CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	August 16, 2017, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once at 09^{30}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$20 passed; $\geq$27 grade 4; $\geq$32 grade 5,
GU $\geq$20 grade G; $\geq$ 29 grade VG.

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1. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Give a definition for what a dynamical system is.

b) A nonautonomous system can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)\,,$$

i.e. the flow $\boldsymbol{f}$ depends explicitly on time. Is a nonautonomous system a dynamical system? Explain your answer.

c) What does a transcritical bifurcation mean?

d) What are the stable manifolds of a fixed point?

e) Give an example of how the knowledge of stable manifolds of a fixed point could be used to understand the dynamics in a dynamical system.

f) What is a quasiperiodic flow? Give an example!

g) In the problem sets the Lyapunov exponents were evaluated using a QR-decomposition method. Why is this method preferred over direct numerical evaluation of the eigenvalues of $\mathbb{M}^{\mathrm{T}}\mathbb{M}$ where $\mathbb{M}$ is the deformation matrix, or over evaluation of the Lyapunov exponent using separations between a number of particles?

h) Sketch the typical shape of the generalized dimension spectrum D_q against q for a mono fractal and for a multi fractal.

2. Quadfurcation [2 points]

a) Give/construct an example of a one-dimensional dynamical system showing a pitchfork bifurcation as a parameter r passes 0.

b) Sketch the bifurcation diagram for your system in subtask a).

c) Pitchfork bifurcations are examples of 'trifurcations', meaning a division into three branches of fixed points as r passes 0. Construct an example of a 'quadfurcation', in which no fixed points exist for $r < 0$ and four fixed points exist for $r > 0$.

Solution
One example is

$$\dot{x} = r - (x-1)^2(x+1)^2$$

No fixed point for $r < 0$. Four fixed points for $r > 0$. Two simultaneous saddle-node bifurcations at $x = -1$ and $x = 1$

d) Sketch the bifurcation diagram for your system in subtask c).

3. Phase portrait [2 points] Consider the system

$$\begin{aligned} \dot{x} &= x(ax - y) \\ \dot{y} &= y(2x - y)\,. \end{aligned} \tag{1}$$

a) Find all fixed points of the system (1).

Solution
For all values of a we have a fixed point at the origin $(x^*, y^*) = (0, 0)$. For $a = 2$ we have a line of fixed points along $y = 2x$.

b) What does linear stability analysis predict about the fixed point(s)?

Solution
The Jacobian is

$$\mathbb{J} = \begin{pmatrix} 2ax - y & -x \\ 2y & 2x - 2y \end{pmatrix}.$$

For all fixed points in subtask a), the eigenvalues evaluated at the fixed point vanish. Linear stability theory is therefore inconclusive.

c) For $a = 2$, sketch the nullclines and the phase-plane dynamics (phase portrait) in the region $-2 \le x \le 2$ and $-2 \le y \le 2$.

4. Trapping regions for the van der Pol oscillator [2 points] Consider the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{2}$$

with μ a real parameter.

a) Give physical interpretations or explanations of the different terms in Eq. (2).

Solution
See Lecture notes.

b) Consider the dynamics in the phase-plane (x, y) with $y = \dot{x}$. Knowing that this dynamical system shows an attractive limit cycle when $\mu > 0$, show that it has a repelling limit cycle when $\mu < 0$.

Solution
The dynamics in the phase-plane is

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\mu(x^2 - 1)y - x\,.\end{aligned}$$

These equations are invariant under the simultaneous change $\mu \to -\mu$, $y \to -y$, and $t \to -t$. Thus, the dynamics with flipped sign of μ corresponds to a time reversal and a flip of the y-coordinate. Since we know that the system has an attracting limit cycle when $\mu > 0$, and since the time reversal changes the stability of all attractors (trajectories running backwards), we conclude that the system with $\mu < 0$ must have a repelling limit cycle (the flip of the y-coordinate just mirrors the system, but does not affect existence or stability of the attractors).

c) Let $r = \sqrt{x^2 + y^2}$ and derive an equation for $\dot{r}$ in terms of x and y.

