CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	August 22, 2018, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Jan Meibohm, 072-579 7068 (mobile), visits once around 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$18 passed; $\geq$26 grade 4; $\geq$31 grade 5,
GU $\geq$18 grade G; $\geq$ 28 grade VG.

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1. Multiple choice questions [2 points] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Classify the fixed point of the two-dimensional dynamical system:

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x}\,, \qquad \text{where } \mathbb{A} = \begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}.$$

A. It is a saddle point.

B. It is a stable spiral.

C. It is an unstable spiral.

D. It is a stable node.

E. It is an unstable node.

b) Which of the following kinds of fixed points do you typically encounter in Hamiltonian dynamical systems

A. Nodes

B. Saddles

C. Spirals

D. Centers

E. Conservative systems do not have fixed points.

c) May the following dynamical system exhibit chaos?

$$\begin{aligned}\dot{x} &= x^2 + 2xy \\ \dot{y} &= x - y + xy\,. \\ \dot{z} &= z^2\end{aligned}$$

A. Yes, because of the dimensionality of the system.

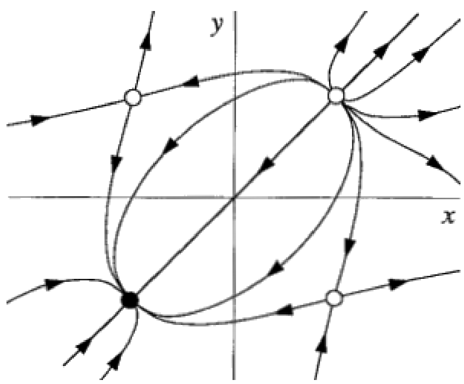
B. Yes, because the system is non-linear.

C. Yes, because the maximal Lyapunov exponent is positive.

D. Yes, because the system is mixing.

E. No.

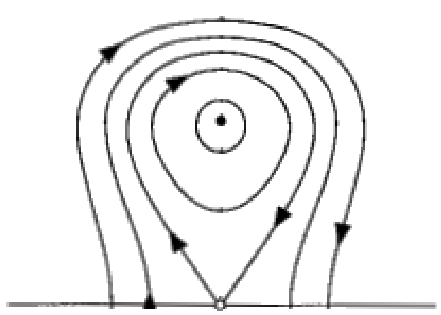
d) The figure below shows a phase portrait of a flow with four fixed points:



What is the index of a curve surrounding this phase portrait?

A. -2 B. -1 C. 0 D. 1 E. 2

e) The image below shows a section of a phase portrait:



Which of the following statements are true for the trajectory that originates and ends at the unstable fixed point?

A. It is a heteroclinic orbit.

B. It is a homoclinic orbit.

C. It is a separatrix.

D. Its index is +1.

E. Its index is not defined.

f) The figure below shows the first few generations in the construction of a fractal. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.

S_1 $\qquad$ S_2 $\qquad$ S_3

Which of the following alternatives describe the box-counting dimension of the fractal above?

A. $\dfrac{\log(2)}{\log(3)}$ B. $\dfrac{\log(3)}{\log(2)}$ C. $\dfrac{\log(3)}{\log(4)}$ D. $\dfrac{\log(4)}{\log(3)}$ E. $\dfrac{3}{2}$

2. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Assume that you are given a dynamical system which you simulate on your computer. How can you determine whether the obtained solutions are structurally stable?

b) Explain the Poincaré-Bendixon theorem.

c) Explain what kinds of dynamics one can obtain from a system of two uncoupled oscillators on the torus:

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 = \text{const.} \\ \dot{\theta}_2 &= \omega_2 = \text{const.} \end{aligned} .$$

d) Explain what a Hopf bifurcation is.

e) Explain the difference of chaotic motion in a volume-conserving system and in a dissipative system.

f) What is the significance of the parameter q in the generalized dimension spectrum D_q?

3. Normal forms of bifurcations [2.5 points] The normal forms of typical bifurcations for dynamical systems of dimensionality one are the following:

Type	saddle-node	transcritical	supercrit. pitchfork	subcrit. pitchfork
Normal form	$\dot{x} = r + x^2$	$\dot{x} = rx - x^2$	$\dot{x} = rx - x^3$	$\dot{x} = rx + x^3$

a) Discuss why normal forms of bifurcations are useful in the context of dynamical systems.

b) Consider the system

$$\dot{x} = \frac{x}{x+1} - ax\,,$$

where a is a real parameter. The system undergoes a bifurcation at $x = 0$ as the parameter a changes. Identify the bifurcation point and type of bifurcation by writing the system on normal form close to the bifurcation.

c) Use the normal forms above to construct a dynamical system of dimensionality one, $\dot{x} = f(x)$, with a single bifurcation parameter a such that the system undergoes a supercritical pitchfork bifurcation at $a = 0$ and $x = 0$, and a subcritical pitchfork bifurcation at $a = 1$ and $x = 1$. *Hint: To simplify, you can start from the ansatz $f(x) = c_0(a) + c_1(a)x + c_2x^2 + c_3x^3 + c_4x^4$, where c_i are coefficients to be determined and where only $c_0(a)$ and $c_1(a)$ depend on a.*

d) Sketch the bifurcation diagram of your system in subtask c).

4. Laser model [2 points] A simple model for a laser is provided by

$$\begin{aligned} \dot{n} &= GnN - kn \\ \dot{N} &= -GnN - fN + p\,, \end{aligned} \tag{1}$$

where $N(t)$ is the number of excited atoms and $n(t)$ is the number of photons in the laser field. The parameters G, k, and f are positive and p can take either sign.

a) Introduce suitable dimensionless units and write the system on dimensionless form in terms of two dimensionless parameters.

b) Find all the fixed points of the system (1) and determine their stability.

c) Make a plot over the two-dimensional parameter space spanned by the two dimensionless parameters from subtask a). Plot any curves where regular bifurcations occur and label them with their type. Also label the regions separated by bifurcation curves with the number of stable fixed points and the number of unstable fixed points.

5. Van der Pol relaxation oscillator [1.5 points] The van der Pol oscillator is governed by the following dynamics:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0\,, \tag{2}$$

where $x(t)$ is a dynamical variable and μ is a positive parameter, $\mu > 0$.

a) Introduce $y = \dot{x}/\mu + x^3/3 - x$ and derive a dynamical system for x and y.

b) Classify the fixed points (determine stability and type) of the dynamical system in subtask a) for general positive values of μ.

c) Explain the dynamics of the van der Pol oscillator in the limit of large values of μ.

6. Lyapunov exponents [2 points] Consider the dynamical system

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= (c - a)x - xz + cy \\ \dot{z} &= xy - bz \end{aligned} \tag{3}$$

where x, y and z are dynamical variables and a, b and c are parameters.

a) For $a = 40$, $b = 3$, and $c = 28$ the system (3) has no stable fixed points. In this limit, calculate the sum of the Lyapunov exponents of the system.

b) Given that the maximal Lyapunov exponent in the system (3) is positive, discuss what long-term behavior you expect from the system for the parameter values quoted in subtask a).

c) For $a = 3$ and $b = c = 1$ the system (3) has a single fixed point at the origin $x = y = z = 0$ which is stable and attracts the full phase-volume. Determine the Lyapunov exponents of the system.
Hint: For this system the Lyapunov exponents are equal to the real part of the stability exponents of separations.

d) Discuss how the dynamics in subtask *c*) can turn into the dynamics in subtask b) as the parameters change.