# Solutions/answers to selected problems of the exam 12:th of April in Dynamical Systems 2017

**CHALMERS, GÖTEBORGS UNIVERSITET**

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

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| **Time:** | April 12, 2017, at $14^{00} - 18^{00}$ |
| **Place:** | Johanneberg |
| **Teachers:** | Kristian Gustafsson, 070-050 2211 (mobile), visits once at $15^{00}$ |
| **Allowed material:** | Mathematics Handbook for Science and Engineering |
| **Not allowed:** | any other written material, calculator |

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Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
**CTH** $\geq$20 passed; $\geq$27 grade 4; $\geq$32 grade 5,
**GU** $\geq$20 grade G; $\geq$ 29 grade VG.

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**1. Short questions [2 points]** For each of the following questions give a concise answer within a few lines per question.

a) What defines a conservative dynamical system?

b) What is the difference between a conservative dynamical system and a Hamiltonian dynamical system?

c) Explain the main differences between a supercritical and a subcritical bifurcation.

d) Explain what a Hopf bifurcation is.

e) State three properties of the index of a curve, $I_C$.

f) Explain what a fractal (strange) attractor is.

g) What conditions must be satisfied for a system to show a fractal (strange) attractor?

h) What is the significance of the parameter $q$ in the generalized dimension spectrum $D_q$?

**2. Bifurcation [2 points]** Consider the dynamical system

$$\begin{aligned} \dot{x} &= ax + y + x^3 \\ \dot{y} &= x - y\,, \end{aligned} \tag{1}$$

with a real parameter $a$.

a) Find all fixed points of the system (1) and give conditions on $a$ for which the fixed points exist.

**Solution**
Fixed points:
$(x_1^*, y_1^*) = (0, 0)$
$(x_\pm^*, y_\pm^*) = \pm\sqrt{-a-1} \cdot (1, 1)$ if $a < -1$.

b) Use linear stability analysis to classify the fixed points you found in subtask a) as functions of the parameter $a$.

**Solution**
The Jacobian of the system (1) is

$$\begin{aligned} \mathbb{J} &= \begin{pmatrix} a + 3x^2 & 1 \\ 1 & -1 \end{pmatrix} \\ \mathrm{tr}\mathbb{J} &= a + 3x^2 - 1 \\ \det \mathbb{J} &= -a - 3x^2 - 1 \end{aligned}$$

Evaluated at the fixed points we have

$$\begin{aligned} \mathrm{tr}\mathbb{J}_1 &= a - 1 \\ \det \mathbb{J}_1 &= -a - 1 \\ \mathrm{tr}\mathbb{J}_\pm &= -2a - 4 \\ \det \mathbb{J}_\pm &= 2a + 2 \end{aligned}$$

The eigenvalues of all the fixed points are real for all values of $a$.

The fixed point at $(0, 0)$ is a stable node if $a < -1$ and a saddle point if $a > -1$.

The other two fixed points are saddle points when $a < -1$.

c) Plot the bifurcation diagram for one of the components of the fixed points, for example $x^*$, against the parameter $a$. Label each branch plotted with the type of fixed point you found in the classification in subtask b). What kind of bifurcation(s) do you obtain?

**Solution**
Subcritical pitchfork bifurcation at $a = -1$.

**3. Non-linear stability analysis and phase portrait [2 points]** Consider the system

$$\begin{aligned}\dot{x} &= y - xy^2 \\ \dot{y} &= -x + yx^2 \,.\end{aligned} \tag{2}$$

a) Find all fixed points of the system (2).

**Solution**
One fixed point at the origin $(x^*, y^*) = (0, 0)$ and curve of fixed points at $(x, 1/x)$.

b) What does linear stability analysis predict about the fixed point(s)?

**Solution**
The Jacobian is

$$\mathbb{J} = \begin{pmatrix} -y^2 & 1 - 2xy \\ -1 + 2xy & x^2 \end{pmatrix} .$$

For the fixed point at the origin linear stability theory predicts a center.

