CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	Test exam
Place:	
Teachers:	
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$18 passed; $\geq$26 grade 4; $\geq$31 grade 5,
GU $\geq$18 grade G; $\geq$ 28 grade VG.

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1. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) What is a dynamical system?

Solution
A dynamical system consists of a set of quantities and a rule how these evolve in time. These rules can be continuous (system of ordinary differential equations, flow) or discrete (system of recurrence equations, map)

b) Give three examples of dynamical systems.

Solution
Example examples: Pendulum, van Der Pol oscillator, models of lasers, ...

c) What is a nullcline?

Solution
A geometric shape (line in two-dimensional systems) formed by the condition that the flow vanishes for one coordinate, $\dot{x}_i = 0$. Nullclines partition space into regions where the flow has definite signs and the intersections between nullclines give the fixed points of a system.

d) Give two examples of applications of nullclines in the analysis of dynamical systems.

Solution
To help drawing phase portraits. To find locations of fixed points. To understand how phase portrait changes when system parameters are modified.

e) What is the index of a fixed point?

Solution
The index of a fixed point is equal to the index obtained from a curve C encircling the fixed point and no other fixed points. This is a unique number and tells how many counter-clockwise revolutions the vector field does as the curve C is encircled counter-clockwise.

f) What does the Poincaré-Bendixon theorem state?

Solution
A trapping region is a closed region where the flow does not point outwards anywhere. The Poincaré-Bendixon theorem applies to dimensionality two systems. It states that for any trapping region that does not contain any fixed point, trajectories inside the trapping region must end up on a closed orbit.

g) What does the maximal Lyapunov exponent characterize?

Solution
The maximal Lyapunov exponent λ_1 characterizes the large-time average exponential growth or contraction rate of separations $\boldsymbol{\delta}(t)$ between two close-by particles:

$$\lambda_1 \equiv \lim_{|\boldsymbol{\delta}(0)|\to 0} \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\boldsymbol{\delta}(t)|}{|\boldsymbol{\delta}(0)|}$$

h) What is the correlation dimension?

Solution
The correlation dimension is defined as the scaling with small ϵ of the probability to find two points within distance ϵ, $P(|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \epsilon) \sim \epsilon^{D_2}$. The correlation dimension is equal to the generalized dimension D_q with $q = 2$.

2. Imperfect bifurcation [2 points] Consider the system

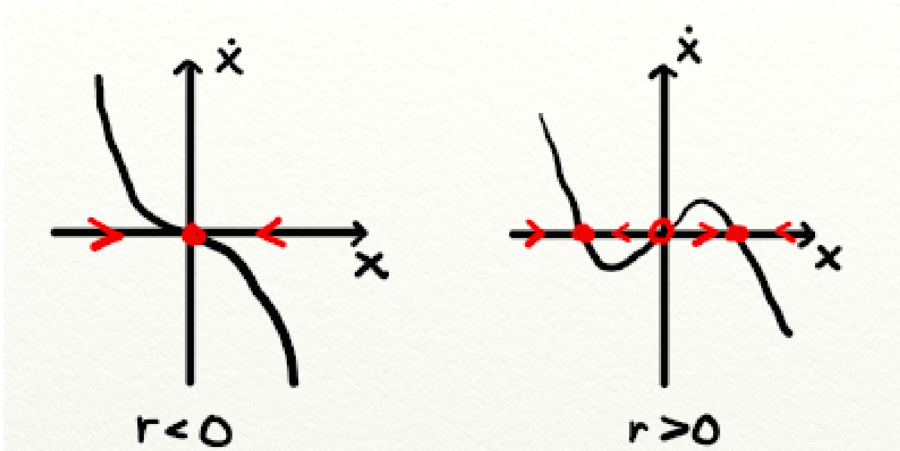
$$\dot{x} = rx + ax^2 - x^3$$

where a is a real parameter.