Solution

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{xy + y(-\mu(x^2 - 1)y - x)}{r} = \frac{-\mu(x^2 - 1)y^2}{\sqrt{x^2 + y^2}}$$

d) When $\mu < 0$, show that there exist 'trapping regions' in the form of circles of radii $r < r_\mathrm{c}$ such that all solutions starting from initial conditions inside these circles tend to the origin. Determine r_c.

Solution
Consider the dynamics of r:

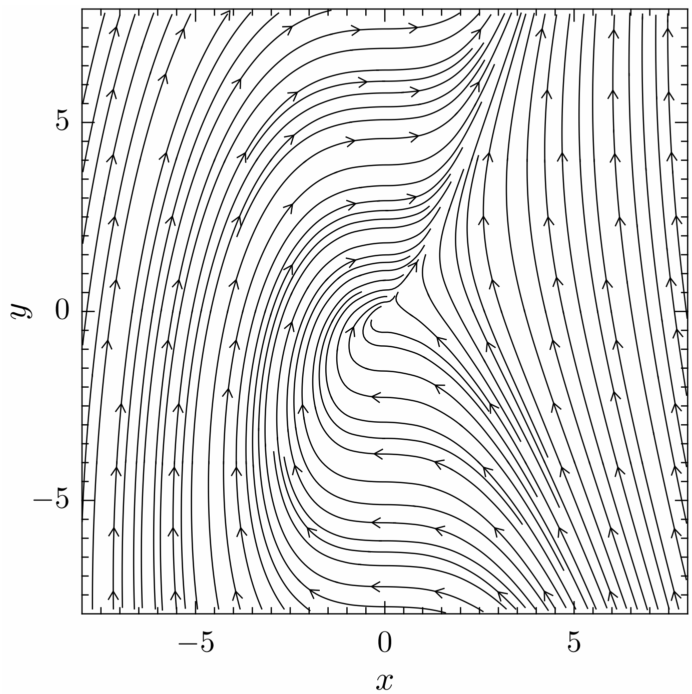
$$\dot{r} = \underbrace{\frac{-\mu y^2}{\sqrt{x^2 + y^2}}}_{>0}(x^2 - 1)$$

The first factor is positive since $\mu > 0$. The second factor is negative if $|x| < 1$. Thus, the radial flow through all circles of radius smaller than $r_\mathrm{c} = 1$ is negative. Therefore, initial conditions starting with $r < r_\mathrm{c}$ tend to the origin.

5. Indices and bifurcations [2 points] The phase portraits of two dynamical systems are plotted in subtasks a) and b) below.

a) What is the index of the fixed point of the following dynamical system?

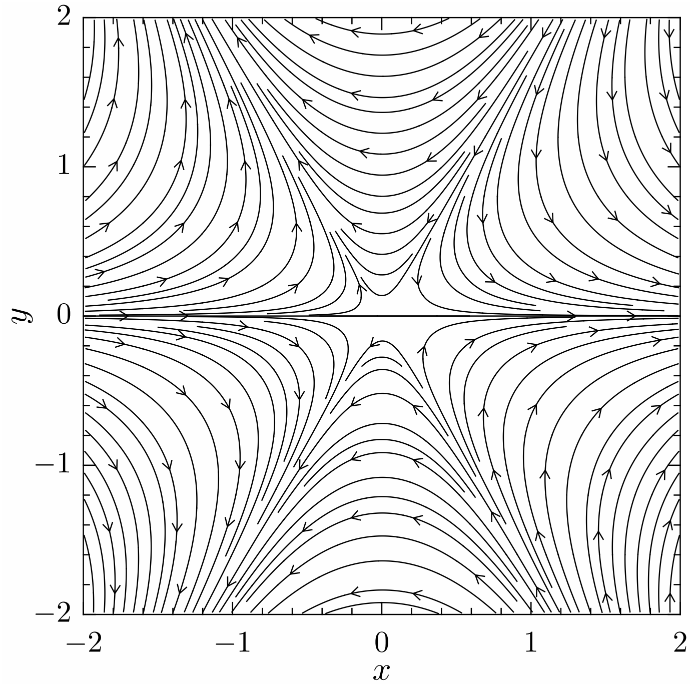
$$\dot{x} = y - x$$
$$\dot{y} = x^2$$



Solution
The index is $I = 0$.

b) What is the index of the fixed point of the following dynamical system?

