CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	August 22, 2018, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Jan Meibohm, 072-579 7068 (mobile), visits once around 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$18 passed; $\geq$26 grade 4; $\geq$31 grade 5,
GU $\geq$18 grade G; $\geq$ 28 grade VG.

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1. Multiple choice questions [2 points] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Classify the fixed point of the two-dimensional dynamical system:

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x}\,, \qquad \text{where } \mathbb{A} = \begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}.$$

A. It is a saddle point.

B. It is a stable spiral.

C. It is an unstable spiral.

D. It is a stable node.

E. It is an unstable node.

b) Which of the following kinds of fixed points do you typically encounter in Hamiltonian dynamical systems

A. Nodes

B. Saddles

C. Spirals

D. Centers

E. Conservative systems do not have fixed points.

c) May the following dynamical system exhibit chaos?

$$\dot{x} = x^2 + 2xy$$
$$\dot{y} = x - y + xy\,.$$
$$\dot{z} = z^2$$

A. Yes, because of the dimensionality of the system.

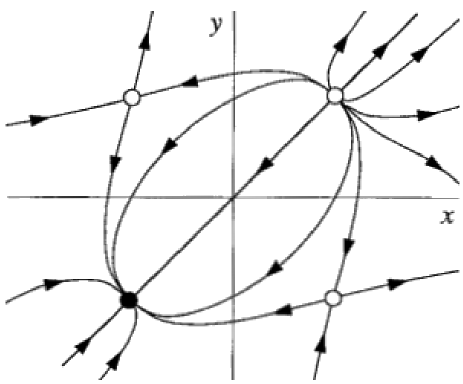
B. Yes, because the system is non-linear.

C. Yes, because the maximal Lyapunov exponent is positive.

D. Yes, because the system is mixing.

E. No.

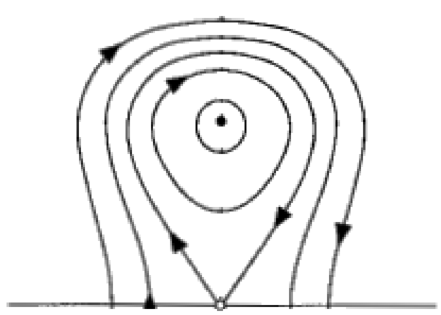
d) The figure below shows a phase portrait of a flow with four fixed points:



What is the index of a curve surrounding this phase portrait?

A. -2 B. -1 C. 0 D. 1 E. 2

e) The image below shows a section of a phase portrait:



Which of the following statements are true for the trajectory that originates and ends at the unstable fixed point?

A. It is a heteroclinic orbit.

B. It is a homoclinic orbit.

C. It is a separatrix.

D. Its index is +1.

E. Its index is not defined.

f) The figure below shows the first few generations in the construction of a fractal. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.

S_1 S_2 S_3

Which of the following alternatives describe the box-counting dimension of the fractal above?

A. $\dfrac{\log(2)}{\log(3)}$ B. $\dfrac{\log(3)}{\log(2)}$ C. $\dfrac{\log(3)}{\log(4)}$ D. $\dfrac{\log(4)}{\log(3)}$ E. $\dfrac{3}{2}$

2. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Assume that you are given a dynamical system which you simulate on your computer. How can you determine whether the obtained solutions are structurally stable?

b) Explain the Poincaré-Bendixon theorem.

c) Explain what kinds of dynamics one can obtain from a system of two uncoupled oscillators on the torus:

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 = \text{const.} \\ \dot{\theta}_2 &= \omega_2 = \text{const.} \end{aligned} .$$

d) Explain what a Hopf bifurcation is.

e) Explain the difference of chaotic motion in a volume-conserving system and in a dissipative system.

f) What is the significance of the parameter q in the generalized dimension spectrum D_q?

3. Normal forms of bifurcations [2.5 points] The normal forms of typical bifurcations for dynamical systems of dimensionality one are the following:

Type	saddle-node	transcritical	supercrit. pitchfork	subcrit. pitchfork
Normal form	$\dot{x} = r + x^2$	$\dot{x} = rx - x^2$	$\dot{x} = rx - x^3$	$\dot{x} = rx + x^3$

a) Discuss why normal forms of bifurcations are useful in the context of dynamical systems.

Solution
The normal forms describe the universal forms of dynamical systems close to bifurcations. They can therefore be used to analytically identify bifurcations and their type, or to construct dynamical systems with desired bifurcation properties.

b) Consider the system

$$\dot{x} = \frac{x}{x+1} - ax\,,$$

where a is a real parameter. The system undergoes a bifurcation at $x = 0$ as the parameter a changes. Identify the bifurcation point and type of bifurcation by writing the system on normal form close to the bifurcation.

Solution
Series expand the flow $\dot{x} = f(x)$ around $x = 0$ to get

$$f(0) \approx (1-a)x - x^2 + x^3\,.$$

Comparison to the normal forms show that we have a transcritical bifurcation at the bifurcation point $a_c = 1$.

c) Use the normal forms above to construct a dynamical system of dimensionality one, $\dot{x} = f(x)$, with a single bifurcation parameter a such that the system undergoes a supercritical pitchfork bifurcation at $a = 0$ and $x = 0$, and a subcritical pitchfork bifurcation at $a = 1$ and $x = 1$.
Hint: To simplify, you can start from the ansatz $f(x) = c_0(a) + c_1(a)x + c_2x^2 + c_3x^3 + c_4x^4$, *where* c_i *are coefficients to be determined and where only* $c_0(a)$ *and* $c_1(a)$ *depend on* a.

Solution
We use the normal forms for the pitchfork bifurcations to construct the system. Series expansions to third order around $x = 0$ gives

$$f(x=0) \approx c_0(a) + c_1(a)x + c_2x^2 + c_3x^3$$

Comparison of this equation to the normal form of the supercritical pitchfork $rx - x^3$ we get $c_0(a) = O(a)$, $c_1(a) = a + O(a^2)$, $c_2 = 0$, and $c_3 = -1$, i.e. the ansatz reduces to $f(x) = c_0(a) + c_1(a)x - x^3 + c_4x^4$. Series expansions to third order around $x = 1$ now gives

$$\begin{aligned}
f(x=1) &\approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f'''(1)(x-1)^3 \\
&= c_0(a) + c_1(a) - 1 + c_4 + (c_1 - 3 + 4c_4)(x-1) \\
&\quad + (-3 + 6c_4)(x-1)^2 + (-1 + 4c_4)(x-1)^3
\end{aligned}$$

We compare this equation to the normal form of the subcritical pitchfork at $x = 1$ and $r = 1$: $(r-1)(x-1) + (x-1)^3$. Both the $(x-1)^2$

and $(x-1)^3$ terms give $c_4 = 1/2$. Choosing $c_1 = a$ and $c_0 = -a^2/2$ (in agreement with the result above) the behaviour around $x = 1$ and $a = 1$ becomes

$$f(x=1) \approx -\frac{(a-1)^2}{2} + (a-1)(x-1) + (x-1)^3 \approx (a-1)(x-1) + (x-1)^3$$

in agreement with the normal form.

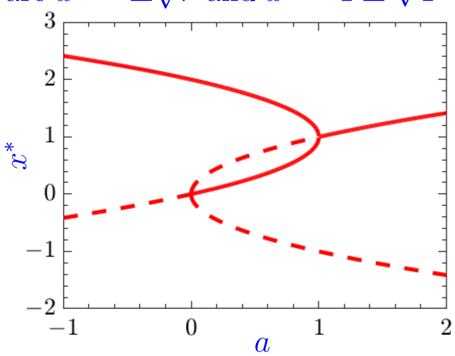
In conclusion, our constructed system becomes

$$\dot{x} = -\frac{a^2}{2} + ax - x^3 + \frac{1}{2}x^4 \,.$$

d) Sketch the bifurcation diagram of your system in subtask c).

Solution

The following figure shows a sketch of the bifurcation diagram (it is not necessary to solve the problem analytically, but the fixed points are $x^* = \pm\sqrt{r}$ and $x^* = 1 \pm \sqrt{1-r}$):



4. Laser model [2 points] A simple model for a laser is provided by

$$\begin{aligned} \dot{n} &= GnN - kn \\ \dot{N} &= -GnN - fN + p \,, \end{aligned} \tag{1}$$

where $N(t)$ is the number of excited atoms and $n(t)$ is the number of photons in the laser field. The parameters G, k, and f are positive and p can take either sign.

a) Introduce suitable dimensionless units and write the system on dimensionless form in terms of two dimensionless parameters.

Solution

Let $t = t_0\tau$, $n = n_0 x$ and $N = N_0 y$ to rewrite Eq. (1) as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= \frac{t_0}{n_0}[n_0 N_0 Gxy - n_0 kx] = t_0 N_0 Gxy - t_0 kx \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= \frac{t_0}{N_0}[-n_0 N_0 Gxy - N_0 fy + p] = -t_0 n_0 Gxy - t_0 fy + \frac{t_0}{N_0}p \,, \end{aligned}$$

Choose $t_0 = 1/k$, $N_0 = n_0 = k/G$ to get

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= xy - x \\
\frac{\mathrm{d}y}{\mathrm{d}\tau} &= -xy - \underbrace{\frac{f}{k}}_{\equiv\alpha} y + \underbrace{\frac{Gp}{k^2}}_{\equiv\beta} .
\end{aligned}$$

with $\alpha > 0$ and β can take either sign.

b) Find all the fixed points of the system (1) and determine their stability.

Solution

The first equation has nullclines at either $x = 0$ or $y = 1$. The first case gives the fixed point $(x_1^*, y_1^*) = (0, \beta/\alpha)$, the second case gives $(x_1^*, y_1^*) = (\beta - \alpha, 1)$.

The Jacobian becomes

$$\begin{aligned}
\mathbb{J} &= \begin{pmatrix} y-1 & x \\ -y & -\alpha - x \end{pmatrix} \\
\mathbb{J}(x_1^*, y_1^*) &= \begin{pmatrix} \beta/\alpha - 1 & 0 \\ -\beta/\alpha & -\alpha \end{pmatrix} \\
\operatorname{tr}\mathbb{J}(x_1^*, y_1^*) &= \beta/\alpha - 1 - \alpha \\
\det\mathbb{J}(x_1^*, y_1^*) &= \alpha - \beta \\
\mathbb{J}(x_2^*, y_2^*) &= \begin{pmatrix} 0 & \beta - \alpha \\ -1 & -\beta \end{pmatrix} \\
\operatorname{tr}\mathbb{J}(x_2^*, y_2^*) &= -\beta \\
\det\mathbb{J}(x_2^*, y_2^*) &= \beta - \alpha
\end{aligned}$$

The first fixed point is unstable (saddle-point) if $\alpha < \beta$ since $\det\mathbb{J}(x_1^*, y_1^*) < 0$. When $\beta < \alpha$, $\operatorname{tr}\mathbb{J}(x_1^*, y_1^*) = \beta/\alpha - 1 - \alpha < \alpha/\alpha - 1 - \alpha < 0$, i.e. the first fixed point is stable (stable node).

The second fixed point is unstable (saddle-point) if $\alpha > \beta$ since $\det\mathbb{J}(x_2^*, y_2^*) < 0$. When $\beta > \alpha > 0$, $\operatorname{tr}\mathbb{J}(x_1^*, y_1^*) = -\beta > 0$, it is stable (node or spiral depending on the parameters).

c) Make a plot over the two-dimensional parameter space spanned by the two dimensionless parameters from subtask a). Plot any curves where regular bifurcations occur and label them with their type. Also label the regions separated by bifurcation curves with the number of stable fixed points and the number of unstable fixed points.

Solution

The number of fixed points remain constant for all parameters, but their stability changes along the line $\alpha = \beta$ in the α-β parameter space (transcritical bifurcation).

5. Van der Pol relaxation oscillator [1.5 points] The van der Pol oscillator is governed by the following dynamics:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0\,, \tag{2}$$

where $x(t)$ is a dynamical variable and μ is a positive parameter, $\mu > 0$.

a) Introduce $y = \dot{x}/\mu + x^3/3 - x$ and derive a dynamical system for x and y.

Solution
The dynamical system corresponding to Eq. (2) is

$$\begin{aligned}\dot{x} &= \mu[y - (x^3/3 - x)]\\ \dot{y} &= \ddot{x}/\mu + (x^2 - 1)\dot{x} = -x/\mu\,.\end{aligned}$$

b) Classify the fixed points (determine stability and type) of the dynamical system in subtask a) for general positive values of μ.

Solution
The system has a single fixed point at the origin. The Jacobian evaluated at the origin becomes

$$\begin{aligned}\mathbb{J} &= \begin{pmatrix} \mu & \mu \\ -1/\mu & 0 \end{pmatrix}\\ \operatorname{tr}\mathbb{J} &= \mu\\ \det\mathbb{J} &= 1\end{aligned}$$

The fixed point is an unstable spiral for $0 < \mu < 2$ and an unstable node for $\mu > 2$. When $\mu = 2$, $\mathbb{J}$ is not a multiple of the unit matrix and the fixed point is therefore an unstable degenerate node.

c) Explain the dynamics of the van der Pol oscillator in the limit of large values of μ.

Solution
Relaxation oscillations, see Lecture notes 8.0.1

6. Lyapunov exponents [2 points] Consider the dynamical system

$$\begin{aligned}\dot{x} &= a(y - x)\\ \dot{y} &= (c - a)x - xz + cy\\ \dot{z} &= xy - bz\end{aligned} \tag{3}$$

where x, y and z are dynamical variables and a, b and c are parameters.

a) For $a = 40$, $b = 3$, and $c = 28$ the system (3) has no stable fixed points. In this limit, calculate the sum of the Lyapunov exponents of the system.

Solution

The Jacobian becomes

$$\mathbb{J} = \begin{pmatrix} -a & a & 0 \\ c-a-z & c & -x \\ y & x & -b \end{pmatrix}$$
$$\operatorname{tr}\mathbb{J} = c - b - a = -15 < 0$$

Since $\operatorname{tr}\mathbb{J}$ is independent of the position, the sum of the Lyapunov exponents becomes $\lambda_1 + \lambda_2 + \lambda_3 = -15$

b) Given that the maximal Lyapunov exponent in the system (3) is positive, discuss what long-term behavior you expect from the system for the parameter values quoted in subtask a).

Solution
Since the system is dissipative $\operatorname{tr}\mathbb{J} < 0$ everywhere, since there are no stable fixed points, and since $\lambda_1 > 0$ we expect the dynamics to be chaotic with a strange attractor.

c) For $a = 3$ and $b = c = 1$ the system (3) has a single fixed point at the origin $x = y = z = 0$ which is stable and attracts the full phase-volume. Determine the Lyapunov exponents of the system.
Hint: For this system the Lyapunov exponents are equal to the real part of the stability exponents of separations.

Solution

For a system with a globally attracting fixed point, all trajectories approach it and consequently the Lyapunov exponents become equal to the real part of the stability exponents of the Jacobian at the fixed point:

$$\mathbb{J} = \begin{pmatrix} -3 & 3 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

One eigenvalue is $\sigma_3 = -1$, the other two are obtained from the eigenvalues of the upper left 2×2 matrix with trace $\tau = -2$ and determinant $\Delta = -3 + 6 = 3$:

$$\sigma_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta}) = -1 \pm \mathrm{i}\sqrt{2}$$

Consequently, all three Lyapunov exponents are equal to -1.

d) Discuss how the dynamics in subtask *c*) can turn into the dynamics in subtask b) as the parameters change.

Solution
Discussion based on Lecture notes 13.1.