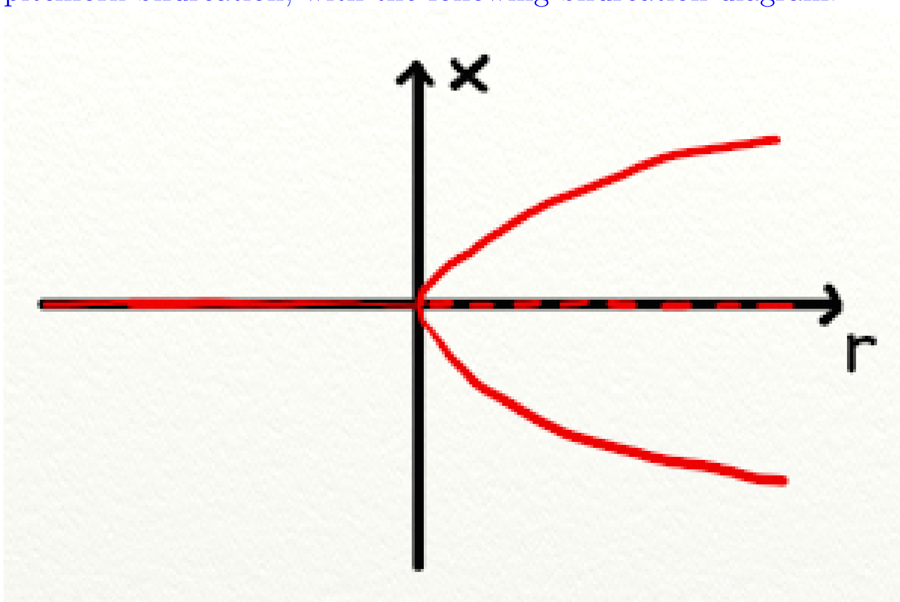
a) When $a = 0$ a bifurcation occurs when r passes zero. What kind of bifurcation is it (explain why)? Sketch the bifurcation diagram.

Solution
Determine the fixed points and their stability when $a = 0$ by sketching

$\dot{x}$ against x with $r < 0$ and with $r > 0$:



Thus, i) when $r < 0$ we have a single stable fixed point at the origin and when ii) $r > 0$ there are two stable fixed points and one unstable at the origin. The case $a = 0$ is the normal form of a supercritical pitchfork bifurcation, with the following bifurcation diagram:



b) For each value of non-zero a we obtain a different bifurcation diagram x^* against r. Sketch all the qualitatively different bifurcation diagrams that can be obtained by varying a (you can skip the case $a = 0$).

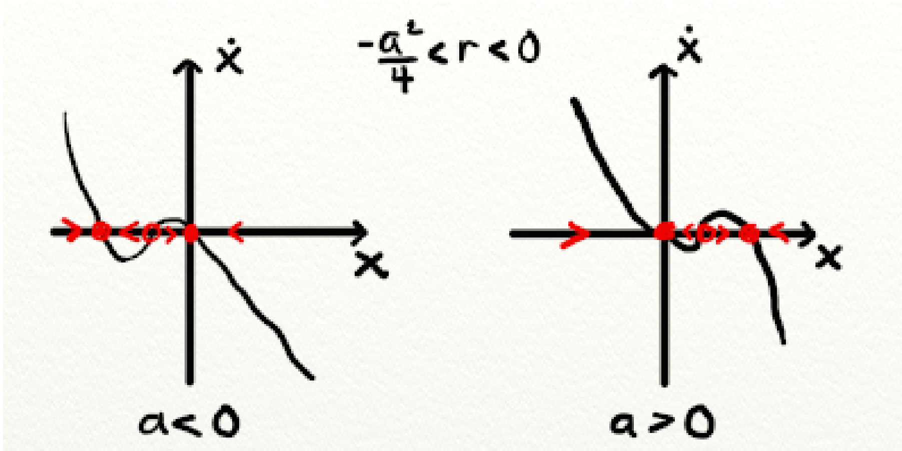
Solution

The system has fixed points at

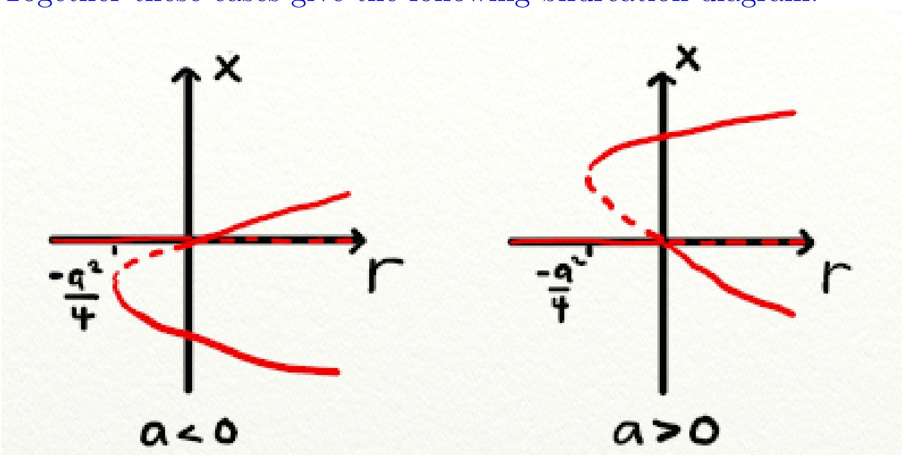
$$x_1^* = 0\,,\ x_2^* = \frac{a + \sqrt{a^2 + 4r}}{2}\,,\ x_3^* = \frac{a - \sqrt{a^2 + 4r}}{2}\,.$$

