critical small parameters I and α such that $\Delta H = 0$ and consequently $\Delta y = 0$ for the perturbed trajectory ('perturbed' = governed by Eq. (1) with non-zero α and I).

The change in H when going from $\theta = -\pi - \text{asin}(I)$ to $\theta = \pi - \text{asin}(I)$ (the upper homoclinic orbit) along perturbed trajectory is given by (Melnikov integral, upper integration bound t_{f} is the time at which $\theta = \pi - \text{asin}(I)$ is reached (assuming $I \geq I_{\text{c}}$ so that $\theta =$

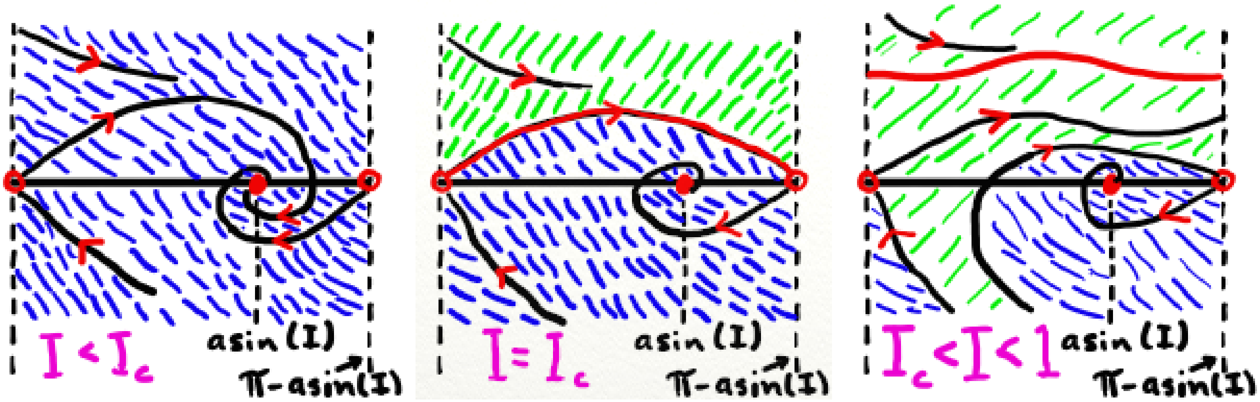
$\pi - \text{asin}(I)$ is reached at all):

$$
\begin{aligned}
\Delta H &= \int_{-\infty}^{t_\mathrm{f}} \mathrm{d}t' \dot{H} = \int_{-\infty}^{t_\mathrm{f}} \mathrm{d}t' [\underbrace{\dot{y}}_{-\alpha y - \sin\theta + I} y + \underbrace{\dot{\theta}}_{y} \sin\theta] \\
&= \int_{-\infty}^{t_\mathrm{f}} \mathrm{d}t' y(I - \alpha y) = \int_{-\infty}^{t_\mathrm{f}} \mathrm{d}t' \frac{\mathrm{d}\theta}{\mathrm{d}t'} (I - \alpha y) \\
&= \int_{-\pi - \text{asin}(I)}^{\pi - \text{asin}(I)} \mathrm{d}\theta (I - \alpha y) = [\text{Use } y = y_\mathrm{h}^+ = \sqrt{2 + 2\cos\theta} + O(I, \alpha)] \\
&= \int_{-\pi - \text{asin}(I)}^{\pi - \text{asin}(I)} \mathrm{d}\theta (I - \alpha\sqrt{2 + 2\cos\theta}) + O(\alpha^2, \alpha I) \\
&= 2\pi I - 8\alpha + O(\alpha^2, \alpha I)
\end{aligned}
$$

Thus $\Delta H = 0$ is a simple zero for $I_\mathrm{c} = 4\alpha/\pi + O(\alpha^2)$. We have found an orbit that has the same value of H after one lap on the cylinder, and which is isolated (ΔH is simple zero, any modification of I_c gives $\Delta H \neq 0$) $\Rightarrow$ the found orbit is a homoclinic orbit.

The same evaluation for the lower homoclinic orbit (going from $\theta = \pi - \text{asin}(I)$ to $\theta = -\pi - \text{asin}(I)$) gives $\Delta H = -2\pi I - 8\alpha$, i.e. $\Delta H = 0$ cannot be satisfied $\Rightarrow$ the lower homoclinic orbit is always broken if $\alpha > 0$ and $I > 0$.

In conclusion, for small α and I we have a homoclinic bifurcation at $I_\mathrm{c} \approx 4\alpha/\pi$:



For $I < I_\mathrm{c}$ the damping overtakes the driving and the motion is damped to zero.

