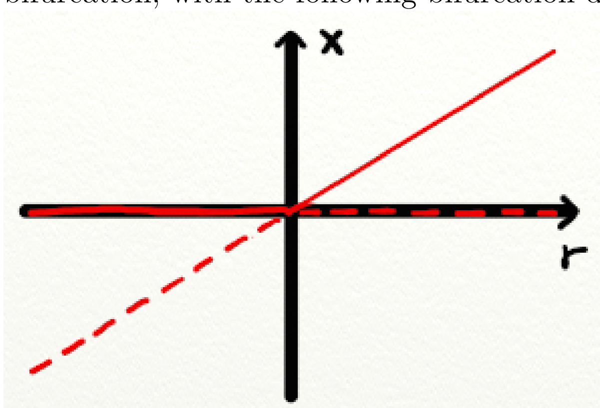
Solutions/answers to selected problems of the exam 9:th of January in Dynamical Systems 2017

2. Imperfect bifurcation [2.5 points] Consider the system

$$\dot{x} = rx - x^2 + hx^3 \tag{1}$$

where h and r are real parameters.

a) When $h = 0$ a bifurcation occurs when r passes zero. What kind of bifurcation is it (explain why)? Sketch the bifurcation diagram.
Answer: When $h = 0$ there are two fixed points $x^* = 0$ and $x^* = r$. Since $f'(x) = r - 2hx$ the fixed point at $x^* = 0$ is stable when $r < 0$ and unstable when $r > 0$. The fixed point at $x^* = r$ is unstable when $r < 0$ and stable when $r > 0$. This is the normal form of a transcritical bifurcation, with the following bifurcation diagram:



b) For each value of non-zero h we obtain a different bifurcation diagram x^* against r for the dynamical system in Eq. (1). Use analytical calculations and/or sketches of the flow to determine the fixed points and their stability for generic values of r and h. Use the results from the analysis to sketch one typical bifurcation diagram for each of the two cases $h < 0$ and $h > 0$.
Answer: The system has fixed points at

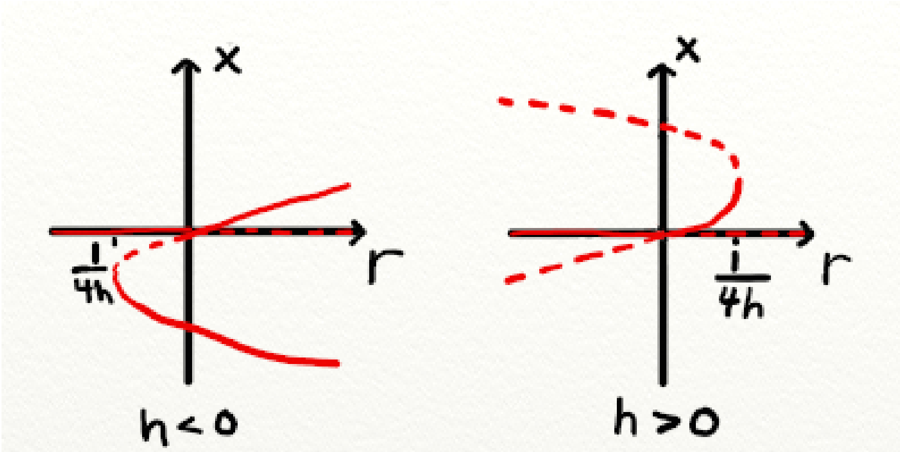
$$x_1^* = 0\,,\ x_2^* = \frac{1 + \sqrt{1 - 4hr}}{2h}\,,\ x_3^* = \frac{1 - \sqrt{1 - 4hr}}{2h}\,.$$

The fixed point at the origin is stable if $r < 0$ and unstable if $r > 0$.

- If $hr > 1/4$ we have a single fixed point located at the origin.
- If $hr < 0$ the square root is larger than 1 and we have one fixed point to the left and one to the right of the fixed point in the origin. These must have the opposite stability compared to the fixed point in the origin, i.e. when $r < 0$ they are unstable, when $r > 0$ they are stable.

