CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	January 08, 2018, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once at 09^{30}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$18 passed; $\geq$26 grade 4; $\geq$31 grade 5,
GU $\geq$18 grade G; $\geq$ 28 grade VG.

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1. Multiple choice questions [2 points] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Classify the fixed point of the two-dimensional dynamical system:

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x}\,, \qquad \text{where } \mathbb{A} = \begin{pmatrix} 3 & 4 \\ -4 & 2 \end{pmatrix}.$$

A. It is a saddle point.

B. It is a stable spiral.

C. It is an unstable spiral.

D. It is a stable node.

E. It is an unstable node.

b) Which of the following statements are true in general for smooth one-dimensional dynamical systems (flows on the line)?

A. They can be solved using separation of variables.

B. They can have periodic solutions with finite period time.

C. They can be chaotic.

D. Around any non-infinite initial position a unique solution exists within a non-empty time interval.

E. The only possible non-infinite attractors are fixed points.

c) Which of the following kinds of fixed points do you typically encounter in conservative dynamical systems

A. Nodes

B. Saddles

C. Spirals

D. Centers

E. Conservative systems do not have fixed points.

d) A three-dimensional dynamical system has the following Lyapunov exponents: $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = -1$. Which of the following statements are true?

A. The system may be chaotic.

B. The system may be volume preserving.

C. The system may be Hamiltonian.

D. The system may have a globally attracting limit cycle.

E. The Lyapunov spectrum is unchanged if time changes sign.

e) The existence and uniqueness theorem implies that trajectories cannot intersect if the flow is smooth enough. But in many phase portraits of smooth systems different trajectories appear to intersect at fixed points, for example close to saddle points. Is this a contradiction? Answer with one of the following alternatives.

A. No, this is an artefact of projecting higher-dimensional trajectories onto two dimensions.

B. No, since no trajectory can pass the fixed point, trajectories do not intersect.

C. No, since flows are non-smooth at fixed points.

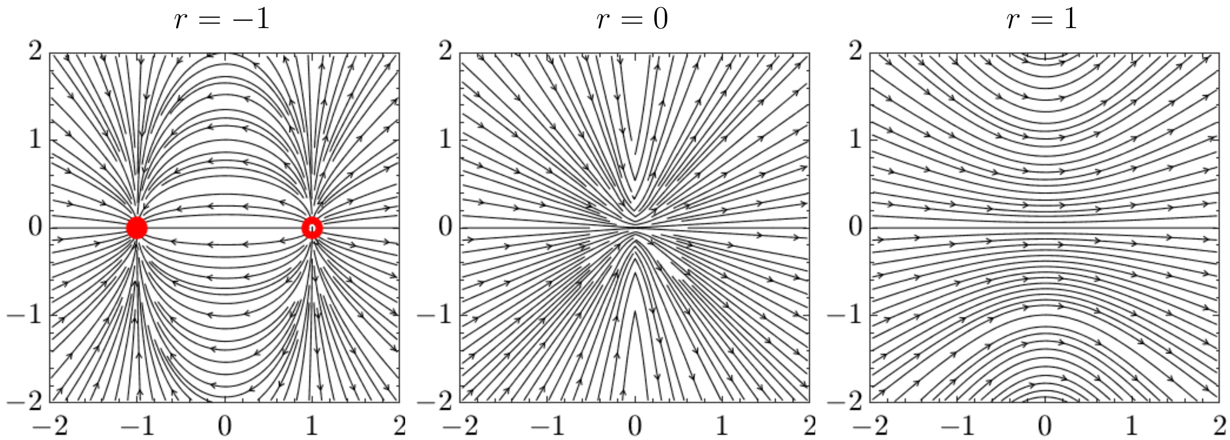
D. No, near the fixed point the density of trajectories become too high for the resolution of the phase portrait. Therefore trajectories appear to intersect even though they do not.