At $I = I_\mathrm{c}$ a limiting trajectory (the homoclinic orbit) is created that allows the pendulum to make one full lap, but in infinite time.

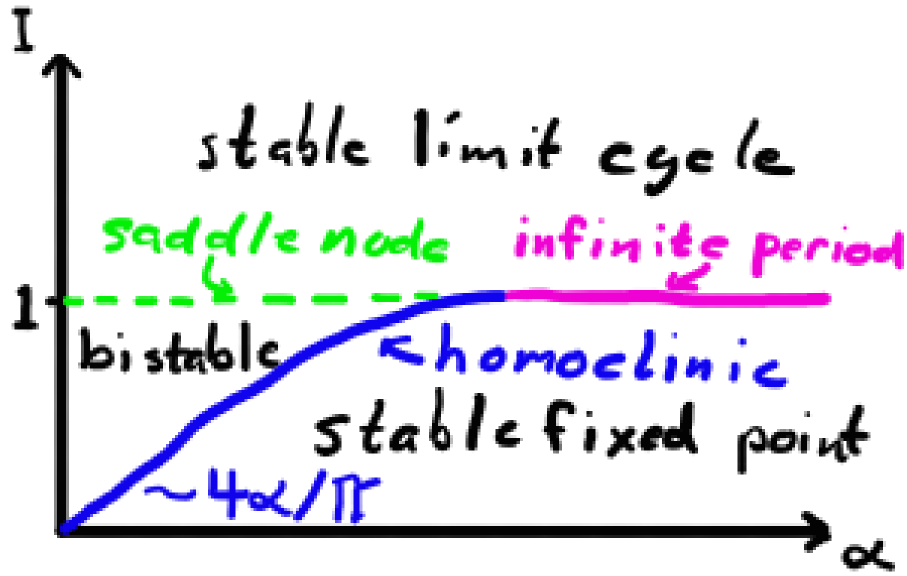
For $I > I_c$ the system is bistable, it shows a stable limit cycle (full lap rotations) and a stable spiral decaying the motion to zero. Which attractor is reached depends on whether if the initial position is in the basin of attraction of the limit cycle (green hashed area above) or the spiral (blue hashed), separated by the stable/unstable manifolds of the saddle point.

9.1.2 Larger α and I

When $I > 1$ no fixed points exists. Librations are not possible (Index theory: closed orbits must encircle a fixed point). It is possible to show (Strogatz 8.5) that a unique attracting limit cycle in the form of a rotation exists (the strong torque give the pendulum a stationary full-lap rotation).

For larger values of α it is possible to show that the limit cycle form in an infinite-time bifurcation (Strogatz 4.6), i.e. not a homoclinic bifurcation as for small values of α.

Using numerical simulations one obtain a phase-diagram (Strogatz Fig. 8.5.10)



Here lines show bifurcations, black text denote system attractor(s).

9.2 Coupled oscillators (Strogatz 8.6)

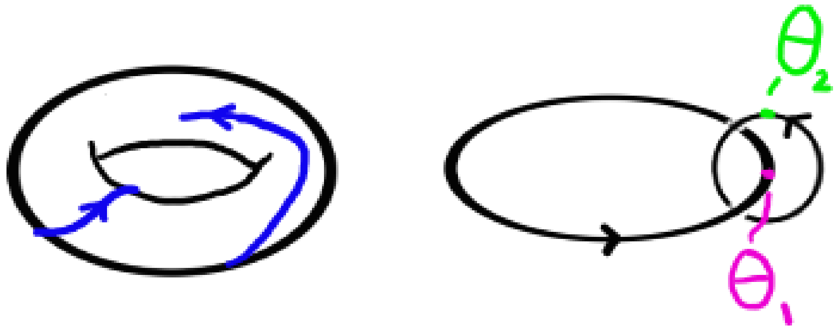
As we saw in the previous example the topology of the dynamics matters: the flow on a cylinder allows for new type of periodic orbit (rotations) not encircling a fixed point.
In a system with two periodic coordinates we have a flow on a torus.

