APPENDIX A

THE OPPOSITE- AND SAME-SPIN ELECTRON-ELECTRON INTERACTION

In this appendix we derive formal Fermi-golden-rule estimates for both the opposite- and same-spin intersubband scattering rates. The opposite-spin scattering $(\alpha_{i_1}, \alpha_{i_2}) \to (\alpha_{f_1}, \alpha_{f_2})$ is given alone by a so-called direct contribution. In contrast the same-spin scattering comprises both a direct and an exchange contribution.

For the opposite-spin scattering we derive the formal Fermi-golden-rule estimate evaluated in Chapter IV. In particular, we express the opposite-spin direct contribution as a q-sum over dimensionless Coulomb interaction matrix element $U_{\alpha_{i_1}\alpha_{f_1},\alpha_{i_2}\alpha_{f_2}}(q)$ weighted by the dimensionless measure, $P_{\alpha_{i_1}\alpha_{i_2}\to\alpha_{f_1}\alpha_{f_2}}(q)$, of the scattering phase-space at in-plane momentum transfer q.

For the same-spin scattering we demonstrate that the exchange and direct contribution will exactly cancel if the q-variation of the dimensionless matrix element $U_{\alpha_{i_1}\alpha_{f_1},\alpha_{i_2}\alpha_{f_2}}(q)$ can be ignored. This motivates our focus on the opposite-spin scattering in Chapter IV.

Section A.1 The intersubband Coulomb interaction

We describe the Coulomb interaction between an electron at $\vec{R}_1$ and one at $\vec{R}_2$ in terms of the Hamiltonian

$$H_{e-e}(\vec{R}_1, \vec{R}_2) = \left(\frac{e^2}{\epsilon_0}\right) \frac{\exp(-k_s\sqrt{(x_1 - x_2)^2 + (\vec{r}_1 - \vec{r}_2)^2})}{\sqrt{(x_1 - x_2)^2 + (\vec{r}_1 - \vec{r}_2)^2}}, \tag{1}$$

where $\vec{r}$ denoteds the position in the in-plane directions. The wave vector k_s in Eq. (1) represents a convergence factor and the unscreened interaction results in the limit $k_s \to 0$.

A scattering event in which two electrons — initially in the subband states $|\alpha_1, \vec{k}_1, s_1\rangle$ and $|\alpha_2, \vec{k}_2, s_2\rangle$ — scatters to states in the final subbands, α_3 and α_4, with an in-plane momentum transfer $\vec{q}$ is described by the matrix element

$$V^0_{\alpha_2\alpha_4,\alpha_1\alpha_3}(q) \equiv A\langle \alpha_4, \vec{k}_2 + \vec{q}, s_2; \alpha_3, \vec{k}_1 - \vec{q}, s_1|\hat{H}_{e-e}|\alpha_2, \vec{k}_2, s_2; \alpha_1, \vec{k}_1, s_1\rangle, \tag{2}$$

where A denotes the in-plane area. Following the discussion in Section IV.A we scale this matrix element by the characteristic intersubband momentum transfer $q_\Delta \equiv \sqrt{2m_e^*\Delta}$,

$$V^0_{\alpha_2\alpha_4,\alpha_1\alpha_3}(q) = \left(\frac{e^2}{\epsilon_0 q_\Delta}\right) U_{\alpha_4\alpha_2,\alpha_3\alpha_1}(q), \tag{3}$$

and thus introduce the dimensionless interaction matrix element

$$\begin{aligned} U_{\alpha_2\alpha_4,\alpha_1\alpha_3}(q) = 2\pi \int dx_1 \int dx_2 \Psi^*_{\alpha_4}(x_2)\Psi_{\alpha_2}(x_2) \times \\ \frac{q_\Delta \exp(-\sqrt{k_s^2 + q^2}|x_1 - x_2|)}{\sqrt{k_s^2 + q^2}} \Psi^*_{\alpha_3}(x_1)\Psi_{\alpha_1}(x_1). \end{aligned} \tag{4}$$

Again the unscreened interaction results in the limit $k_s \to 0$.

Section A.2 Direct and exchange scattering

In this section we discuss the rate of scattering $\gamma^{s_{i_1},s_{i_2}}_{\alpha_{i_1}\alpha_{i_2}\rightarrow\alpha_{f_1}\alpha_{f_2}}(\vec{k}_{i_1},\vec{k}_{i_2})$ out from a pair of *specified* initially occupied states, $|\alpha_{i_{1,2}},\vec{k}_{i_{1,2}},s_{i_{1,2}}\rangle$ and into available final states in subbands $\alpha_{f_{1,2}}$. The opposite-spin scattering results alone in a so-called direct term whereas the same-spin scattering comprises *both* a direct and an exchange contribution. We demonstrate, however, that the exchange and direct contributions will exactly cancel *in* the same-spin scattering if the momentum dependence of the intersubband Coulomb matrix element, Eq. (4), can be ignored.