- If $0 < hr < 1/4$ there are two new possibilities: the two non-zero fixed points both either lie to the left ($h < 0 \Rightarrow r < 0$) or to the right ($h > 0 \Rightarrow r > 0$) of the origin.

Together these cases give the following bifurcation diagrams:



When $h \neq 0$ we have a saddle-node bifurcation at $hr = 1/4$ and a transcritical bifurcation at $r = 0$.
Note that when h approaches zero, the saddle-node bifurcation is moved to $-\infty$ ($h < 0$) or $+\infty$ ($h > 0$) and the bifurcation diagram takes the form in problem a). This can be explicitly seen by expanding the locations of the fixed points in terms of small h. For small values of h the fixed points are located at

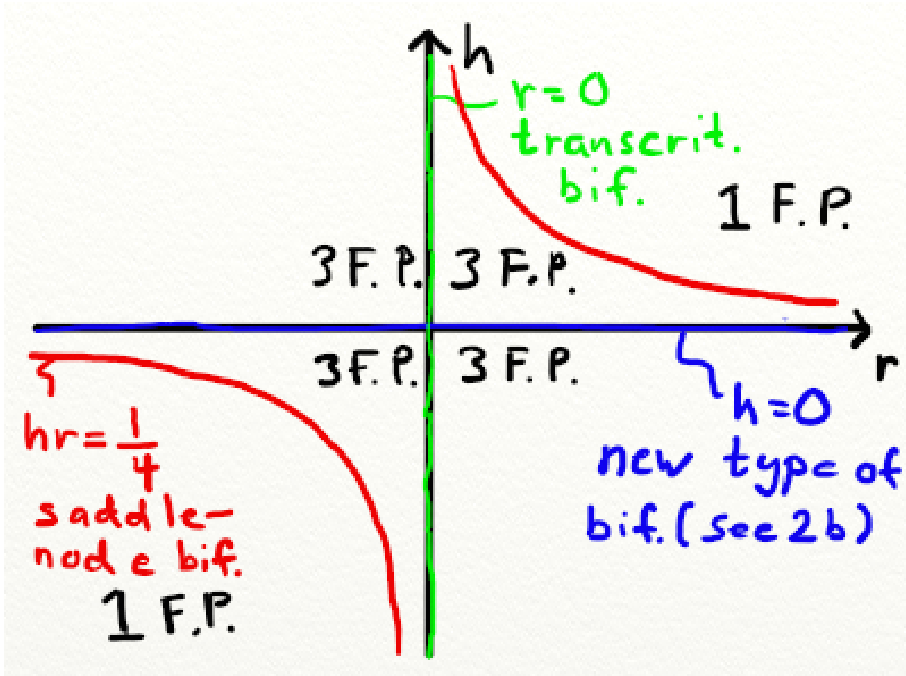
$$x_1^* = 0\,, \; x_2^* \sim \frac{1}{h} - r\,, \; x_3^* \sim r\,.$$

with gradients

$$f(x_1^*) = r\,, \; f(x_2^*) \sim \frac{1}{h} - 3r\,, \; f(x_3^*) \sim -r\,.$$

The bifurcation that occurs as h passes zero is of a type we have not encountered in the course: when $h < 0$ the second fixed point x_2^* is stable, as $h \to 0^-$ it vanishes at $-\infty$ (the system (1) becomes a second-order polynomial and there exists only two solutions), and when h becomes positive the second fixed point reappears as an unstable fixed point.

c) Divide the (r,h)-plane into regions separated by lines, where each line is determined by the condition that the dynamics in Eq. (1) has exactly two separate fixed points. Label each region with the number of fixed points in that region. Label each line with the type of bifurcation that occurs if the line is passed to a different region.
Answer: There are three lines that give two separate fixed points: $h = 0$, $r = 0$ and $4hr = 1$. According to the analysis for the cases $h < 0$, $h = 0$ and $h > 0$ above, we obtain the following bifurcation diagram:



3. Stability analysis and phase portrait [1.5 points] Consider the system

$$\begin{aligned}\dot{x} &= xy \\ \dot{y} &= x^2 - y\,.\end{aligned} \tag{2}$$

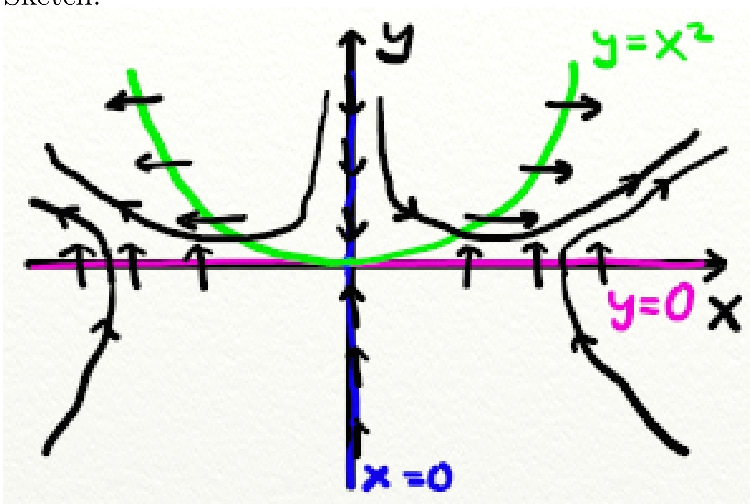
a) Find all fixed points of this system.
Answer:
The only fixed point is $(x^*, y^*) = (0, 0)$.

b) What does linear stability analysis predict about the fixed point(s)?
Answer:
The system has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0$ and corresponding eigenvectors $\boldsymbol{v}_1 = (0, 1)$ and $\boldsymbol{v}_2 = (1, 0)$. This is a marginal case between stable and unstable dynamics. If no non-linear terms were present, we would have a line of fixed points at $y = 0$.

c) Consider the linear system $\dot{\boldsymbol{x}} = \mathbb{J}(\mathbf{0})\boldsymbol{x}$ with $\boldsymbol{x} = (x, y)$ and $\mathbb{J}(\mathbf{0})$ being the Jacobian evaluated at the origin. This system corresponds to the system obtained in problem b) by linearization around the fixed point at the origin. How many fixed points do the linear system $\dot{\boldsymbol{x}} = \mathbb{J}(\mathbf{0})\boldsymbol{x}$ have? Is this consistent with your findings in a)? Why?
Answer:
The linear system has a line of fixed points, while the system in problem a) has a single isolated fixed point. The reason this is possible is because the linear system is structurally unstable; the non-linear terms changes the properties of the fixed point.

d) In order to classify the fixed point at the origin for the system (2), sketch the nullclines and make a qualitative phase diagram. What kind of fixed point do you obtain?

Answer:
Nullclines with $\dot{x} = 0$: $x = 0$ and $y = 0$.
Nullclines with $\dot{y} = 0$: $y = x^2$.
Sketch:

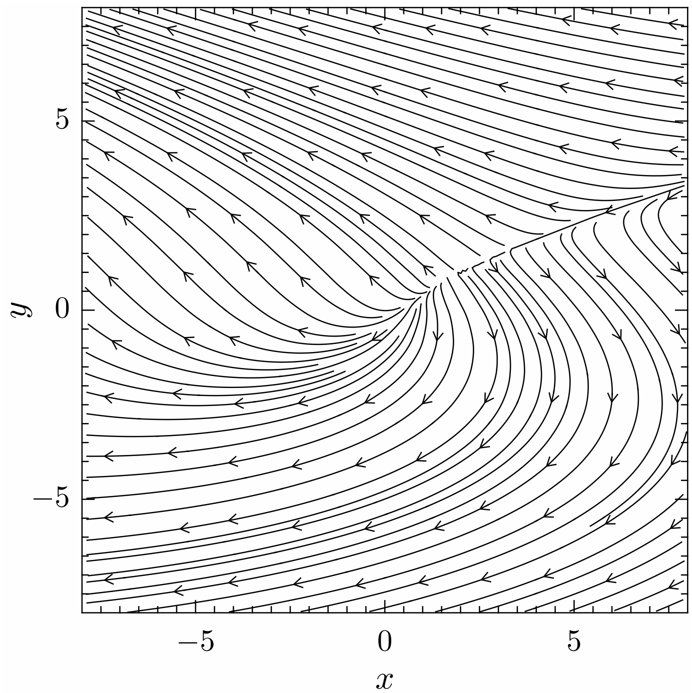


The fixed point behaves as a saddle point: it is unstable, but it has a stable direction along the y-axis. This is also consistent with the index $I = -1$ of the fixed point.

