Aspects of Non-perturbative String Theory

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April 19, 1993

Preface

In my thesis work I have mainly considered two-dimensional gravity and string theory, treated with matrix models and with two-dimensional topological quantum field theory. The main emphasis has been laid on the case of pure gravity, where calculations are less involved, though still voluminous.

When the string moves through the embedding space it spans out a two-dimensional surface, the string world sheet, and the embedding space induces a metric on the world sheet. The "conventional" approach to string theory is to use a path integral/functional integral formulation whereby a *dynamical* metric, independent of the induced metric, is introduced on the world sheet. In order to calculate correlators (scattering amplitudes) of the theory the dependence on the dynamical metric is integrated over in a path integral. To do this integration is, at best, a difficult task, and with this method only correlators for low genus (number of handles) have been found. The aim of string theory is to at least find the results for any genus world sheet, from which a description perturbative in genus number can be found, or even to find a non-perturbative description from which the true vacuum expectation value, corresponding to "our" world, might be found. This, however, is a task for the future.

There are (at least) two other ways to consider string theory in order to avoid the troublesome functional integral mentioned for the "conventional" approach. They have different starting points, but can eventually be shown to be equivalent, at least under certain conditions. These two methods and illuminating the equivalence between them is what my work was concentrated on.

One method, the method of matrix models, is based on a discrete approximation of the string world sheet by a dynamical triangulation — the world sheet is approximated by a net of equilateral (equal side lengths) triangles. The partition function (vacuum expectation value) for the dynamical triangulation is described using $N \times N$-matrices. In the continuum limit (i.e. infinitely many triangles in the triangulation), with the additional requirement that N grows in a controlled way — "the double scaling limit" — a non-linear differential equation, the string equation, is derived. The differential equation turns out to be part of the Korteweg–deVries (KdV) hierarchy with boundary conditions. A different way to express the same is to say that certain constraints, so-called Virasoro constraints, on the partition function are found in the double scaling limit. The τ-functions of the KdV-hierarchy satisfy all but one of the Virasoro constraints; since the τ-functions are non-perturbative quantities (can not be treated genus-by-genus) this illustrates that the matrix model approach is indeed non-perturbative.

The other method makes use of two-dimensional topological quantum field theory. The correlators in topological quantum field theory are constructed in such a way that they have no dependence on the above-mentioned dynamical metric of the string world sheet, hence there is no functional integral of the metric. The matter part of this theory (i.e. before coupling to gravity) exhibits a BRST-like symmetry through the symmetry operator Q. When the matter part of the topological string theory is coupled to gravity this Q-symmetry is preserved, and in the case of pure gravity (no matter sector, but the Q-symmetry still remains) one finds a recursion relation for the correlators. The recursion relation can be rewritten as constraints on the partition function, namely the same Virasoro constraints as in the matrix model approach, thereby illuminating the equivalence of the two methods, at least at the pure gravity level.

The structure of the thesis is as follows: First a short introduction to string theory and conformal field theory is given in Chapter 1, kept down to the minimum required by the following chapters. In Chapter 2 the matrix model approach is presented, along with the Virasoro constraints and a short introduction to the KdV hierarchy, its generalization the Kadomtsev–Petviashvili (KP) hierarchy and the tau-functions. As indicated above, these hierarchies and constraints are the sources of the equivalence between the two models.

Finally, topological quantum field theory is treated in Chapter 3. The matter part of topological string theory, where the Q-symmetry is introduced, is the main theme of this chapter. Though in the end I want to consider only the pure gravity case, the Q-symmetry found in the matter sector is important also for pure gravity. At the end of the chapter it is shown how to obtain the correlator recursion relations, mentioned above, from the Virasoro constraints.

Contents

Chapter 1

Introduction

In this chapter I will, in Section 1.1, give a (very) short introduction to string theory, and in Section 1.2 results from two dimensional conformal field theory, needed in the subsequent chapters, are presented.

1.1 String Theory

String theory is an attempt to unify the two physical theories *quantum theory* and *the general theory of relativity.* Quantum theory is a theory of matter and energy, in which three of the four fundamental forces can be unified, namely the electromagnetic, the weak and the strong force. The gravitational force can not be included in quantum theory without renormalizing. The general theory of relativity describes the gravitational force, and string theory is an attempt to unite the two theories, thereby unifying all four fundamental forces, including an infinite number of not yet observed forces.

A string can be thought of as a "rubber band" moving around in a space-time background, described by a differentiable manifold. As it moves around, the string traces out its *world sheet,* a two-dimensional manifold (Figure 1.1), the same manner in which a particle traces out a one-dimensional *world line* in particle physics. String theory is concerned with the description of the world sheet of the string.

The bosonic string is the simplest type of string, and the one I will consider here. The most simple generally covariant action is in flat (Minkowskian) space-time

$$S_{\mathrm{M}} = -\frac{1}{4\pi}\int d^2\xi\sqrt{g}(g^{ab}(z,\bar{z})\eta_{\mu\nu}\partial_a X^\mu(z,\bar{z})\partial_b X^\nu(z,\bar{z}) + 4\pi\beta - \tilde{\gamma}R(z,\bar{z})) \quad (1.1)$$

with suitable normalization (the string tension can be absorbed in the fields X^μ); here $(\xi^1,\xi^2) = (\sigma,\tau)$ are coordinates on the world sheet, $(z,\bar{z}) = (e^{\tau+i\sigma}, e^{\tau-i\sigma})$, and g^{ab}, $(a,b = 1,2)$ is the metric on the world sheet. g_{ab} is *NOT* the induced metric $h_{ab}(z,\bar{z}) = \eta_{\mu\nu}\partial_a X^\mu(z,\bar{z})\partial_b X^\nu(z,\bar{z})$, the metric is totally independent of

Figure 1.1: The string world sheet in a space-time background. World sheet coordinates are (σ, τ) or $(z, \bar{z}) = (e^{\tau+i\sigma}, e^{\tau-i\sigma})$, and the fields $X^\mu(z,\bar{z})$, $\mu = 0, 1, \ldots, D-1$ are vector-fields in the D-dimensional manifold in which the string is embedded, with endpoints on the strings given by $(z, \bar{z})$. Here I assumed Euclidean space-time.

the embedding space-time, and will exist even if the background space does not exist (that is, has zero dimension). In the above action R is the Ricci scalar, and $\tilde{\gamma}$ and β are constants, the latter can be identified with the (bare 2-dimensional) cosmological constant.

A generalization of the above would be to replace the flat background metric $\eta_{\mu\nu}$ by a general, curved metric $G(X)_{\mu\nu}$ — different choices of background metrics $G(X)_{\mu\nu}$ lead to different conformal field theories (or possibly non-conformal theories) for the matter sector of the string theory. Here, I will only use the flat background metric, and after Wick-rotating into Euclidean space $\tau_{\mathrm{M}} \to -i\tau_{\mathrm{E}}$, $X^0_{\mathrm{M}} \to -iX^0_{\mathrm{E}}$, $S_{\mathrm{M}} \to iS_{\mathrm{E}}$, the action becomes

$$S_{\mathrm{E}} = \frac{1}{4\pi} \int d^2z \sqrt{g} (g^{ab}(z,\bar{z}) \partial_a X^\mu(z,\bar{z}) \partial_b X^\mu(z,\bar{z}) + 4\pi\beta - \tilde{\gamma} R(z,\bar{z})) \tag{1.2}$$

For a fixed topology, the right-most term in (1.2) is a constant — it depends only on the topology — and it can (and often will) be discarded. However, the string can be described using a perturbative expansion (an $\hbar$-expansion, which is in string theory an expansion in genus) Figure 1.2, therefore the topology can not be considered fixed and the right-most term in (1.2) has to be included. In other words, if one wants a description where a finite number of terms only in the perturbation series is not sufficient, the last term in the action (1.2) must be included.

Hence to obtain expectation values, specifically the vacuum expectation value or partition function, one could wish to sum over all such two-dimensional world sheet metrics g^{ab}, as suggested by Polyakov [3], and the partition function takes

Figure 1.2: Perturbative series for the string

the form

$$Z = \sum_{\text{topologies}} \int_{\substack{\text{world sheet} \\ \text{metrics}}} \mathcal{D}g \int_{\text{embeddings}} \mathcal{D}X e^{-S_{\mathrm{E}}} \tag{1.3}$$

where it is assumed that only connected surfaces contribute.

To evaluate the functional integral over the geometries is at best a difficult task. It leads, after much work, to quantum Liouville theory, this is the "conventional" approach to string theory. I will concentrate on two other approaches, the matrix models and the two-dimensional topological field theory.

The aim of string theory is, at least, to find expectation values (in particular the *vacuum* expectation value, the partition function) for world sheets of any genus, from which a perturbative (in genus number) description can be found. Perturbative string theory is found using *conformal* field theory on the Riemann surfaces making up the world sheets for each genus, adding up genus-by-genus. However, for a non-perturbative description, non-conformal field theories must be used. In the topological field theory approach we shall see how, by perturbing the theory away from a topological *conformal* field theory to generally non-conformal topological field theories, a description is found in which *all* terms in a genus expansion are needed (namely, a relation for the correlation functions recursive in genus number is found). One can not be sure that *all* non-conformal topological theories can be reached by perturbing away from the topological conformal field theories the way described in Chapter 3, and it would be desirable to find a more general result. The partition function (or rather, the square of the partition function) turns out to be closely connected to the τ-function, which is indeed a non-perturbative quantity, thus giving the desired non-perturbative description, at least for pure topological gravity.

In the matrix model approach the starting point is a sum over all Riemann surfaces of all genera, thus per definition the matrix approach is non-perturbative.

1.2 Two-dimensional Conformal Field Theory

The following is a short review of two-dimensional conformal field theory, with emphasis on the results needed in other parts of the thesis. For a more careful introduction and further results, see for example [4, 6].

Weyl transformations are transformations that leave the metric invariant up to a change of scale

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \lambda(x) g_{\mu\nu}(x) \tag{1.4}$$

with $\mu, \nu = 0, 1, \ldots, D-1$, $x = (x^0, \ldots, x^{D-1})$, where D is the dimension of the space, or in two dimensions

$$g_{ab}(z, \bar{z}) \to g'_{ab}(z', \bar{z}') = \lambda(z, \bar{z}) g_{ab}(z, \bar{z}) \tag{1.5}$$

with the two coordinates $z \equiv z_1 + iz_2$, $\bar{z} \equiv z_1 - iz_2$ and the two-dimensional metric

$$g_a{}^b = \left(\begin{smallmatrix} g_z{}^z & g_{\bar{z}}{}^z \\ g_z{}^{\bar{z}} & g_{\bar{z}}{}^{\bar{z}} \end{smallmatrix}\right), \quad g^{zz} = g^{\bar{z}\bar{z}} = 0, \quad g^{z\bar{z}} = g^{\bar{z}z} = \frac{1}{\sqrt{g}}.$$

Conformal transformations, a special case of Weyl transformations, preserve angles, and in two dimensions conformal transformations are equivalent to analytical (holomorphic) and anti-analytical transformations

$$z \to w(z) \quad , \qquad \bar{z} \to \bar{w}(\bar{z}) \tag{1.6}$$

i.e. transformations that transform z and $\bar{z}$ separately. A (two-dimensional) primary conformal field $\phi^{(h,\bar{h})}(z, \bar{z})$ is a field that transforms as follows under the conformal transformation $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$

$$\phi^{(h,\bar{h})}_{\{z\}}(z, \bar{z}) = \left(\frac{dw(z)}{dz}\right)^h \left(\frac{d\bar{w}(\bar{z})}{d\bar{z}}\right)^{\bar{h}} \phi^{(h,\bar{h})}_{\{w\}}(w(z), \bar{w}(\bar{z})), \tag{1.7}$$

subscripts $\{z\}$ and $\{w\}$ denote which coordinate system is considered. The field $\phi^{(h,\bar{h})}(z, \bar{z})$ is said to have *conformal weight* (or *conformal dimension*) h with respect to z and conformal weight $\bar{h}$ with respect to $\bar{z}$. The bar on $\bar{h}$ does not mean that it is the complex conjugate of h, the bar is only meant to remind us that $\bar{h}$ is associated with $\bar{z}$. Both h and $\bar{h}$ are (in a subset of the) real numbers. Some references (e.g. [6]) define the *conformal weight* or *scaling dimension* of $\phi^{(h,\bar{h})}(z, \bar{z})$ to be $h + \bar{h}$, and the *conformal spin* or the *response to rotation* to be $h - \bar{h}$.

Via the analytical and anti-analytical transformations (1.6), the behaviour of the conformal fields under two-dimensional conformal transformations splits into two similar transformations of each of the two variables z and $\bar{z}$, thus most of the time it will be enough to consider the behaviour of the fields under the transformation of z. The z-part of $\phi^{(h,\bar{h})}(z, \bar{z})$ is called the *leftmoving* part, and the $\bar{z}$-part is the right-moving part (this is a matter of convention).

Restricting the discussion to the left-moving part mostly, an infinitesimal transformation $w(z) = z + \epsilon(z)$ results in

$$\begin{aligned}
\phi^{(h)}_{\{z\}}(z) &= \left(\frac{dw(z)}{dz}\right)^h \phi^{(h)}_{\{w\}}(w(z)) \\
&= (1 + \partial_z \epsilon(z))^h \left(\phi^{(h)}_{\{w\}}(z) + (w - z)\partial_w \phi^{(h)}_{\{w\}}(z) + \ldots\right) \\
&= (1 + h\partial_z \epsilon(z)) \left(\phi^{(h)}_{\{w\}}(z) + \epsilon(z)\partial_z \phi^{(h)}_{\{w\}}(z)\right) + \mathcal{O}(\epsilon^2) \\
&= \phi^{(h)}_{\{w\}}(z) + h\partial_z \epsilon(z)\phi^{(h)}_{\{w\}}(z) + \epsilon(z)\partial_z \phi^{(h)}_{\{w\}}(z) + \mathcal{O}(\epsilon^2) \qquad (1.8)
\end{aligned}$$

assuming $\partial_z \epsilon(z) \sim \mathcal{O}(\epsilon)$. Re-writing (1.8) and keeping only terms up to first order in ϵ:

$$\delta_\epsilon \phi^{(h)}(z) = h\partial_z \epsilon(z)\phi^{(h)}(z) + \epsilon(z)\partial_z \phi^{(h)}(z) \qquad (1.9)$$

A conformal field theory is a field theory invariant under conformal transformations, a scale-invariant theory. In a conformal field theory the trace of the symmetric energy-momentum tensor T_{ab} vanishes

$$\begin{aligned}
0 &= \frac{\delta S}{\delta \lambda} = \sum_{a,b} \frac{\delta g_{ab}}{\delta \lambda}\frac{\delta S}{\delta g_{ab}} = \sum_{a,b} g_{ab} T^{ab} \frac{1}{\sqrt{g}} \\
&= \frac{1}{\sqrt{g}} \sum_a T_a{}^a = \frac{1}{\sqrt{g}} (T_z{}^z + T_{\bar{z}}{}^{\bar{z}}) \qquad (1.10)
\end{aligned}$$

and also

$$\begin{aligned}
T_z{}^z &= T_{zz} g^{zz} + T_{z\bar{z}} g^{\bar{z}z} = 0 + \frac{1}{\sqrt{g}} T_{z\bar{z}} \\
T_{\bar{z}}{}^{\bar{z}} &= T_{\bar{z}z} g^{z\bar{z}} + T_{\bar{z}\bar{z}} g^{\bar{z}\bar{z}} = \frac{1}{\sqrt{g}} T_{\bar{z}z} + 0
\end{aligned}$$

hence $T_{z\bar{z}} = -T_{\bar{z}z}$. The symmetry of the energy-momentum tensor gives $T_{z\bar{z}} = T_{\bar{z}z} = 0$. Also

$$\begin{aligned}
T_{\bar{z}}{}^z &= \frac{1}{\sqrt{g}} T_{\bar{z}\bar{z}} \\
T_z{}^{\bar{z}} &= \frac{1}{\sqrt{g}} T_{zz} \ ,
\end{aligned}$$

and $\partial_z T_{\bar{z}\bar{z}} = \partial_{\bar{z}} T_{zz} = 0$; define $T(z) \equiv T_{zz}$ and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$. The operator product expansion (OPE) of a conformal field $\phi^{(h)}(z)$ with $T(z)$ is

$$T(z')\phi^{(h)}(z) = \frac{1}{z' - z}\partial_z \phi^{(h)}(z) + \frac{h}{(z' - z)^2}\phi^{(h)}(z) + \text{regular terms} \qquad (1.11)$$

and the OPE of $T(z)$ with itself is

$$T(z')T(z) = \frac{1}{z'-z}\partial_z T(z) + \frac{2}{(z'-z)^2}T(z) + \frac{c/2}{(z'-z)^4} + \text{regular terms} \quad (1.12)$$

where c is the *central charge*. $T(z)$ is not a primary field (it has the extra fourth order term in (1.12)), but we will still say that $T(z)$ has conformal dimension $(h,\bar{h}) = (2,0)$. The OPE for $\bar{T}(\bar{z})$ is similar, with the *same* central charge c.