The electron occupation at in-plane momentum $\vec{k}$ within subband α is described by the generally nonequilibrium distribution function $f^<_\alpha(\vec{k})$ which we assume independent of both the spin and of the direction of $\vec{k}$ (i.e., $f^<_\alpha(\vec{k}) = f^<_\alpha(k=|\vec{k}|)$.) To simplify the formal expressions below we also introduce the distribution $f^>_\alpha(\vec{k})$ of *available,* i.e., empty, subband-α states.

The Fermi-golden-rule estimate of $\gamma^{s_{i_1},s_{i_2}}_{\alpha_{i_1}\alpha_{i_2}\rightarrow\alpha_{f_1}\alpha_{f_2}}(\vec{k}_{i_1},\vec{k}_{i_2})$ is given directly by the squared transition matrix elements, of the Hamiltonian, Eq. (1), evaluated between the anti-symmetric pair-states which describe the initial and final electron configuration. Specifically, we describe the pair of initially occupied states using

$$|\alpha_{i_1},\vec{k}_{i_1},s_{i_1};\alpha_{i_2},\vec{k}_{i_2},s_{i_2}\rangle_A \equiv \frac{1}{\sqrt{2}}\Big\{|\alpha_{i_1},\vec{k}_{i_1},s_{i_1}\rangle|\alpha_{i_2},\vec{k}_{i_2},s_{i_2}\rangle - |\alpha_{i_2},\vec{k}_{i_2},s_{i_2}\rangle|\alpha_{i_1},\vec{k}_{i_1},s_{i_1}\rangle\Big\} \tag{5}$$

and use a similar pair state to describe the set of possible final pair-states within subbands $\alpha_{f_{1,2}}$. The intersubband scattering rate resulting from the Coulomb interaction, Eq. (1), is then given by

$$
\begin{aligned}
\gamma^{s_{i_1},s_{i_2}}_{(\alpha_{i_1}\alpha_{i_2})\to(\alpha_{f_1},\alpha_{f_2})}(\vec{k}_{i_1},\vec{k}_{i_2}) = \frac{1}{2} \sum_{\alpha_4,\vec{k}_4,s_4} f^{>}_{\alpha_4}(\vec{k}_4) \sum_{\alpha_3,\vec{k}_3,s_3} f^{>}_{\alpha_3}(\vec{k}_3)\times \\
\left[\delta_{\alpha_4,\alpha_{f_2}}\delta_{\alpha_3,\alpha_{f_1}} + \delta_{\alpha_4,\alpha_{f_1}}\delta_{\alpha_3,\alpha_{f_2}}\right] \times \\
|\langle \alpha_3,\vec{k}_3,s_3;\alpha_4,\vec{k}_4,s_4|H_{e-e}|\alpha_{i_1},\vec{k}_{i_1},s_{i_1};\alpha_{i_2},\vec{k}_{i_2},s_{i_2}\rangle|^2\times \\
2\pi\delta\Big([E_{\alpha_{i_1}} - E_{\alpha_3} + E_{||}(\vec{k}_{i_1}) - E_{||}(\vec{k}_3)] \\
+ [E_{\alpha_{i_2}} - E_{\alpha_4} + E_{||}(\vec{k}_{i_2}) - E_{||}(\vec{k}_4)]\Big),
\end{aligned}
\tag{6}
$$

where the projection (second line) restricts all possible final states to *available* states actually in subbands $\alpha_{f_{1,2}}$. The expression within the delta-function in Eq. (6) is just the energy difference between the final and initial pair states.

The interaction matrix elements in Eq. (6) is given by

$$
\begin{aligned}
\sum_{s_3,s_4} |\langle \alpha_4,\vec{k}_4,s_4;\alpha_3,\vec{k}_3,s_3|\hat{H}_{e-e}|\alpha_1,\vec{k}_1,s_1;\alpha_2,\vec{k}_2,s_2\rangle_A|^2 = \\
\frac{1}{A^2}\sum_{q_a,q_b} \delta_{\vec{q}_a,\vec{k}_4-\vec{k}_2}\delta_{\vec{q}_b,\vec{k}_3-\vec{k}_2}\delta_{\vec{q}_a+\vec{q}_b,\vec{k}_1-\vec{k}_2} \\
\left(\frac{e^2}{\epsilon_0 q_\Delta}\right)^2 \Big[|U_{\alpha_4\alpha_2,\alpha_3\alpha_1}(q_a)|^2 + |U_{\alpha_3\alpha_2,\alpha_4\alpha_1}(q_b)|^2 \\
-\delta_{s_1,s_2} 2\mathrm{Re}\{U_{\alpha_4\alpha_2,\alpha_3\alpha_1}(q_a)U_{\alpha_3\alpha_2,\alpha_4\alpha_1}(q_b)^*\}\Big],
\end{aligned}
\tag{7}
$$