- If $r > 0$ the square root is larger than a and we have one fixed point to the left and one to the right of the fixed point in the origin (case ii) above, stability must be the same because the $x \to \pm\infty$ behaviour does not change when a is introduced).

- If $r < -a^2/4$ we have a single, stable fixed point in the origin, i.e. case i) above.
- If $-a^2/4 < r < 0$ there are two new possibilities: the two non-zero fixed points both either lie to the left ($a < 0$) or to the right ($a > 0$) of the origin:



Together these cases give the following bifurcation diagram:

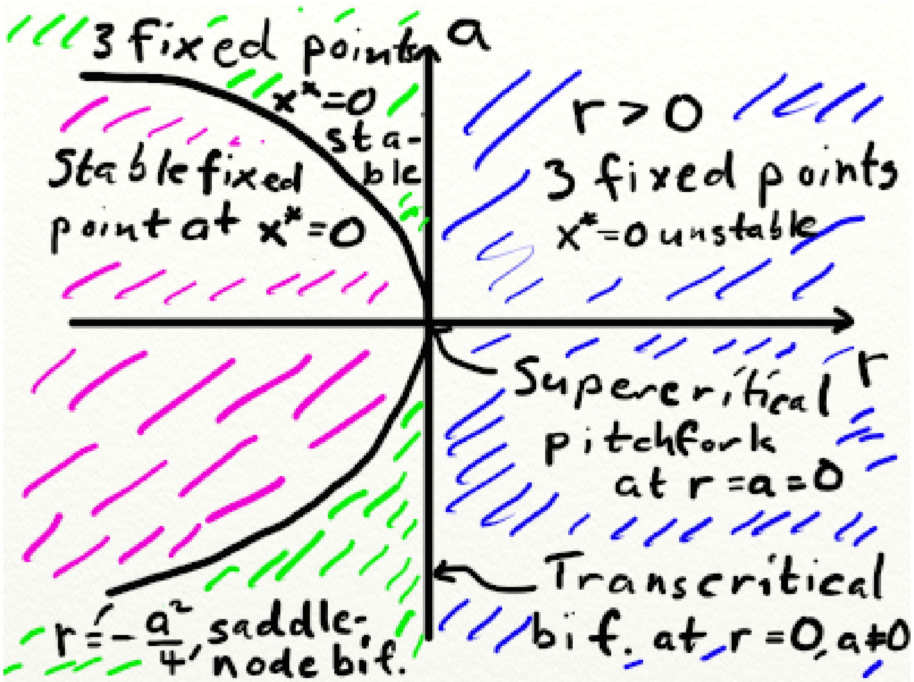


When $a \neq 0$ we have a saddle-node bifurcation at $r = -a^2/4$ and a transcritical bifurcation at $r = 0$.

c) Sketch the regions in the (r,a)-plane that correspond to the bifurcation diagrams in a) and b) and label the bifurcations that occur when you pass from one region to another.

Solution

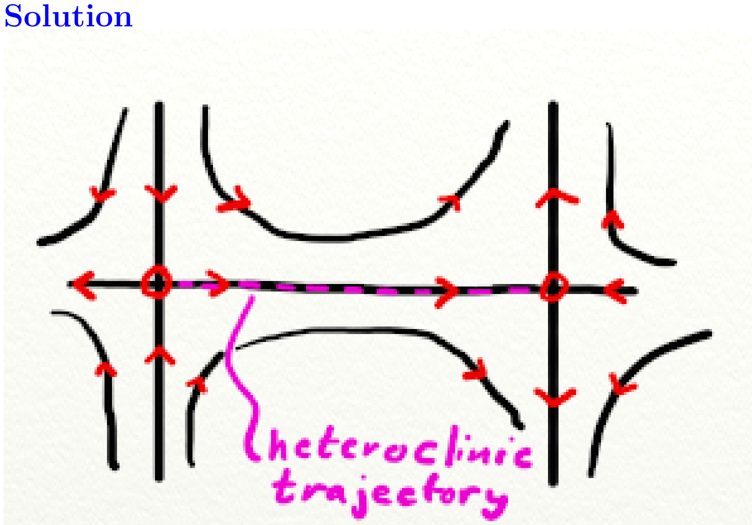
Summarizing the bifurcation diagrams for $a < 0$, $a = 0$ and $a > 0$ we get:



3. Phase portrait [1 point] A system with dimensionality two is known to have exactly two fixed points. Both of these are saddle points.

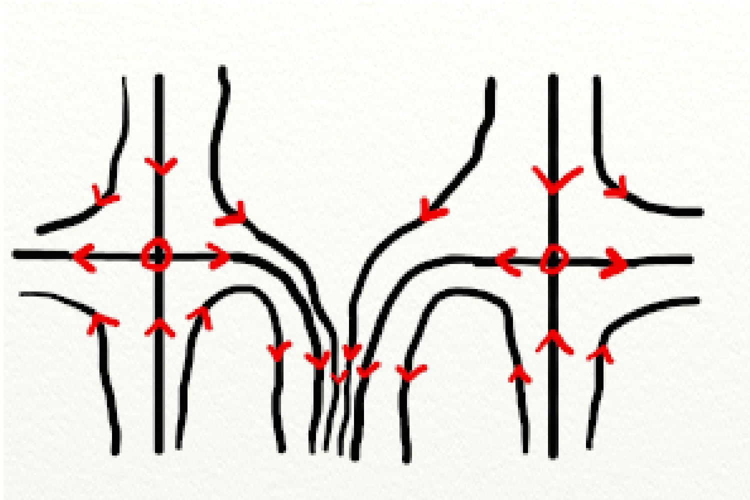
a) Sketch a phase portrait in which a single trajectory connects the two saddles.

Solution



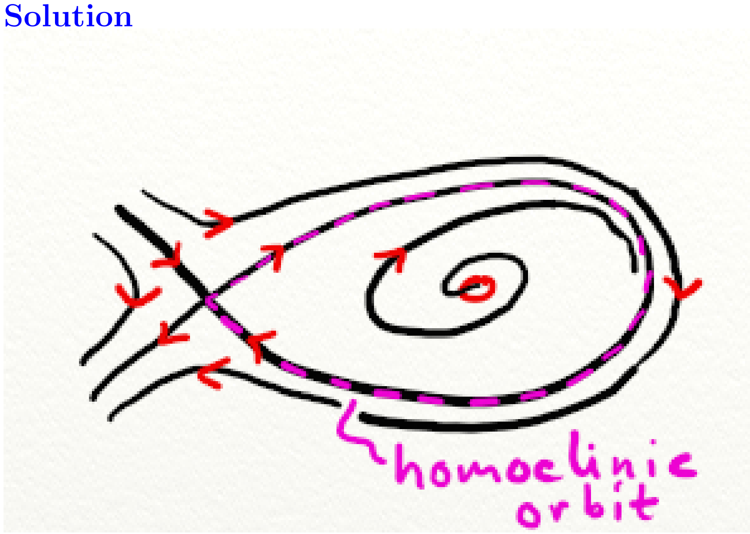
b) Sketch a phase portrait in which no trajectory connects the saddles.

Solution



c) Now change one of the fixed points to a spiral. Sketch a phase portrait with a homoclinic orbit.

Solution



4. Stability analysis [1 point] Classify the fixed point(s) for the system

$$\dot{x} = -y + ax^3$$
$$\dot{y} = x + ay^3$$

for any real value of the parameter a.

Solution

The system has a single fixed point $(x^*, y^*) = (0, 0)$. The Jacobian evaluated at this fixed point is

$$\mathbb{J} = \begin{pmatrix} 3ax^2 & -1 \\ 1 & 3ay^2 \end{pmatrix}\Bigg|_{x=y=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This matrix has eigenvalues $\pm$i, i.e. linear stability analysis gives a center. When $a = 0$, there are no non-linear contributions and the fixed point is a linear center.