The Laurent expansion of a field $\phi^{(h)}(z)$ may often be expressed as

$$\phi^{(h)}(z) = \sum_{n\in\mathsf{Z}} \phi_n z^{-n-h} \qquad \phi_n = \oint_0 \frac{dz}{2\pi i}\phi^{(h)}(z) z^{n+h-1} \ , \quad (1.13)$$

the mode-expansion usually has $n+h \in \mathsf{Z}$, thus if the field has conformal dimension $h = \frac{1}{2}$ then the sum is often over half-integral values of n. In particular, the mode-expansion of the energy-momentum tensor is

$$T(z) = \sum_{n\in\mathsf{Z}} L_n z^{-n-2} \qquad L_n = \oint_0 \frac{dz}{2\pi i} T(z) z^{n+1} \ . \quad (1.14)$$

Write a small variation with $\epsilon(z)$, from [4]

$$\delta_\epsilon \phi^{(h)}(z) = [L_\epsilon, \phi^{(h)}(z)] = \oint_z \frac{dw}{2\pi i}\epsilon(w)T(w)\phi^{(h)}(z) \quad (1.15)$$

from which the algebra

$$[L_{\epsilon_1}, L_{\epsilon_2}] = L_{\epsilon_2\epsilon_1' - \epsilon_1\epsilon_2'} + \frac{c}{12}\oint_0 \frac{dz}{2\pi i}\epsilon_2(z)\epsilon_1'''(z) \quad (1.16)$$

is obtained. Using (1.16), the modes (1.14) with the OPE (1.12), and $L_\epsilon = L_n$ with $\epsilon_n = z^{n+1}$, give the *Virasoro algebra*

$$\begin{aligned} [L_m, L_n] &= L_{\epsilon_n\epsilon_m' - \epsilon_m\epsilon_m'} + \frac{c}{12}\oint_0 \frac{dz}{2\pi i}\epsilon_n(z)\epsilon_m'''(z) \\ &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\oint_0 \frac{dz}{2\pi i} z^{n+m-1} \\ &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \end{aligned} \quad (1.17)$$

from

$$\epsilon_n\epsilon_m' - \epsilon_m\epsilon_m' = (m+1)z^{n+1+m} - (n+1)z^{m+1+n} = (m-n)z^{m+n+1} \quad (1.18)$$

and

$$\epsilon_n\epsilon_m''' = (m+1)m(m-1)z^{n+1+m-2} = m(m^2-1)z^{n+m-1} \ . \quad (1.19)$$

We will also need to know that for a field with conformal dimension h

$$[L_m, \phi_n] = (m(h-1) - n)\phi_{n+m} \tag{1.20}$$

which can be found by similar calculations, using (1.15) and the OPE (1.11).

Define a state by $|\phi^{(h)}\rangle \equiv \lim_{z\to 0} \phi^{(h)}(z)|0\rangle$, where $|0\rangle$ is the vacuum of the theory,

$$L_{-n}|0\rangle = 0 \qquad \text{for} \quad n > 0 \tag{1.21}$$

then

$$L_0|\phi^{(h)}\rangle = h|\phi^{(h)}\rangle \tag{1.22}$$

from the OPE (1.11), where L_0 picks out the second order term with $z = 0$. $|\phi^{(h)}\rangle$ is a *highest weight state*

$$L_n|\phi^{(h)}\rangle = 0 \qquad \text{for} \quad n > 0 \ , \tag{1.23}$$

by acting with L_{-m}, $m > 0$, upon the state $|\phi^{(h)}\rangle$ the eigenvalue h with respect to L_0 is increased by an amount m to $h + m$,

$$L_0(L_{-m}|\phi^{(h)}\rangle) = [L_0, L_{-m}]|\phi^{(h)}\rangle + L_{-m}L_0|\phi^{(h)}\rangle = (m+h)(L_{-m}|\phi^{(h)}\rangle) \ . \tag{1.24}$$

1.2.1 Vertex Operators

The operator expansion of a bosonic operator $\phi(z)$ is defined by

$$\phi(z) = \hat{q} + \hat{p}\log z + \sum_{k\neq 0} \frac{a_k}{k} z^{-k} \tag{1.25}$$

with

$$[a_n, a_m] = n\delta_{n+m,0} \qquad\qquad [\hat{p}, \hat{q}] = 1 \tag{1.26}$$

The definition differs from the "usual" definition by factors of i [4], but the above is the definition that will be used later (in Chapter 2). Normal ordering is such that $:\hat{p}\hat{q}:= \hat{q}\hat{p}$, a choice along with (1.26) consistent with the choice of vacuum states $\langle q = 0|$ and $|p = 0\rangle$

$$\langle q = 0| : \hat{p}\hat{q} : |p = 0\rangle = \langle q = 0|\hat{q}\hat{p}|p = 0\rangle = 0 \tag{1.27}$$

and

$$\begin{aligned}
&\langle q = 0|\phi(z)\phi(w)|p = 0\rangle \\
&= \langle q = 0|(\hat{q} + \hat{p}\log z + \sum_{k\neq 0}\frac{a_k}{k}z^{-k})(\hat{q} + \hat{p}\log w + \sum_{l\neq 0}\frac{a_l}{l}w^{-l})|p = 0\rangle \\
&= \langle q = 0|\hat{p}\hat{q}|p = 0\rangle \log z + \sum_{l\neq 0}\langle q = 0|\hat{p}a_l|p = 0\rangle\frac{w^{-l}}{l}\log z
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{k\neq 0} \langle q=0|a_k \hat{q}|p=0\rangle \frac{z^{-k}}{k} + \sum_{k\neq 0, l\neq 0} \langle q=0|a_k a_l|p=0\rangle \frac{z^{-k} w^{-l}}{kl} \\
= \; & \langle q=0|1|p=0\rangle \log z + \sum_{k>0, l>0} \langle q=0|a_{-l} a_k + k\delta_{k-l,0}|p=0\rangle \frac{z^{-k} w^{l}}{-kl} \\
= \; & \log z - \sum_{l>0} \frac{1}{l} (\frac{w}{z})^l \\
= \; & \log z + \log(1 - \frac{w}{z}) \\
= \; & \log(z-w) \qquad (1.28)
\end{aligned}
$$

(the usual choice is $\langle X(z)X(w)\rangle \sim -\log(z-w)$; the sign difference arises from the difference in factors of i in the expansions of $X(z)$ and of $\phi(z)$ as defined above).

In two dimensions it is possible to exponentiate a bosonic operator and obtain a fermionic operator after normalizing

$$
: e^{\alpha\phi(z)} : = e^{\alpha \sum_{k>0} \frac{a_{-k}}{(-k)} z^k} e^{\alpha \hat{q}} e^{\alpha \hat{p} \log z} e^{\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}} \qquad (1.29)
$$

with α a parameter to be determined. With the energy-momentum tensor $T(z) \equiv \beta : (\partial_z \phi(z))^2 :$ the OPE with $: e^{\alpha\phi(z)} :$ fixes the choice of β and α. Define for convenience $a_0 \equiv -\hat{p}$ so that $\partial_z \phi(z) = -\sum_{k\in \mathsf{Z}} a_k z^{-k-1}$, and the OPE becomes

$$
T(w) : e^{\alpha\phi(z)} : = \beta : \partial_w \phi(w) \partial_w \phi(w) :: \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \phi(z)^n : \qquad (1.30)
$$

The singular part of the n'th term comes from $2n$ single contractions and $n(n-1)$ double contractions

$$
\begin{aligned}
& \beta : \partial_w \phi(w) \partial_w \phi(w) :: \phi(z)^n : \\
& \quad = \beta 2n \langle \partial_w \phi(w) \phi(z) \rangle : \partial_w \phi(w) \phi(z)^{n-1} : \\
& \qquad + \beta n(n-1) \langle \partial_w \phi(w) \phi(z) \rangle \langle \partial_w \phi(w) \phi(z) \rangle : \phi(z)^{n-2} : \\
& \qquad + \text{regular terms} \qquad (1.31)
\end{aligned}
$$

Using $\partial_z \langle \phi(w)\phi(w)\rangle = \partial_w \log(w-z) = \frac{1}{w-z}$ and the expansion

$$
\partial_w \phi(w) = \partial_z \phi(z) + (w-z)\partial_z^2 \phi(z) + \mathcal{O}((w-z)^2)
$$

the singular terms of (1.31) become

$$
\begin{aligned}
& 2n\beta \frac{1}{w-z} : \partial_z \phi(z)\, \phi(z)^{n-1} : + \beta n(n-1) \frac{1}{(w-z)^2} : \phi(z)^{n-2} : \\
& \qquad = \frac{2\beta}{w-z} \partial_z : \phi(z)^n : + \beta \frac{n(n-1)}{(w-z)^2} : \phi(z)^{n-2} : \qquad (1.32)
\end{aligned}
$$

thus the full OPE is

$$T(w):e^{\alpha\phi(z)}:=\frac{2\beta}{w-z}\partial_z:e^{\alpha\phi(z)}:+\frac{\alpha^2\beta}{(w-z)^2}:e^{\alpha\phi(z)}:+\text{reg. terms} \qquad (1.33)$$

Comparing to (1.11) we see that $\beta=\frac{1}{2}$, and that the conformal dimension is $h=\alpha^2\beta=\alpha^2/2$.

A last thing to mention is that I will take all correlators to have implicit "time"-ordering, in order to have the fields in the correlator $\langle\psi(z_1)\psi(z_2)\ldots\rangle$ ordered such that $|z_1|>|z_2|>\ldots$. In radial coordinates, $z=e^{i\sigma+\tau}$, with τ the "time-coordinate". We have $|z|^2=zz^*=e^{2\tau}$, hence the name "time-ordering".

Chapter 2

Matrix Model Approach

The matrix model approach to string theory makes use of a discretization of the string world sheet, using a dynamical triangulation, a discretization where the world sheet is constructed from equilateral triangles (Figure 2.1). The discretization has been used in computer simulations of gravity and of general surfaces [8], but here only analytic aspects will be considered.

Below I will introduce the discretization. In Section 2.1 I will describe how the triangulation can be expressed using matrix integrals, and how the model behaves in the continuum limit and in the so-called "double scaling" limit. Section 2.2 uses the technique of orthogonal polynomials to carry out the matrix integral, and to find certain recursion relations. These recursion relations give rise to differential equations (the Painlevé equations) well-known from the theory of KdV-hierarchies. This motivates an attempt in Section 2.3 to solve the matrix integral using differential operators similar to the differential operators of the KdV-hierarchy. The presentation of the method will be short, because a more general link to the KdV-/KP-hierarchies will be presented in Section 2.4. This link is the identification of the partition function of the matrix model (or rather, the pure gravity free energy) with τ-functions of the KP-hierarchy satisfying certain constraints, the Virasoro constraints. These are the same constraints satisfied by the partition function of *topological* gravity, thereby illuminating the conjecture (which for pure gravity is more than a conjecture) that the matrix model approach and the topological field theory approach are in fact the same. But first an introduction to triangulations of the string world sheet, where I will mostly follow [12].

In the case of pure gravity, where the dimension of the embedding space of the string is zero, there is no integration over the fields X^μ in the partition function (1.3) with the action (1.2), and the partition function simplifies to

$$Z = \sum_h \int_{\substack{\text{world sheet}\\ \text{metrics}}} \mathcal{D}g\, e^{-\beta A + \tilde{\gamma}\chi} \tag{2.1}$$

where the sum over topologies is represented by a sum over genus h, and where

Figure 2.1: A triangulation of the string world sheet.

the action consists in a coupling to the area $A = \int d^2\xi\sqrt{g}$ with what we can take to be the bare cosmological constant, and to the Euler character $\chi = \frac{1}{4\pi}\int d^2\xi\sqrt{g}R = 2-2h$. In a triangulation of the world sheet the Gaussian curvature of the surface resides in the vertices of the triangles; because the triangles all have the same side lengths there is zero curvature in vertices where six triangles meet (a parabolic point), whereas the curvature is positive in vertices with less than six incident triangles (an elliptic point) and negative in vertices with more than six incident triangles (a hyperbolic point). The sum over all such triangulations is the discrete analog of the integral over all possible geometries and the sum over topologies in (2.1),

$$\sum_{\text{all triangulations}} e^{-\beta A+\tilde{\gamma}\chi} \tag{2.2}$$

and the next step is to find how the sum of contributions from each triangulation can be represented (in terms of matrix integrals).

2.1 Triangulations Described by Matrices

It is a well-known fact [7] that scalar φ^4-theory gives rise to Feynman diagrams of four-point vertices, via the expansion in powers of the coupling constant g

$$Z = \sum_{n=1}^{\infty} g^n Z_n \ ; \tag{2.3}$$

the term $g^n Z_n$ corresponds to the diagrams with n vertices. This generalizes to m-point vertices in φ^m-theory, in particular φ^3-theory gives rise to diagrams with three-point vertices. The diagrams thus formed resemble two dimensional surfaces, but do not yet have enough structure to describe a 2D Riemann surface. For this, the propagators in the φ^3-theory, represented by a line, have to be replaced by a propagator, represented by a "ribbon", Figure 2.2. Such propagators are obtained when replacing the scalar φ by an $N \times N$ hermitean matrix M;

Figure 2.2: (a) Propagator and vertex in a scalar theory (φ^3). (b) Propagator and vertex in a theory of hermitean matrices ($\mathrm{Tr}M^3$).

the indices of M are represented by the arrows on the propagators [9]. Then a triangulated Riemann surface can be described by its dual Feynman diagram (Figure 2.3) from $\mathrm{Tr}M^3$-theory; the sum over surfaces becomes in this language a matrix integral

$$e^Z = \frac{1}{N!\,\mathrm{Vol}U(N)} \int dM\, e^{-\frac{1}{2}\mathrm{Tr}M^2 + \frac{g}{\sqrt{N}}\mathrm{Tr}M^3} \tag{2.4}$$

with g the coupling constant of the three-point vertices (not to be confused with the metric g appearing in (2.1)), and with the measure

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{1\leq i<j\leq N} d(\mathrm{Re}M_{ij})d(\mathrm{Im}M_{ij}). \tag{2.5}$$

(M hermitean, thus $\mathrm{Re}M_{ji} = \mathrm{Re}M_{ij}$, $\mathrm{Im}M_{ji} = -\mathrm{Im}M_{ij}$). The integral is defined by analytical continuation to negative values of g. The connection to (2.1) will be discussed shortly. As usual in quantum field theory [7], Z as defined above includes only contributions from connected diagrams (corresponding to connected Riemann surfaces), unlike e^Z which includes the disconnected diagrams as well.

2.1.1 Continuum and Double Scaling Limits

In the expansion (2.3) of the partition function, n counts the number of vertices in the Feynman diagrams. Each vertex in the Feynman diagram corresponds to a triangle in its dual (triangulated) Riemann surface, thus n also counts the number of triangles in the triangulation. The continuum limit of the theory is obtained when the number of triangles diverges (triangles of vanishing size if the total area of the surface is to be kept fixed), that is, when $\langle n\rangle \to \infty$.

We now want to argue that $\langle n\rangle \to \infty$ is connected to a singular behaviour of $Z(g)$ regarded as an analytic function of g. Consider a typical singularity at $g = g_c$ of the form [10]

$$Z(g) \sim \frac{1}{(g_c - g)^{-(2-\gamma)\chi/2}} \quad \text{(singular for genus} > 1) \tag{2.6}$$

It turns out that for pure gravity $\gamma = -\frac{1}{2}$ [12]. Now, the mean value of n is

$$\begin{aligned}
\langle n \rangle &= \frac{\sum n g^n Z_n}{Z(g)} = \frac{g\frac{\partial}{\partial g} Z(g)}{Z(g)} \\
&= g\frac{\partial}{\partial g} \log Z(g) \sim g\frac{\partial}{\partial g}\left(((2-\gamma)\chi/2)\log(g_c - g)\right) \\
&= (2-\gamma)\frac{\chi}{2}\frac{g}{g_c - g} \qquad (2.7)
\end{aligned}$$

thus it has a pole for $g \to g_c$, and the continuum limit $\langle n \rangle \to \infty$ is the same as the limit $g \to g_c$.

By temporarily rescaling $M \to \sqrt{N}M$ (2.4) becomes (ignoring for a moment the normalization)

$$e^Z = \int dM e^{N(-\frac{1}{2}\mathrm{Tr}M^2 + g\mathrm{Tr}M^3)} \qquad (2.8)$$

One reads off from this that each vertex in the Feynman diagram contributes a factor Ng, a propagator (crosses an edge in the triangulation) contributes a factor N^{-1} (the inverse of the coefficient of the $\mathrm{Tr}M^2$ term), and a closed loop (corresponding to a triangle vertex) contributes a factor N from the summation index. Thus, if $F(=n)$ is the number of triangles, E the number of edges and V the number of triangle vertices then, by the Gauss-Bonnet theorem ($V - E + F = \chi = 2 - 2h$), each triangulated surface, or its dual Feynman diagram, contributes a factor (ignoring symmetries of the Feynman diagrams)

$$(gN)^F N^{-E} N^V = g^n N^\chi = g^n N^{2-2h} \qquad (2.9)$$

Hence an expansion of the partition function Z in powers of N^2 becomes a genus expansion, a perturbative expansion (ignore normalization)

$$Z(g,N) = \sum_{h\geq 0} N^{2-2h} \sum_n g^n Z_{n,h} \qquad (2.10)$$

We see that for fixed g (in such a way that $\sum_n g^n Z_{n,h}$ is finite) only genus zero surfaces ($\sum_n Z_{n,0}$) will contribute in the large N limit. However, for genus $h > 1$, $\sum_n g^n Z_{n,h}$ diverges for $g \to g_c$, so we might be able to tune $N \to \infty$ in the continuum limit $g \to g_c$ such that surfaces of genera higher than zero also contribute. Using (2.6) we can re-write (2.10)

$$\begin{aligned}
Z(g,N) &= \sum_{h\geq 0} f_h (g_c - g)^{(2-\gamma)\chi/2} N^\chi \\
&= \sum_{h\geq 0} f_h [(g_c - g)^{(2-\gamma)/2} N]^\chi \qquad (2.11)
\end{aligned}$$

then if $\kappa^{-1} \equiv (g_c - g)^{(2-\gamma)/2} N$ is kept fixed in the double limit $N \to \infty$, $g \to g_c$ all genera might contribute. This limit is called the *double scaling limit.* For pure gravity ($\gamma = -\frac{1}{2}$) we have $\kappa^{-1} = (g_c - g)^{5/4} N$.