where we have included the trivial sum over the final electron spin configuration. Note that the in-plane momentum is conserved in the scattering events since the specified condition, $\vec{q}_a + \vec{q}_b = \vec{k}_1 - \vec{k}_2$, along with the definitions, $\vec{q}_a \equiv \vec{k}_4 - \vec{k}_2$ and $\vec{q}_b \equiv \vec{k}_3 - \vec{k}_2$, is equivalent to the conservation rule $\vec{k}_3 + \vec{k}_4 = \vec{k}_1 + \vec{k}_2$.

If the two scattering electrons in the initial pair-state Eq. (5) has *opposite* spins the scattering is given alone by the so-called *direct* contribution. The formal expression for this opposite-spin direct scattering is

$$\begin{aligned}
\gamma^{\uparrow\downarrow}_{(\alpha_{i_1},\alpha_{i_2})\rightarrow(\alpha_{f_1}\alpha_{f_2})}(\vec{k}_{i_1},\vec{k}_{i_2}) &\equiv \gamma^{D}_{(\alpha_{i_1},\alpha_{i_2})\rightarrow(\alpha_{f_1},\alpha_{f_2})}(\vec{k}_{i_1},\vec{k}_{i_2}) = \\
&\left(\frac{e^2}{\epsilon_0 q_\Delta}\right)^2 \frac{1}{A^2}\sum_{\vec{q}} f^{>}_{\alpha_{f_1}}(\vec{k}_{i_1}-\vec{q}) f^{>}_{\alpha_{f_2}}(\vec{k}_{i_2}+\vec{q})\{2|U_{\alpha_{f_2}\alpha_{i_2},\alpha_{f_1}\alpha_{i_1}}(q)|^2\} \\
&2\pi\delta\Big([E_{\alpha_{i_1}} - E_{\alpha_{f_1}} + E_{||}(\vec{k}_{i_1}) - E_{||}(\vec{k}_{i_1}-\vec{q})] \\
&+ [E_{\alpha_{i_2}} - E_{\alpha_{f_2}} + E_{||}(\vec{k}_{i_2}) - E_{||}(\vec{k}_{i_2}+\vec{q})]\Big).
\end{aligned} \tag{8}$$

In contrast, if the spins of the scattering electrons are identical we obtain again the direct contribution $\gamma^{D}_{(\alpha_{i_1},\alpha_{i_2})\rightarrow(\alpha_{f_1},\alpha_{f_2})}(\vec{k}_{i_1},\vec{k}_{i_2})$ *and* a so-called exchange contribution corresponding to the last term in Eq. (7). That is, the same-spin scattering rate is given by

$$\begin{aligned}
\gamma^{\uparrow\uparrow}_{(\alpha_{i_1}\alpha_{i_2})\rightarrow(\alpha_{f_1},\alpha_{f_2})}(\vec{k}_{i_1},\vec{k}_{i_2}) &= \gamma^{D}_{(\alpha_{i_1},\alpha_{i_2})\rightarrow(\alpha_{f_1},\alpha_{f_2})}(\vec{k}_{i_1},\vec{k}_{i_2}) \\
&- \gamma^{Ex}_{(\alpha_{i_1},\alpha_{i_2})\rightarrow(\alpha_{f_1},\alpha_{f_2})}(\vec{k}_{i_1},\vec{k}_{i_2}),
\end{aligned} \tag{9}$$