For the line of fixed points we obtain

$$\mathbb{J} = \begin{pmatrix} -1/x^2 & -1 \\ 1 & x^2 \end{pmatrix} .$$

with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = (x^4 - 1)/x^2$. One eigenvalue being zero is consistent with a non-isolated fixed points. The second eigenvalue is either positive ($|x| > 1$) or negative ($|x| < 1$).

c) Sketch the phase-plane dynamics in the region $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$. In order to do this, it may be helpful to express the dynamics in polar coordinates.

**Solution**
Trajectories lie on circles centered at the origin. When the radius becomes large enough it intersects the curve of fixed points. The flow changes direction at intersections.

d) Classify the fixed point at the origin for the non-linear system (2).

**Solution**
Using symmetry properties of the flow it can be shown that we must have a non-linear center.

An alternative approach is to use polar coordinates

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = 0$$

i.e. all trajectories are circles.

**4. Infinite-period bifurcation [2 points]** Consider a dynamical system in spherical coordinates

$$\dot{r} = r - r^3$$
$$\dot{\theta} = \mu - \sin\theta\,,$$

where $r > 0$ and $\mu$ is a real parameter.

a) For $\mu < 1$ and for $\mu > 1$, find all attractors of the corresponding Cartesian dynamical system (you do not need to change to Cartesian coordinates if you do not want to).

**Solution**
Since the dynamics of $r$ and $\theta$ are uncoupled we may analyze the dynamics as two one-dimensional systems. Plotting the dynamics shows that the radial dynamics has a repelling fixed point at $r^* = 0$ and an attracting fixed point at $r^* = 1$.

For the case $\mu > 1$, the dynamics of $\theta$ has no fixed points. For this case the Cartesian dynamical system has a globally attracting limit cycle.

For the case $\mu < 1$, the dynamics of $\theta$ has two fixed points located at $\theta^* = \text{asin}(\mu)$ and $\theta^* = \pi - \text{asin}(\mu)$. Plotting the dynamics shows that the fixed point at $\theta^* = \text{asin}(\mu)$ is stable and the other fixed point is unstable. The dynamics is therefore globally attracted to the point $(r^*, \theta^*) = (1, \text{asin}(\mu))$.

b) Describe the bifurcation that happens as $\mu$ passes unity.

**Solution**
Infinite-period bifurcation, see lecture notes.

c) For any closed orbit(s) of the system, estimate the dependence of the period time on $\mu$ (up to a prefactor) close to $\mu = 1$.

**Solution**
The bifurcation in the $\theta$-dynamics is a saddle-node bifurcation. Before the fixed points are formed, the dynamics in $\theta$ will slow down close to $\theta = \pi/2$ where the bifurcation occurs. Close to a saddle-node bifurcation, the period time is dominated by the time to pass the 'ghost' of the fixed points. The time to pass this region scales as $T \sim 1/\sqrt{\mu - 1}$, see lecture notes.

d) Give a motivation of why the time dependence you calculated in subtask c) may be useful.

**Solution**
It could be useful for analysis of experimental data: Measuring how the experimental period time changes with some control parameter close to the bifurcation may give information of possible dynamics underlying the experiment (different types of bifurcations have different dependence on the control parameter).

**5. The deformation matrix [2 points]** Consider the dynamical system

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + \nu r^2 \end{aligned} \tag{3}$$

with a real parameter $\mu$.
In the problem sets you were supposed to show that the corresponding dynamical system in Cartesian coordinates $x = r\cos\theta$ and $y = r\sin\theta$ is

$$\begin{aligned} \dot{x} &= \mu x - y\omega - x^3 - \nu y^3 - \nu x^2 y - xy^2 \\ \dot{y} &= \omega x + \mu y + \nu x^3 - y^3 - x^2 y + \nu x y^2\,. \end{aligned} \tag{4}$$

The deformation matrix $\mathbb{M}$ is defined as the matrix projecting an initial infinitesimal separation vector $\boldsymbol{\delta}(0)$ to an infinitesimal separation $\boldsymbol{\delta}(t)$ at $t$:

$$\boldsymbol{\delta}(t) = \mathbb{M}(t)\boldsymbol{\delta}(0)\,.$$

The stability exponents of separations are defined as

$$\tilde{\sigma}_i \equiv \lim_{t\to\infty} \frac{1}{t} \ln m_i$$

where $m_i$ is the $i$:th eigenvalue of $\mathbb{M}$.

a) For the case $\mu > 0$, find the radius and period time of the attracting limit cycle in the system Eq. (3).