When $a \neq 0$ we must consider the effect of non-linear terms to characterize the fixed point. This can be done in several ways. For example, evaluating the divergence $\mathrm{tr}\mathbb{J} = 3a(x^2+y^2)$ shows that the flow is contracting everywhere (except the point $x = y = 0$) if $a < 0$ and expanding if $a > 0$. We conclude that the fixed point must be a stable spiral if $a < 0$, and an unstable spiral if $a > 0$.
Answer The fixed point at the origin is a stable spiral if $a < 0$, a linear center if $a = 0$, and an unstable spiral if $a > 0$. NB: This problem is very similar to one example in Lecture 4.

5. Pendulum [2 points] Consider a damped, rigid pendulum with constant driving

$$\ddot{\theta} = -\frac{\gamma}{m}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{\tau}{I_0} \tag{1}$$

where θ is the angle to the gravitational acceleration $\boldsymbol{g}$ with magnitude $g = |\boldsymbol{g}|$, m is a point mass, l is the distance of the point mass from the pendulum center, τ is a constant torque applied to the pendulum, and I_0 is the moment of inertia with respect to the center.

a) Find a condition on the parameters in Eq. (1) that allows to use the overdamped limit:

$$0 = -\frac{\gamma}{m}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{\tau}{I_0}\,.$$

Solution
Go to dimensionless units $t = t_0 t'$ and multiply the equation by l/g to make the entire equation dimensionless:

$$\frac{l}{t_0^2 g}\frac{\mathrm{d}^2\theta}{\mathrm{d}t'^2} = -\frac{1}{t_0}\frac{\gamma l}{mg}\frac{\mathrm{d}\theta}{\mathrm{d}t'} - \sin\theta + \frac{\tau l}{I_0 g}$$

Choose the time scale $t_0 = \gamma l/(mg)$ to put the prefactor of $\frac{\mathrm{d}\theta}{\mathrm{d}t'}$ to unity (this term we want to keep). The dimensionless system becomes

$$\frac{m^2 g}{\gamma^2 l}\frac{\mathrm{d}^2\theta}{\mathrm{d}t'^2} = -\frac{\mathrm{d}\theta}{\mathrm{d}t'} - \sin\theta + \frac{\tau l}{I_0 g}\,.$$

If $\epsilon = m^2 g/(\gamma^2 l) \ll 1$ we can neglect the angular acceleration (except for initial transients), and the overdamped limit applies.
Answer: One condition where the overdamped limit applies is $\epsilon \ll 1$.

Note that if we would have multiplied the equation by I_0/τ instead of l/g upon dedimensionalization, we would have obtained a different time scale t_0 and a different condition on the parameters, $\epsilon\tau l/(gI_0) \ll 1$, for which the overdamped limit applies. In fact there are infinitely many possible combinations to combine ϵ, $\tau l/(gI_0)$ and the choice of t_0 to form conditions for the overdamped limit. Which of the different possibilities *one should choose depends on the specific purpose one has in mind, and are related by rescaling of time in the overdamped system.*

Another remark: The overdamped limit is a new dynamical system that is approached after a short initial transient. The angular acceleration in the new dynamical system is not equal to the acceleration of the original dynamical system for general phase-space coordinates $(\theta, \dot\theta)$. The overdamped system is a one-dimensional system

$$\dot\theta = -\frac{mg}{\gamma l}\sin\theta + \frac{m\tau}{\gamma I_0} = f(\theta)\,.$$

Extending this system with angular velocity $y = \dot\theta$ gives a two-dimensional system

$$\begin{aligned}\dot\theta &= f(\theta)\\ \dot y &= \frac{\mathrm{d}f}{\mathrm{d}t}(\theta) = y\frac{\partial f}{\partial\theta}(\theta)\end{aligned}$$

which has a different form compared to the original system corresponding to Eq. (1). However, along the actual phase-space trajectories in the overdamped system, the angular acceleration $\dot y = y\frac{\partial f}{\partial\theta}(\theta)$ is equal to the neglected angular acceleration in the original system.

b) Determine the fixed points and their stability for the overdamped pendulum.

