Solutions/answers to selected problems of the exam 12:th of April in Dynamical Systems 2017

CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	April 12, 2017, at $14^{00} - 18^{00}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once at 15^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$20 passed; $\geq$27 grade 4; $\geq$32 grade 5,
GU $\geq$20 grade G; $\geq$ 29 grade VG.

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1. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) What defines a conservative dynamical system?

b) What is the difference between a conservative dynamical system and a Hamiltonian dynamical system?

c) Explain the main differences between a supercritical and a subcritical bifurcation.

d) Explain what a Hopf bifurcation is.

e) State three properties of the index of a curve, I_C.

f) Explain what a fractal (strange) attractor is.

g) What conditions must be satisfied for a system to show a fractal (strange) attractor?

h) What is the significance of the parameter q in the generalized dimension spectrum D_q?

2. Bifurcation [2 points] Consider the dynamical system

$$\begin{aligned} \dot{x} &= ax + y + x^3 \\ \dot{y} &= x - y \,, \end{aligned} \tag{1}$$

with a real parameter a.

a) Find all fixed points of the system (1) and give conditions on a for which the fixed points exist.

b) Use linear stability analysis to classify the fixed points you found in subtask a) as functions of the parameter a.

c) Plot the bifurcation diagram for one of the components of the fixed points, for example x^*, against the parameter a. Label each branch plotted with the type of fixed point you found in the classification in subtask b). What kind of bifurcation(s) do you obtain?

3. Non-linear stability analysis and phase portrait [2 points] Consider the system

$$\begin{aligned} \dot{x} &= y - xy^2 \\ \dot{y} &= -x + yx^2 \,. \end{aligned} \tag{2}$$

a) Find all fixed points of the system (2).

b) What does linear stability analysis predict about the fixed point(s)?

c) Sketch the phase-plane dynamics in the region $-2 \le x \le 2$ and $-2 \le y \le 2$. In order to do this, it may be helpful to express the dynamics in polar coordinates.

d) Classify the fixed point at the origin for the non-linear system (2).

4. Infinite-period bifurcation [2 points] Consider a dynamical system in spherical coordinates

$$\begin{aligned} \dot{r} &= r - r^3 \\ \dot{\theta} &= \mu - \sin\theta \,, \end{aligned}$$

where $r > 0$ and μ is a real parameter.

a) For $\mu < 1$ and for $\mu > 1$, find all attractors of the corresponding Cartesian dynamical system (you do not need to change to Cartesian coordinates if you do not want to).

b) Describe the bifurcation that happens as μ passes unity.

c) For any closed orbit(s) of the system, estimate the dependence of the period time on μ (up to a prefactor) close to $\mu = 1$.

d) Give a motivation of why the time dependence you calculated in subtask c) may be useful.

5. The deformation matrix [2 points] Consider the dynamical system

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + \nu r^2 \end{aligned} \tag{3}$$

with a real parameter μ.
In the problem sets you were supposed to show that the corresponding dynamical system in Cartesian coordinates $x = r\cos\theta$ and $y = r\sin\theta$ is

$$\begin{aligned} \dot{x} &= \mu x - y\omega - x^3 - \nu y^3 - \nu x^2 y - x y^2 \\ \dot{y} &= \omega x + \mu y + \nu x^3 - y^3 - x^2 y + \nu x y^2 \,. \end{aligned} \tag{4}$$

The deformation matrix $\mathbb{M}$ is defined as the matrix projecting an initial infinitesimal separation vector $\boldsymbol{\delta}(0)$ to an infinitesimal separation $\boldsymbol{\delta}(t)$ at t:

$$\boldsymbol{\delta}(t) = \mathbb{M}(t)\boldsymbol{\delta}(0)\,.$$

The stability exponents of separations are defined as

$$\tilde{\sigma}_i \equiv \lim_{t\to\infty} \frac{1}{t} \ln m_i$$

where m_i is the i:th eigenvalue of $\mathbb{M}$.