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = -2xy$$



Solution
The index is $I = -2$.

c) Add a perturbation term μ to the x-component of the flow in subtask b). Describe the bifurcation (if any) that occurs when μ passes through zero in the perturbed system:

$$\dot{x} = x^2 - y^2 + \mu$$
$$\dot{y} = -2xy\,.$$

Solution
If $\mu > 0$ we have two fixed points at $(x^*, y^*) = (0, \pm\sqrt{\mu})$. The Jacobian is

$$J = \begin{pmatrix} 2x & -2y \\ -2y & -2x \end{pmatrix}.$$

Fixed points are saddle points, both with $\lambda_{1,2} = \pm 2\sqrt{\mu}$.

If $\mu < 0$ the fixed points are located at $(x^*, y^*) = (\pm\sqrt{-\mu}, 0)$. Also in this case both fixed points are saddle points.

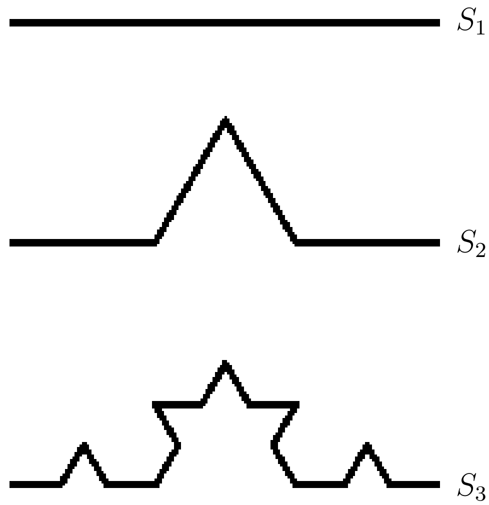
In conclusion, as μ passes 0 two saddle points collide and reemerge as two saddle points. It may look like nothing has happened, but by calculation of the eigendirections, one finds that the stable|unstable manifolds discontinuously change direction by $\pi/4$ in the bifurcation as μ passes zero.

d) Is the bifurcation in subtask c) consistent with the indices of involved fixed points and with the result you obtained in subtask b)?

Solution
Yes, since saddle points have index $I = -1$. At $\mu = 0$ these join into a fixed point of index $I = -2$.

6. Box-counting dimension [2 points] The two figures below show the first few generations in the construction of two fractals. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.

a) Analytically find the box-counting dimension D_0 (explicitly if possible, otherwise implicitly) of the Koch curve, obtained by at each generation replacing the middle third interval of all lines of length L with two new lines. The two replacing lines both have length $L/3$ and form a wedge:

