CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	Test exam
Place:	
Teachers:	
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$18 passed; $\geq$26 grade 4; $\geq$31 grade 5,
GU $\geq$18 grade G; $\geq$ 28 grade VG.

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1. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) What is a dynamical system?

b) Give three examples of dynamical systems.

c) What is a nullcline?

d) Give two examples of applications of nullclines in the analysis of dynamical systems.

e) What is the index of a fixed point?

f) What does the Poincaré-Bendixon theorem state?

g) What does the maximal Lyapunov exponent characterize?

h) What is the correlation dimension?

2. Imperfect bifurcation [2 points] Consider the system

$$\dot{x} = rx + ax^2 - x^3$$

where a is a real parameter.

a) When $a = 0$ a bifurcation occurs when r passes zero. What kind of bifurcation is it (explain why)? Sketch the bifurcation diagram.

b) For each value of non-zero a we obtain a different bifurcation diagram x^* against r. Sketch all the qualitatively different bifurcation diagrams that can be obtained by varying a (you can skip the case $a = 0$).

c) Sketch the regions in the (r,a)-plane that correspond to the bifurcation diagrams in a) and b) and label the bifurcations that occur when you pass from one region to another.

3. Phase portrait [1 point] A system is known to have exactly two fixed points Both of these are saddles.

a) Sketch a phase portrait in which a single trajectory connects the two saddles.

b) Sketch a phase portrait in which no trajectory connects the saddles.

c) Now change one of the fixed points to a spiral. Sketch a phase diagram with a homoclinic orbit.

4. Stability analysis [1 point] Classify the fixed point(s) for the system

$$\dot{x} = -y + ax^3$$
$$\dot{y} = x + ay^3$$

for any real value of the parameter a.

5. Pendulum [2 points] Consider a damped, rigid pendulum with constant driving

$$\ddot{\theta} = -\frac{\gamma}{m}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{\tau}{I_0} \tag{1}$$

where θ is the angle to the gravitational acceleration $\boldsymbol{g}$ with magnitude $g = |\boldsymbol{g}|$, m is a point mass, l is the distance of the point mass from the pendulum center, τ is a constant torque applied to the pendulum, and I_0 is the moment of inertia with respect to the center.

a) Find a condition on the parameters in Eq. (1) that allows to use the overdamped limit:

$$0 = -\frac{\gamma}{m}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{\tau}{I_0}\,.$$

b) Determine the fixed points and their stability for the overdamped pendulum.

c) Does the overdamped pendulum have a constant of motion? If so, what is it?

d) Can you find a condition on the parameters for the driven pendulum in Eq. (1) such that it has a constant of motion? If so, what is it?

6. Bifurcations and Lyapunov exponents [2 points] Consider the dynamical system from the third hand-in:

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + \nu r^2 \end{aligned} \tag{2}$$

and answer the following questions.

a) What kind of bifurcation occurs as μ passes zero? Explain why it is that bifurcation.

b) What is the radius and period time of the stable limit cycle when $\mu > 0$?

c) Can you determine which Lyapunov exponents are positive/negative/zero for the system (2) when $\mu < 0$ and when $\mu > 0$?

7. Fractal dimensions [2 points] The generalized fractal dimension D_q is defined by

$$D_q \equiv \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln I(q,\epsilon)}{\ln(1/\epsilon)} \tag{3}$$

with

$$I(q,\epsilon) = \sum_{k=1}^{N_{\text{box}}} p_k^q(\epsilon)\,.$$

where p_k is the probability to be in the k:th occupied box and N_{box} is the total number of occupied boxes.

a) Show that D_q is constant for a mono-fractal, i.e. a fractal where all occupied boxes have equal probability.

b) In Eq. (3) the limit $\epsilon \to 0$ is taken. This raises some problems in numerical evaluations of D_q from numerical or experimental data. Discuss the potential problems and discuss how they can be resolved.

c) Evaluate the limit $q \to 1$ (the information dimension).

d) Why is the limit $q \to 1$ specifically sensitive to normalization of probabilities?

e) Does the resolution you discussed in problem b) apply to the limit $q \to 1$?