Figure 2.3: A triangulation and its dual Feynman diagram.

To make contact between the expressions (2.1) and (2.4), take the area of each triangle to be unity (i.e. area of triangle not rescaled like above), then the total area is $A = n$. If we then take $g = e^{-\beta}$ contact is made to the β-part of (2.1) by $g^n = g^A = e^{-\beta A}$. From (2.9) we see that by identifying $N = e^{\tilde{\gamma}}$ contact is made to the other part of (2.1), $N^\chi = e^{\tilde{\gamma}\chi}$, hence $g^n N^\chi = e^{-\beta A + \tilde{\gamma}\chi}$.

2.1.2 Generalizations of Discretization

The generalization of the above triangulations to discretizations of the world sheet involving general p-polygons is similarly found to give the polynomial action

$$V(M) = \sum_{p \geq 2} \frac{g_p}{N^{\frac{p}{2}-1}} \mathrm{Tr} M^p \tag{2.12}$$

with $g_2 \equiv \frac{1}{2}$, and the free energy

$$e^Z = \frac{1}{N!\,\mathrm{Vol}\,U(N)} \int dM e^{-V(M)}. \tag{2.13}$$

The powers of N in $V(M)$ are such that the argument leading to the genus expansion (2.10) is still valid.

The matrix integral (2.13) factorizes into an integral over the eigenvalues λ_i, $i = 1, 2, \ldots, N$ of M and an integral over the N-dimensional unitary group. Namely, consider a matrix integral

$$\int dM\, f(M) \tag{2.14}$$

with $f(M)$ a function invariant under unitary transformations

$$M \rightarrow UMU^{-1}\ . \tag{2.15}$$

The volume form of dM, taken to be given by the metric $\|dM\|^2 = \mathrm{Tr}(dM\, dM) = \sum_{i,j} |dM_{ij}|^2$, is also invariant under unitary transformations (2.15), due to the cyclic properties of the trace.

Let U be a unitary matrix that diagonalizes M, $M = U\Lambda U^{-1}$. Then

$$\begin{aligned} dM &= dU\Lambda U^{-1} + Ud\Lambda U^{-1} - U\Lambda U^{-1}dUU^{-1} \\ &= U\{(U^{-1}dU)\Lambda + d\Lambda - \Lambda(U^{-1}dU)\}U^{-1} \end{aligned} \tag{2.16}$$

using the standard trick

$$\begin{aligned} \mathbf{1} = UU^{-1} \quad &\Rightarrow \quad 0 = (dU)U^{-1} + Ud(U^{-1}) \\ &\Rightarrow \quad -U^{-1}(dU)U^{-1} = d(U^{-1})\ . \end{aligned} \tag{2.17}$$

If U is close to unity, then

$$U + dU = U(\mathbf{1} + U^{-1}dU) = U(\mathbf{1} + \delta U) \tag{2.18}$$

with δU an infinitesimal anti-hermitean matrix. If U is not close to unity, then at least $U \simeq U_0$ for U_0 some unitary matrix, and $U_0 + dU_0 = U_0(\mathbf{1} + U_0^{-1}dU_0)$ shows that $\tilde{U} = \mathbf{1} + U_0^{-1}dU_0 = \mathbf{1} + \delta\tilde{U}$ is close to unity. The arguments below, for δU, can then be used on $\delta\tilde{U}$. With (2.18), the volume form of dM becomes

$$\begin{aligned} \|dM\|^2 &= \mathrm{Tr}(dM\, dM) \\ &= \mathrm{Tr}\,\{(\delta U\Lambda + d\Lambda - \Lambda\delta U)(\delta U\Lambda + d\Lambda - \Lambda\delta U)\} \\ &= \sum_{i,j}(\delta U_{ij}\lambda_j + d\lambda_j\delta_{ij} - \lambda_i\delta U_{ij})(\delta U_{ji}\lambda_i + d\lambda_i\delta_{ij} - \lambda_j\delta U_{ji}) \\ &= -\sum_{i,j}\delta U_{ij}\delta U_{ji}(\lambda_j - \lambda_i)^2 + \sum_j (d\lambda_j)^2 \\ &= \sum_{i,j}|\delta U_{ij}|^2(\lambda_j - \lambda_i)^2 + \sum_j (d\lambda_j)^2 \\ &= \sum_{i,j}(\mathrm{Re}\delta U_{ij})^2(\lambda_j - \lambda_i)^2 \\ &\qquad\qquad + \sum_{i,j}(\mathrm{Im}\delta U_{ij})^2(\lambda_j - \lambda_i)^2 + \sum_j (d\lambda_j)^2 \end{aligned} \tag{2.19}$$

— cross-terms $\sum_{i,j}\delta U_{ij}(\lambda_j - \lambda_i)d\lambda_i\delta_{ij}$ vanish and $\delta U_{ji} = -\delta U_{ij}^*$ (anti-hermitean). From the volume form the metric can be read off, and one finds that

$$dM = \prod_{i\neq j}(\mathrm{Re}\delta U_{ij}\,\mathrm{Im}\delta U_{ij}\,|\lambda_i - \lambda_j|)\prod_k d\lambda_k \tag{2.20}$$

hence

$$\begin{aligned} \int dMf(M) &= \int\prod_{i,j}(\mathrm{Re}\delta U_{ij}\mathrm{Im}\delta U_{ij})\prod_{i<j}(\lambda_i - \lambda_j)^2\prod_k d\lambda_k f(\Lambda) \\ &= \mathrm{Vol}\,U(N)\int\prod_k d\lambda_k\Delta^2(\lambda)f(\Lambda) \end{aligned} \tag{2.21}$$

with the Vandermonde determinant

$$\Delta(\lambda) = \prod_{1\le i<j\le N} (\lambda_j - \lambda_i) = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \dots & \lambda_N^{N-1} \end{vmatrix} = \det{\lambda_i}^{j-1} \tag{2.22}$$

Thus, because $e^{-V(M)}$ is invariant under unitary transformations, the matrix integral (2.13) becomes

$$e^Z = \frac{1}{N!}\int \left(\prod_{i=1}^{N} d\lambda_i\right) \Delta^2(\lambda) e^{-\sum_{j=1}^{N} V(\lambda_j)} \tag{2.23}$$

an integral over the (real) eigenvalues of M.

2.2 Matrix Integrals by the Technique of Orthogonal Polynomials

To solve (2.23) the standard method is, as explained in several references [6, 10, 11, 12], to introduce an infinite set of n'th order polynomials $P_n(\lambda)$, $n \in \mathsf{N}_0$, that are orthogonal with respect to the measure $d\lambda e^{-V(\lambda)}$

$$\int_{-\infty}^{\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{n,m} \tag{2.24}$$

and they are normalized such that the coefficient of the leading term λ^n is equal to unity,

$$\begin{aligned} P_n(\lambda) &= \lambda^n + \dots \\ &= \sum_i p_{n,i}\lambda^i, \qquad p_{n,n} = 1, \qquad p_{n,i} = 0 \text{ for } i > n. \end{aligned} \tag{2.25}$$

The aim is to rewrite the partition function in terms of the (squared) norms h_n of the polynomials P_n. The norms and the polynomials contain via (2.24) information about the potential $V(\lambda)$, and thus about the coupling constants $g_2, g_3, \dots$ in $V(\lambda)$.

Now, the orthogonal polynomials can be used to rewrite the Vandermonde determinant (2.22), by appropriate use of rules for adding linear combinations of the matrix whose determinant appears in (2.22)

$$\Delta(\lambda) = \det P_{j-1}(\lambda_i) = \sum_{\text{permut.}} (-1)^{\mathcal{P}} \prod_k P_{i_k-1}(\lambda_k). \tag{2.26}$$

$\mathcal{P}$ is the parity of the permutation $k \mapsto i_k$. Then, when substituting (2.26), the integral (2.23) becomes

$$e^Z = \frac{1}{N!}\int\left(\prod_{i=1}^{N} d\lambda_i\right)\left(\sum_{\text{permut.}}(-1)^{\mathcal{P}}\prod_{k} P_{i_k-1}(\lambda_k)\right)\times \\ \left(\sum_{\text{permut.}}(-1)^{\mathcal{P}'}\prod_{k'} P_{i_{k'}-1}(\lambda_{k'})\right) e^{-\sum_{j=1}^{N} V(\lambda_j)} \tag{2.27}$$

From the orthogonality of the polynomials the terms in the above integral expand into integrals of the type (2.24) where the individual integrals contribute a factor

$$\int d\lambda_l P_{i_l-1}(\lambda_l)P_{i_l-1}(\lambda_l)e^{-V(\lambda_l)} = h_{i_l-1} \tag{2.28}$$

Each term in the sums in (2.27) thus contributes

$$\prod_{i=1}^{N} h_{i-1}\ ; \tag{2.29}$$

the number of non-vanishing terms is $N!$ (N choices of polynomial pairs for variable λ_1, $N-1$ choices left for variable λ_2 etc.) and the partition function, not forgetting the normalization $1/N!$ in (2.27), reduces to

$$\begin{aligned} e^Z &= \frac{1}{N!}\left(N!\prod_{i=1}^{N} h_{i-1}\right) = \prod_{i=0}^{N-1} h_i \\ &= h_0^N \prod_{i=1}^{N}\left(\frac{h_i}{h_{i-1}}\right)^{N-i} = h_0^N \prod_{i=1}^{N} f_i^{N-i} \end{aligned} \tag{2.30}$$

with $f_i \equiv h_i/h_{i-1}$. "All" we need now in order to find the (discrete) partition function is to determine the factors h_i or f_i. The orthogonal polynomials determine the factors h_i, which in turn determine the partition function Z. However, we will be more concerned with the relations arising from the h_i's.

2.2.1 Recursion Relations for Polynomials $P_n(\lambda)$

The set of orthogonal polynomials $\{P_n(\lambda)\}_{n\in\mathbb{N}_0}$ forms a complete set of basis vectors in the space of polynomials. Expand the product $\lambda P_n(\lambda)$ on this set of basis vectors

$$\lambda P_n(\lambda) = \sum_{i=0}^{n+1} a_i P_i(\lambda)\ . \tag{2.31}$$

The coefficients are

$$a_{n+1} = 1\ ; \qquad a_i = h_i^{-1}\int d\lambda e^{-V(\lambda)}\lambda P_n(\lambda)P_i(\lambda) \tag{2.32}$$

as is seen from

$$\begin{aligned}
\int d\lambda e^{-V(\lambda)} \lambda P_n(\lambda) P_i(\lambda) &= \int d\lambda e^{-V(\lambda)} \left(\sum_{j=0}^{n+1} a_j P_j(\lambda) \right) P_i(\lambda) \\
&= \sum_{j=0}^{n+1} a_j h_j \delta_{i,j} \\
&= a_i h_i \qquad (2.33)
\end{aligned}$$

From the orthogonality of the polynomials all coefficients a_i for $i \leq n-2$ vanish, e.g. for $i = n-2$

$$\begin{aligned}
a_{n-2} &= \frac{1}{h_{n-2}} \int d\lambda e^{-V(\lambda)} P_n(\lambda) \lambda P_{n-2}(\lambda) \\
&= \frac{1}{h_{n-2}} \int d\lambda e^{-V(\lambda)} P_n(\lambda) \left(\lambda^{n-1} + \mathcal{O}(\lambda^{n-2}) \right) \\
&= \frac{1}{h_{n-2}} \int d\lambda e^{-V(\lambda)} P_n(\lambda) \left(P_{n-1}(\lambda) + \mathcal{O}(\lambda^{n-2}) \right) \\
&= 0 \qquad (2.34)
\end{aligned}$$

and what is left of the expansion (2.31) is

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + s_n P_n(\lambda) + r_n P_{n-1}(\lambda) \qquad (2.35)$$

where the new coefficients $s_n (= a_n)$ and $r_n (= a_{n-1})$ were introduced. This recursion relation (or actually its specialization to even potentials) will be used extensively. The coefficients r_n and s_n contain, via the polynomials $P_n(\lambda)$, information about the coupling constants $g_2, g_3, \ldots$ in $V(\lambda)$.

The coefficient r_n is closely related to $f_n = h_n / h_{n-1}$ of the partition function, the connection is established from the integral

$$\begin{aligned}
&\int d\lambda e^{-V(\lambda)} P_n(\lambda) \{\lambda P_{n-1}(\lambda)\} \\
&= \int d\lambda e^{-V(\lambda)} P_n(\lambda) \{P_n(\lambda) + s_{n-1} P_{n-1}(\lambda) + r_{n-1} P_{n-2}(\lambda)\} \\
&= \int d\lambda e^{-V(\lambda)} P_n(\lambda) P_n(\lambda) \\
&= h_n \qquad (2.36)
\end{aligned}$$

and when substituting the recursion relation for $P_n(\lambda)\lambda$

$$\begin{aligned}
&\int d\lambda e^{-V(\lambda)} \left(P_n(\lambda) \lambda \right) P_{n-1}(\lambda) \\
&= \int d\lambda e^{-V(\lambda)} \{P_{n+1}(\lambda) + s_n P_n(\lambda) + r_n P_{n-1}(\lambda)\} P_{n-1}(\lambda) \\
&= r_n h_{n-1} \qquad (2.37)
\end{aligned}$$

hence the coefficients r_n are connected to the partition function in (2.30) via $r_n = h_n / h_{n-1} = f_n$. Notice that the coefficient s_n does not enter the partition function (2.30).

2.2.2 Restriction to Even Potentials

Up until now no restrictions were put on the polynomial potential $V(\lambda) = \sum_{p\geq 2} \bar{g}_p \lambda^p$, with $\bar{g}_p \equiv g_p/N^{\frac{p}{2}-1}$. It is costumery at this point to restrict the discussion to *even* potentials, assuming that what follows can be generalized to any polynomial potential. As will be discussed (superficially) later this seems to be a valid assumption, except that the restriction to even potentials causes a factor of 2 difference between the partition function thus obtained compared to the partition function that would be obtained for a general polynomial potential. In the following the potential $V(\lambda)$ is assumed to be even.

First I want to show that the orthogonal polynomials with even indices, $P_{2n}(\lambda)$, are even functions of λ and that the polynomials with odd indices $P_{2n+1}(\lambda)$ are odd functions of λ, given that $V(\lambda)$ is even. Note that if $V(\lambda)$ is odd then $e^{-V(\lambda)}$ causes problems in the integral for $\lambda \to \infty$ or $\lambda \to -\infty$ because $V(\lambda) \to -\infty$.

The proof is inductive. Clearly $P_0(\lambda) = 1$ is an even function. $P_1(\lambda) = \lambda + \text{constant}$ is odd:

$$\begin{aligned} 0 &= \int d\lambda e^{-V(\lambda)} P_1(\lambda) P_0(\lambda) \\ &= \int d\lambda e^{-V(\lambda)} (\lambda + \text{constant}) \\ &= \text{constant} \int d\lambda e^{-V(\lambda)} (= \text{constant } \cdot h_0) \end{aligned} \tag{2.38}$$

— in the first line I used that $P_1(\lambda)$ and $P_0(\lambda)$ are orthogonal, in the second line the first term vanishes because the integrand $e^{-V(\lambda)}\lambda$ is odd (and the integration is for $\lambda \in]-\infty, \infty[$). In the third line the integrand is an even, positive function and hence does not vanish when integrated, therefore the constant must be zero, thus $P_1(\lambda)$ is an odd function.

Similar reasoning shows that $P_{2n}(\lambda)$ is an even function. First assume it has been shown that $P_{2i-2}(\lambda)$ is an even function and that $P_{2i-1}(\lambda)$ is an odd function for $i \leq n$. Expand $P_{2n}(\lambda)$

$$\begin{aligned} P_{2n}(\lambda) &= \lambda^{2n} + b_{2n-1}P_{2n-1}(\lambda) + \ldots \\ &\quad + b_{2i-1}P_{2i-1}(\lambda) + \ldots + b_1 P_1(\lambda) + b_0 P_0(\lambda) \end{aligned} \tag{2.39}$$

then like before

$$\begin{aligned} 0 &= \int d\lambda e^{-V(\lambda)} P_{2n}(\lambda) P_{2i-1}(\lambda) \\ &= \int d\lambda e^{-V(\lambda)} \{\lambda^{2n} + b_{2n-1}P_{2n-1}(\lambda) + \ldots + b_{2i-1}P_{2i-1}(\lambda) + \ldots \\ &\qquad + b_1 P_1(\lambda) + b_0 P_0(\lambda)\} P_{2i-1}(\lambda) \ . \end{aligned} \tag{2.40}$$

Again, for parity reasons the first term vanishes

$$\int d\lambda e^{-V(\lambda)} \lambda^{2n} P_{2i-1}(\lambda) = 0 \tag{2.41}$$

and all other terms except $\int d\lambda e^{-V(\lambda)} b_{2i-1} P_{2i-1}(\lambda) P_{2i-1}(\lambda) = b_{2i-1} h_{2i-1}$ vanish because the polynomials are orthogonal, and what is left of (2.40) is

$$0 = b_{2i-1} h_{2i-1} \tag{2.42}$$

Thus in the expansion of $P_{2n}(\lambda)$ (2.39) the coefficients b_{2i-1}, $i \leq n$ vanish, i.e. only even polynomials contribute in the expansion of $P_{2n}(\lambda)$, hence I have proven that $P_{2n}(\lambda)$ is an even function of λ. The proof of $P_{2n+1}(\lambda)$ odd follows the same lines as above, now integrating $P_{2n+1}(\lambda) P_{2i}(\lambda)$, $i \leq n$ (including index $2n$ now that $P_{2n}(\lambda)$ has also been shown to be even).