with the *exchange* contribution defined as

$$\begin{aligned}
\gamma^{Ex}_{(\alpha_{i_1},\alpha_{i_2})\rightarrow(\alpha_{f_1}\alpha_{f_2})}(\vec{k}_{i_1},\vec{k}_{i_2}) &= \left(\frac{e^2}{\epsilon_0 q_\Delta}\right)^2 \frac{1}{A}\sum_{\vec{q}_a}\frac{1}{A}\sum_{\vec{q}_b} \\
&\delta_{\vec{q}_a+\vec{q}_b,\vec{k}_{i_1}-\vec{k}_{i_2}} f^{>}_{\alpha_{f_1}}(\vec{k}_{i_1}-\vec{q}_a) f^{>}_{\alpha_{f_2}}(\vec{k}_{i_2}+\vec{q}_a)\times \\
&2\mathrm{Re}\{U_{\alpha_{f_2}\alpha_{i_2},\alpha_{f_1}\alpha_{i_1}}(q_a) U_{\alpha_{f_1}\alpha_{i_2},\alpha_{f_2}\alpha_{i_1}}(q_b)^*\}\times \\
&2\pi\delta\Big([E_{\alpha_{i_1}} - E_{\alpha_{f_1}} + E_{||}(\vec{k}_{i_1}) - E_{||}(\vec{k}_{i_1}-\vec{q}_a)] \\
&+[E_{\alpha_{i_2}} - E_{\alpha_{f_2}} + E_{||}(\vec{k}_{i_2}) - E_{||}(\vec{k}_{i_2}+\vec{q}_a)]\Big).
\end{aligned} \tag{10}$$

There are three intersubband scattering processes with a net decay to the lower subband, namely (a) the near-equilibrium scattering $(2,1) \longrightarrow (1,1)$ of a single upper-subband electron by an equilibrium lower-subband Fermi sea, (b) the nonequilibrium scattering $(2,2) \longrightarrow (1,1)$ in which two upper-subband electrons both decays to the lower subband, and (c) the nonequilibrium scattering

$(2,2) \longrightarrow (2,1)$ in which two upper-subband electrons scatters but only one decays to subband E_1 whereas the other electron scatters within subband E_2. In all three cases we have

$$U_{\alpha_{f_2}\alpha_{i_2},\alpha_{f_1}\alpha_{i_1}}(q) = U_{\alpha_{f_1}\alpha_{i_2},\alpha_{f_2}\alpha_{i_1}}(q), \tag{11}$$

and there will be no same-spin scattering if the interaction matrix element be approximated as constant: The exchange contribution, Eq. (10), then cancels exactly the direct contribution, Eq. (9).

Provided still that the q-variation of the Coulomb matrix elements can be ignored, the exact cancellations holds of course also for the *total* same-spin scattering, given by summing the results, Eqs. (9) and (10), over all initially occupied pair-states (in subbands $\alpha_{i_{1,2}}$.)

Of course, the actual intersubband Coulomb matrix elements exhibits some variation, and for example the unscreened matrix element $U^0_{22,21}(q)$ will diverge in the limit $q \to 0$. *However,* within the domain of possible in-plane momentum transfers q we find in Chapter IV only a very moderate q-variation of the Coulomb matrix elements. For that reason we expect a significant cancellation in the same-spin scattering and have, as a consequence, in this dissertation only evaluated the (dominant) opposite-spin direct scattering.

A.3 The total opposite-spin scattering

The *total* rate of opposite-spin scattering $(\alpha_{i_1}, \alpha_{i_2}) \to (\alpha_{f_1}, \alpha_{f_2})$ is given by summing the direct contribution, Eq. (8) over all initially occupied (opposite-spin) pair-states in subbands $\alpha_{i_{1,2}}$:

$$\gamma^{D}_{\alpha_{i_1}\alpha_{i_2}\to\alpha_{f_1}\alpha_{f_2}} \equiv \frac{1}{2}\sum_{\alpha_1,\vec{k}_1,s_1}\sum_{\alpha_2,\vec{k}_2,s_2\neq s_1}\left[\delta_{\alpha_{i_1}\alpha_1}\delta_{\alpha_{i_2}\alpha_2}+\delta_{\alpha_{i_1}\alpha_2}\delta_{\alpha_{i_2}\alpha_1}\right]\times \\ f^{<}_{\alpha_1}(\vec{k}_1)f^{<}_{\alpha_2}(\vec{k}_2)\gamma^{s_1,s_2}_{(\alpha_1,\alpha 2)\to(\alpha_{f_1},\alpha_{f_2})}(\vec{k}_1,\vec{k}_2). \tag{12}$$