4. Indices and bifurcations [2 points] The phase portraits of two dynamical systems are plotted in subproblems a) and b) below.

a) What is the index of the fixed point of the following dynamical system?

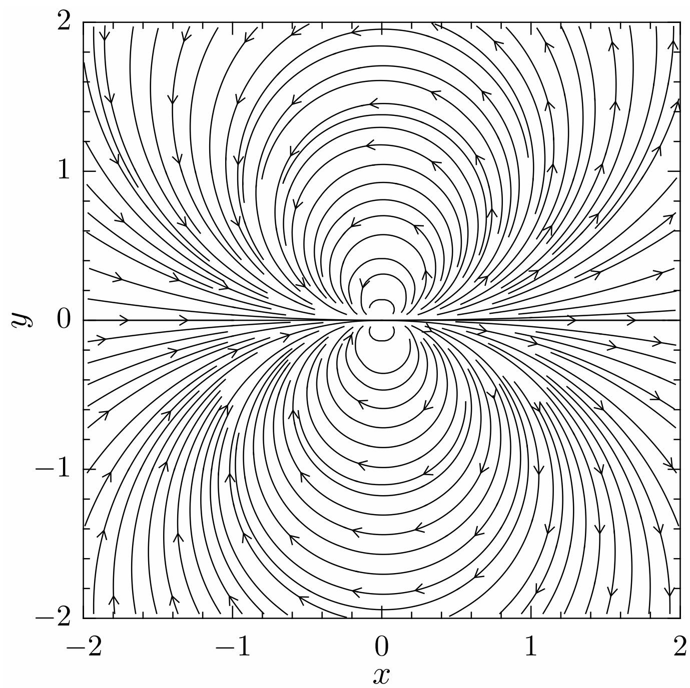
$$\dot{x} = x - y^2 - 1$$
$$\dot{y} = -x + 2y$$



Answer:
The index is $I = 0$

b) What is the index of the fixed point of the following dynamical system?

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = 2xy$$



Answer:
The index is $I = 2$

Now add perturbations to the x-components of the flows.

c) What is the bifurcation that occurs when μ passes through zero in the perturbed system below? Explain why it is that bifurcation.

$$\begin{aligned}\dot{x} &= x - y^2 - 1 + \mu \\ \dot{y} &= -x + 2y\end{aligned}$$

Answer:
The fixed points are at $(x^*, y^*) = (2,1)(1 \pm \sqrt{\mu})$. The Jacobian is

$$\mathbb{J} = \begin{pmatrix} 1 & -2y \\ -1 & 2 \end{pmatrix}$$

with eigenvalues $\lambda_{1,2} = (3 \pm \sqrt{1+8y})/2$. Thus, when $\mu < 0$ we have no fixed points. When $\mu > 0$ we have two fixed points with eigenvalues $\lambda_{1,2} = (3 \pm \sqrt{9 + 8\sqrt{\mu}})/2$ (saddle) and $\lambda_{1,2} = (3 \pm \sqrt{9 - 8\sqrt{\mu}})/2$ (unstable node). We thus have a saddle-node bifurction at $\mu = 0$.

Without calculations:
Draw the nullclines $x = 2y$ and $x = y^2 + 1 + \mu$ and see that they have zero overlap for $\mu < 0$ and double fixed point for $\mu > 0$.

d) What is the bifurcation that occurs when μ passes through zero in the perturbed system below? Explain why it is that bifurcation.

$$\begin{aligned}\dot{x} &= x^2 - y^2 + \mu \\ \dot{y} &= 2xy\end{aligned}$$

Answer:
Case $\mu > 0$: We have fixed points at $x = 0$ and $y = \pm\sqrt{\mu}$. The Jacobian is

$$J = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$\Rightarrow$ Both fixed points are centers with $(\lambda_{1,2} = \pm 2i\sqrt{\mu})$.