E. No, it just appears that way due to numerical errors when plotting the phase portrait.

The following sequence of images shows the phase portraits for the system

$$\begin{aligned} \dot{x} &= r + x^2 \\ \dot{y} &= xy \end{aligned} \tag{1}$$

for three values of r:



f) What is the sum of the indices of the two fixed points in the leftmost panel with $r = -1$?
A. -2 B. -1 C. 0 D. 1 E. 2

2. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Consider once again the system in Eq. (1) above. Is the index preserved in the bifurcation as r passes 0? Explain.

Solution
No, the index is not preserved. For $r < 0$ the index is 2, after the bifurcation the index is 0. At the bifurcation point $r = 0$ the line $x = 0$ consists of non-isolated fixed points. Therefore index theorem does not apply when $r = 0$ and the index does not need to be conserved.

b) What is meant by a ghost or slow passage in the context of fixed points?

Solution
Close to for example a saddle-node bifurcation of a one-dimensional dynamical system the magnitude of the flow is small where the fixed points used to be. This creates a bottleneck in the dynamics with slow passing time (typically of the order $1/\sqrt{\mu}$ where μ is the bifurcation parameter of a bifurcation on normal form).

c) Explain what a limit cycle is. Give an example of a system with a limit cycle.

Solution
A limit cycle is a closed orbit that is isolated (at least from one side). It can be stable, unstable or half-stable. An example of a system with a limit cycle is the van der Pol oscillator.

d) What is meant by structural stability?

Solution

Structural stability means that the topology of the flow does not change as the vector field is weakly perturbed. For example, linear centers are nt structurally stable since the flow surrounding them may become either attracting or repelling under a small non-linear perturbation.

e) In the problem sets you calculated the Lyapunov exponents for a continuous dynamical system (the Lorenz equations) and for a discrete dynamical system (the Hénon map). Contrast any similarities or differences in the two approaches.

Solution

For the continuous system a discretisation of the dynamics was used to calculate the deformation matrix $\mathbb{M}$ as a product of discrete matrices for short time intervals. Therefore the calculation of the Lyapunov exponents was basically identical to that of a discrete system, with the difference that the discrete matrix in the continuous case was given by $\mathbb{I} + \mathbb{J}\delta t$, while it for the discrete system was given simply by $\mathbb{J}$.

f) The generalized fractal dimension D_q is defined by

$$D_q \equiv \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln I(q,\epsilon)}{\ln(1/\epsilon)} \tag{2}$$

with

$$I(q,\epsilon) = \sum_{k=1}^{N_{\text{box}}} p_k^q(\epsilon)\,.$$

Here p_k is the probability to be in the k:th occupied box and N_{box} is the total number of occupied boxes. Briefly explain an appropriate method to calculate D_q from numerical or experimental data.

Solution

A suitable method is to plot $\ln I(q,\epsilon)$ against $\ln \epsilon$ to see in which range we have a power-law scaling $I(q,\epsilon) \sim A_q \epsilon^{(q-1)D_q}$. In this range the slope of the curve gives D_q independent from A_q:

$$D_q = \frac{1}{q-1} \frac{\Delta \ln I(q,\epsilon)}{\Delta \ln \epsilon}\,.$$

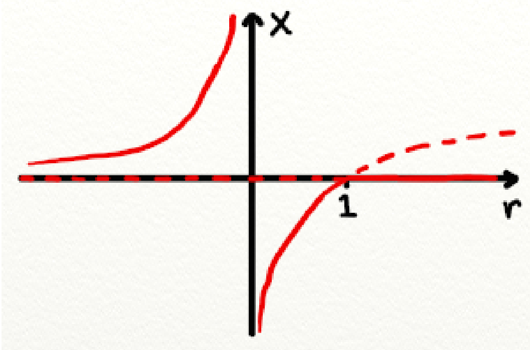
3. Bifurcations [2 points]

a) Sketch the bifurcation diagram for the dynamical system

$$\dot{x} = x + rx(x-1)\,.$$

Determine all values r_c where bifurcations occur and determine the kind(s) of bifurcation(s).

Solution
Fixed points where $x + rx(x-1) = 0$, i.e. at $x^* = 0$ and $x^* = 1 - 1/r$. Linear stability analysis $f'(x) = 1 - r + 2rx$ shows that $x^* = 0$ is stable if $r > 1$ and unstable otherwise and that $x^* = 1 - 1/r$ is stable if $r < 1$ and unstable otherwise. We have a transcritical bifurcation at $r_c = 1$.

