CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	January 09, 2017, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once at 09^{30}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$20 passed; $\geq$27 grade 4; $\geq$32 grade 5,
GU $\geq$20 grade G; $\geq$ 29 grade VG.

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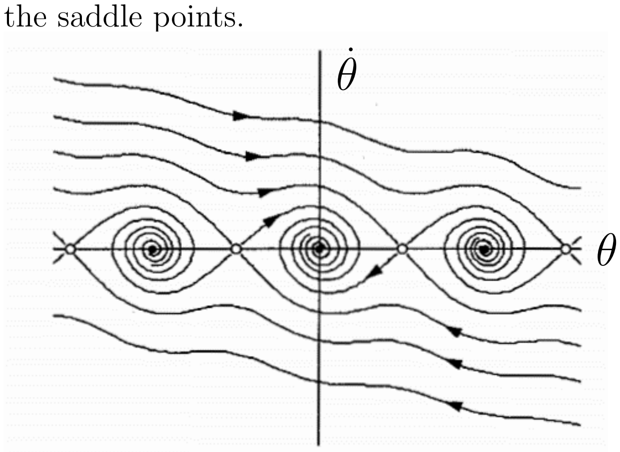
1. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) What is a Hamiltonian dynamical system?

b) Give three examples of Hamiltonian dynamical systems.

c) What is the difference between a libration and a rotation?

d) The figure below shows the phase portrait of a damped pendulum. Identify and give a physical interpretation of the stable manifolds of the saddle points.



e) Identify and give a physical interpretation of the stable manifolds of the spirals in the figure above.

f) The existence and uniqueness theorem implies that trajectories cannot intersect if the flow is smooth enough. But in many phase portraits different trajectories appear to intersect at fixed points. Is this a contradiction?

g) What does the Lyapunov time characterize? Give a rough estimate (no calculation) for the Lyapunov time for a dynamical system of your choice.

h) What does the Lyapunov dimension characterize? How is it related to the generalized dimension spectrum D_q?

2. Imperfect bifurcation [2.5 points] Consider the system

$$\dot{x} = rx - x^2 + hx^3 \tag{1}$$

where h and r are real parameters.

a) When $h = 0$ a bifurcation occurs when r passes zero. What kind of bifurcation is it (explain why)? Sketch the bifurcation diagram.

b) For each value of non-zero h we obtain a different bifurcation diagram x^* against r for the dynamical system in Eq. (1). Use analytical calculations and/or sketches of the flow to determine the fixed points and their stability for generic values of r and h. Use the results from the analysis to sketch one typical bifurcation diagram for each of the two cases $h < 0$ and $h > 0$.

c) Divide the (r,h)-plane into regions separated by lines, where each line is determined by the condition that the dynamics in Eq. (1) has exactly two separate fixed points. Label each region with the number of fixed points in that region. Label each line with the type of bifurcation that occurs if the line is passed to a different region.

3. Stability analysis and phase portrait [1.5 points] Consider the system

$$\begin{aligned}\dot{x} &= xy\\ \dot{y} &= x^2 - y\,.\end{aligned} \tag{2}$$

a) Find all fixed points of this system.

b) What does linear stability analysis predict about the fixed point(s)?

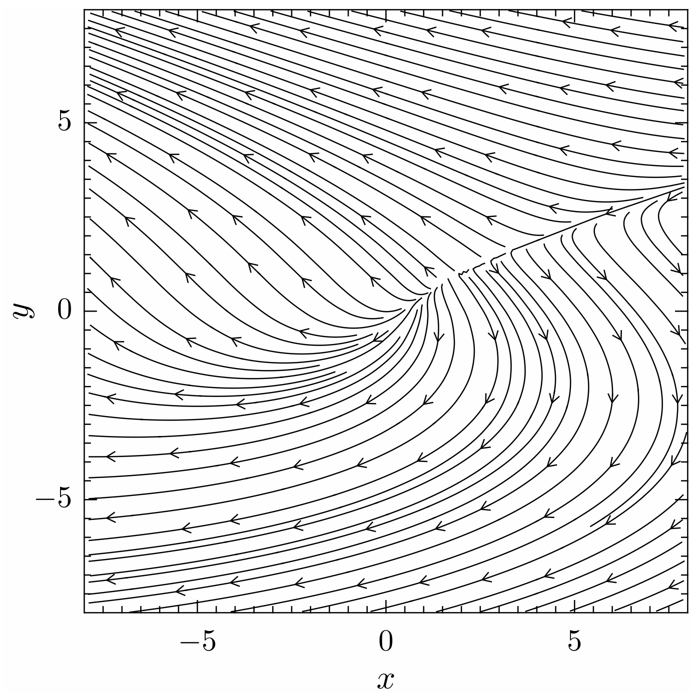
c) Consider the linear system $\dot{\boldsymbol{x}} = \mathbb{J}(\mathbf{0})\boldsymbol{x}$ with $\boldsymbol{x} = (x, y)$ and $\mathbb{J}(\mathbf{0})$ being the Jacobian evaluated at the origin. This system corresponds to the system obtained in problem b) by linearization around the fixed point at the origin. How many fixed points do the linear system $\dot{\boldsymbol{x}} = \mathbb{J}(\mathbf{0})\boldsymbol{x}$ have? Is this consistent with your findings in a)? Why?

d) In order to classify the fixed point at the origin for the system (2), sketch the nullclines and make a qualitative phase diagram. What kind of fixed point do you obtain?