a) For the case $\mu > 0$, find the radius and period time of the attracting limit cycle in the system Eq. (3).

b) Analytically calculate the stability exponents of separations when $\mu < 0$ (OBS: different limit compared to subtask a)) for the system (4) in Cartesian coordinates.

c) Analytically calculate the stability exponents of separations when $\mu < 0$ for the system (3) in polar coordinates.

d) In the problem sets you were supposed to use a relation for the transformation of the deformation matrix under a general non-singular coordinate transformation $\boldsymbol{x} = \boldsymbol{G}(\boldsymbol{y})$:

$$\mathbb{M}_{\boldsymbol{y}}(t) = \mathbb{J}_G^{-1}(\boldsymbol{y}(t))\mathbb{M}_{\boldsymbol{x}}(t)\mathbb{J}_G(\boldsymbol{y}(0))\,. \tag{5}$$

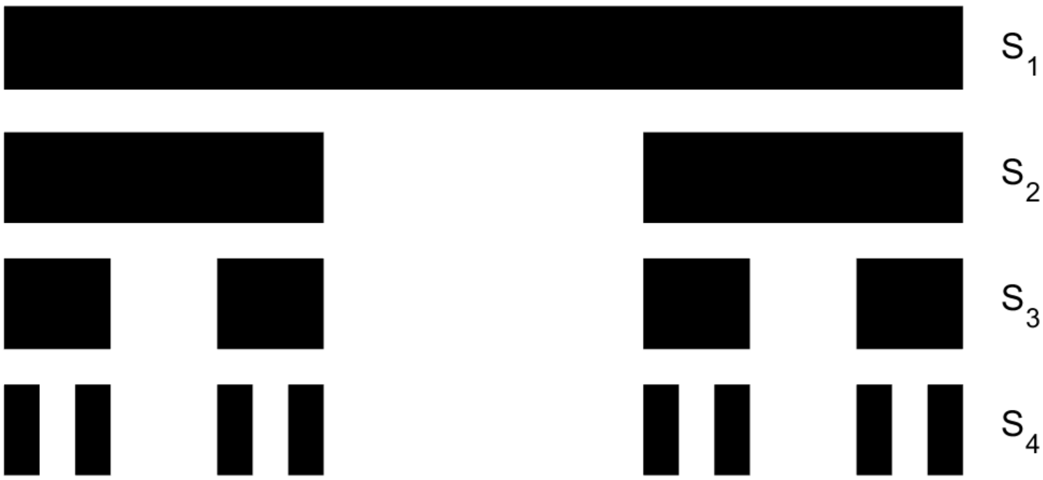
Here $\mathbb{M}_{\boldsymbol{x}}(t)$ is the deformation matrix in the original coordinates, $\mathbb{M}_{\boldsymbol{y}}(t)$ is the deformation matrix in the transformed coordinates, and $\mathbb{J}_G(\boldsymbol{y}(t))$ is the gradient matrix of the transformation $\boldsymbol{G}$ with components

$$[J_G(\boldsymbol{y}(t))]_{ij} = \frac{\partial x_i}{\partial y_j}\,.$$

Does the relation (5) apply to your results in subtasks b) and c)?

6. Box-counting dimension [2 points] The figures below show the first few generations S_1, S_2, S_3 and S_4 in the construction of modified versions of the *middle thirds Cantor set*. For each figure the fractal set is obtained by iterating to generation S_n with $n \to \infty$.

a) Start by a two-dimensional strip of finite width and height. Analytically find the box-counting dimension D_0 of the fractal set, obtained by at each generation removing the middle third horizontal interval out of three equally sized horizontal intervals:



b) Start by a two-dimensional strip of finite width and height. Analytically find the box-counting dimension D_0 of the fractal set, obtained by at each generation removing both the middle third horizontal and vertical intervals out of three equally sized horizontal and vertical intervals:

c) Discuss how the results in subtasks a) and b) are related to the box-counting dimension of the middle-third Cantor set.