9.2.1 Flow on a torus

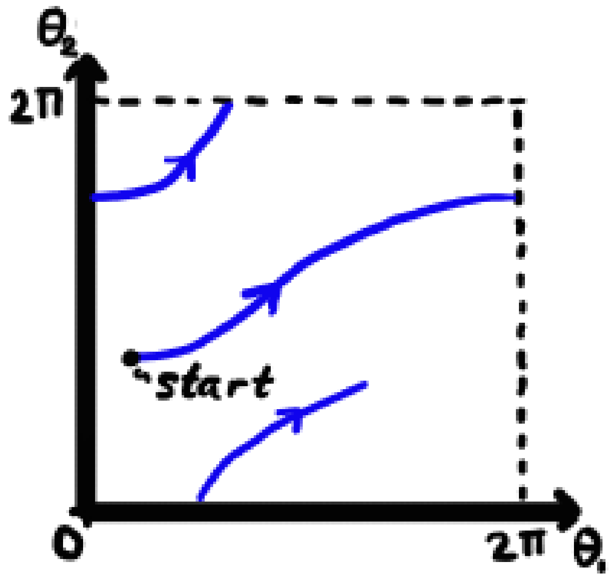
The dynamics

$$\dot{\theta}_1 = f_1(\theta_1, \theta_2)$$
$$\dot{\theta}_2 = f_2(\theta_1, \theta_2)$$

represents a flow on a torus (θ_1 and θ_2 both 2π periodic):

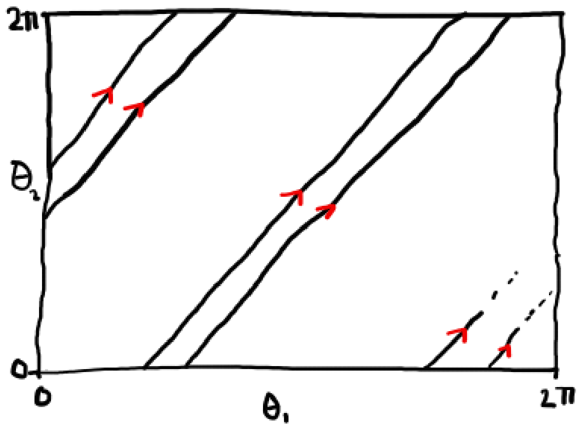


Easier to visualise on a square with periodic boundaries (opposite end-points are the same points)



9.2.2 Example: Uncoupled oscillators

Simplest case $f_1 = \omega_1 =$ const. and $f_2 = \omega_2 =$ const.. Uncoupled dynamics:



The lines have slope: $\mathrm{d}\theta_2/\mathrm{d}\theta_1 = \dot{\theta}_2/\dot{\theta}_1 = \omega_2/\omega_1$.
Two special cases:

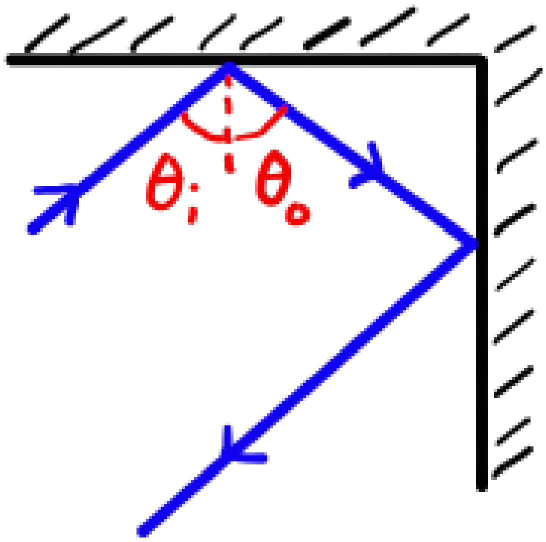
- ω_2/ω_1 rational (can be written as a ratio p/q with p and q integers) $\Rightarrow$ closed orbits
- ω_2/ω_1 irrational $\Rightarrow$ trajectory never closes, cover entire torus (quasiperiodic, new long-term behaviour . Only appears on the torus.)

9.2.3 Other examples

- Phase locking and synchronization (Strogatz 8.6).
- Integrable hamiltonian systems: Can be solved using 'action-angle coordinates', the solutions are simply uncoupled oscillators on (hyper) tori. If there is time, this will be discussed in a later lecture, but a simplified version, integrable billiard systems comes next.

9.3 Billiards

In billiard systems a particle has constant velocity until it hits a boundary, where it bounces using the law of reflection (incident angle θ_{i}=outgoing angle θ_{o}):



Billiards are important as simple models of problems in optics, statistical physics (gases), electrons in metal (Lorentz gas), etc.