The purpose of the derivation of the parity of the polynomials is to show that when the potential $V(M)$ is assumed even, then the coefficient s_n in (2.35) vanishes

$$s_n = a_n = \int d\lambda \underbrace{e^{-V(\lambda)}}_{\text{even}} \lambda \underbrace{P_n(\lambda) P_n(\lambda)}_{\text{even}} = 0 \tag{2.43}$$

and the recursion relation (2.35) for the polynomials reduces to

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + r_n P_{n-1}(\lambda). \tag{2.44}$$

Before restricting to an even potential we saw that the coefficient s_n of the recursion relation (2.35) for a general polynomial potential did not enter in the expression of the partition function (2.30). This gives a clue that it might not be very important for the discussion if the potential is kept general or restricted to even polynomials.

2.2.3 Recursion Relations for the Coefficients r_n

The coefficients r_n can, as already mentioned, be used to evaluate the partition function via $r_n = h_n / h_{n-1}$. In order to find a relation among the r_n's consider the integral $\int d\lambda e^{-V(\lambda)} P_n'(\lambda) \lambda P_n(\lambda)$ where the prime means derivative with respect to λ. The highest order term of the product $P_n'(\lambda)\lambda$ is

$$\begin{aligned} P_n'(\lambda)\lambda &= \frac{d}{d\lambda}\left(\lambda^n + \mathcal{O}(\lambda^{n-2})\right)\lambda \\ &= n\lambda^n + \mathcal{O}(\lambda^{n-2}) \\ &= nP_n(\lambda) + \mathcal{O}(P_{n-2}(\lambda)) \end{aligned} \tag{2.45}$$

then the integral is

$$\begin{aligned} \int d\lambda e^{-V(\lambda)} P_n'(\lambda) \lambda P_n(\lambda) &= \int d\lambda e^{-V(\lambda)} \left\{ nP_n(\lambda) + \mathcal{O}(P_{n-2}(\lambda)) \right\} P_n(\lambda) \\ &= nh_n. \end{aligned} \tag{2.46}$$

On the other hand, when inserting the recursion relation (2.44) into the integral and integrating by parts

$$\begin{aligned}
&\int d\lambda e^{-V(\lambda)} P_n'(\lambda)\lambda P_n(\lambda) \\
&= \int d\lambda e^{-V(\lambda)} \underbrace{P_n'(\lambda)}_{\mathcal{O}(\lambda^{n-1})} \{P_{n+1}(\lambda) + r_n P_{n-1}(\lambda)\} \\
&= r_n \int d\lambda e^{-V(\lambda)} P_n'(\lambda) P_{n-1}(\lambda) \\
&= -r_n \int d\lambda \left(e^{-V(\lambda)} P_{n-1}(\lambda)\right)' P_n(\lambda) \\
&= -r_n \int d\lambda (-V'(\lambda) e^{-V(\lambda)} P_{n-1}(\lambda) + e^{-V(\lambda)} \underbrace{P_{n-1}'(\lambda)}_{\mathcal{O}(\lambda^{n-2})}) P_n(\lambda) \\
&= r_n \int d\lambda e^{-V(\lambda)} V'(\lambda) P_{n-1}(\lambda) P_n(\lambda). \qquad (2.47)
\end{aligned}$$

In the third line, when integrating by parts, I ignored the boundary term

$$\left[e^{-V} P_n P_{n-1}\right]_{\lambda\to-\infty}^{\infty} \qquad (2.48)$$

but if the highest order term in $V(\lambda)$ is even (which is evidently the case when the whole potential is even) then $e^{-V(\lambda)} \to 0$ for $\lambda \to \pm\infty$, of course faster than the polynomial $P_n(\lambda)P_{n-1}(\lambda)$ goes to $\pm\infty$. The relation finally reads

$$n h_n = r_n \int d\lambda e^{-V(\lambda)} V'(\lambda) P_{n-1}(\lambda) P_n(\lambda). \qquad (2.49)$$

With the even potential given by [1]

$$V(\lambda) = \sum_{p\geq 1} \frac{g_{2p}}{N^{p-1}} \lambda^{2p} = \sum_{p\geq 1} \bar{g}_{2p} \lambda^{2p} \qquad (2.50)$$

and differentiated with respect to λ

$$V'(\lambda) = \sum_{p\geq 1} 2p\bar{g}_{2p} \lambda^{2p-1} \qquad (2.51)$$

the relation (2.49) will consist of terms with the factors [2]

$$h_n \alpha_n^{[2p-1]} \equiv \int d\lambda e^{-V(\lambda)} \lambda^{2p-1} P_{n-1}(\lambda) P_n(\lambda) \qquad (2.52)$$

and relation (2.49) becomes

$$n = r_n \sum_{p\geq 1} 2p\bar{g}_{2p} \alpha_n^{[2p-1]} \qquad (2.53)$$

[1] Here the sum has been changed to even terms only, as opposed to the sum in the general potential.

[2] As will be clear shortly the r.h.s. contains a factor of h_n for all values of $p = 1, 2, 3, ...$

Figure 2.4: The "stairs" illustrating the evaluation of $\alpha_n^{[3]}$.

To evaluate the factors $h_n\alpha_n^{[2p-1]}$ insert the recursion relation $\lambda P_n = P_{n+1} + r_n P_{n-1}$ in the integral (2.52) for every factor of λ; one obtains 2^{2p-1} terms of the form

$$\text{factor} \cdot \int d\lambda e^{-V} P_m P_{n-1} \tag{2.54}$$

but only terms with $m = n-1$ do not vanish.

As an example, calculate $h_n\alpha_n^{[3]}$, i.e. with $p = 2$ in (2.52) and $q \equiv 2p-1 = 3$.

$$\begin{aligned}
h_n\alpha_n^{[3]} &= \int d\lambda e^{-V} \lambda^3 P_n P_{n-1} \\
&= \int d\lambda e^{-V} \lambda^2 \{P_{n+1} + r_n P_{n-1}\} P_{n-1} \\
&= \int d\lambda e^{-V} \lambda \{P_{n+2} + r_{n+1} P_n + r_n P_n + r_n r_{n-1} P_{n-2}\} P_{n-1} \\
&= \int d\lambda e^{-V} \{P_{n+3} + r_{n+2} P_{n+1} + (r_{n+1} + r_n) P_{n+1} \\
&\qquad + (r_{n+1} + r_n) r_n P_{n-1} + r_n r_{n-1} P_{n-1} + r_n r_{n-1} r_{n-2} P_{n-3}\} P_{n-1} \\
&= \{(r_{n+1} + r_n) r_n + r_n r_{n-1}\} \int d\lambda e^{-V} P_{n-1} P_{n-1} \\
&= r_n \{r_{n+1} + r_n + r_{n-1}\} h_{n-1} \\
&= h_n \{\underbrace{r_{n+1} + r_n + r_{n-1}}_{\text{3 terms}}\}
\end{aligned} \tag{2.55}$$

where for the last step I used $r_n h_{n-1} = h_n$.

The above calculation can be depicted as steps up and down staircases (Figure 2.4) — each factor of λ with P_n gives, when using the recursion relation, two polynomials P_{n+1} and P_{n-1} with some coefficients. This way, the highest degree polynomial after eliminating q factors of λ is P_{n+q}, depicted as going *up* the stairs all q steps, and the lowest degree polynomial is P_{n-q}, obtained by going *down* the stairs for all q steps. Any of the orthogonal polynomials of degree between $n-q$ and $n+q$ can be reached in one or (usually) more ways by combining the up and down steps for each factor of λ. However, only integrals of the polynomial P_{n-1} (with the other P_{n-1}) do not vanish, thus only combinations with *net* up and down steps -1 contribute.

In the λ^3 case (Figure 2.4) the three possible "routes" from P_n to P_{n-1} are marked. Each step down (that is, the $r_m P_{m-1}$ term in the recursion relation) contributes a factor r_m, whereas each step up (the P_{m+1} term in the recursion relation) contributes only a factor 1. The three "routes" give the factors $1 \cdot r_{n+1} \cdot r_n$, $r_n \cdot 1 \cdot r_n$ and $r_n \cdot r_{n-1} \cdot 1$, times h_{n-1} from the remaining integral $\int d\lambda e^{-V} P_{n-1} P_{n-1}$; compare to the explicit calculation (2.55).

For higher powers of λ in (2.52) there are more steps to go, and therefore more routes giving non-vanishing contribution. That complicates the counting of factors of r_m, but the principle for counting remains the same. For a factor λ^q, there are $q = 2p-1$ steps to go, of these p must be steps down (giving an r factor) in order to end up with a polynomial of one degree less. This gives $\binom{2p-1}{p}$ different routes, or terms; $\binom{q}{p} = \frac{q!}{p!(q-p)!}$ are the binomial coefficients.

For $p = 3$ (a factor λ^5 in (2.52)) using Figure 2.5 the counting gives the 10 terms ($\binom{5}{3} = 10$ different routes from **A** to **B**)

$$\begin{aligned} h_n \alpha_n^{[5]} &= (r_{n+2} r_{n+1} r_n + r_{n+1} r_{n+1} r_n + r_{n+1} r_n r_n \\ &\quad + r_{n+1} r_n r_{n-1} + r_n r_{n+1} r_n + r_n r_n r_n + r_n r_n r_{n-1} \\ &\quad + r_n r_{n-1} r_n + r_n r_{n-1} r_{n-1} + r_n r_{n-1} r_{n-2}) h_{n-1} \\ &= h_n (r_{n+2} r_{n+1} + r_{n+1}^2 + 2 r_{n+1} r_n + r_{n+1} r_{n-1} \\ &\quad + r_n^2 + 2 r_n r_{n-1} + r_{n-1}^2 + r_{n-1} r_{n-2}) \end{aligned} \tag{2.56}$$

For $p = 1$ (a factor λ in (2.52)) the result is simple

$$h_n \alpha_n^{[1]} = \int d\lambda e^{-V} \lambda P_n P_{n-1} = r_n h_{n-1} = h_n \tag{2.57}$$

Returning to the relation (2.53) again, it is now *in principle* possible to rewrite the relation in terms of r_n's, but as soon as the potential $V(\lambda)$ is of degree higher than 4 or 6 the actual counting of factors becomes tedious. For the simple potential

$$V(\lambda) = \tfrac{1}{2}\lambda^2 + \frac{g_4}{N}\lambda^4 \; ; \qquad V'(\lambda) = \lambda + \frac{4g_4}{N}\lambda^3 \tag{2.58}$$

the relation (2.53) becomes

$$n = r_n \left(1 + \frac{4g_4}{N}(r_{n+1} + r_n + r_{n-1})\right). \tag{2.59}$$

Figure 2.5: The "stairs" illustrating the evaluation of $\alpha_n^{[5]}$.

2.2.4 Double Scaling Limit of r_n Recursion Relation

The following is a short sketch of how the recursion relation for the coefficients r_n (2.53) behaves in the double scaling limit. After several rescalings and changes of variables in the double scaling limit of (2.53) (or rather the more simple relation (2.59)) one ends up with a set of non-linear differential equations. For more details, see [12].

First, in (2.59) make the following change of variables

$$\frac{n}{N} \to \xi \, , \qquad \frac{1}{N} \to \epsilon, \qquad \frac{r_{n\pm 1}}{N} \to r(\xi \pm \epsilon) \tag{2.60}$$

then (2.59) reads

$$\xi = r(\xi) \left\{1 + 4g_4(r(\xi - \epsilon) + r(\xi) + r(\xi + \epsilon))\right\}. \tag{2.61}$$

Notice that no limits were introduced yet, the variables ξ and ϵ are rational numbers, but for N some large, however finite number, an approximate expansion of $r(\xi)$ in powers of the by-now-small number ϵ will be used. But first rescale $\bar{r}(\xi) \equiv g_4 r(\xi)$ — and drop the bars again immediately —

$$g_4\xi = r(\xi) \left\{1 + 4(r(\xi - \epsilon) + r(\xi) + r(\xi + \epsilon))\right\} \tag{2.62}$$

(this also works for a higher order potential, e.g. for a sixth order potential a term of the form $g_6 r(\xi \pm q) r(\xi \pm q')$ within the curly brackets would rescale to $b\bar{r}(\xi \pm q)\bar{r}(\xi \pm q')$ with $b \equiv g_6/(g_4)^2$, so that there is still an extra, free variable — namely b — compared to the fourth order potential).

Now define the function $W(r)$ to be the zero'th order terms on the r.h.s. of (2.62)

$$W(r) \equiv r + 12r^2 \tag{2.63}$$

(for the generalization to higher order potentials see later) and define also r_c and g_c by

$$W'(r)|_{r=r_c} = 0 \ , \qquad g_c \equiv W(r_c) \ . \tag{2.64}$$

Clearly $W(r) = g_4\xi$ to leading order in the ϵ-expansion. Scaling arguments show that g_c as defined here is indeed the critical value of g_4 leading to the continuum limit, in the same way as $g \to g_c$ of the third order potential in (2.4) lead to the continuum limit. In the limit $N \to \infty$ the variable ξ becomes a continuous variable $0 \le \xi \le 1$; in the double scaling limit $\xi \approx 1$ dominates [12].

Now approximate

$$r(\xi \pm \epsilon) \approx r(\xi) \pm r'(\xi)\epsilon + \tfrac{1}{2}r''(\xi)\epsilon^2 \pm \tfrac{1}{3!}r'''(\xi)\epsilon^3 + \tfrac{1}{4!}r^{(4)}(\xi)\epsilon^4 + \mathcal{O}(\epsilon^5) \tag{2.65}$$

and insert in (2.62)

$$\begin{aligned} g_4\xi &= r(\xi)\{1 + 4(r(\xi) + \underbrace{2r(\xi) + 2\cdot\tfrac{1}{2}r''(\xi)\epsilon^2 + 2\cdot\tfrac{1}{4!}r^{(4)}(\xi)\epsilon^4 + O(\epsilon^5)}_{\text{From } r(\xi-\epsilon)+r(\xi+\epsilon)})\} \\ &= W(r) + 4r(\xi)\left\{r''(\xi)\epsilon^2 + \tfrac{1}{12}r^{(4)}(\xi)\epsilon^4\right\} + O(\epsilon^5) \\ &= g_c + \tfrac{1}{2}\, W''(r)|_{r=r_c}\,(r(\xi) - r_c)^2 + \tfrac{1}{3!}\, W'''(r)|_{r=r_c}\,(r(\xi) - r_c)^3 \\ &\quad + \mathcal{O}((r(\xi) - r_c)^4) + 4r(\xi)\left\{r''(\xi)\epsilon^2 + \tfrac{1}{12}r^{(4)}(\xi)\epsilon^4\right\} + O(\epsilon^5) \end{aligned} \tag{2.66}$$

In the continuum limit $g_4 \to g_c$. In the double scaling limit κ was shown (page 16) to be fixed with $\kappa^{-1} = (g - g_c)^{5/4}N$. Hence introduce a scaling parameter (related to N) by

$$g_4 - g_c \sim \kappa^{-4/5}N^{4/5} \sim \kappa^{-4/5}a^2 \tag{2.67}$$

In other words, the limit $a \to 0$ with $\epsilon = \frac{1}{N} = a^{5/2}$ and the above definitions is the double scaling limit.

The next step is to change variables from ξ to z by

$$g_c - g_4\xi = a^2 z \tag{2.68}$$

then from $g_c - g_4 \sim \frac{1}{2}W''(r)|_{r=r_c}(r(\xi) - r_c)^2$ in the region of $\xi \approx 1$ one finds that $r(\xi)$ scales as

$$r(\xi) - r_c = au(z) \tag{2.69}$$

with $u(z) \sim \sqrt{z}$. One also finds $\epsilon\frac{\partial}{\partial\xi} = \epsilon\frac{\partial z}{\partial \xi}\frac{\partial}{\partial z} = a^{5/2}(-g_4)a^{-2}\frac{\partial}{\partial z} = -g_4 a^{1/2}\frac{\partial}{\partial z}$ and $\epsilon^2 r''(\xi) = g_4^2 a^2 u''(z)$, $\epsilon^4 r^{(4)}(\xi) = g_4^4 a^3 u^{(4)}(z)$. Using these new variables (2.66) becomes

$$-a^2 z = \tfrac{1}{2}W''a^2u^2 + \tfrac{1}{3!}W'''a^3u^3 + 4(au + r_c)\{a^2 g_4^2 u'' + \tfrac{1}{12}a^3 g_4^4 u^{(4)}\} + \mathcal{O}(a^4) \tag{2.70}$$

with, for short, $W'' \equiv W''(r)|_{r=r_c}$ etc. For the second and third power of a-terms one finds

$$-z = \tfrac{1}{2}W''u^2 + 4r_c g_4^2 u'' \tag{2.71}$$

$$0 = \tfrac{1}{3!}W'''u^3 + 4uu'' + \tfrac{1}{3}r_c g_4^4 u^{(4)} \tag{2.72}$$

the first of which, appropriately rescaled, turns out to be a differential equation known from the theory of KdV hierarchies, the first of the Painlevé equations, which is also called the *string equation.* This gives the first hint that the matrix models might be connected to the theory of KdV hierarchies, or more generally the KP hierarchies and the theory of τ-functions. These differential hierarchies will be introduced in Section 2.4.