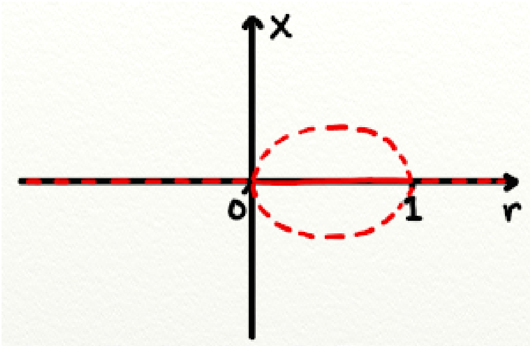


b) Sketch the bifurcation diagram for the dynamical system

$$\dot{x} = x^3 + r^2 x - rx$$

Determine all values r_c where bifurcations occur and determine the kind(s) of bifurcation(s).

Solution
Fixed points where $x^3 + r(r-1)x = 0$, i.e. at $x^* = 0$ and $x^* = \pm\sqrt{r(1-r)}$ (these exists if $0 \leq r \leq 1$). We have $f'(x) = 3x^2 + r^2 - r$, i.e. $x^* = 0$ is stable if $0 < r < 1$ and unstable otherwise, while $x^* = \pm\sqrt{r(1-r)}$ are unstable when they exist. The system has two subcritical pitchfork bifurcations at $r_c = 0$ and $r_c = 1$.



4. Construction of degenerate fixed points [2 points] Both stars and degenerate nodes are fixed points of linear systems whose Jacobian evaluated at the fixed point has two equal eigenvalues. For a star all vectors are eigenvectors, while a degenerate node only has a single eigenvector.

a) Construct an example of a linear dynamical system for x and y that has a stable star at the point $(x^*, y^*) = (1, 2)$.

Solution
The condition that any vector $\boldsymbol{\xi}$ is an eigenvector, $\mathbb{J}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$, implies

that $\mathbb{J}$ is $\lambda\mathbb{I}$, where $\mathbb{I}$ is the identity matrix and λ is the eigenvalue. As an example, the system

$$\begin{aligned}\dot{x} &= 1 - x\\ \dot{y} &= 2 - y\end{aligned}$$

has a fixed point at $(x^*, y^*) = (1, 2)$ that is a stable star (eigenvalue $\lambda = -1$).

b) Construct an example of a linear dynamical system for x and y that has a stable degenerate node at the origin.

Solution
Let $\dot{\boldsymbol{x}} = \mathbb{J}\boldsymbol{x}$ The eigenvalues are equal with negative value $\lambda = \operatorname{tr}\mathbb{J}/2$ if $\operatorname{tr}\mathbb{J} = -2\sqrt{\det\mathbb{J}}$:

$$J_{11} + J_{22} = -2\sqrt{J_{11}J_{22} - J_{12}J_{21}}\,.$$

As an example, choose $J_{21} = 0$, $J_{12} = 1$, $J_{11} = J_{22} = -1$:

$$\mathbb{J} = \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$$

with eigenvalue $\lambda = -1$. Since this choice of $\mathbb{J}$ is not a multiple of the unit matrix, we can only have one eigenvector corresponding to the single eigenvalue $\lambda = -1$, as required. Explicitly, the system becomes

$$\begin{aligned}\dot{x} &= -x + y\\ \dot{y} &= -y\end{aligned}$$

c) Determine the stable and unstable manifolds for your example system in subtask b).

Solution
The stable manifold is given by the entire phase plane. In contrast, the stable direction of the fixed point, $\boldsymbol{v}$, such that $\mathbb{J}\boldsymbol{v} = -\boldsymbol{v}$:

$$\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}\boldsymbol{v} = \begin{pmatrix}-v_x + v_y\\ -v_y\end{pmatrix} = -\begin{pmatrix}v_x\\ v_y\end{pmatrix}$$

We can take as eigenvector $\boldsymbol{v} = (1, 0)$ and the stable direction is the x-axis. The unstable manifold is simply the fixed point $(x^*, y^*) = (0, 0)$ (also 'does not exist' is an acceptable answer).

d) Construct a nonlinear system with a single fixed point at $(x^*, y^*) = (0, 0)$. The system should be such that linear stability analysis predicts a line of fixed points, but the contribution from non-linear terms results in a single fixed point that is a stable node.