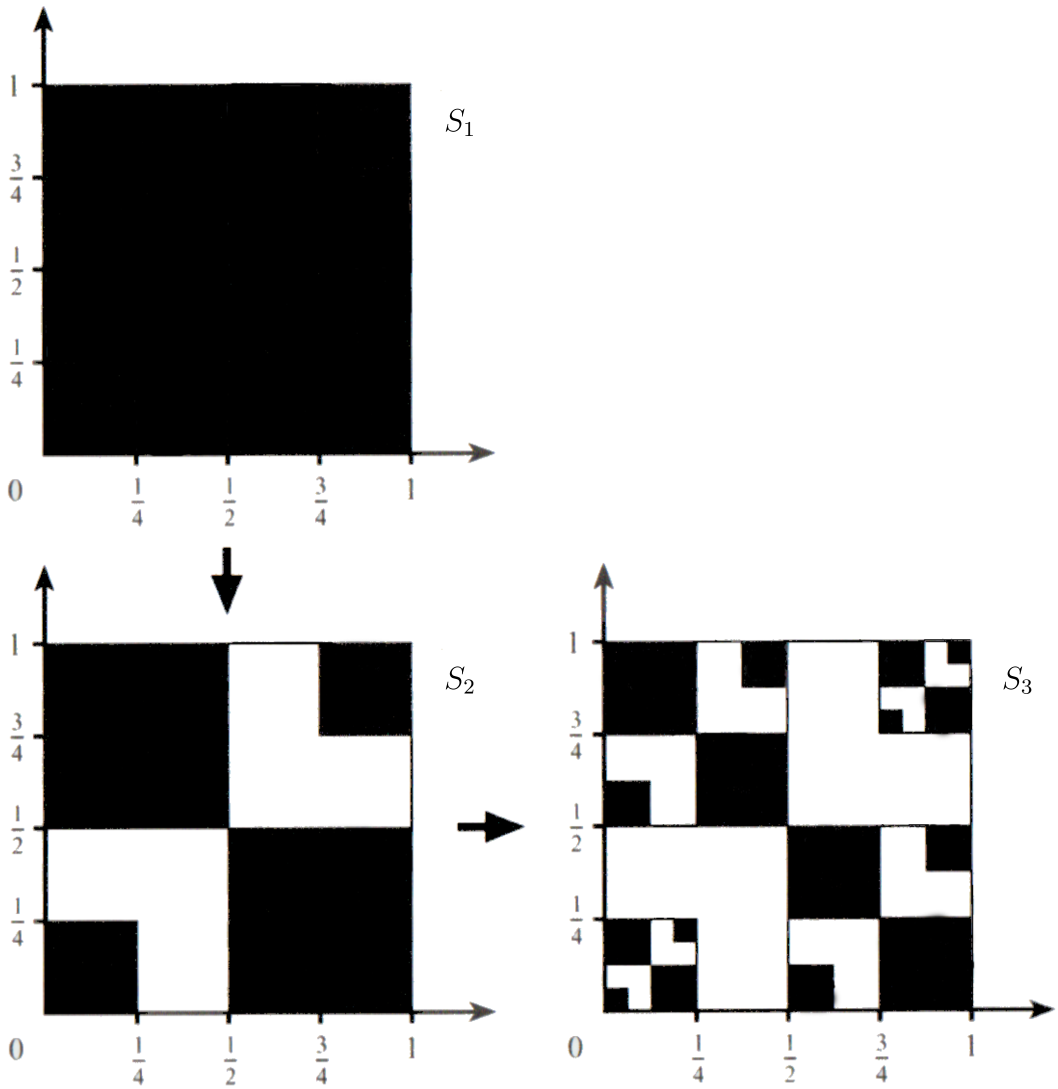


Solution

The box-counting dimension is

$$D_0 = \frac{\ln 4}{\ln 3}$$

b) Analytically find the box-counting dimension D_0 (explicitly if possible, otherwise implicitly) of the fractal constructed by infinite iteration of the sequence illustrated below:



Solution
This fractal resembles the asymmetric third fourth's Cantor set. Choose length scales $\lambda_a = 1/2$ and $\lambda_b = 1/4$. Have $N(\epsilon) = 2N_a(\epsilon) + 2N_b(\epsilon)$. Furthermore $N_a = N(\epsilon/\lambda_a)$ and $N_b = N(\epsilon/\lambda_b)$ gives

$$
\begin{aligned}
N(\epsilon) &= 2N(\epsilon/\lambda_a) + 2N(\epsilon/\lambda_b) \\
A\epsilon^{-D_0} &= 2A\epsilon^{-D_0}\lambda_a^{D_0} + 2A\epsilon^{-D_0}\lambda_b^{D_0} \\
1 &= 2 \cdot 2^{-D_0} + 2 \cdot 4^{-D_0}
\end{aligned}
$$

The box-counting dimension is given implicitly as the solution to the transcendental equation

$$
\begin{aligned}
1 &= 3^{D_0}/5^{D_0} + 1/5^{D_0} \\
1 + 3^{D_0} &= 5^{D_0}
\end{aligned}
$$