For a general, but even potential (2.50) the definition of the quantity $W(r)$ generalizes. $W(r)$ is defined as all the zero'th order terms in the ϵ-expansion on the r.h.s. of the generalization of (2.62). To find $W(r)$, first consider the general form of the recursion relation for the r_n's (2.53)

$$\frac{n}{N} = \frac{r_n}{N}\sum_{p\geq 1} 2p\frac{g_{2p}}{N^{p-1}}\alpha_n^{[2p-1]}$$

Although the exact evaluation of $\alpha_n^{[2p-1]}$ is practically impossible one can still extract what will be the zero'th order in ϵ behaviour after the rescaling (2.60). From the "staircase principle" it is known that $\alpha_n^{[2p-1]}$ has $\binom{2p-1}{p}$ non-vanishing terms, each term with p r_m-factors (m some unspecified index $n-1$, n, $n+1$ etc.) times h_{n-1}. Using $r_n h_{n-1} = h_n$, $\alpha_n^{[2p-1]}$ has the form

$$\alpha_n^{[2p-1]} \sim \binom{2p-1}{p} r_m^{p-1} \tag{2.73}$$

Insert into (2.53)

$$\frac{n}{N} \sim \frac{r_n}{N}\sum_{p\geq 1} 2p\frac{g_{2p}}{N^{p-1}}\binom{2p-1}{p} r_m^{p-1} \tag{2.74}$$

(again with m an unspecified index) and rescale using (2.60)

$$\xi \sim r(\xi)\sum_{p\geq 1} 2pg_{2p}\binom{2p-1}{p} r(\xi)^{p-1} = \sum_{p\geq 1} 2g_{2p}\frac{(2p-1)!}{((p-1)!)^2} r(\xi)^p \tag{2.75}$$

— to zero'th order in ϵ. The last rescaling needed is, like in the simple case, $r(\xi) = \bar{r}(\xi)/g_4$ (and drop bars again)

$$g_4\xi \sim \sum_{p\geq 1} 2\frac{(2p-1)!}{((p-1)!)^2}\frac{g_{2p}}{g_4^{p-1}} r(\xi)^p \tag{2.76}$$

and therefore define

$$W(r) = \sum_{p\geq 1} 2\frac{(2p-1)!}{((p-1)!)^2}\frac{g_{2p}}{g_4^{p-1}} r^p \tag{2.77}$$

in accordance with the previous definition (2.63) since the coefficients of r and r^2 are

$$\text{coef.}(r) = 2\frac{g_2}{g_4^0} = 2g_2 = 1\ ; \quad \text{coef.}(r^2) = 2\cdot 3!\frac{g_4}{g_4^{2-1}} = 12.$$

With a higher order potential (2.50) one might tune the $m-2$ first of the extra parameters $g_6, g_8, \ldots$ such that for a certain value of r, defined as r_c, the first $m-1$ derivatives of $W(r)$ vanish, $W'(r)|_{r=r_c} = W''(r)|_{r=r_c} = \ldots = W^{(m-1)}(r)|_{r=r_c} = 0$, $W^{(m)}(r)|_{r=r_c} \neq 0$, and still $g_c \equiv W(r_c)$. This point in the space of couplings is the *m'th critical point*, it corresponds to the coupling of gravity to $(2, 2m-1)$-minimal conformal matter.

With an analysis similar to the above, more of the Painlevé equations might be found.

2.3 Differential Operator Approach

The differential operator approach to the matrix model might be motivated by the following. First define polynomials $\Pi_n(\lambda)$ normalized such that

$$\Pi_n(\lambda) = P_n(\lambda)/\sqrt{h_n}\ . \tag{2.78}$$

Then the recursion relation $\lambda P_n = P_{n+1} + r_n P_{n-1}$ becomes

$$\begin{aligned} \lambda\Pi_n &= \frac{\sqrt{h_{n+1}}}{\sqrt{h_n}}\Pi_{n+1} + r_n\frac{\sqrt{h_{n-1}}}{\sqrt{h_n}}\Pi_{n-1} \\ &= \sqrt{r_{n+1}}\Pi_{n+1} + \sqrt{r_n}\Pi_{n-1} \\ &\equiv Q_{nm}\Pi_m \end{aligned} \tag{2.79}$$

with

$$Q_{nm} = \sqrt{r_m}\delta_{m,n+1} + \sqrt{r_n}\delta_{m+1,n} \tag{2.80}$$

Thus Q_{nm} corresponds to an insertion of λ in an integral with polynomials Π_n. Also, define the matrix A_{nm} as the matrix that corresponds to the insertion of $\frac{\partial}{\partial\lambda}$ into the integral. Then, of course, the two matrices have the commutator

$$[A, Q] = 1\ . \tag{2.81}$$

In the double scaling limit the two matrices become differential operators [12, 14], with

$$[P, Q] = 1 \tag{2.82}$$

where P is the operator corresponding to the anti-symmetrized version of A, and Q is, for the one matrix model, a second order differential operator $Q = \partial^2 + 2u_2(z)$. $2u_2(z)$ is the universal part of $r(\xi)$ in the previous section, and $\partial = \frac{\partial}{\partial z}$.

It is convenient at this point to introduce a pseudo-differential operator L

$$L(z, \partial) = \partial + u_2(z)\partial^{-1} + u_3(z)\partial^{-2} + \ldots \tag{2.83}$$

which is also introduced in the theory of KdV-hierarchies in Section 2.4.3. Then $Q = (L^2)_+ = \partial^2 + 2u_2(z)$, where $(\cdot)_+$ means truncation to non-negative powers of ∂. For $(p,q) = (2l-1, 2)$ conformal matter let $P = (Q^{p/q})_+ = (Q^{l-\frac{1}{2}})_+ = (L^{2l-1})_+$ and define the polynomial (in u_2 and derivatives thereof) R_l as (half) the residue of $Q^{l-\frac{1}{2}}$,

$$2R_l \equiv (Q^{l-\frac{1}{2}})_{-1} = (L^{2l-1})_{-1} \ , \tag{2.84}$$

otherwise stated

$$L^{2l-1} = P + 2R_l\partial^{-1} + \mathcal{O}(\partial^{-3}) \ . \tag{2.85}$$

One has

$$0 = [Q^{l-\frac{1}{2}}, Q] = [(Q^{l-\frac{1}{2}})_+, Q] + [(Q^{l-\frac{1}{2}})_-, Q] \tag{2.86}$$

or

$$[(Q^{l-\frac{1}{2}})_+, Q] = [Q, (Q^{l-\frac{1}{2}})_-] \tag{2.87}$$

but the l.h.s. has only non-negative powers of ∂ whereas the r.h.s.

$$\begin{aligned} [Q, (Q^{l-\frac{1}{2}})_-] &= [Q, (L^{2l-1})_-] \\ &= [\partial^2 + 2u_2, 2R_l\partial^{-1} + \mathcal{O}(\partial^{-3})] \\ &= \partial^2(2R_l\partial^{-1}) - 2R_l\partial + \mathcal{O}(\partial^{-1}) \\ &= 2R_l' + 2R_l' + 2R_l\partial - 2R_l\partial + \mathcal{O}(\partial^{-1}) \\ &= 4R_l' + \mathcal{O}(\partial^{-1}) \end{aligned} \tag{2.88}$$

only has the $4R_l'$-term and negative power terms. Therefore (2.87) reduces to

$$[(Q^{l-\frac{1}{2}})_+, Q] = 4R_l' \tag{2.89}$$

hence (2.82) becomes

$$1 = [P, Q] = [(Q^{l-\frac{1}{2}})_+, Q] = 4R_l' \ . \tag{2.90}$$

R_l can be found to be the Gelfand-Dikii polynomials of the KdV- hierarchy using similar manipulations to set up a recursion relation for the R_l's. Their explicit form, for the first few l's, is with $u = -2u_2$ [12, 13],

$$\begin{aligned} R_0 &= \frac{1}{2} \\ R_1 &= -\frac{1}{4}u \\ R_2 &= \frac{3}{16}u^2 - \frac{1}{16}u'' \\ R_3 &= -\frac{5}{32}u^3 + \frac{5}{32}(uu'' + \frac{1}{2}u'^2) - \frac{1}{64}u^{(4)} \end{aligned} \tag{2.91}$$

Thus (2.90) for $l = 2$ is the differentiated version of the string equation (2.71), with proper re-scalings,

$$z = \frac{3}{4}u^2 - \frac{1}{4}u'' \ . \tag{2.92}$$

The next section show a different method to connect to the KdV-hierarchies.

2.4 Virasoro Constraints

First a remark about the terminology used in the following. As mentioned in Section 2.1 the partition function Z as found in (2.4)

$$e^Z = \frac{1}{N! \operatorname{Vol} U(N)} \int dM e^{-\frac{1}{2}\operatorname{Tr}M^2 + \frac{g}{\sqrt{N}}\operatorname{Tr}M^3} \tag{2.93}$$

only includes contributions from *connected* Feynman diagrams, or the dual *connected* Riemann surfaces, whereas e^Z also includes the disconnected diagrams. In the literature it is common to use the term "partition function" about the *matrix model* partition function $Z_{\text{matrix}} = e^{Z_{\text{grav}}}$ and the term "free energy" about the gravity partition function in the double scaling limit, thus the generating function of connected surfaces is the "free energy". This is only a matter of words, but it might cause some confusion; in the following I will change my terminology accordingly. In topological field theory, presented in Chapter 3, there is a similar use of the terms "partition function" and "free energy" (cf. Section 3.2.4).

In [14] it was suggested that the partition function of the one-matrix model is related to the τ-function of the KdV hierarchy. It has later been suggested [18] that for the one-matrix models considered, namely the ones with an *even* potential, the connection in the double scaling limit (in particular: the continuum limit) is that the square root of the (matrix model) partition function is a τ-function of the KdV hierarchy,

$$Z = \tau^2 \tag{2.94}$$

However, the partition function is not uniquely determined, being linked to a τ-function of the KdV-hierarchy, since there are more τ-functions in the hierarchy, but with appropriate constraints the ambiguity is removed (see Section 2.4.3 below).

As we shall see in the following there is a set of constraints on the matrix model partition function in the double scaling limit (or its corresponding τ-function), the so-called Virasoro constraints

$$\mathcal{L}_n \tau = 0, \quad n \geq -1 \tag{2.95}$$

where the $\mathcal{L}_n$'s are differential operators that satisfy (part of) the Virasoro algebra with vanishing central charge (compare to (1.17))

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}. \tag{2.96}$$

Actually, one can show (Section 2.4.3) that the string equation can be written as the constraint

$$\mathcal{L}_{-1}\sqrt{Z} = 0 \tag{2.97}$$

and the fact (then to be shown) that $\sqrt{Z}$ is a τ-function of the KdV-hierarchy imposes the other constraints

$$\mathcal{L}_n\sqrt{Z} = 0, \quad n \geq 0. \tag{2.98}$$

As mentioned in the introduction, these Virasoro constraints turn out to be the same as the constraints imposed on the topological pure gravity partition function [25, 27].

The Virasoro constraints thus characterize the system. They generalize to constraints involving operators of W-algebras, so-called W-constraints, when coupling gravity to minimal matter using certain multimatrix models [19, 20]. Here, I will only consider pure gravity and the Virasoro constraints.

For one-matrix models with general (by which I mean non-symmetric, but still polynomial) potentials in the double scaling limit, the partition function will be the square root of the partition function of an even potential, $Z^2_{\text{gen. pot.}} = Z_{\text{even pot.}}$ [18], suggesting an even closer relationship to the integrable hierarchies if only we would extend the analysis to general potentials.

However, the models with even potentials remain easier to deal with than general potentials, and some models are at present only soluble if restricted to even potentials. Most of the following will, after introducing the more general case, be restricted to a discussion of even potentials. For the one-matrix model with hermitean matrices this is sometimes called the *reduced hermitean model,* "reduced" because all couplings to odd powers in the general potential are put equal to zero.

To find the Virasoro constraints on the one-matrix partition function in the double scaling limit, there are (at least) two paths to follow. One can either go along the lines of [18] and enter the continuum limit before solving for the constraints. This way one arrives at the Virasoro constraints via the loop equation for the kth critical point. Or one can solve for the *discrete* counterparts of the Virasoro constraints, namely constraints on the one-matrix partition function not yet taken to the double scaling limit. Then, in the double scaling limit, these constraints are explicitly carried over into constraints on the partition function, the *continuous* Virasoro constraints [19, 21]. Either way the same Virasoro constraints on the double scaling partition function emerge. Note that in the references [19, 21] by the term "continuum limit" is actually meant the double scaling limit.

I will here pursue the second idea, thus first, in Section 2.4.1, solve for the discrete constraints and then, in Section 2.4.2, I shall briefly discuss how to carry the constraints over into the continuum (double scaling) limit. The continuum limit is still a rather unclear point, and right now it is discussed how to understand the limit, and if this way to enter the continuum limit is sensible [22]. At any rate, this is a controversial topic. A short review of τ-functions and the KP- and KdV-hierarchies is given in Section 2.4.3 along with a derivation of the string equation from the constraint $\mathcal{L}_{-1}\tau = 0$.

2.4.1 Discrete Virasoro Constraints

In this section I will present how to solve for the discrete Virasoro constraints. The presentation follows in parts Marshakov [20] and the references [19, 21], though in more detail, and carefully keeping track of the constants. The notation is slightly different in order to conform at least somewhat to the notation used in the rest of the thesis.

The basic idea is to establish a quantity $\psi_N^G\{t\}$ that by construction satisfies a series of (discrete Virasoro) constraints,

$$L_n \psi_N^G\{t\} = 0, \quad n \geq -1$$

$$L_n = \sum_{k \geq 0} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{a+b=n} \frac{\partial^2}{\partial t_a \partial t_b} \tag{2.99}$$

and then to show that $\psi_N^G\{t\}$ equals the partition function of the hermitean one-matrix model with $N \times N$-matrices.

Introduce a free bosonic operator by [3]

$$\phi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$$

$$\phi(z) = \hat{q} + \hat{p} \log z + \sum_{k \neq 0} \frac{a_k}{k} z^{-k}$$

$$[a_n, a_m] = n \delta_{n+m,0} \qquad\qquad [\hat{p}, \hat{q}] = 1 \tag{2.100}$$

compactified on a circle with a radius such that momentum is quantized in units of $\sqrt{2}$. Define the vacuum states

$$\hat{p}|0\rangle = 0 \qquad\qquad \langle N|\hat{p} = \sqrt{2}N\langle N|$$

$$a_k|0\rangle = \langle N|a_{-k} = 0 \qquad \text{for} \qquad k > 0. \tag{2.101}$$

the values of N are integer. The energy-momentum tensor and its mode expansion are found to be (see Chapter 1) [4]

$$T(z) \equiv \tfrac{1}{2} : (\partial\phi(z))^2 := \sum_{n \in \mathsf{Z}} T_n z^{-n-2} \tag{2.102}$$

with

$$T_n|0\rangle = 0, \qquad n \geq -1. \tag{2.103}$$

The modes T_n are functions of the operators a_k and $\hat{p}$ as will be shown below in eq. (2.115). Finally define the "Hamiltonian"

$$H(t) \equiv -\frac{1}{\sqrt{2}} \oint_0 \frac{dz}{2\pi i} V(z) \partial\phi(z) = \frac{1}{\sqrt{2}} \sum_{k \geq 1} t_k a_k \tag{2.104}$$

[3] This definition differs slightly from the definition in [19].