Solution
One example of such a system is

$$\dot{x} = -x$$
$$\dot{y} = -y^3$$

We have a single fixed point attracting from all directions without swirl (stable node) and linear stability analysis predicts a line of fixed points because one eigenvalue of the Jacobian $\mathbb{J}(0,0)$ is zero:

$$\mathbb{J} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} .$$

5. Biased van der Pol oscillator [2 points] Consider the van der Pol oscillator biased with a constant force F:

$$\ddot{x} = -\mu(x^2 - 1)\dot{x} - x + F \,. \tag{3}$$

Here μ and F are real parameters.

a) Introduce $y = \dot{x}$ and write Eq. (3) as a dynamical system and determine all of its fixed points.

Solution
Let $y = \dot{x}$ and write

$$\dot{x} = y$$
$$\dot{y} = -\mu(x^2 - 1)y - x + F$$

A single fixed point at $(x^*, y^*) = (F, 0)$.

b) Analytically find the values of μ and F for which bifurcations occur in the system. The kind of bifurcations you should look for are bifurcations where the real part of an eigenvalue of the Jacobian at an isolated fixed point passes zero. Identify the types of found bifurcations.

Solution
The Jacobian of the fixed point $(x^*, y^*) = (F, 0)$ is

$$\mathbb{J}(x^*, y^*) = \begin{pmatrix} 0 & 1 \\ -2\mu x^* y^* - 1 & -\mu((x^*)^2 - 1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\mu(F^2 - 1) \end{pmatrix}$$
$$\operatorname{tr} \mathbb{J}(x^*, y^*) = -\mu(F^2 - 1)$$
$$\det \mathbb{J}(x^*, y^*) = 1$$

with eigenvalues

$$\lambda_{1,2} = \frac{-\mu(F^2 - 1) \pm \sqrt{\mu^2(F^2 - 1)^2 - 4}}{2} \,.$$

A bifurcation occurs if the real part of the eigenvalue passes zero. Assuming that the square root is positive, the condition to have a zero eigenvalue is:

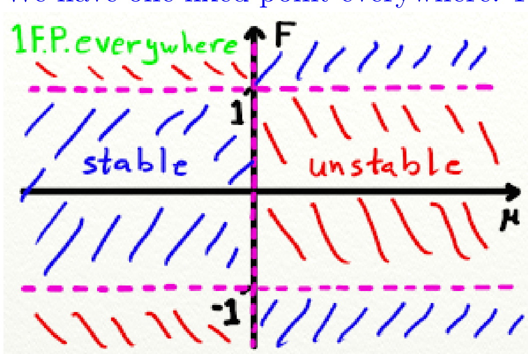
$$\begin{aligned} 0 &= \frac{-\operatorname{tr}\mathbb{J} \pm \sqrt{(\operatorname{tr}\mathbb{J})^2 - 4\det\mathbb{J}}}{2} \\ \operatorname{tr}\mathbb{J} &= \pm\sqrt{(\operatorname{tr}\mathbb{J})^2 - 4\det\mathbb{J}} \\ (\operatorname{tr}\mathbb{J})^2 &= (\operatorname{tr}\mathbb{J})^2 - 4\det\mathbb{J} \\ 0 &= -4\det\mathbb{J}\,. \end{aligned}$$

That is, this condition can only be satisfied if $\det\mathbb{J} = 0$, which is not the case here ($\det\mathbb{J} = 1$). The only possibility is to have a Hopf bifurcation, i.e. if the real part of the eigenvalue crosses zero with a non-zero imaginary part. The imaginary part of the eigenvalue can only come from the square root. The real part crosses zero when either $\mu = 0$ or when $F = \pm 1$. For both these cases, the eigenvalues are purely imaginary, i.e. we have Hopf bifurcations. In conclusion, we have Hopf bifurcations along the curves $\mu = 0$, $F = -1$, or $F = 1$.

c) Plot the curves in (μ, F) space where bifurcations occur and label them with their types. Label the regions between the bifurcation curves with the number of stable fixed points and the number of unstable fixed points.

