Elliptic polarization

Adding two linearly polarized waves that are polarized in orthogonal directions (e.g. the x- and y-direction) in general gives rise to an elliptically polarized wave. In the following we wish to show that by adding the two linearly polarized waves given by the electric fields

$$\mathbf{E}_x = E_x \mathbf{e}_x = \mathbf{e}_x E_{x0} \cos(kz - \omega t) \tag{1}$$
$$\mathbf{E}_y = E_y \mathbf{e}_y = \mathbf{e}_y E_{y0} \cos(kz - \omega t + \varepsilon) \tag{2}$$

the electric field vector $\mathbf{E} = \mathbf{E}_x + \mathbf{E}_y$ of the resulting wave traces out an ellipse in the x-y plane.

First, as in Hecht Chapter 8, rewrite the above two equations

$$\begin{aligned} \frac{E_y}{E_{y0}} &= \cos(kz - \omega t + \varepsilon) \\ &= \cos(kz - \omega t)\cos\varepsilon - \sin(kz - \omega t)\sin\varepsilon \\ &= \frac{E_x}{E_{x0}}\cos\varepsilon - \sqrt{1 - \left(\frac{E_x}{E_{x0}}\right)^2}\sin\varepsilon\,, \end{aligned} \tag{3}$$

rearrange to get rid of the square root

$$\left(\frac{E_y}{E_{y0}} - \frac{E_x}{E_{x0}}\cos\varepsilon\right)^2 = \left(1 - \left(\frac{E_x}{E_{x0}}\right)^2\right)\sin^2\varepsilon \tag{4}$$

and use $\cos^2\varepsilon + \sin^2\varepsilon = 1$ to obtain equation (8.14) in Hecht

$$\left(\frac{E_x}{E_{x0}}\right)^2 + \left(\frac{E_y}{E_{y0}}\right)^2 - 2\frac{E_x E_y \cos\varepsilon}{E_{x0} E_{y0}} = \sin^2\varepsilon\,. \tag{5}$$

Hecht now claims that this is the equation for an ellipse turned by an angle α with respect to the E_x-axis in an E_x-E_y coordinate system where α is given by

$$\tan(2\alpha) = \frac{2E_{x0}E_{y0}\cos\varepsilon}{E_{x0}^2 - E_{y0}^2}\,. \tag{6}$$

Let us now see why this is true.

The formula for an ellipse with one of its semi-axes, say the a semi-axis, along the principal axis of the coordinate system is

$$\frac{\eta^2}{a^2} + \frac{\xi^2}{b^2} = 1 \tag{7}$$

if the axes of the coordinate system are denoted by η and ξ (see Fig. 1). The curve might be described by a parameterization using the parameter (angle) t

$$\eta(t) = a\cos t \qquad\qquad \xi(t) = b\sin t \tag{8}$$

Now imagine a coordinate system in which the ellipse is turned an angle α, the x-y-coordinate system. The two coordinate systems are connected via the transformations (rotations)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} \tag{9}$$

or

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{10}$$

Thus the equation for the ellipse in the x-y-coordinate system is

$$x^2(b^2\cos^2\alpha + a^2\sin^2\alpha) + y^2(a^2\cos^2\alpha + b^2\sin^2\alpha) - 2xy\cos\alpha\sin\alpha(a^2 - b^2) = a^2b^2 \tag{11}$$

The maxima of $x(t)$ and $y(t)$, named x_0 and y_0 in Figure 1, are found by differentiating $x(t) = a\cos\alpha\cos t - b\sin\alpha\sin t$ and $y(t) = a\sin\alpha\cos t + b\cos\alpha\sin t$ with respect to the parameter t

$$\left.\frac{\partial x}{\partial t}\right|_{t=t_{x0}} = -a\cos\alpha\sin t_{x0} - b\sin\alpha\cos t_{x0} = 0 \tag{12}$$

from which we find

$$\tan t_{x0} = -\frac{b}{a}\tan\alpha\,. \tag{13}$$

This value t_{x0} maximizes $x(t)$; $x(t_{x0}) = x_0$. By further noting that

$$\tan t_{x0} = \frac{\sqrt{1-\cos^2 t_{x0}}}{\cos t_{x0}} = \frac{\sin t_{x0}}{\sqrt{1-\sin^2 t_{x0}}} \tag{14}$$

we can write equation (13) as

$$\cos t_{x0} = \frac{a\cos\alpha}{\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}} \quad \text{and} \quad \sin t_{x0} = \frac{-b\sin\alpha}{\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}} \tag{15}$$

(notice that we must choose $\sin t_{x0} = \xi_{x0}/b$ negative, according to Figure 1). Thus we have

$$x_0 = a\cos\alpha\cos t_{x0} - b\sin\alpha\sin t_{x0} = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}\,. \tag{16}$$