[4] I reserve the notation L_n for the Virasoro constraints and use T_n for the modes of the energy-momentum tensor.

with the potential formally defined by

$$V(z) = \sum_{k\geq 1} t_k z^k. \tag{2.105}$$

The two expressions for the Hamiltonian are indeed equal; if we define $a_0 \equiv -\hat{p}$ the current $\partial\phi(z)$ is

$$\partial\phi(z) = \frac{\hat{p}}{z} - \sum_{k\neq 0} a_k z^{-k-1} = -\sum_{k\in\mathsf{Z}} a_k z^{-k-1} \tag{2.106}$$

and the middle part of (2.104) can be written as

$$\begin{aligned}
-\frac{1}{\sqrt{2}}\oint_0 \frac{dz}{2\pi i} V(z)\partial\phi(z) &= -\frac{1}{\sqrt{2}}\oint_0 \frac{dz}{2\pi i}(\sum_{k\geq 1} t_k z^k)(-\sum_{l\in\mathsf{Z}} a_l z^{-l-1}) \\
&= \frac{1}{\sqrt{2}}\sum_{k\geq 1}\sum_{l\in\mathsf{Z}} t_k a_l \delta_{k,l} \\
&= \frac{1}{\sqrt{2}}\sum_{k\geq 1} t_k a_k.
\end{aligned} \tag{2.107}$$

Now we want to construct a quantity satisfying discrete Virasoro constraints. As a first step, consider the state

$$\langle N, \{t\}| \equiv \langle N| e^{H(t)} = \langle N| e^{\frac{1}{\sqrt{2}}\sum_{k\geq 1} t_k a_k} \tag{2.108}$$

We want to show that acting with the operator a_k upon the state (2.108) from the right is the same as acting upon it with a multiplication or a differential operator.

$$\begin{aligned}
\langle N, \{t\}| a_{-k} &= \langle N| e^{H(t)} a_{-k} \qquad k > 0 \\
&= \langle N| [e^{H(t)}, a_{-k}] \\
&= \langle N| e^{H(t)} [H(t), a_{-k}] \\
&= \langle N, \{t\}| \frac{1}{\sqrt{2}} \sum_{l\geq 1} t_l [a_l, a_{-k}] \\
&= \frac{1}{\sqrt{2}} \langle N, \{t\}| \sum_{l\geq 1} t_l l \delta_{l,k} \\
&= \frac{1}{\sqrt{2}} k t_k \langle N, \{t\}|
\end{aligned} \tag{2.109}$$

thus a_{-k}, $k > 0$ is replaced by the multiplication operator $\frac{k}{\sqrt{2}} t_k$. Similarly a_k, $k > 0$ is replaced by the differential operator $\sqrt{2}\frac{\partial}{\partial t_k}$, this can be checked using the identity $[a_k, a_{-k}] = k$:

$$\begin{aligned}
\langle N, \{t\}|[a_k, a_{-k}] &= \langle N, \{t\}|a_k a_{-k} - \langle N, \{t\}|a_{-k} a_k \\
&= \sqrt{2}\frac{\partial}{\partial t_k}(\langle N, \{t\}|a_{-k}) - \frac{k}{\sqrt{2}} t_k \langle N, \{t\}|a_k \\
&= \sqrt{2}\frac{\partial}{\partial t_k}(\frac{k}{\sqrt{2}} t_k \langle N, \{t\}|) - \frac{k}{\sqrt{2}} t_k \sqrt{2}\frac{\partial}{\partial t_k}\langle N, \{t\}| \\
&= (k + k t_k \frac{\partial}{\partial t_k} - k t_k \frac{\partial}{\partial t_k})\langle N, \{t\}| \\
&= k\langle N, \{t\}| \qquad (2.110)
\end{aligned}$$

as expected. The effect of $\frac{\partial}{\partial t_0}$ is defined to be $\frac{\partial}{\partial t_0} Z = -NZ$ such that $\hat{p} \sim \sqrt{2}N \sim -\sqrt{2}\frac{\partial}{\partial t_0}$ and $\hat{q} \sim -\frac{1}{\sqrt{2}} t_0$. Hence any matrix element of the type

$$\langle N, \{t\}| : F(a_k, a_{-k}) : |\alpha\rangle \qquad (2.111)$$

where $|\alpha\rangle$ is an arbitrary state, can be written as

$$\langle N, \{t\}| : F(a_k, a_{-k}) : |\alpha\rangle = F(\sqrt{2}\frac{\partial}{\partial t_k}, \frac{k}{\sqrt{2}} t_k)\langle N, \{t\}|\alpha\rangle \qquad (2.112)$$

in particular

$$\langle N, \{t\}|T_n|\alpha\rangle = L_n \langle N, \{t\}|\alpha\rangle, \qquad n \geq -1 \qquad (2.113)$$

with L_n a differential operator. Because the operators T_n satisfy a Virasoro algebra in conformal field theory, we find that the operators L_n will also satisfy a Virasoro algebra. To establish the connection to the discrete matrix models, the mere existence of the Virasoro differential operators L_n suffices. But for clarity, and for use in the continuum limit, it is worth while going through the exercise of evaluating the explicit form of the L_n's, (2.99).

The modes of the energy-momentum tensor are functions of the free field modes a_k, the explicit dependence follows from simple calculations:

$$\begin{aligned}
T(z) &= \tfrac{1}{2} : (\partial\phi(z))^2 : \\
&= \tfrac{1}{2} : (-\sum_{k \in \mathsf{Z}} a_k z^{-k-1})(-\sum_{l \in \mathsf{Z}} a_l z^{-l-1}) : \\
&= \tfrac{1}{2} \sum_{k,l \in \mathsf{Z}} : a_k a_l : z^{-k-l-2} \qquad (2.114)
\end{aligned}$$

so that the modes of the energy-momentum tensor are

$$\begin{aligned}
T_n &= \tfrac{1}{2} \sum_{\substack{k,l \in \mathsf{Z} \\ k+l=n}} : a_k a_l : \\
&= \tfrac{1}{2} \sum_{\substack{k<0, l \in \mathsf{Z} \\ k+l=n}} : a_k a_l : + \tfrac{1}{2} \sum_{\substack{k \geq 0, l \in \mathsf{Z} \\ k+l=n}} : a_k a_l :
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\sum_{k>0} a_{-k}a_{n+k} + \tfrac{1}{2}\sum_{\substack{k,l\geq 0\\ k+l=n}} a_k a_l + \tfrac{1}{2}\sum_{\substack{l<0,0\leq k\\ k+l=n}} a_l a_k \\
&= \tfrac{1}{2}\sum_{k>0} a_{-k}a_{n+k} + \tfrac{1}{2}\sum_{\substack{k,l\geq 0\\ k+l=n}} a_k a_l + \tfrac{1}{2}\sum_{\substack{l>0\\ l\geq -n}} a_{-l}a_{n+l}, \qquad n \in \mathsf{Z}.
\end{aligned} \tag{2.115}
$$

The first and third sums are equal in the range of n's we are interested in, thus

$$
T_n = \sum_{k>0} a_{-k}a_{n+k} + \tfrac{1}{2}\sum_{\substack{k,l\geq 0\\ k+l=n}} a_k a_l \qquad \text{for} \qquad n \geq -1 \tag{2.116}
$$

Remember one could replace $a_{-k} \to \frac{k}{\sqrt{2}}t_k$, $a_k \to \sqrt{2}\frac{\partial}{\partial t_k}$ when acting on $\langle N, \{t\}|$. With this the expression for the differential operators become

$$
T_n \to L_n = \sum_{k\geq 0} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{a+b=n} \frac{\partial^2}{\partial t_a \partial t_b} \qquad \text{for} \qquad n \geq -1 \tag{2.117}
$$

(no harm done when adding the vanishing $k = 0$ term in the first sum) with the correspondence

$$
\langle N, \{t\}|T_n|\alpha\rangle = L_n\langle N, \{t\}|\alpha\rangle. \tag{2.118}
$$

Now, if $|\alpha\rangle = G|0\rangle$ is a state such that $T_n G|0\rangle = 0$ for $n \geq -1$, or what is enough $[T_n, G] = 0$, then obviously

$$
L_n\langle N, \{t\}|G|0\rangle = \langle N, \{t\}|T_n G|0\rangle = 0, \tag{2.119}
$$

in other words, the correlator $\psi_N^G\{t\} \equiv \langle N, \{t\}|G|0\rangle$ satisfies the discrete Virasoro constraints (2.99). A trivial choice of G is the identity operator $\mathbf{1}$ for which the correlator becomes

$$
\psi_N^{\mathbf{1}}\{t\} = \langle N, \{t\}|0\rangle = \begin{cases} 0 & \text{if } N \neq 0 \text{ (momentum conservation)} \\ \langle 0|e^{\sum_{k>0} a_k t_k}|0\rangle = 1 & \text{if } N = 0 \ (a_k|0\rangle = 0 \text{ for } k > 0) \end{cases} \tag{2.120}
$$

Evidently the Virasoro operators (2.117) vanish on $\psi_N^{\mathbf{1}}\{t\}$. A more instructive choice and, as we shall see, the choice that reveals $\psi_N^G\{t\}$ as the partition function of the $N \times N$ hermitean one-matrix model, is to let G be the exponential of the screening operator Q_+ (see Chapter 1 on vertex operators),

$$
\begin{aligned}
G &= e^{Q_+}, \\
Q_\pm &\equiv \oint \frac{dz}{2\pi i} J_\pm(z) = \oint \frac{dz}{2\pi i} {:}\exp \alpha_\pm \phi(z){:}\,.
\end{aligned} \tag{2.121}
$$

The screening operators are conformally invariant, i.e. have conformal dimension zero, thus $J_\pm(z)$ has conformal dimension 1, which fixes the values of $\alpha_\pm$ in the integrand — it was shown in Chapter 1 that the conformal dimension of the vertex operator $:\exp \alpha_\pm \phi(z):$ is $\alpha_\pm^2/2$, therefore the values of $\alpha_\pm$ are

$$\frac{\alpha_\pm^2}{2} = 1, \quad \alpha_\pm = \pm\sqrt{2}. \tag{2.122}$$

We still need to show that $G = e^{Q_+}$, or Q_+, commutes with T_n for $n \geq -1$. However, it is enough to show that the operator product expansion (OPE) of the whole energy-momentum tensor with $Q_\pm$ vanishes

$$\begin{aligned} T(w)Q_\pm &= \oint_0 \frac{dz}{2\pi i} T(w) J_\pm(z) \\ &= \oint_0 \frac{dz}{2\pi i} \left\{ \frac{1}{w-z}\partial_z J_\pm(z) + \frac{1}{(w-z)^2} J_\pm(z) + \text{regular terms} \right\} \\ &= 0 \end{aligned} \tag{2.123}$$

the integrand is a total derivative of a single valued function in a neighbourhood of $z = 0$, and thus vanishes when integrated along a contour around $z = 0$. The correlator $\langle N, \{t\}|e^{Q_+}|0\rangle$ therefore satisfies the Virasoro constraints (2.99). It now remains to be shown that the correlator is in fact a partition function of the discrete one-matrix models.

The state $\langle N|$ can be written as a coherent state $\langle N| = \langle 0|e^{-\sqrt{2}N\hat{q}}$

$$\begin{aligned} \langle N|\hat{p} &= \langle 0|e^{-\sqrt{2}N\hat{q}}\hat{p} \\ &= \langle 0|[e^{-\sqrt{2}N\hat{q}}, \hat{p}] \\ &= \langle 0|e^{-\sqrt{2}N\hat{q}}[-\sqrt{2}N\hat{q}, \hat{p}] \\ &= \sqrt{2}N\langle N| \; . \end{aligned} \tag{2.124}$$

Furthermore, acting with $e^{\sqrt{2}\hat{q}}$ upon $\langle N|$ (or on $\langle N, \{t\}|$) brings the momentum one unit down

$$\begin{aligned} \langle N, \{t\}|e^{\sqrt{2}\hat{q}} &= \langle 0|e^{-\sqrt{2}N\hat{q}} e^{\frac{1}{\sqrt{2}}\sum_{k>0} t_k a_k} e^{\sqrt{2}\hat{q}} \\ &= \langle 0|e^{-\sqrt{2}(N-1)\hat{q}} e^{\frac{1}{\sqrt{2}}\sum_{k>0} t_k a_k} \\ &= \langle N-1, \{t\}| \; . \end{aligned} \tag{2.125}$$

Thus, since a factor of $e^{\sqrt{2}\hat{q}}$ appears in Q_+

$$: e^{\sqrt{2}\phi(z)} : = e^{\sqrt{2}\sum_{k>0} \frac{a_{-k}}{(-k)} z^k} e^{\sqrt{2}\hat{q}} e^{\sqrt{2}\hat{p}\log z} e^{\sqrt{2}\sum_{k>0} \frac{a_k}{k} z^{-k}} \tag{2.126}$$

using the normal ordering of $\hat{p}$ and $\hat{q}$, $: \hat{p}\hat{q} := \hat{q}\hat{p}$, selection rules on the momentum reduce the correlator $\langle N, \{t\}|e^{Q_+}|0\rangle$ to

$$\langle N, \{t\}|\frac{1}{N!}(: e^{\sqrt{2}\phi(z)} :)^N|0\rangle \tag{2.127}$$

integrated over all N z-coordinates. Acting with $:e^{\sqrt{2}\phi(z)}:$ upon the state $\langle N, \{t\}|$ from the right yields

$$
\begin{aligned}
&\langle N,\{t\}| : e^{\sqrt{2}\phi(z)} : \\
&= \langle N,\{t\}| e^{\sqrt{2}\sum_{k>0}\frac{a_{-k}}{(-k)}z^k} e^{\sqrt{2}\hat{q}} e^{\sqrt{2}\hat{p}\log z} e^{\sqrt{2}\sum_{k>0}\frac{a_k}{k}z^{-k}} \\
&= e^{-\sqrt{2}\sum_{k>0}\frac{1}{\sqrt{2}}t_k z^k}\langle N-1,\{t\}| e^{\sqrt{2}\hat{p}\log z} e^{\sqrt{2}\sum_{k>0}\frac{a_k}{k}z^{-k}} \\
&= e^{-\sqrt{2}\sum_{k>0}\frac{1}{\sqrt{2}}t_k z^k} e^{\sqrt{2}\sqrt{2}(N-1)\log z} e^{\sqrt{2}\sum_{k>0}\frac{\sqrt{2}}{k}\frac{\partial}{\partial t_k}z^{-k}}\langle N-1,\{t\}| \\
&= e^{-\sqrt{2}\sum_{k>0}\frac{1}{\sqrt{2}}t_k z^k} e^{2(N-1)\log z}\langle N-1,\{t_n+\frac{2}{n}z^{-n}\}| \qquad (2.128)
\end{aligned}
$$

Then, with the implicit time-ordering $|z_1| > |z_2| > \ldots > |z_N|$ the correlator $\psi_N^G\{t\} = \langle N,\{t\}|e^{Q_+}|0\rangle$ becomes

$$
\begin{aligned}
&\langle N,\{t\}|e^{Q_+}|0\rangle \\
&= \frac{1}{N!}\oint\left(\prod_{i=1}^N\frac{dz_i}{2\pi i}\right)\langle N,\{t\}| : e^{\sqrt{2}\phi(z_1)} :: e^{\sqrt{2}\phi(z_2)} : \cdots : e^{\sqrt{2}\phi(z_N)} : |0\rangle \\
&= \frac{1}{N!}\oint\left(\prod_{i=1}^N\frac{dz_i}{2\pi i}\right) e^{-\sqrt{2}\sum_{k>0}\frac{1}{\sqrt{2}}t_k z_1^k} e^{2(N-1)\log z_1} \\
&\qquad\qquad \times\langle N-1,\{t_n+\frac{2}{n}z_1^{-n}\}| : e^{\sqrt{2}\phi(z_2)} : \cdots : e^{\sqrt{2}\phi(z_N)} : |0\rangle \\
&= \frac{1}{N!}\oint\left(\prod_{i=1}^N\frac{dz_i}{2\pi i}\right)\left(\prod_{j=1}^N e^{2(N-j)\log z_j}\right) e^{-\sqrt{2}\sum_{k>0}\frac{1}{\sqrt{2}}t_k z_1^k} \\
&\quad \times e^{-\sqrt{2}\sum_{k>0}\frac{1}{\sqrt{2}}(t_k+\frac{2}{k}z_1^{-k})z_2^k} e^{-\sqrt{2}\sum_{k>0}\frac{1}{\sqrt{2}}(t_k+\frac{2}{k}(\sum_{l=1}^{N-1}z_l^{-k}))z_N^k} \\
&\qquad\qquad \times\underbrace{\langle 0,\{t_n+\frac{2}{n}\sum_{l=1}^N z_l^{-n}\}|0\rangle}_{1} \\
&= \frac{1}{N!}\oint\left(\prod_{i=1}^N\frac{dz_i}{2\pi i}\right)\left(\prod_{j=1}^N e^{2(N-j)\log z_j}\right) e^{-\sum_{i>0}V(z_i)} \\
&\qquad\qquad \times e^{-2\sum_{k>0}\frac{1}{k}\sum_{1\le l\le N-1, l<m\le N}z_l^{-k}z_m^k}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{N!}\oint\left(\prod_{i=1}^{N}\frac{dz_i}{2\pi i}\right)\left(\prod_{j=1}^{N} e^{2(N-j)\log z_j}\right) e^{-\sum_{i>0}V(z_i)} \\
&\qquad\qquad\qquad\qquad \times e^{2\sum_{1\le l\le N-1, l<m\le N}\log(1-\frac{z_m}{z_l})} \\
&= \frac{1}{N!}\oint\left(\prod_{i=1}^{N}\frac{dz_i}{2\pi i}\right) e^{-\sum_{i>0}V(z_i)} e^{2\sum_{1\le l\le N-1, l<m\le N}\log(z_l-z_m)} \\
&= \frac{1}{N!}\int\left(\prod_{i=1}^{N} dz_i\right) e^{-\sum_{i>0}V(z_i)}\prod_{l<m}(z_l-z_m)^2
\end{aligned}
\tag{2.129}
$$

using $\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z_j}{z_i}\right)^k = -\log(1-\frac{z_j}{z_i})$ with $|z_j| < |z_i|$, and where the last step is obtained by analytic continuation of the contour integral, to change the integration in the complex plane to an integration over real values of z_i. The correlator (2.129) is indeed the one-matrix discrete partition function $Z_{\mathrm{matrix}} = e^{Z_{\mathrm{grav}}}$ in (2.23) if the variables z_i are taken to be the eigenvalues of the $N \times N$-matrix M, and if $V(z)$ is taken to be the potential of the model, i.e. with the coupling constants $t_k = g_k/N^{\frac{k}{2}-1}$. Note that since the potentials of the one-matrix models in the previous sections were *polynomial* potentials, the expansion (2.105) of $V(z)$ has to break off for some value of k. This completes the proof that $\langle N, \{t\}|e^{Q_+}|0\rangle$ is the partition function of the one-matrix model.