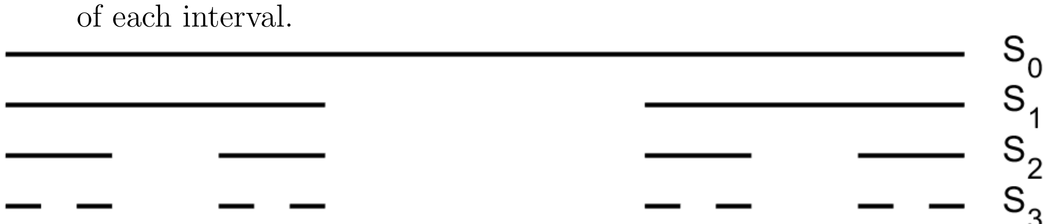
Solution

We have one fixed point everywhere. It is stable when $-\mu(F^2 - 1) < 0$



6. Middle Cantor sets [2 points] The two figures below show the first few generations in the construction of two fractals. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.

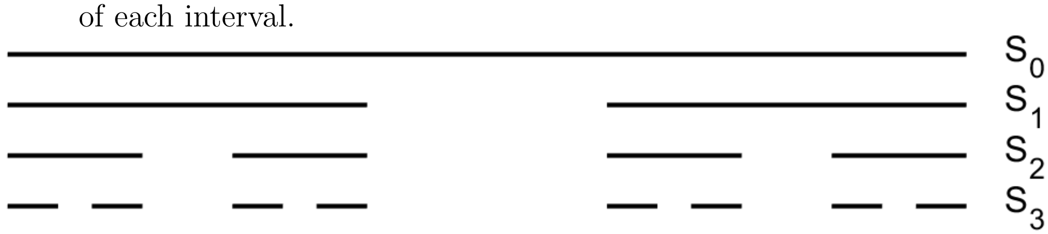
a) Analytically find the box-counting dimension D_0 for the 'Middle third's Cantor set', obtained by at each generation removing the central 1/3 of each interval.



Solution
Using boxes of side length $\epsilon = 3^{-n}$ it takes $N_n = 2^n$ boxes to cover the fractal. Thus

$$D_0 = -\lim_{n\to\infty} \frac{\ln(2^n)}{\ln(3^{-n})} = \frac{\ln 2}{\ln 3}.$$

b) Analytically find the box-counting dimension D_0 for the 'Middle fourth's Cantor set', obtained by at each generation removing the central 1/4 of each interval.



Solution
At generation n we have $N_n = 2^n$ connected intervals of equal length: $l_n = l_{n-1}(1-1/4)/2 = l_{n-1}3/8$. This recursion equation is solved by (using Beta or by inspection) $l_n = l_0(3/8)^n$. Using $\epsilon_n = l_n$ the box-counting dimension becomes

$$D_0 = -\lim_{n\to\infty} \frac{\ln(2^n)}{\ln(l_0(3/8)^n)} = \frac{\ln 2}{\ln(8/3)}.$$

c) Analytically find the box-counting dimension D_0 for the generalized Cantor set obtained by at each generation removing the central fraction q of each interval. To check your result, make sure that D_0 equals your results in a) for $q = 1/3$ and in b) for $q = 1/4$. You can also check that $D_0 \to 1$ as $q \to 0$ and $D_0 \to 0$ as $q \to 1$.

Solution
Similar to subtask b), at generation n we have $N_n = 2^n$ connected intervals of equal length: $l_n = l_{n-1}(1-q)/2$. This gives $l_n = l_0((1-q)/2)^2$ and the fractal dimension becomes

$$D_0 = -\lim_{n\to\infty} \frac{\ln(2^n)}{\ln(l_0((1-q)/2)^n)} = \frac{\ln 2}{\ln(2/(1-q))}.$$