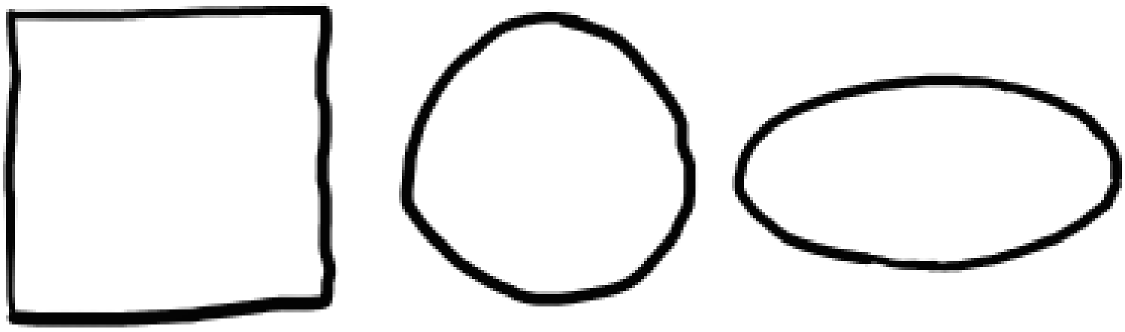
Billiard systems constitute examples of Hamiltonian systems:

$$H(\boldsymbol{x},\boldsymbol{p}) = \frac{p^2}{2m} + V(\boldsymbol{x}), \text{ with } V(\boldsymbol{x}) = \begin{cases} 0 & \text{Inside billiard area} \\ \infty & \text{At walls} \end{cases}$$

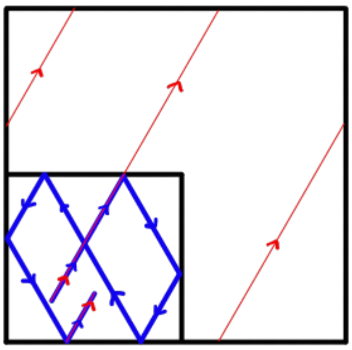
It is possible to show (later) that a Hamiltonian system is integrable if there exists d independent integrals of motion. These systems show periodic (or quasiperiodic) motion (regular patterns)

The type of dynamics depend on the geometry of the billiard table.

Integrable billiards -



- Rectangle: p_x^2 and p_y^2 are constants of motion. The dynamics closely resembles that of uncoupled oscillators on the torus in Section 9.2.2:



The billiard trajectory (blue) is obtained by mirroring of the uncoupled-oscillator trajectory (red) around the centers of the vertical and horizontal axes. Depending on the ratio v_y/v_x we get a closed orbit (rational ratio) or a quasiperiodic orbit (irrational ratio).

- Circle: Kinetic energy $p^2/(2m)$ and angular momentum w.r.t. center are constants of motion.

- Ellipse: Kinetic energy $p^2/(2m)$ and product of angular momentum w.r.t. the 2 foci is conserved

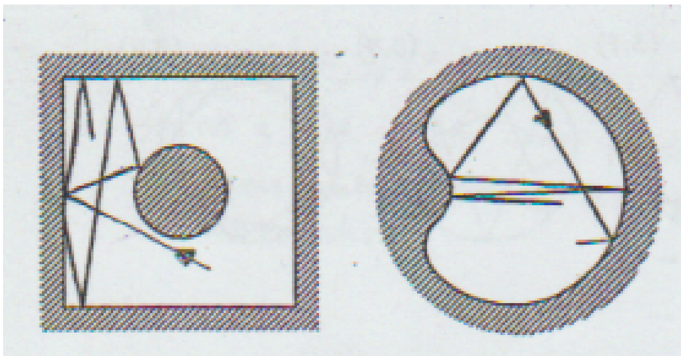
Integrable motion is always predictable. Two close-by initial conditions stay together for a large time (they may separate, but not exponentially fast).

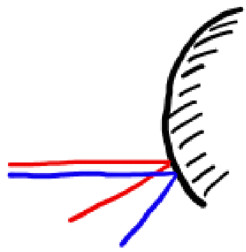
Note: If the billiard table is two-dimensional, we have a four-dimensional dynamical system (x,y,v_x,v_y). We plot a two-dimensional projection ⇒ it may look like trajectories cross, but they do not in the four-dimensional phase space.

Now consider cases where one symmetry is broken

9.4 Non-integrable billiards

Introduce circle in center of rectangular billiard (left).





Energy $(p_x^2 + p_y^2)/(2m)$ still conserved, but not the individual components p_x^2 and p_y^2 ⇒ non-integrable system for most initial conditions.

Convex surface causes closeby trajectories with closeby angles to separate ⇒ trajectories with small initial distance show exponential separation after many bounces. Further: (most) trajectories are aperiodic (infinitely long!) and densely fills space.

These systems are chaotic: (almost all) trajectories depend sensitively on the initial condition.
Any disturbance (in physical system) or numerical inaccuracy (in numerical simulations) is exponentially amplified ⇒ long-term prediction impossible.