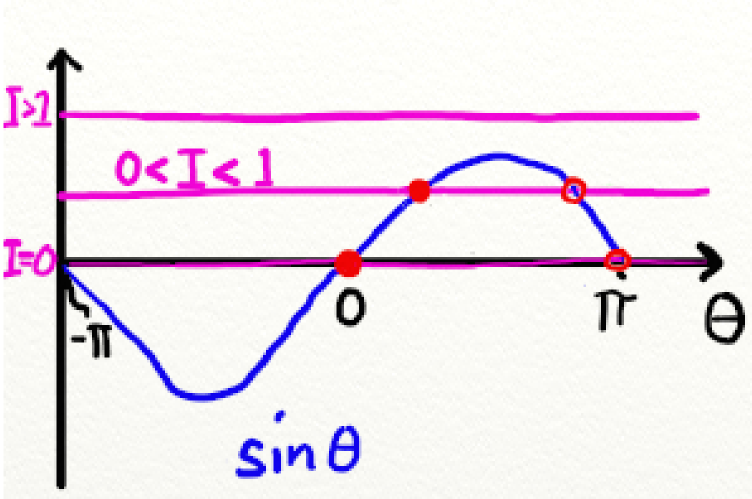
Solution
Solve

$$0 = -\frac{\gamma}{m}\dot\theta - \frac{g}{l}\sin\theta + \frac{\tau}{I_0}\,,$$

with $\dot\theta = 0$ to obtain $\sin\theta^* = l\tau/(gI_0)$ with explicit solutions

$$\begin{aligned}\theta_1^* &= \arcsin\left(\frac{\mathrm{l}\tau}{\mathrm{gI_0}}\right)\\ \theta_2^* &= \pi - \arcsin\left(\frac{\mathrm{l}\tau}{\mathrm{gI_0}}\right).\end{aligned}$$

The stability of these can be obtained by inspection of $I = l\tau/(gI_0)$ and $\sin\theta$:



When $I = 0$, $\theta_1^* = 0$ is stable and $\theta_2^* = \pi$ is unstable (damped pendulum). By continuity, θ_1^* must be stable and θ_2^* must be unstable for all values of $-1 < I < 1$ (no bifurcations occur in this interval). This can also be seen by noting that the flow is proportional to $I - \sin\theta$. Finally, at $I = \pm 1$ there is a saddle-node bifurcation where the fixed points are half-stable.

c) Does the overdamped pendulum have a conserved quantity? If so, what is it?

Solution
The overdamped pendulum is a first-order differential equation in one variable θ. This means that any conserved quantity must satisfy $\theta =$ const.. We can not find such θ for general initial conditions, meaning the overdamped pendulum does not have a conserved quantity.

d) Can you find a condition on the parameters for the driven pendulum in Eq. (1) such that it has a conserved quantity? If so, what is it?

Solution
We know that a Hamiltonian system has one conserved quantity (the energy). When $\gamma = 0$, Eq. (1) is equivalent to a Hamiltonian dynamical system. Therefore, a condition on the parameters to have a conserved quantity is $\gamma = 0$.

To find the corresponding conserved quantity, assume $\gamma = 0$ and multiply Eq. (1) by $\dot\theta$ and integrate

$$
\begin{aligned}
0 = \int_0^T \mathrm{d}t\dot\theta\left[\ddot\theta + \frac{g}{l}\sin\theta - \frac{\tau}{I_0}\right] &= \int_0^T \mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\dot\theta^2 - \frac{g}{l}\cos\theta - \frac{\tau}{I_0}\theta\right] \\
&= \frac{1}{2}\dot\theta_T^2 - \frac{g}{l}\cos\theta_T - \frac{\tau}{I_0}\theta_T - \left(\frac{1}{2}\dot\theta_0^2 - \frac{g}{l}\cos\theta_0 - \frac{\tau}{I_0}\theta_0\right)
\end{aligned}
$$

In conclusion $E = \frac{1}{2}\dot\theta^2 - \frac{g}{l}\cos\theta - \frac{\tau}{I_0}\theta$ is a conserved quantity. *If multiplied by $I_)$, the terms represent in order: Kinetic energy, gravitational potential energy, and work applied on the pendulum from the torque (c.f. Work= $F \cdot x$ for a one-dimensional constant force).*

6. Bifurcations and Lyapunov exponents [2 points] Consider the dynamical system from the third hand-in:

$$\begin{aligned}\dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + \nu r^2\end{aligned} \tag{2}$$

and answer the following questions.

a) What kind of bifurcation occurs in the corresponding Cartesian system as μ passes zero? Explain why it is that bifurcation.