**Solution**
The radial dynamics has a stable fixed point at $r^* = \sqrt{\mu}$ (the radius of the limit cycle). At this radius, the angular dynamics reads $\dot{\theta} = \omega + \nu\mu$, i.e. the angular frequency is $\omega + \nu\mu$ and the period time is $T = 2\pi/(\omega + \nu\mu)$.

b) Analytically calculate the stability exponents of separations when $\mu < 0$ (OBS: different limit compared to subtask a)) for the system (4) in Cartesian coordinates.

**Solution**
In general, the separation vector $\boldsymbol{\delta}$ is governed by the linearized dynamics

$$\dot{\boldsymbol{\delta}}(t) = \mathbb{J}(t)\boldsymbol{\delta}(t)\,.$$

When $\mu < 0$ the system has a globally attracting fixed point at $(x^*, y^*) = (0, 0)$. We are interested in the dynamics at large $t$. In this limit we are close to the fixed point and the Jacobian approaches a constant matrix

$$\mathbb{J}(t \to \infty) = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

with eigenvalues $\sigma_{1,2} = \mu \pm \mathrm{i}\omega$.

In this limit we solve

$$\boldsymbol{\delta}(t) = \underbrace{e^{\mathbb{J}(t\to\infty)t}}_{\mathbb{M}(t)} \boldsymbol{\delta}(0)\,.$$

The eigenvalues of $\mathbb{M}$ thus becomes

$$m_\pm(t) = e^{\sigma_{1,2}t} = e^{(\mu\pm\mathrm{i}\omega)t}\,.$$

Using the definition the stability exponents of separations become

$$\tilde{\sigma}_\pm = \mu \pm \mathrm{i}\omega\,.$$

c) Analytically calculate the stability exponents of separations when $\mu < 0$ for the system (3) in polar coordinates.

**Solution**

When $\mu < 0$ the polar system is attracted to $r^* = 0$, but $\theta$ changes linearly with time.

The Jacobian of the polar system for large time $t$ (i.e. evaluated at $r = 0$) is

$$\mathbb{J}(t\to\infty) = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}$$

The eigenvalues of $\mathbb{M}$ thus becomes $m_1(t) = e^{\mu t}$ and $m_2(t) = 1$ with corresponding stability exponents of separations

$$\tilde{\sigma}_1 = 0$$
$$\tilde{\sigma}_2 = \mu$$

d) In the problem sets you were supposed to use a relation for the transformation of the deformation matrix under a general non-singular coordinate transformation $\boldsymbol{x} = \boldsymbol{G}(\boldsymbol{y})$:

$$\mathbb{M}_{\boldsymbol{y}}(t) = \mathbb{J}_G^{-1}(\boldsymbol{y}(t))\mathbb{M}_{\boldsymbol{x}}(t)\mathbb{J}_G(\boldsymbol{y}(0))\,. \tag{5}$$