With a similar derivation for $y(t)$ we find

$$y_0 = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha} \tag{17}$$

and thus

$$x_0^2 - y_0^2 = (a^2 - b^2)(\cos^2\alpha - \sin^2\alpha) = (a^2 - b^2)\cos(2\alpha) \tag{18}$$

and

$$\begin{aligned} x_0^2 y_0^2 &= (a^2\cos^2\alpha + b^2\sin^2\alpha)(a^2\sin^2\alpha + b^2\cos^2\alpha) \\ &= (b^4 + a^4)\sin^2\alpha\cos^2\alpha + a^2b^2(1 - 2\sin^2\alpha\cos^2\alpha) \\ &= a^2b^2 + (a^2 - b^2)^2\frac{\sin^2(2\alpha)}{2} \end{aligned} \tag{19}$$

Inserting equations (16,17,18,19) into the equation for the rotated ellipse, equation (11), we find

$$x^2y_0^2 + y^2x_0^2 - 2xy\frac{\sin(2\alpha)}{2}\frac{x_0^2 - y_0^2}{\cos(2\alpha)} = x_0^2y_0^2 - \frac{(x_0^2 - y_0^2)^2}{\cos^2(2\alpha)}\frac{\sin^2(2\alpha)}{2}\,. \tag{20}$$

If we now, by some lucky inspiration, decide to define a quantity ε by

$$\cos\varepsilon = \frac{\tan(2\alpha)\,(x_0^2 - y_0^2)}{2x_0y_0} \tag{21}$$

we find

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{y}{y_0}\right)^2 - 2\frac{xy}{x_0y_0}\cos\varepsilon = 1 - \cos^2\varepsilon = \sin^2\varepsilon \tag{22}$$

which is the equation (5) for the x and y components of the electric field. We also see from (21) that the rotation angle is given by

$$\tan(2\alpha) = \frac{2x_0y_0\cos\varepsilon}{x_0^2 - y_0^2}\,. \tag{23}$$

Hence, adding two linearly polarized waves does in general give rise to an elliptically polarized wave.

Elsebeth Schröder, Chalmers, February 2001.

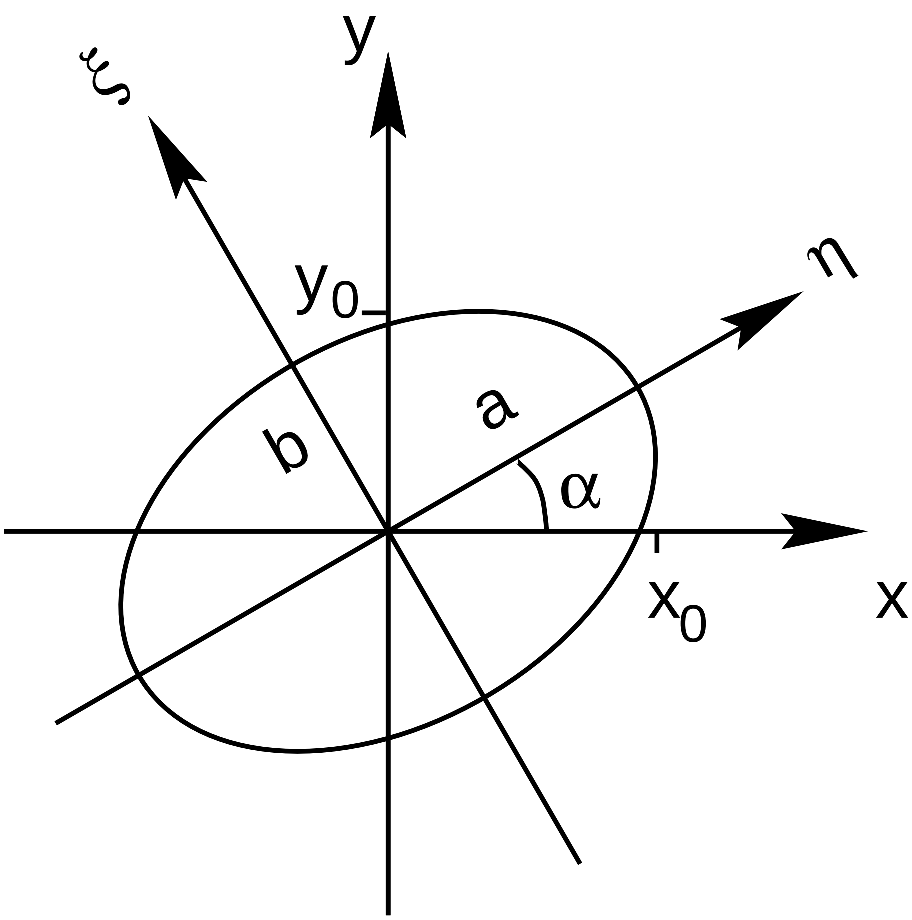

Figure 1: The rotated ellipse.