Solution
As shown in problem 2a) the r-dynamics undergoes a supercritical pitchfork bifurcation at $\mu = 0$ (now r is constrained to be positive). Determine the eigenvalues of the Jacobian close to the origin (skip non-linear terms in x and y):

$$\begin{aligned}\dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \approx \mu r\cos\theta - r\omega\sin\theta = \mu x - \omega y \\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \approx \mu r\sin\theta + r\omega\cos\theta = \mu y + \omega x\,.\end{aligned}$$

The corresponding Jacobian becomes

$$\mathbb{J} = \begin{pmatrix}\mu & -\omega \\ \omega & \mu\end{pmatrix}$$

with eigenvalues $\lambda_{1,2} = \mu \pm \mathrm{i}\omega$. The eigenvalues form a complex pair as the bifurcation occurs. Since the r-dynamics undergoes a supercritical pitchfork bifurcation, a stable limit cycle is formed after the bifurcation, meaning that we have a supercritical Hopf bifurcation.

b) What is the radius and period time of the stable limit cycle when $\mu > 0$?

Solution
The limit cycle forms at the positive stable fixed point of the system $\dot{r} = 0$. Solving $0 = \mu r - r^3$ for positive r gives $r^* = \sqrt{\mu}$. Thus, the radius of the cycle is $\sqrt{\mu}$.
At the limit cycle $r = \sqrt{\mu}$, the angular dynamics becomes

$$\dot{\theta} = \omega + \nu\mu\,,$$

i.e. θ has a constant angular frequency $\omega + \nu\mu$ and the period time becomes $T = 2\pi/(\omega + \nu\mu)$.

c) Can you determine which Lyapunov exponents are positive/negative/zero for the system (2) when $\mu < 0$ and when $\mu > 0$?

Solution
When $\mu < 0$ the system has one globally attracting fixed point at $x^* = y^* = 0$. It follows that for large times any trajectory approaches the fixed point, and all Lyapunov exponents are therefore negative.
When $\mu > 0$ we have an attracting limit cycle, for which one Lyapunov exponent must be zero (for dynamics along the cycle) and one Lyapunov exponent must be negative (because cycle is attracting).

7. Fractal dimensions [2 points] The generalized fractal dimension D_q is defined by

$$D_q \equiv \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln I(q, \epsilon)}{\ln(1/\epsilon)} \tag{3}$$

with

$$I(q, \epsilon) = \sum_{k=1}^{N_{\text{box}}} p_k^q(\epsilon) \,.$$

Here p_k is the probability to be in the k:th occupied box and N_{box} is the total number of occupied boxes.

a) Show that D_q is constant for a mono-fractal, i.e. a fractal where all occupied boxes have equal probability.

Solution
If all boxes have equal probability $p_k \sim 1/N_{\text{box}}$, D_q becomes

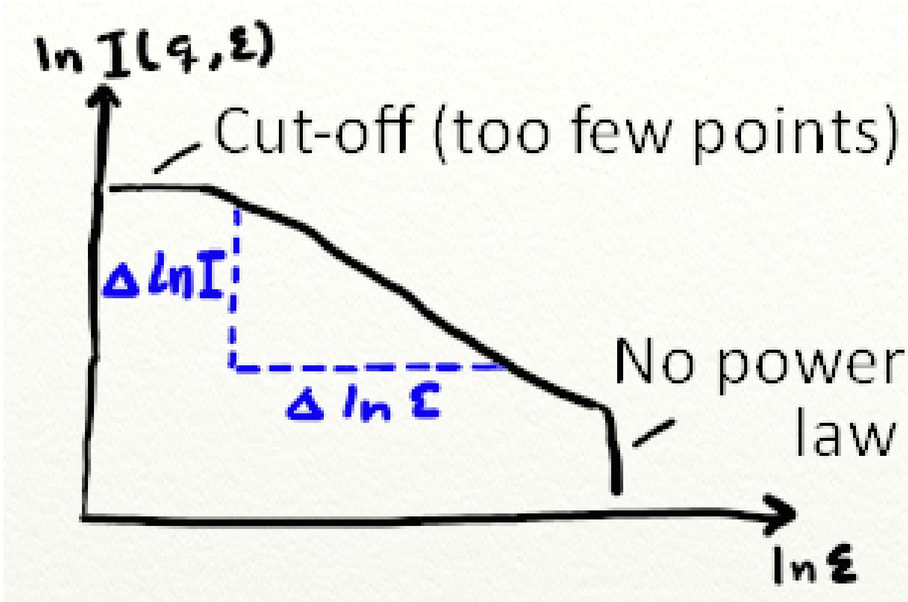
$$\begin{aligned} D_q &= \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln\left(\sum_{k=1}^{N_{\text{box}}} p_k^q\right)}{\ln(1/\epsilon)} \\ &= \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln(N_{box}(1/N_{\text{box}})^q)}{\ln(1/\epsilon)} \\ &= \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln(N_{box}^{1-q})}{\ln(1/\epsilon)} = \lim_{\epsilon \to 0} \frac{\ln(N_{box})}{\ln(1/\epsilon)} = D_0 \end{aligned}$$

i.e. $D_q = D_0 =$ const. Q.E.D.