Case $\mu < 0$: We have fixed points at $x = \pm\sqrt{-\mu}$ and $y = 0$. Fixed points are star nodes with $(\lambda_{1,2} = -2\sqrt{-\mu})$ (stable) and $(\lambda_{1,2} = 2\sqrt{-\mu})$ (unstable).

Does non-linear terms destroy the types of the fixed points?
Spirals are not allowed due to the symmetry properties of the flow: The symmetry under reversibility $(t, x) \to (-t, -x)$ 'protects' closed orbits around the centers on the y-axis from forming spirals. The symmetry under $y \to -y$ prevents the star nodes to deform into spirals.

We have no reason to expect that the star nodes do not transform into regular nodes when the non-linear terms are added (changing to spherical coordinates centered around one of the fixed points show that we indeed have an angular dependence in the dynamics), i.e. the star nodes are regular nodes when non-linear terms are taken into account.

In conclusion: The bifurcation at $\mu = 0$ consists of a pair of one stable node and one unstable node forming two centers.

e) Are the bifurcations in problems c) and d) consistent with the indices of involved fixed points and with the results in problems a) and b)? Explain your answer.
Answer:
For problem a): Yes, saddles have index $I = -1$ and nodes have index $I = +1$. Upon bifurcation these joins into a fixed point of index $I = 0$. After the bifurcation, no fixed points exists and the index remains zero. For problem b): Yes, nodes and centers both have indices $I = +1$. Upon bifurcation these joins into a fixed point of index $I = 2$.

5. Homoclinic bifurcation [2 points] Consider the dynamical system

$$\begin{aligned} \dot{x} &= \mu x + y - x^2 \\ \dot{y} &= -x + \mu y + 2x^2 \,. \end{aligned} \tag{3}$$

a) What kind of bifurcation occurs at the origin as μ passes zero? Explain why it is that bifurcation. If you happen to find that the bifurcation is a Hopf bifurcation, you can use the following condition on a to determine whether the bifurcation is supercritical ($a < 0$) or subcritical ($a > 0$):

$$\begin{aligned} 16a = &f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &- f_{xy}(f_{xx} + f_{yy}) + g_{xy}(g_{xx} + g_{yy}) + f_{xx}g_{xx} - f_{yy}g_{yy} \,. \end{aligned}$$

Here subscripts denote partial derivatives evaluated at the origin.
Answer:
The Jacobian at the origin is

$$J = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$$

with eigenvalues $\lambda_{1,2} = \mu \pm \mathrm{i}$. Thus, the real part of the eigenvalues pass zero with a finite imaginary part $\Rightarrow$ we have a <u>Hopf bifurcation</u>.

Evaluation of the criterion gives

$$16a = f_{xx}g_{xx} = (-2) \cdot 4 = -8$$

i.e. $a < 0$ and the bifurcation is <u>supercritical</u>.

b) The system (3) has a homoclinic bifurcation at $\mu = \mu_c \approx 0.066$. Explain what happens close to the homoclinic bifurcation by sketching the dynamics for the three cases of μ slightly below μ_c, μ equal to μ_c, and μ slightly above μ_c (no bifurcation occurs in the interval $0 < \mu < \mu_c$).
Answer:

From the fact that we have a supercritical Hopf bifurcation at $\mu = 0$, we now there exists a stable limit cycle when $0 < \mu < \mu_c$. At $\mu = \mu_c$ this collides with the saddle point of the system, forming a homoclinic orbit. At $\mu > \mu_c$ the homoclinic orbit is destroyed and the system no longer has a limit cycle. It could be noted that one of the stable manifolds still connects to the spiral at the origin. See your own hand-ins for sketches of these cases.