This approach is a constructive approach, which has been used to construct more complicated models (more complicated backgrounds) [19, 20]. This, however, lies beyond the aim of my thesis work.

2.4.2 Continuum Virasoro Constraints

After having obtained the discrete Virasoro constraints (2.99) in the previous section, the aim of this section is to find the continuum (i.e. double scaling limit) Virasoro constraints for the one-matrix model with hermitean $N \times N$-matrices M and an *even* potential[5]

$$
\begin{aligned}
\mathcal{L}_n^{\mathrm{cont}} &= \sum_{k\ge 0}\left(k+\tfrac{1}{2}\right)T_k\frac{\partial}{\partial T_{k+n}} + G\sum_{0\le k\le n-1}\frac{\partial^2}{\partial T_k\partial T_{n-k-1}} \\
&\quad + \frac{\delta_{n,0}}{16} + \frac{\delta_{n+1,0}T_0^2}{16G}
\end{aligned}
\tag{2.130}
$$

[5]This is with the notation of [19, 21], correspondence with the notation of [18] and of the references used in Chapter 3 on topological pure gravity, is obtained by $T_{\mathrm{matrix}} = t_{\mathrm{topo}}\frac{\sqrt{2G}}{\lambda}$ (and not by $T_{\mathrm{matrix}} = t_{\mathrm{topo}}\frac{\sqrt{G}}{\sqrt{2\lambda}}$ as claimed in [19]). G, or λ, might be fixed at the same value as in topological gravity.

However, this is a lengthy task, which has been undertaken by the authors of [21]. In their 55-page paper they show in great detail how the continuum limit of the Virasoro constraints is obtained, via results for the complex one-matrix model

$$Z_N^{\mathrm{c}}(t) = \int \mathcal{D}M\mathcal{D}M^\dagger e^{-\mathrm{Tr}\sum_{k\geq 0} t_{2k}(MM^\dagger)^k} \tag{2.131}$$

where M is now an $N \times N$-complex matrix.

In this section I will give the main lines of this from [21], not claiming to explain *all* the reasons for the many changes of variables, changes between models etc. There is also a shorter (and in my opinion slightly confusing) review of how to obtain the continuum constraints in [19].

The starting point is the hermitean one-matrix model with a general potential $\mathrm{Tr}\sum_{k\geq 0} t_k M^k$ and partition function

$$Z_N(t) = \int \mathcal{D}M e^{-\mathrm{Tr}\sum_{k\geq 0} t_k M^k} \tag{2.132}$$

One then wants to show that the *reduced* (that is, with $t_{\mathrm{odd}} = 0$) hermitean model with $2N \times 2N$ matrices — almost — corresponds to the square of the $N \times N$ complex model in the double scaling limit

$$Z_{2N,\mathrm{red}} \sim (Z_N^{\mathrm{c}})^2 \tag{2.133}$$

The complex partition function Z_N^{c} satisfies the constraints

$$L_n^{\mathrm{c}} Z_N^{\mathrm{c}} = 0 \quad \text{for } n \geq 0;$$

$$L_n^{\mathrm{c}} = \sum_{k\geq 0} k t_{2k} \frac{\partial}{\partial t_{2(k+n)}} + \sum_{0\leq k\leq n} \frac{\partial^2}{\partial t_{2k}\partial t_{2(n-k)}} \tag{2.134}$$

which, after appropriate change of the couplings $\{t\} \to \{T\} \to \{\tilde{T}\}$, turn into

$$\begin{aligned}
\tilde{\mathcal{L}}_{-1} &= \sum_{k\geq 1} \left(k + \tfrac{1}{2}\right) T_k \frac{\partial}{\partial \tilde{T}_{k-1}} + \frac{T_0^2}{16G} \\
\tilde{\mathcal{L}}_0 &= \sum_{k\geq 0} \left(k + \tfrac{1}{2}\right) T_k \frac{\partial}{\partial \tilde{T}_k} \\
\tilde{\mathcal{L}}_n &= \sum_{k\geq 0} \left(k + \tfrac{1}{2}\right) T_k \frac{\partial}{\partial T_{k+n}} \\
&\quad + G \sum_{0\leq k\leq n-1} \frac{\partial^2}{\partial T_k \partial T_{n-k-1}} - (-1)^n \frac{1}{16a^n} \qquad \text{for } n \geq 1
\end{aligned} \tag{2.135}$$

with $a \to 0$, $\tilde{T} \to T$ and G fixed in the double scaling limit, as will be discussed in a moment. Thus in the double scaling limit these are *almost* the operators

(2.130), so that $\tilde{\mathcal{L}}_n Z_N^{\rm c} \sim \tilde{\mathcal{L}}_n \sqrt{Z_{2N,\rm red}} = 0$ are *almost* the constraints sought after, in the double scaling limit. However, there are a few "almosts" to be taken care of in the above. This is what I will loosely discuss in the following. For details and/or further explanation, see [21].

First consider the relation (2.133). The Virasoro constraints (2.99) acting on Z_{2N} are, when rewritten to act on $\log Z_{2N}$

$$\left[\sum_k k t_k \frac{\partial}{\partial t_{n+k}} + \sum_m \frac{\partial^2}{\partial t_m \partial t_{n-m}}\right] \log Z_{2N} + \sum_m \left[\frac{\partial \log Z_{2N}}{\partial t_m} \frac{\partial \log Z_{2N}}{\partial t_{n-m}}\right] = 0 \quad (2.136)$$

For the reduced partition function $Z_{2N,\rm red}$ one has

$$\frac{\partial}{\partial t_{\rm odd}} \log Z_{2N,\rm red} = 0 \quad (2.137)$$

and (2.136) becomes

$$\left[\sum_k 2k t_{2k} \frac{\partial}{\partial t_{2n+2k}} + \sum_m \frac{\partial^2}{\partial t_{2m} \partial t_{2n-2m}} + \sum_m \frac{\partial^2}{\partial t_{2m+1} \partial t_{2n-2m-1}}\right] \log Z_{2N,\rm red}$$
$$+ \sum_m \left[\frac{\partial \log Z_{2N,\rm red}}{\partial t_{2m}} \frac{\partial \log Z_{2N,\rm red}}{\partial t_{2n-2m}}\right] = 0 \quad (2.138)$$

Now, **if** in the double scaling limit the following holds

$$\sum_m \frac{\partial^2 \log Z_{2N,\rm red}}{\partial t_{2m+1} \partial t_{2n-2m-1}} \sim \sum_m \frac{\partial^2 \log Z_{2N,\rm red}}{\partial t_{2m} \partial t_{2n-2m}} \quad (2.139)$$

then

$$\left[\sum_k k t_{2k} \frac{\partial}{\partial t_{2n+2k}} + \sum_m \frac{\partial^2}{\partial t_{2m} \partial t_{2n-2m}}\right] \log Z_{2N,\rm red}$$
$$+ \tfrac{1}{2} \sum_m \left[\frac{\partial \log Z_{2N,\rm red}}{\partial t_{2m}} \frac{\partial \log Z_{2N,\rm red}}{\partial t_{2n-2m}}\right] = 0 \quad (2.140)$$

which can be written, using $\frac{1}{2} \log Z_{2N,\rm red} = \log \sqrt{Z_{2N,\rm red}}$ and returning to constraints directly on $\sqrt{Z_{2N,\rm red}}$ instead of $\log \sqrt{Z_{2N,\rm red}}$, as

$$\left[\sum_k k t_{2k} \frac{\partial}{\partial t_{2n+2k}} + \sum_m \frac{\partial^2}{\partial t_{2m} \partial t_{2n-2m}}\right] \sqrt{Z_{2N,\rm red}} = 0\ . \quad (2.141)$$

These are the same constraints (2.134) satisfied by the complex partition function $Z_N^{\rm c}$, hence

$$Z_{2N,\rm red} \sim (Z_N^{\rm c})^2 \quad (2.142)$$

provided (2.139) holds in the double scaling limit, which is actually not the case, as I will return to in a moment. But first assume that (2.139) holds, and return to the complex model, to see how the complex model continuum constraints (2.135) come about.

Consider the following change of couplings $\{t\} \to \{T\}$

$$T_m = \sum_{k \geq m} \frac{\sqrt{G} a^{m+\frac{1}{2}} h_k \Gamma(k+\frac{1}{2})}{(k-m)! \Gamma(m+\frac{3}{2})} \tag{2.143}$$

or the reversed

$$h_k \equiv k t_{2k} = \sum_{m \geq k} (-1)^{k-m} \frac{a^{-m-\frac{1}{2}} T_m \Gamma(m+\frac{3}{2})}{\sqrt{G}(m-k)! \Gamma(k+\frac{1}{2})} \tag{2.144}$$

where in the double scaling limit $a \to 0$ with G fixed (namely the a and G which appeared in (2.135)). The reason for this change of variables is, in short, that whereas the couplings t_{2k} of the complex model couple to $\mathrm{Tr}(M\, M^\dagger)^k$, and in the hermitean reduced model to $\mathrm{Tr} M^{2k}$, the operators of interest are linear combinations of them, [15, 17] and [16, sect. 4], and the new couplings (2.143) are associated with these operators [21].

Actually an extra change $\{T\} \to \{\tilde{T}\}$

$$T_m = \tilde{T}_m + a \frac{m}{m+\frac{1}{2}} \tilde{T}_{m-1} \qquad \text{for } m \geq 1 \tag{2.145}$$

and

$$T_0 = \tilde{T}_0 - 4N\sqrt{Ga} \tag{2.146}$$

is necessary in order to get variables (couplings) $\tilde{T}$ that in the double scaling limit can be related to the KdV-flows [21]. Clearly $\tilde{T}_m \to T_m$ in the double scaling limit $a \to 0$. With these changes $\{t\} \to \{T\} \to \{\tilde{T}\}$ the operators of (2.134) (now with a superscript to denote couplings) might be written (see [21, Sect. 6.2])

$$L_n^{\{t\}} = \sum_{0 \leq k \leq n} a^{k-1} \binom{n}{k} \mathcal{L}_{k-1}^{\{\tilde{T}\}} \tag{2.147}$$

or

$$\mathcal{L}_n^{\{\tilde{T}\}} = a^{-n} \sum_{0 \leq k \leq n+1} (-1)^{k+n+1} \binom{n+1}{k} L_k^{\{t\}} \tag{2.148}$$

with

$$\begin{aligned}
\mathcal{L}_{-1}^{\{\tilde{T}\}} &= \sum_{k \geq 1} (k+\tfrac{1}{2}) T_k \frac{\partial}{\partial \tilde{T}_{k-1}} + aN^2 \\
\mathcal{L}_n^{\{\tilde{T}\}} &= \sum_{k \geq -n} (k+\tfrac{1}{2}) T_k \frac{\partial}{\partial \tilde{T}_{k+n}} \\
&\quad + G \sum_{0 \leq k \leq n-1} \frac{\partial^2}{\partial \tilde{T}_k \partial \tilde{T}_{n-k-1}} - (-1)^n a^{-n} N^2 \qquad \text{for } n \geq 0
\end{aligned} \tag{2.149}$$

and still with

$$\mathcal{L}_n^{\{\tilde{T}\}} Z_N^{\mathrm{C}} = 0 \quad \text{for } n \geq -1 \; . \tag{2.150}$$

Note that both sets of variables T and $\tilde{T}$ appear in (2.149). Even when ignoring any terms with powers (positive or negative) of a, and when taking $\tilde{T} = T$, (2.149) is not quite the same as the continuum constraints (2.130) one wants to arrive at (which are the same constraints found in the topological pure gravity case) — the first sum starts at a different value of the summation variable k, and the two last terms in (2.130) do not appear in (2.149).

Now, rescaling the partition function

$$Z_N^{\mathrm{C}} \to \tilde{Z}_N^{\mathrm{C}} = e^{-\frac{1}{2}A_{mn}\tilde{T}_m\tilde{T}_n} Z_N^{\mathrm{C}} \tag{2.151}$$

with

$$A_{mn} = \frac{(-1)^{n+m}\Gamma(n+\frac{3}{2})\,\Gamma(m+\frac{3}{2})G^{-1}a^{-m-n-1}}{2\Gamma^2(\frac{1}{2})\,(n+m+1)(n+m+2)n!\,m!} \tag{2.152}$$

and the operators

$$\mathcal{L}_n^{\{\tilde{T}\}} \to \tilde{\mathcal{L}}_n^{\{\tilde{T}\}} = e^{-\frac{1}{2}A_{mn}\tilde{T}_m\tilde{T}_n} \mathcal{L}_n^{\{\tilde{T}\}} e^{\frac{1}{2}A_{mn}\tilde{T}_m\tilde{T}_n} \tag{2.153}$$

gives

$$\begin{aligned}
\tilde{\mathcal{L}}_n^{\{\tilde{T}\}} \tilde{Z}_N^{\mathrm{C}} &= e^{-\frac{1}{2}A_{mn}\tilde{T}_m\tilde{T}_n} \mathcal{L}_n^{\{\tilde{T}\}} e^{\frac{1}{2}A_{mn}\tilde{T}_m\tilde{T}_n} e^{-\frac{1}{2}A_{mn}\tilde{T}_m\tilde{T}_n} Z_N^{\mathrm{C}} \\
&= e^{-\frac{1}{2}A_{mn}\tilde{T}_m\tilde{T}_n} \mathcal{L}_n^{\{\tilde{T}\}} Z_N^{\mathrm{C}} = 0
\end{aligned} \tag{2.154}$$

where the explicit form of the constraints becomes (2.135). Comparing to the continuum constraints (2.130) in the double scaling limit $a \to 0$ (where also $\tilde{T} \to T$) the only two discrepancies are the terms $\delta_{n,0}\frac{1}{16}$ and, more important $-(-1)^n\frac{1}{16a^n}$; the latter does not vanish in the $a \to 0$ limit, in contrary, it blows up! Hence the complex one-matrix model does not have a well-defined continuum (double scaling) limit.

Returning to the reduced hermitean model, the connection (2.133) would indicate that also this model would have problems in the continuum. But fortunately there were problems with the "relation" (2.139), these problems save the continuum of the (reduced) hermitean model. Namely, the difference of the two sides in (2.139) changes the form of the constraints (2.141) by an extra term, this extra term turns out after tedious calculations [21, Sect. 4.2] to exactly cancel the term $-(-1)^n\frac{1}{16a^n}$ of (2.135) and to add the term $\delta_{n,0}\frac{1}{16}$, whereby the continuum constraints (2.130) are found in the double scaling limit of the thus revised constraints (2.135)!

However, it is still unclear if this approach is valid [22], and the whole business of negative powers of a in the $a \to 0$ limit, even when they are removed as above, is a controversial one. There might also more generally be problems when restricting to *even* potentials $\mathrm{Tr}\sum_{k\geq 0} t_{2k}M^{2k}$.

2.4.3 τ-functions of the KdV-hierarchy, the String Equation

The main point in comparing the matrix model approach of pure gravity to the topological field theory approach (Chapter 3) is to show that (the square root of) their partition functions satisfy the same Virasoro constraints. But there is another interesting connection already mentioned, namely that the same Virasoro constraints except the $\mathcal{L}_{-1}$ constraint are also satisfied by τ-functions of the Korteweg–deVries (KdV) hierarchy. Exactly which of these τ-functions correspond to the square root of the partition functions (now ignoring the above-mentioned problems) is determined by the $\mathcal{L}_{-1}$ constraint $\mathcal{L}_{-1}\tau = 0$, which will be shown to be the string equation.