Here $\mathbb{M}_{\boldsymbol{x}}(t)$ is the deformation matrix in the original coordinates, $\mathbb{M}_{\boldsymbol{y}}(t)$ is the deformation matrix in the transformed coordinates, and $\mathbb{J}_G(\boldsymbol{y}(t))$ is the gradient matrix of the transformation $\boldsymbol{G}$ with components

$$[J_G(\boldsymbol{y}(t))]_{ij} = \frac{\partial x_i}{\partial y_j}\,.$$

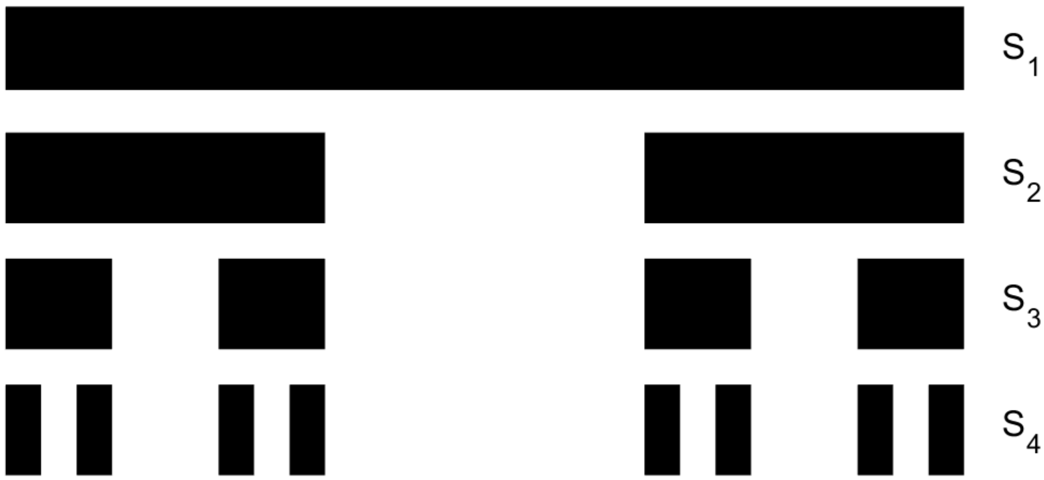
Does the relation (5) apply to your results in subtasks b) and c)?

**Solution**

We can view a fixed point as a closed orbit of zero period time. The relation (5) should therefore hold for all times $t$. However, as the results in subtasks b) and c) show, this is not true (the eigenvalues of the deformation matrices in b) and c) differ). The reason for this is that the coordinate transformation $\mathbb{J}_G$ has the eigenvalues 0 and $\cos\theta$ as $r \to 0$ and is therefore singular.

**6. Box-counting dimension [2 points]** The figures below show the first few generations $S_1$, $S_2$, $S_3$ and $S_4$ in the construction of modified versions of the *middle thirds Cantor set*. For each figure the fractal set is obtained by iterating to generation $S_n$ with $n \to \infty$.

a) Start by a two-dimensional strip of finite width and height. Analytically find the box-counting dimension $D_0$ of the fractal set, obtained by at each generation removing the middle third horizontal interval out of three equally sized horizontal intervals:




**Solution**
Covering the fractal with boxes of side length $\epsilon_k = 3^{-k}$ the total number of boxes is proportional to $N_k = 3^k 2^k$. The box-counting dimension becomes $D_0 = \ln(3^k 2^k)/\ln(3^k) = 1 + \ln(2)/\ln(3)$.

b) Start by a two-dimensional strip of finite width and height. Analytically find the box-counting dimension $D_0$ of the fractal set, obtained by at each generation removing both the middle third horizontal and vertical intervals out of three equally sized horizontal and vertical intervals:


**Solution**
Covering the fractal with boxes of side length $\epsilon_k = 3^{-k}$ the total number

of boxes is proportional to $N_k = 4^k$. The box-counting dimension becomes $D_0 = 2\ln(2)/\ln(3)$.

c) Discuss how the results in subtasks a) and b) are related to the box-counting dimension of the middle-third Cantor set.

**Solution**
The middle third Cantor set has box-counting dimension $D_0^{\text{cantor}} = \ln 2/\ln 3$ (see lecture notes). In subtask a) we add one dimension to the Cantor set and the box-counting dimension becomes $D_0 = 1 + D_0^{\text{cantor}}$. In subtask b) the total dimension of the fractal set is the sum of the Cantor sets: $D_0 = 2D_0^{\text{cantor}}$.