b) In Eq. (3) the limit $\epsilon \to 0$ is taken. This raises some problems in numerical evaluations of D_q from numerical or experimental data. Discuss the potential problems and discuss how they can be resolved.

Solution
Two problems in evaluating Eq. (3) with a small value of ϵ for numerical data are that it is unclear whether we have enough points in our data to resolve the chosen value of ϵ, and that there is an unknown coefficient A_q in the scaling law corresponding to Eq. (3): $I(q, \epsilon) \sim A_q \epsilon^{(q-1)D_q}$.

These problems can be resolved by plotting $\ln I(q, \epsilon)$ against $\ln \epsilon$ to see in which range we have a power-law scaling:



In this range the scaling $I(q,\epsilon) \sim A_q \epsilon^{(q-1)D_q}$ holds, and the slope of the curve gives D_q independent from A_q:

$$D_q = \frac{1}{q-1}\frac{\Delta \ln I(q,\epsilon)}{\Delta \ln \epsilon}\,.$$

c) Evaluate the limit $q \to 1$ (the information dimension).

Solution

Expand $\ln I(q,\epsilon)$ around $q = 1$

$$\ln I(q,\epsilon) = \ln\left(\sum_{k=1}^{N_{\text{box}}} p_k^q\right)$$

$\left[\text{From Beta: } a^x \sim 1 + x\ln a \Rightarrow p_k^{q-1} \sim 1 + (q-1)\ln p_k \Rightarrow p_k^q \sim p_k(1+(q-1)\ln p_k) \text{ for } q \approx 1\right]$

$$= \ln\left(\sum_{k=1}^{N_{\text{box}}} p_k + \sum_{k=1}^{N_{\text{box}}} p_k(q-1)\ln p_k)\right)$$

$\Bigg[\text{Use norm: } \sum_{k=1}^{N_{\text{box}}} p_k = 1 \text{ and from Beta:}$

$\ln(1+x) \sim x \Rightarrow \ln(1+(q-1)A) \approx (q-1)A \text{ for } q \approx 1 \text{ with } A = \sum_{k=1}^{N_{\text{box}}} p_k \ln p_k\Bigg]$

$$= \sum_{k=1}^{N_{\text{box}}} p_k(q-1)\ln p_k\,.$$

Using this expression we get the limit $q \to 1$:

$$
\begin{aligned}
\lim_{q\to 1} D_q &= \lim_{q\to 1}\lim_{\epsilon\to 0} \frac{1}{1-q}\frac{\ln\left(I(q,\epsilon)\right)}{\ln(1/\epsilon)} \\
&= \lim_{q\to 1}\lim_{\epsilon\to 0} \frac{1}{1-q}\frac{(q-1)\sum_{k=1}^{N_{\text{box}}} p_k \ln p_k}{\ln(1/\epsilon)} \\
&= \lim_{\epsilon\to 0} \frac{\sum_{k=1}^{N_{\text{box}}} p_k \ln p_k}{\ln(\epsilon)}\,.
\end{aligned}
$$

d) Why is the limit $q \to 1$ specifically sensitive to normalization of probabilities?

Solution
If p_k is not normalized to unity, $\sum_{k=1}^{N_{\text{box}}} p_k = 1$, the lowest-order contribution in the expansion of $\ln I(q,\epsilon)$ is not proportional to $q-1$, meaning that D_1 becomes (wrongly) infinite.

e) Does the resolution you discussed in problem b) apply to the limit $q \to 1$?

Solution
Yes, it applies with some modifications. From problem c) we have $D_1 \ln(\epsilon) = \sum_{k=1}^{N_{\text{box}}} p_k \ln p_k = \ln(\exp(\sum_{k=1}^{N_{\text{box}}} p_k \ln p_k))$ for small values of ϵ. I.e. if we replace I_1 by $\exp(\sum_{k=1}^{N_{\text{box}}} p_k \ln p_k)$ the resolution in problem b) applies.