In terms of the dimensionless Coulomb matrix element, Eq. (3), the formal result Eq. (12) reads

$$\gamma^{D}_{\alpha_{i_1}\alpha i_2\to\alpha_{f_1}\alpha_{f_2}} = \left(\frac{e^2}{\epsilon_0 q_\Delta}\right)^2\sum_{\vec{q}}\left|U_{\alpha_{f_2}\alpha_{i_2},\alpha_{f_1}\alpha_{i_1}}(q)\right|^2 F^{D}_{(\alpha_{i_1},\alpha_{i_2})\to(\alpha_{f_1},\alpha_{f_2})}(q), \tag{13}$$

where we have introduced the phase-space contribution

$$F^{D}_{\alpha_{i_1}\alpha_{i_2}\to\alpha_{f_1},\alpha_{f_2}}(q) = \frac{1}{2}\left(\frac{2}{A}\sum_{\vec{k}_1} f^{<}_{\alpha_{i_1}}(\vec{k}_1)f^{>}_{\alpha_{f_1}}(\vec{k}_1-\vec{q})\right)\times \\ \left(\frac{2}{A}\sum_{\vec{k}_2} f^{<}_{\alpha_{i_2}}(\vec{k}_2)f^{>}_{\alpha_{f_4}}(\vec{k}_2+\vec{q})\right)\times \\ 2\pi\delta\Big([E_{\alpha_{i_1}}-E_{\alpha_{f_1}}+E_{||}(\vec{k}_1)-E_{||}(\vec{k}_1-\vec{q})] \\ +[E_{\alpha_{i_2}}-E_{\alpha_{f_2}}+E_{||}(\vec{k}_2)-E_{||}(\vec{k}_2+\vec{q})]\Big). \tag{14}$$

Both the squared transition matrix elements and the phase-space factor, Eq. (14), depends as indicated only on the length, $q\equiv|\vec{q}|$ of the in-plane momentum transfer.

We emphasize that the formal result, Eq. (13), constitute an estimate for the *total* opposite-spin scattering within the in-plane area A (containing a total of $An_{2/1}$ upper/lower-subband electrons.) In contrast the rates Γ and $\Gamma_{22\to21}$ evaluated in Chapter IV represent the total scattering rate *per* upper-subband electron, An_2.

Specifically, in estimating the rates, Γ and $\Gamma_{22\to21}$, we assume a complete upper-subband occupation below emitter chemical potential μ_L, which then naturally defines an upper-subband quasi-Fermi level

$$\mu_2 \equiv \mu_L - E_2, \tag{15}$$

and a corresponding wave vector

$$k_{\mu_2} \equiv \sqrt{2m_e^* \mu_2}. \tag{16}$$

The total upper-subband occupation density is

$$n_2 = N_L = \frac{k_{\mu_2}^2}{2\pi} \propto \mu_2. \tag{17}$$

We furthermore assume that the lower subband is empty and obtain a formal estimate for scattering rate $\Gamma \equiv \Gamma_{22\to 11}$ defined through

$$\{AN_L\}\Gamma = \sum_{\vec{q}} \left(\frac{e^2}{\epsilon_0 q_\Delta}\right) |U_{21,21}(q)|^2 F_{22\to 11}^D(q), \tag{18}$$

where the phase-space contribution, Eq. (14), reduces to

$$\begin{aligned} F_{22\to 11}^D(q) &= N_L^2 \mu_2^2 P(q) \\ &\equiv \tfrac{1}{2}\left(\frac{2}{A}\right)^2 \sum_{k,k'} \Theta(\mu_2 - E_{||}(k))\Theta(\mu_2 - E_{||}(k'))2\pi\delta(E_f - E_i). \end{aligned} \tag{19}$$

The delta-function argument in Eq. (19) is specified by the conservation of energy,

$$E_f - E_i = 2E_{||}(q) + \vec{q}\cdot(\vec{k} - \vec{k}')/m_e^* - 2\Delta. \tag{20}$$

Similarly, for the scattering rate $\Gamma_{22\to 21}$ we have

$$\{AN_L\}\Gamma_{22\to 21} = \sum_{\vec{q}} \left(\frac{e^2}{\epsilon_0 q_\Delta}\right) |U_{22,21}(q)|^2 F_{22\to 21}^D(q), \tag{21}$$

given by the phase-space contribution,

$$\begin{aligned} F_{22\to 21}^D(q) &= N_L^2 \mu_2^2 P_{22\to 21}(q) \\ &\equiv \tfrac{1}{2}\left(\frac{2}{A}\right)^2 \sum_{k,k'} \Theta(\mu_2 - E_{||}(k))\times \\ &\quad \left[\Theta(\mu_2 - E_{||}(k'))\Theta(E_{||}(\vec{k}' - \vec{q}) - \mu_2)\right] 2\pi\delta(E_f - E_i), \end{aligned} \tag{22}$$

for which the criteria of energy conservation is expressed in

$$E_f - E_i = 2E_{||}(q) + \vec{q} \cdot (\vec{k} - \vec{k}')/m_e^* - \Delta. \tag{23}$$