c) In order to find out how the period of a closed orbit scales as a homoclinic bifurcation is approached, it is useful to first estimate the time it takes for an orbit to pass a saddle point. To estimate the time to pass a general saddle point, consider the linearized dynamics $\dot{x} = \lambda_u x$ and $\dot{y} = \lambda_s y$, where λ_u and λ_s are the eigenvalues of the unstable and stable directions respectively ($\lambda_u > 0$ and $\lambda_s < 0$). Let a trajectory start from the point $(x, y) = (\gamma, 1)$ where γ is small. Find an analytical expression for the time to escape from the saddle to $x(t) = 1$.
Answer:
The escape time is $T \sim -\ln(\gamma)/\lambda_u$

d) Find an analytical expression for λ_u suitable for the system in Eq. (3).
Answer:
We have a saddle point at $(x^*, y^*) = ((1 + \mu^2)/(2 + \mu), \dots)$. Jacobian and its invariants evaluated at the saddle point:

$$
\begin{aligned}
J(x^*) &= \begin{pmatrix} \mu - 2x^* & 1 \\ -1 + 4x^* & \mu \end{pmatrix} \\
\mathrm{tr}\mathbb{J}(\mathrm{x}^*) &= 2\frac{2\mu - 1}{2 + \mu} \\
\det\mathbb{J}(\mathrm{x}^*) &= -\frac{\mu + \mu^3 + 2 + 2\mu^2}{2 + \mu} = -1 - \mu^2 \,.
\end{aligned}
$$

The unstable eigenvalue is

$$
\begin{aligned}
\lambda_u &= \frac{1}{2}\left(\mathrm{tr}\mathbb{J} + \sqrt{(\mathrm{tr}\mathbb{J})^2 - 4\det\mathbb{J}}\right) = \frac{2\mu - 1}{2 + \mu} + \sqrt{\frac{(2\mu - 1)^2}{(2 + \mu)^2} + (1 + \mu^2)} \\
&= \frac{2\mu - 1 + \sqrt{5 + 9\mu^2 + 4\mu^3 + \mu^4}}{2 + \mu} \,.
\end{aligned}
$$

e) Estimate the time of the periodic orbit, T_μ, just below the homoclinic bifurcation. You can assume that this time is completely determined by the passage of the saddle point, i.e. it depends only on λ_μ and γ. You can assume $\gamma \sim A(\mu - \mu_c)^a$ with $A = 3.2$ and $a = 0.7$.
Answer:
Putting everything together

$$
T_u \sim -\ln(A(\mu - \mu_c)^a)/\lambda_u
$$

With A and a from problem formulation and λ_u from part d).

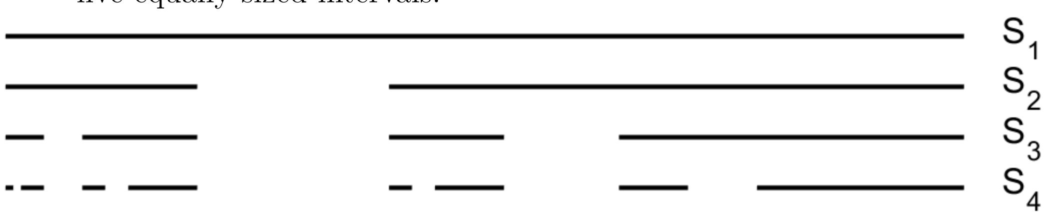
6. Box-counting dimension [2 points] The two figures below show the first few generations in the construction of a) the *even fifths Cantor set* and b) the *second fifths Cantor set*. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.

a) Analytically find the box-counting dimension D_0 of the even fifths Cantor set, obtained by at each generation removing intervals 2 and 4 out of five equally sized intervals:

Answer:
The box-counting dimension is

$$D_0 = \frac{\ln 3}{\ln 5}$$

b) Analytically find the box-counting dimension D_0 of the second fifths Cantor set, obtained by at each generation removing interval 2 out of five equally sized intervals:



Answer:
The box-counting dimension is given implicitly as the solution to the transcendental equation

$$1 + 3^{D_0} = 5^{D_0}$$