4. Indices and bifurcations [2 points] The phase portraits of two dynamical systems are plotted in subproblems a) and b) below.

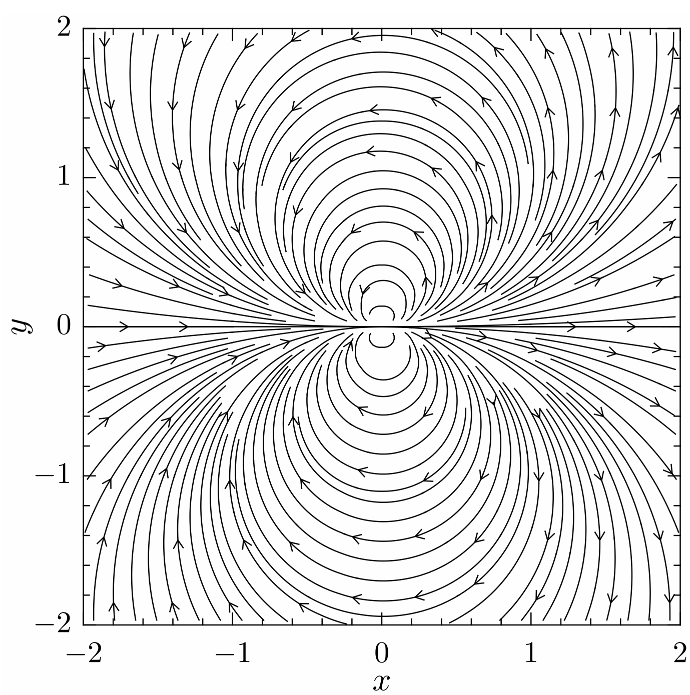
a) What is the index of the fixed point of the following dynamical system?

$$\dot{x} = x - y^2 - 1$$
$$\dot{y} = -x + 2y$$



b) What is the index of the fixed point of the following dynamical system?

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = 2xy$$



Now add perturbations to the x-components of the flows.

c) What is the bifurcation that occurs when μ passes through zero in the perturbed system below? Explain why it is that bifurcation.

$$\begin{aligned}\dot{x} &= x - y^2 - 1 + \mu \\ \dot{y} &= -x + 2y\end{aligned}$$

d) What is the bifurcation that occurs when μ passes through zero in the perturbed system below? Explain why it is that bifurcation.

$$\begin{aligned}\dot{x} &= x^2 - y^2 + \mu \\ \dot{y} &= 2xy\end{aligned}$$

e) Are the bifurcations in problems c) and d) consistent with the indices of involved fixed points and with the results in problems a) and b)?

5. Homoclinic bifurcation [2 points] Consider the dynamical system

$$\begin{aligned}\dot{x} &= \mu x + y - x^2 \\ \dot{y} &= -x + \mu y + 2x^2\,.\end{aligned} \tag{3}$$

a) What kind of bifurcation occurs at the origin as μ passes zero? Explain why it is that bifurcation. If you happen to find that the bifurcation is a Hopf bifurcation, you can use the following condition on a to determine whether the bifurcation is supercritical ($a < 0$) or subcritical ($a > 0$):

$$\begin{aligned}16a =& f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &- f_{xy}(f_{xx} + f_{yy}) + g_{xy}(g_{xx} + g_{yy}) + f_{xx}g_{xx} - f_{yy}g_{yy}\,.\end{aligned}$$

Here subscripts denote partial derivatives evaluated at the origin.

b) The system (3) has a homoclinic bifurcation at $\mu = \mu_c \approx 0.066$. Explain what happens close to the homoclinic bifurcation by sketching the dynamics for the three cases of μ slightly below μ_c, μ equal to μ_c, and μ slightly above μ_c (no bifurcation occurs in the interval $0 < \mu < \mu_c$).

c) In order to find out how the period of a closed orbit scales as a homoclinic bifurcation is approached, it is useful to first estimate the time it takes for an orbit to pass a saddle point. To estimate the time to pass a general saddle point, consider the linearized dynamics $\dot{x} = \lambda_u x$ and $\dot{y} = \lambda_s y$, where λ_u and λ_s are the eigenvalues of the unstable and stable directions respectively ($\lambda_u > 0$ and $\lambda_s < 0$). Let a trajectory start from the point $(x, y) = (\gamma, 1)$ where γ is small. Find an analytical expression for the time to escape from the saddle to $x(t) = 1$.

d) Find an analytical expression for λ_u suitable for the system in Eq. (3).

e) Estimate the time of the periodic orbit, T_μ, just below the homoclinic bifurcation. You can assume that this time is completely determined by the passage of the saddle point, i.e. it depends only on λ_μ and γ. You can assume $\gamma \sim A(\mu - \mu_c)^a$ with $A = 3.2$ and $a = 0.7$.

6. Box-counting dimension [2 points] The two figures below show the first few generations in the construction of a) the *even fifths Cantor set* and b) the *second fifths Cantor set*. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.

a) Analytically find the box-counting dimension D_0 of the even fifths Cantor set, obtained by at each generation removing intervals 2 and 4 out of five equally sized intervals:

b) Analytically find the box-counting dimension D_0 of the second fifths Cantor set, obtained by at each generation removing interval 2 out of five equally sized intervals: