CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	January 14, 2019, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once around 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$18 passed; $\geq$26 grade 4; $\geq$31 grade 5,
GU $\geq$18 grade G; $\geq$ 28 grade VG.

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1. Multiple choice questions [2 points] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Classify the fixed point of the two-dimensional dynamical system:

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x}\,, \qquad \text{where } \mathbb{A} = \begin{pmatrix} -3 & 4 \\ -4 & 2 \end{pmatrix}.$$

A. It is a saddle point.

B. It is a stable spiral.

C. It is an unstable spiral.

D. It is a stable node.

E. It is an unstable node.

b) The normal forms of typical bifurcations for dynamical systems of dimensionality one are the following:

Type	saddle-node	transcritical	supercrit. pitchfork	subcrit. pitchfork
Normal form	$\dot{x} = r + x^2$	$\dot{x} = rx - x^2$	$\dot{x} = rx - x^3$	$\dot{x} = rx + x^3$

How does the stability time of the fixed points close to a subcritical pitchfork bifurcation scale with the bifurcation parameter r?

A. $\sim \frac{1}{r}$ B. $\sim \frac{1}{\sqrt{r}}$ C. ~ 1 D. $\sim \sqrt{r}$ E. $\sim r$

c) Which type(s) of bifurcation(s) does the system $\dot{x} = 5 - re^{-x^2}$ have?

A. Saddle-node bifurcation

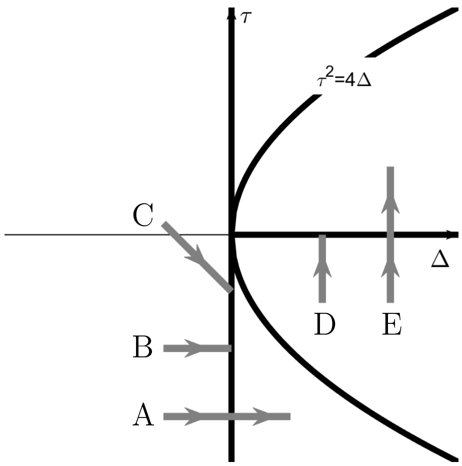
B. Transcritical bifurcation

C. Supercritical pitchfork bifurcation

D. Subcritical pitchfork bifurcation

E. No bifurcation occurs

d) Each path A–E in the Δ-τ diagram below (Δ and τ are the determinant and trace of the stability matrix of dimensionality two) is obtained from τ and Δ of a fixed point upon increasing a parameter.

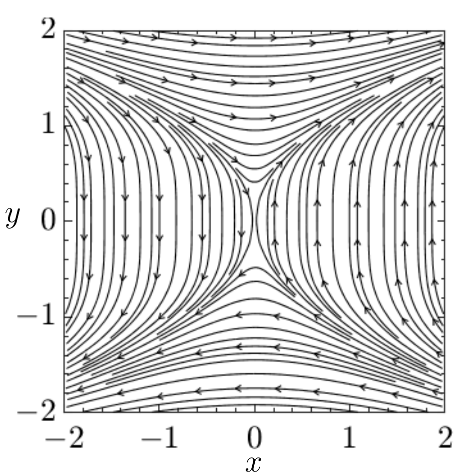


Which of the paths corresponds to one of the fixed points in a normal form of the saddle-node bifurcation in dimensionality two?

A. B. C. D. E.

e) The figure below shows the phase portrait of the system:

$$\dot{x} = y^3$$
$$\dot{y} = x$$



What is the index of the fixed point at the origin?

A. -2 B. -1 C. 0 D. 1 E. 2

f) Which properties hold in general for the Lyapunov exponents in a continuous dynamical system of dimensionality three with a strange attractor that is globally attracting?

A. One Lyapunov exponent is negative

B. One Lyapunov exponent is zero

C. One Lyapunov exponent is positive

D. The sum of all Lyapunov exponents is negative

E. The sum of all Lyapunov exponents is zero

2. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Write down the equations for a Hamiltonian dynamical system of your choice.

Solution
For example, starting from Newton's second law of motion, $F = m\ddot{x}$, we have the Hamiltonian system

$$\dot{x} = \frac{p}{m}, \quad \dot{p} = F$$

b) What is meant by a catastrophe in the context of bifurcation theory?

Solution
A catastrophe is a sudden change in the state of the system as a parameter is changed. For example, after a saddle-node bifurcation or a subcritical pitchfork bifurcation, the system quickly shifts to a distant attractor.

c) Explain what the difference between a global and a local bifurcation is. Give two examples of global bifurcations.

Solution
Bifurcations of cycles, infinite-period bifurcation, bifurcation of heteroclinic trajectories, bifurcation of homoclinic orbit, ...

d) Explain what is meant by a secular term in perturbation theory.

Solution
When performing perturbation theory in a small parameter ϵ in a time-dependent problem, it is common that the perturbation coefficients in ϵ grow without bound as $t \to \infty$. Such terms growing without bound are denoted secular terms.

e) What value does the maximal Lyapunov exponent of a stable limit cycle take? Explain why.

Solution
The maximal Lyapunov exponent is zero. For a stable limit cycle

closeby trajectories are attracted, meaning all Lyapunov exponents are negative, except for the Lyapunov exponent along the cycle which must be zero due to periodicity (separations can neither grow nor shrink in the long run because after one revolution of the limit cycle, the separation is back to the original length).

f) Explain why the transition from regular dynamics to chaos is typically very different in dissipative and in Hamiltonian dynamical systems.

Solution
In dissipative systems the transitions usually occurs though a sequence of bifurcations of where attractors becomes unstable. In Hamiltonian systems there are no attractors and the thransition instead occurs in bifurcations where closed orbits break up.

3. Imperfect bifurcations [2.5 points] Consider the system

$$\dot{x} = 2(4 + a + 3r) + (12 + a + 5r)x + (6 + r)x^2 + x^3 \tag{1}$$

where r and a are real parameters.

a) The system (1) has one fixed point x^* which is independent of the parameters a and b. Find the value of this fixed point.

Solution
Since one fixed point is independent of the parameters we can set the parameters to any value, for instance $a = -3r-4$ (removes the constant term) and furthermore $r = -4$ (removes the linear term), to obtain

$$\dot{x} = 2x^2 + x^3$$

Solving this flow equal zero gives $x^* = -2$ and $x^* = 0$. Inserting these values in the original system, we find that $x^* = -2$ is a fixed point for all parameter values.

b) Find and classify all bifurcations that occur in the system (1) when $a = 0$. Sketch the bifurcation diagram.
Hint: It may be helpful to know that the shifted coordinate $\xi = x + x^*$, where x^* is the parameter-independent fixed point in subtask a), has the following dynamics

$$\dot{\xi} = (a + r)\xi + r\xi^2 + \xi^3 \,.$$

Solution
The shifted system has fixed points at

$$\xi_1^* = 0\,,\ \xi_2^* = \frac{1}{2}\left(-r - \sqrt{r^2 - 4a - 4r}\right)\,,\ \xi_3^* = \frac{1}{2}\left(-r + \sqrt{r^2 - 4a - 4r}\right)\,.$$

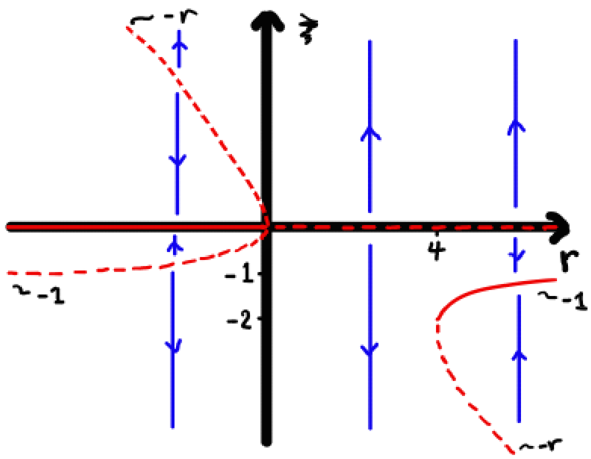
When $a = 0$ we have three fixed points if $r < 0$ or if $r > 4$, otherwise we have one fixed point.

For the case $r = 0$, we have a triple root: $\xi_1^* = \xi_2^* = \xi_3^* = 0$.
For the case $r = 4$, we have a double root: $\xi_2^* = \xi_3^* = -2$.
Since the number of fixed points changes, we can conclude that we have a pitchfork bifurcation between all three fixed points at $r_c = 0$ and a saddle-node bifurcation between ξ_2^* and ξ_3^* at $r_c = 4$.

Differentiating the shifted flow with respect to ξ gives

$$\frac{\partial \dot{\xi}}{\partial \xi} = a + r + 2r\xi + 3\xi^2 .$$

ξ_1^* is stable when $r < -a$ and unstable for $r > -a$. As a consequence, the pitchfork bifurcation at $a = 0$ is subcritical and the upper branch of the saddle-node bifurcation is stable. Sketching the bifurcation diagram (asymptotes $x^* \sim 1 - r$ and $x^* \sim -1$) verifies that the system has no other bifurcations (the corresponding bifurcation diagram for x is simply shifted by 2):



c) Find and classify all bifurcations that occur in the system (1) when $a = -1$. Sketch the bifurcation diagram.

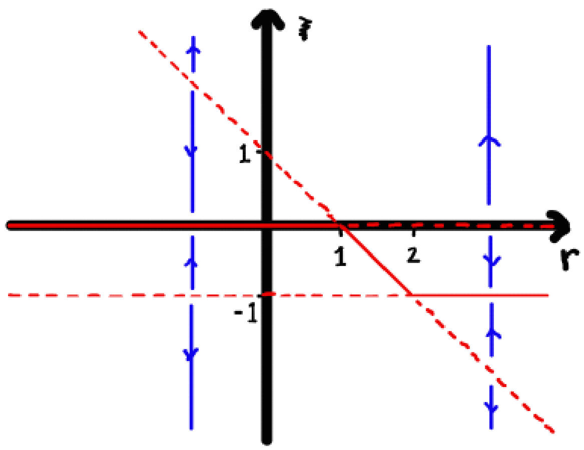
Solution
When $a = -1$ the fixed points are located at

$$\xi_1^* = 0\,,\ \xi_2^* = \frac{1}{2}\left(-r - |r-2|\right)\,,\ \xi_3^* = \frac{1}{2}\left(-r + |r-2|\right)\,.$$

Equivalently, we can reorder the second and third fixed points at $r = 2$ to obtain

$$\xi_1^* = 0\,,\ \xi_2^* = 1 - r\,,\ \xi_3^* = -1\,.$$

Sketch the bifurcation diagram:



We find that the system has two transcritical bifurcations, one at $r_c = 1$ and one at $r_c = 2$.

4. Linear stability analysis and phase portrait [1.5 points] Consider the following dynamical system

$$\begin{aligned} \dot{x} &= ax^2 - xy \\ \dot{y} &= -y + x^2 \end{aligned} \tag{2}$$

where $0 \leq a \leq 1$ is a real parameter.

a) Identify all fixed points of the system (2) and classify them according to linear stability analysis for the parameter range $0 \leq a \leq 1$.

Solution

Fixed points: $(x_1^*, y_1^*) = (0, 0)$ and $(x_1^*, y_1^*) = (a, a^2)$. Jacobian matrix

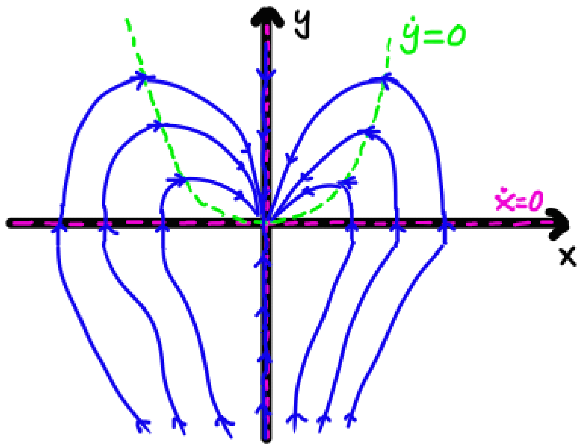
$$\begin{aligned} \mathbb{J}(x,y) &= \begin{pmatrix} 2ax - y & -x \\ 2x & -1 \end{pmatrix} \\ \mathbb{J}(x_1^*, y_1^*) &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \\ \operatorname{tr} \mathbb{J}(x_1^*, y_1^*) &= -1 \\ \det \mathbb{J}(x_1^*, y_1^*) &= 0 \\ \mathbb{J}(x_2^*, y_2^*) &= \begin{pmatrix} a^2 & -a \\ 2a & -1 \end{pmatrix} \\ \operatorname{tr} \mathbb{J}(x_2^*, y_2^*) &= a^2 - 1 \\ \det \mathbb{J}(x_2^*, y_2^*) &= a^2 \end{aligned}$$

According to linear stability analysis, the first fixed point is stable in the y-direction and marginally stable in the x-direction (in a linear system we would have had a line of stable fixed points). For the second fixed point, the discriminant $(a^2-1)^2 - 4a^2$ has a relevant zero when $a^2 = 3 - \sqrt{8}$, i.e. when $a = \sqrt{2} - 1$. According to linear stability analysis, the second fixed point is therefore a center if $a = 1$, a stable spiral if $\sqrt{2} - 1 < a < 1$, a stable degenerate node if $a = \sqrt{2} - 1$ ($\mathbb{J}(x_2^*, y_2^*)$ is not multiple of the unit matrix), a stable node if $0 < a < \sqrt{2} - 1$ and the system only has one fixed point when $a = 0$.

b) Using the nullclines as a guide, sketch the phase portrait of the system (2) for the case $a = 0$. Describe in words how trajectories behave close to the fixed point.

Solution

When $a = 0$, the nullclines corresponding to $\dot{x} = 0$ are $x = 0$ and $y = 0$ and the nullclines corresponding to $\dot{y} = 0$ are $y = x^2$:



When $y < 0$ then $\text{sign}(\dot{x}) = \text{sign}(x)$ and $\dot{y} > 0$, i.e. all trajectories with $x \neq 0$ will cross the line $y = 0$ trajectories, moving away from the fixed point in the x-direction. When $y > 0$ then $\text{sign}(\dot{x}) = -\text{sign}(x)$ and all trajectories will eventually get arbitrarily close to the fixed point. In conclusion, the fixed point is globally attracting, but all trajectories approach it from positive values of y.

5. Hopf bifurcation [2 points] Consider the following dynamical system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \mu y - x^2 y - 2y^3 \end{aligned} \tag{3}$$

where μ is a real parameter.

a) Show that the system (3) undergoes a Hopf bifurcation as μ passes zero.

Solution

The system has a single fixed point at the origin. The Jacobian evaluated at the origin becomes

$$\begin{aligned} \mathbb{J} &= \begin{pmatrix} 0 & 1 \\ -1 - 2xy & \mu - x^2 - 6y^2 \end{pmatrix}\Bigg|_{x=y=0} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \\ \text{tr}\,\mathbb{J} &= \mu \\ \det \mathbb{J} &= 1\,. \end{aligned}$$

The corresponding eigenvalues $\lambda_{\pm} = (\mu \pm \sqrt{\mu^2 - 4})/2$ passes from negative to positive real part with a non-zero imaginary part as μ passes zero. Thus, we have a Hopf bifurcation when μ passes zero.

b) Consider the case $\mu = 0$ in the system (3) and classify the fixed point at the origin.

Solution
When $\mu = 0$ linear stability analysis implies that the fixed point at the origin is a center. However, non-linear terms may destroy the center and to classify the fixed point we must consider the effect of the non-linear terms. Consider the time evolution of the radial coordinate $r = \sqrt{x^2 + y^2}$:

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{xy + y(-x + \mu y - x^2 y - 2y^3)}{r} = \frac{y^2}{r}(\mu - x^2 - 2y^2)\,.$$

When $\mu = 0$, we have $\dot{r} < 0$ for any non-zero values of r. Consequently, the origin is a stable spiral.

c) Consider the case $\mu = 1$. Using the Poincaré-Bendixon theorem, show that the system (3) has at least one closed orbit.

Solution
Consider once again the time evolution of the radial coordinate $r = \sqrt{x^2 + y^2}$, now with $\mu = 1$:

$$\dot{r} = \frac{y^2}{r}(1 - x^2 - 2y^2)\,.$$

First, rewriting $\dot{r} = \frac{y^2}{r}(1 - r^2 - y^2)$ and let $r = 1$ gives $\dot{r} = -\frac{y^4}{r} \leq 0$ at $r = 1$. Second, rewriting $\dot{r} = \frac{y^2}{r}(1 + x^2 - 2r^2)$ and let $r = \frac{1}{\sqrt{2}}$ gives $\dot{r} = \frac{x^2 y^2}{r} \geq 0$ at $r = \frac{1}{\sqrt{2}}$. Thus, no trajectory starting within $1/\sqrt{2} \leq r \leq 1$ can leave this region (a trapping region) and since there are no fixed points in the region, the Poincaré-Bendixon theorem gives that we must have at least one closed orbit in the region.

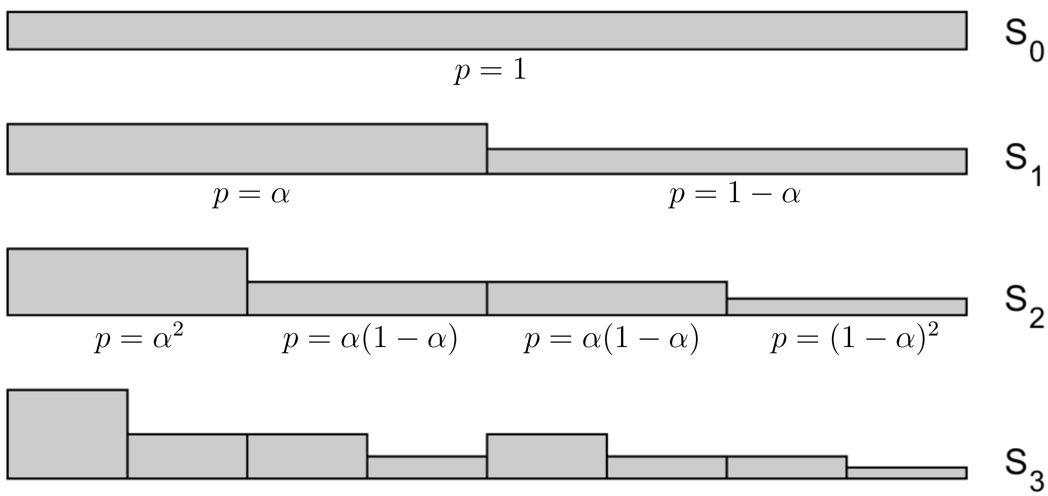
6. Fractal dimension of a weighted Cantor set [2 points] The generalized fractal dimension D_q is defined by

$$D_q \equiv \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln I(q,\epsilon)}{\ln(1/\epsilon)}$$

with

$$I(q,\epsilon) = \sum_{k=1}^{N_{\text{box}}} p_k^q(\epsilon)\,.$$

Here p_k is the probability to be in the k:th occupied box (box with $p_k \neq 0$) and N_{box} is the total number of occupied boxes.
Consider a set S_n where n labels the generation. Start with S_0 being the unit interval. S_n is obtained by dividing each interval in the set S_{n-1} into two subintervals. Upon each division, allocate a fraction α (assume $0 \leq \alpha \leq 1$) of the probability to be in the original interval to the left subinterval, and a fraction $1-\alpha$ to the right subinterval. The figure below illustrates the first few generations S_0, S_1, S_2 and S_3:



The probability to be in different intervals is displayed in the text below the intervals. The height of an interval illustrates the relative probability to be in that interval for the case $\alpha = 2/3$. In what follows, consider the fractal dimension of the set S_∞ obtained by iterating $n \to \infty$.

a) Evaluate the generalized fractal dimension D_q of S_∞ for the case $\alpha = 1$.

Solution
When $\alpha = 1$, the leftmost interval has all probability at each generation. The set becomes a single point with fractal dimension $D_q = 0$.

b) Evaluate the generalized fractal dimension D_q of S_∞ for the case $\alpha = \frac{1}{2}$.

Solution
When $\alpha = \frac{1}{2}$, all intervals have equal probability at each generation. Hence S_n is equivalent to the unit interval at any generation and the fractal dimension becomes $D_q = 1$.

c) Evaluate the box-counting dimension D_0 of S_∞ for $0 < \alpha < 1$.

Solution
Since all intervals have non-zero probability when $0 < \alpha < 1$, the box counting dimension that does not take into account of the probability to be at different intervals, must be equal to unity, i.e. $D_0 = 1$ for $0 < \alpha < 1$.

d) Evaluate the generalized fractal dimension D_q of S_∞ for general values of α.
Hint: To verify your result, you can check that the results in subtasks a), b) and c) come out correctly and that your result is symmetric upon replacing $\alpha \to 1 - \alpha$.

Solution
Consider intervals of length $\epsilon = 2^{-n}$ at generation n. At each iteration, subintervals takes either takes probability α times previous probability, or $1 - \alpha$ times previous probability. It follows that there will be $\binom{n}{m}$ intervals having the probability $\alpha^m(1-\alpha)^{n-m}$ with $m = 0, \ldots, n$. We use this information to evaluate

$$I(q,\epsilon) = \sum_{k=1}^{N_{\text{box}}} p_k^q(\epsilon) = \sum_{m=0}^{n} \binom{n}{m} [\alpha^q]^m [(1-\alpha)^q]^{n-m} = (\alpha^q + (1-\alpha)^q)^n \,.$$

We obtain

$$D_q \equiv \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln I(q,\epsilon)}{\ln(1/\epsilon)} = \frac{1}{1-q} \lim_{n\to\infty} \frac{\ln\left[(\alpha^q + (1-\alpha)^q)^n\right]}{\ln(2^n)} = \frac{1}{1-q} \frac{\ln\left[\alpha^q + (1-\alpha)^q\right]}{\ln 2} \,.$$

Consistency check when $\alpha = 0$:

$$D_q = \frac{1}{1-q}\frac{\ln 1}{\ln 2} = 0$$

Consistency check when $\alpha = 1/2$:

$$D_q = \frac{1}{1-q} \frac{\ln\left[2^{-q} + 2^{-q}\right]}{\ln 2} = \frac{1}{1-q} \frac{\ln\left[2^{1-q}\right]}{\ln 2} = \frac{\ln 2}{\ln 2} = 1$$

Consistency check for $q = 0$ and $0 < \alpha < 1$:

$$D_q = \frac{\ln\left[1+1\right]}{\ln 2} = 1 \,.$$