There are several ways to define the τ-functions of the Kadomtsev–Petviashvili (KP) hierarchy (which can be reduced to the KdV-hierarchy), reflecting the historical evolution of the theory. I shall here use the definition used in [23]. There (in short), one first defines a wavefunction $w(t,k)$ as the eigenfunction of the pseudo-differential operator $L(t,\partial)$

$$L(t,\partial)w(t,k) = kw(t,k) \tag{2.155}$$

where $t = (t_1, t_2, t_3, \ldots)$ is an infinite set of variables, L is given by

$$L(t,\partial) = \partial + u_2(t)\partial^{-1} + u_3(t)\partial^{-2} + \ldots \tag{2.156}$$

and $\partial = \frac{\partial}{\partial t_1}$. The constraint $\frac{\partial}{\partial t_n}L = [(L^n)_+, L]$ determines the functions $u_2, u_3, \ldots$, and imposing the restriction $L^2 = \partial^2 + 2u_2(t)$ reduces the KP-hierarchy to the KdV- hierarchy. The wavefunction $w(t,k)$ satisfies the *bilinear identity*

$$\oint dkw(t,k)w^*(t',k) = 0 \tag{2.157}$$

with the contour around $k = \infty$; $w^*(t,k)$ is the adjoint of $w(t,k)$. The τ-function is defined by the following relation

$$w(t,k) = \frac{\tau(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, t_3 - \frac{1}{3k^3}, \ldots)}{\tau(t_1, t_2, t_3, \ldots)} e^{\sum_{n\geq 1} t_n k^n} . \tag{2.158}$$

One would now like to define a quantity using a free fermion formalism and then show that this quantity is a τ-function of the KP-hierarchy. First define the free fermions

$$\psi(k) = \sum_{n\in \mathsf{Z}} \psi_n k^n \ ; \qquad \psi^*(k) = \sum_{n\in \mathsf{Z}} \psi^*_n k^{-n} \tag{2.159}$$

with

$$\psi_{-n}|0\rangle = 0 \text{ for } n > 0, \qquad \psi^*_n|0\rangle = 0 \text{ for } n \geq 0 \tag{2.160}$$

(here I use the indices of [23] for the fermions, this is not an important point), and define the "Hamiltonian of the KP-hierarchy"

$$H(t) = \sum_{l\geq 1} t_l H_l \ , \quad H_l = \sum_{n\in \mathsf{Z}} \psi_n \psi^*_{n+l} \tag{2.161}$$

It is easy to check $H(t)|0\rangle = 0$. The evolution of an operator a is defined by $a(t) = e^{H(t)} a e^{-H(t)}$, and the quantity one would like to show to be a τ-function is

$$\tau(t;g) = \langle 0|g(t)|0\rangle = \langle 0|e^{H(t)}g|0\rangle \tag{2.162}$$

where g is some not fully specified operator, yet fulfilling certain restrictions [23]. The operator g satisfies

$$g\psi_n = \sum_{m\in \mathsf{Z}} \psi_m g a_{mn} \ , \qquad \psi^*_n g = \sum_{m\in \mathsf{Z}} g\psi^*_m a_{nm} \tag{2.163}$$

(there is no need in the following to specify a_{nm}). The various operators g make up the hierarchy, g determines the τ-function. But first one needs to show that (2.162) is indeed a τ-function.

Define the quantities

$$w(t,k) = \frac{\langle 1|e^{H(t)}\psi(k)g|0\rangle}{\langle 0|e^{H(t)}g|0\rangle} \tag{2.164}$$

and

$$w^*(t,k) = \frac{\langle -1|e^{H(t)}\psi^*(k)g|0\rangle}{k\langle 0|e^{H(t)}g|0\rangle} \tag{2.165}$$

If one can show that $w(t,k)$ as defined above in (2.164) satisfies the bilinear identity (2.157) and that τ and w in (2.162) and (2.164) are connected as in (2.158), then (2.162) is a τ-function.

With the fermions obtained by bosonization, $\psi(k) =: e^{\phi(k)} :$, $\phi(k) = \hat{q} + \hat{p}\log k + \sum_{n\neq 0} \frac{H_n}{n} k^{-n}$ one finds in a way similar to the derivation (2.128) of the discrete Virasoro constraints, but with different constants, that

$$\langle 1|e^{H(t)}\psi(k) = \langle 1,\{t\}|\psi(k) = e^{\sum_{l\geq 1} t_l k^l}\langle 0,\{t_n - \frac{1}{nk^n}\}| \tag{2.166}$$

Hence

$$\begin{aligned} w(t,k) &= \frac{\langle 1|e^{H(t)}\psi(k)g|0\rangle}{\tau(t;g)} \\ &= \frac{\tau(\{t_n - \frac{1}{nk^n}\};g)}{\tau(t;g)} e^{\sum_{l\geq 1} t_l k^l} \end{aligned} \tag{2.167}$$

which is the relation (2.158). The bilinear identity

$$
\begin{aligned}
I &= \oint \frac{dk}{2\pi i} w(t,k) w^*(t',k) \\
&= \oint \frac{dk}{2\pi i} \frac{\langle 1|e^{H(t)}\psi(k)g|0\rangle}{\tau(t;g)} \frac{\langle -1|e^{H(t)}\psi^*(k)g|0\rangle}{k\tau(t;g)} \\
&= \oint \frac{dk}{2\pi i} \frac{\langle 1|e^{H(t)}(\sum_{n\in Z}\psi_n k^n)g|0\rangle}{\tau(t;g)} \frac{\langle -1|e^{H(t)}(\sum_{m\in Z}\psi^*_m k^{-m})g|0\rangle}{k\tau(t;g)} \\
&= \sum_{n\in Z} \frac{\langle 1|e^{H(t)}\psi_n g|0\rangle}{\tau(t;g)} \frac{\langle -1|e^{H(t)}\psi^*_n g|0\rangle}{\tau(t;g)} \qquad (2.168)
\end{aligned}
$$

Using (2.163) to get

$$
\begin{aligned}
&\sum_{n\in Z} \ldots \psi_n g \ldots \psi^*_n g \\
&\quad = \sum_{n,m\in Z} \ldots \psi_n g \ldots g\psi^*_m a_{nm} \\
&\quad = \sum_{n,m\in Z} \ldots \psi_n g a_{nm} \ldots g\psi^*_m \\
&\quad = \sum_{m\in Z} \ldots g\psi_m \ldots g\psi^*_m \qquad (2.169)
\end{aligned}
$$

one finds

$$
I = \sum_{m\in Z} \frac{\langle 1|e^{H(t)}g\psi_m|0\rangle}{\tau(t;g)} \frac{\langle -1|e^{H(t)}g\psi^*_m|0\rangle}{\tau(t;g)} \qquad (2.170)
$$

For a given m either $\psi_m|0\rangle = 0$ or $\psi^*_m|0\rangle = 0$, therefore $w(t,k)$ as defined in (2.164) satisfies the bilinear identity. Thus $\tau(t;g) = \langle 0|e^{H(t)}g|0\rangle$ is a τ-function of the KP-hierarchy.

With a technique similar to the technique used to find the discrete Virasoro constraints, (2.129), one can then show that (2.162) satisfies the $n \geq 0$ (continuum) Virasoro constraints when all even-index variables t_{even} are set to zero (whereby the KdV-reduction is found) and the rest of the variables are re-labeled $t_{2m+1} \leftrightarrow T_m$.

To show that the additional constraint $\mathcal{L}_{-1}\tau = 0$ is the string equation it is convenient to return to the variables $t = (t_1, t_2, \ldots)$ of the KdV-hierarchy, with all $t_{even} = 0$. In terms of these variables the $n = -1$ constraint reads

$$
\mathcal{L}_{-1}\tau = \left(\tfrac{1}{2} \sum_{k\geq 2} k t_k \frac{\partial}{\partial t_{k-2}} + \frac{t_1^2}{4} \right) \tau = 0 \qquad (2.171)
$$

where a definite value of G has been chosen. The sum is over both odd and even values of k, but the terms with even indices vanish anyway. Taking the derivative with respect to t_1 (2.171) becomes

$$\begin{aligned} 0 &= 2\frac{\partial}{\partial t_1}\left(\tau^{-1}\mathcal{L}_{-1}\tau\right) \\ &= \frac{\partial}{\partial t_1}\left(\sum_{k\geq 2} kt_k\frac{\partial\log\tau}{\partial t_{k-2}}+\frac{t_1^2}{2}\right) \\ &= \sum_{k\geq 2} kt_k\frac{\partial^2\log\tau}{\partial t_1\partial t_{k-2}}+t_1 \end{aligned} \tag{2.172}$$

The double derivative of τ can be written in terms of the pseudo-differential operator $L(t,\partial_1)$ (the "inverse" of relation (2.158)) [23]

$$\frac{\partial^2}{\partial t_1\partial t_{k-2}}\log\tau = (L^{k-2})_{-1} \tag{2.173}$$

which in turn can be written as one of the Gelfand–Dikii polynomials using the definition (2.84)

$$(L^{k-2})_{-1} = 2R_{(k-1)/2} \tag{2.174}$$

— remember that only variables t_k with odd indices might be non-vanishing, thus only $k =$ odd is relevant in (2.172). Relation (2.172) then reads

$$\sum_{k\geq 2} 2kt_k R_{(k-1)/2}+t_1 = 0 \tag{2.175}$$

or, when changing labeling,

$$\sum_{m\geq 1} 4(m+\tfrac{1}{2})T_m R_m = -x \tag{2.176}$$

with $T_0 = x$ as is the usual notation. This is the *massive* string equation, the string equation (2.71) is found for $T_2 =$ fixed value $\neq 0$ and $T_m = 0$, $m \neq 2$

$$-x = 10T_2R_2 = 5T_2(\frac{3}{8}u^2-\frac{1}{8}u'') \tag{2.177}$$

again with suitable rescalings of (2.71). The index m of the non-zero variable in (2.176) corresponds to the mth critical point of the matrix model, and to $(2,2m-1)$-minimal matter; $m = 2$ corresponds to pure gravity, as we just saw.

Chapter 3

Topological String Theory

Like in the conventional discussion of the string one can distinguish the matter sector of the topological string theory, and thereafter couple the matter sector to gravity in order to obtain a string theory. The two-dimensional topological field theory, which I will describe first, can be considered being the theory of the matter sector of the topological string, and the string theory appears when coupling to gravity while preserving the special features of topological field theory.

A huge amount of work has been done in the past few years on the subject of topological field theories, see for example [25, 28, 29, 30]. I therefore had to choose where to concentrate my reading, and thereby also the presentation given here. The presentation will mainly be an introduction to two-dimensional topological field theories, i.e. to the matter sector of the topological string. This is the subject of Sections 3.1 and 3.2. At the end of this chapter, in Section 3.3, I will touch upon the subject of coupling the matter sector to gravity, and of obtaining constraints on the partition function of *pure* topological gravity.

3.1 Matter Sector — 2D Topological Field Theory

The usual way to define a quantum field theory is to specify a Lagrangian $\mathcal{L}$. However, more than one Lagrangian might lead to the same quantum field theory, so that two apparently different quantum field theories (different Lagrangians) turn out to be the same. This ambiguity is avoided if the quantum field theories are instead defined by the properties of their correlators. The correlators will, in general, depend on the operators of the theory (in string theory the in- and outgoing states, see Figure 3.1), the positions of the operators, the metric and the topology of the manifold where the operators live etc., e.g. in string theory

$$\langle \tilde{\phi}_1(X^\mu(z_{(1)},\bar{z}_{(1)}))\,\tilde{\phi}_2(X^\mu(z_{(2)},\bar{z}_{(2)}))\cdots\tilde{\phi}_s(X^\mu(z_{(s)},\bar{z}_{(s)}))\rangle$$

string theory topological field theory

Figure 3.1: Mapping of string in- and out states into operators of the topological quantum field theory

String theory is a geometrical theory, and should be invariant under reparametrizations; the above-mentioned general correlators should therefore be integrated over moduli space in order to get physical correlators. This integration has to be carried out explicitly, for example in the Liouville approach to the gravity part of string theory, and implicitly in the matrix model. In topological quantum field theory (hereafter abbreviated to TQFT) this is, in principle, much simpler.

A TQFT in two dimensions can be defined by the following properties of the correlators:

1. The correlators only depend on the operators ϕ_i and the topology, or genus, of the Riemann surface Σ

$$\langle \phi_1(x_1)\phi_2(x_2)\cdots\phi_s(x_s)\rangle_\Sigma = \langle \phi_1\phi_2\cdots\phi_s\rangle_g \tag{3.1}$$

   where g is the genus of the surface Σ and x_r is some generalized coordinate on the surface Σ (clearly with a connection to the coordinates $(z_{(r)}, \bar{z}_{(r)})$ of the string). Hence the correlators are independent of the metric, and thus also of the positions of the operators $\phi_i(x_i)$.

The exact connection between the string theory operators $\tilde{\phi}_r(X^\mu(z_{(r)}, \bar{z}_{(r)}))$ and the operators ϕ_r in topological field theory is a whole topic on its own. Here I will just assume that the connection can be found.

The fact that the correlator independence of the metric implies independence of the operator position is easily seen — by a change in the metric the position of the operators can be changed, but a change in the metric does not change the correlators.

2. Factorization properties. The correlators satisfy two factorization properties,

$$\langle \phi_1\phi_2\cdots\phi_n\cdots\phi_s\rangle_g = \sum_{i,j}\langle \phi_1\phi_2\cdots\phi_n\phi_i\rangle_{g_1}\eta^{ij}\langle \phi_j\phi_{n+1}\cdots\phi_s\rangle_{g_2} \tag{3.2}$$

with $g = g_1 + g_2$, and

$$\langle \phi_1 \phi_2 \cdots \phi_s \rangle_g = \sum_{i,j} \eta^{ij} (-1)^{F_i} \langle \phi_i \phi_j \phi_1 \cdots \phi_s \rangle_{g-1} \tag{3.3}$$

where η^{ij} is the inverse of the two-point function $\eta_{ij} \equiv \langle \phi_i \phi_j \rangle_{g=0}$ and F_i is the fermion number of the field ϕ_i. The sum is over all operators.

There are several remarks to these definitions. First note that the independence of the metric for the correlators is a *definition,* in usual string theories there *is* a dependence of a dynamical metric, but the metric is integrated over, and the resulting correlators do not depend on the metric any more. In TQFT there is no need for this integration, the correlators are independent of the metric by construction.

Second, at this point the operators ϕ_i are all physical operators of the theory, since these are the only operators involved yet. Soon operators that are not physical will be included. The sum in the factorization is over *all* operators in the theory, including *non-physical* operators if present in the theory. This should be kept in mind when extending the theory to non-physical operators in the following subsections. It turns out, however, that for physical correlators (i.e. correlators with only physical operators) only physical operators contribute to the sums in the factorizations.

The factorization process can be illustrated as follows. First, for the sake of the illustration, think of the positions x_i of the operators $\phi_i(x_i)$ as small "holes" cut in the surface; "holes" that can be shrunk to the points x_i — formally, apply a puncture operator $\mathcal{P}_i$ at the position x_i, and fill in the operator ϕ_i. Notice that even though the correlators do not depend on the positions of the operators, the operators are of course still thought of as having a position.

Also, again for the sake of the illustration, distinguish between "in-" and "out-going" operators, labeled by a subscript ϕ_i resp. a superscript ϕ^j. A correlator with the two operators ϕ_i and ϕ^j, $\langle \phi_i \phi^j \rangle_{g=0}$, can then be illustrated as in Figure 3.2, where also a few other correlators are illustrated.

The two factorization properties can now be illustrated as in Figure 3.3 and 3.4, where I considered only bosonic operators. Actually, using the above illustrations it seems plausible that by "sewing" together a number of two- and three-point functions (Figure 3.2) one can obtain any topology with any number of operators. This will be further discussed later.

3.1.1 The Q-symmetry

The way TQFT was defined in the above, via the properties of its correlation functions and with only physical operators present in the theory, no metric was involved. However, in a corresponding Lagrangian formulation of TQFT one

Figure 3.2: Some two- and three-point functions

$$
\begin{aligned}
\langle \phi_1 \phi_2 \phi_3 \rangle_{g=2} &= \sum_{i,j} \langle \phi_1 \phi_2 \phi_i \rangle_{g_1=1} \langle \phi^i \phi^j \rangle_{g'=0} \langle \phi_j \phi_3 \rangle_{g_2=1} \\
&= \sum_{i,j} \langle \phi_1 \phi_2 \phi_i \rangle_{g_1=1} \eta^{ij} \langle \phi_j \phi_3 \rangle_{g_2=1}
\end{aligned}
$$

Figure 3.3: Illustrating the first factorization

$$
\begin{aligned}
\langle\phi_1\phi_2\phi_3\rangle_{g=2} &= \sum_{i,j}\langle\phi^i\phi^j\rangle_{g'=0}\langle\phi_i\phi_j\phi_1\phi_2\phi_3\rangle_{g-1=2-1=1} \\
&= \sum_{i,j}\eta^{ij}\langle\phi_i\phi_j\phi_1\phi_2\phi_3\rangle_1
\end{aligned}
$$

Figure 3.4: Illustrating the second factorization, bosonic operators

needs to introduce a metric g_{ab}, and the energy-momentum tensor T_{ab} vanishes

$$T_{ab} \equiv \frac{1}{\sqrt{g}}\frac{\delta S_{\text{topo}}}{\delta g^{ab}} = 0 \tag{3.4}$$

because the theory is independent of the metric.

Define by $|\phi_i\rangle \equiv \phi_i|0\rangle$ a state in the Hilbert space $\mathcal{H}_{phys}$, corresponding to the physical operator, or local observable, ϕ_i. Now, one can extend $\mathcal{H}_{phys}$ to a Hilbertspace $\mathcal{H} \supset \mathcal{H}_{phys}$ including *non-physical* states. As can be done in the case of gauge theories etc., introduce a fermionic, nilpotent operator Q generating a BRST-like symmetry on the Hilbert space $\mathcal{H}$,

$$\mathcal{H} \xrightarrow{Q} \mathcal{H} \tag{3.5}$$

such that the physical states are the Q-closed states (with a terminology borrowed from the theory of differential geometry)

$$Q|\phi_i\rangle = 0, \qquad |\phi_i\rangle \in \mathcal{H}_{phys}, \tag{3.6}$$

or, in terms of operators

$$[Q,\phi_i] = 0 \tag{3.7}$$

for a bosonic operator ϕ_i and

$$\{Q,\phi_i\} = 0 \tag{3.8}$$

for fermionic operators. Throughout the rest of the presentation of the topological field theory approach **I will restrict the presentation to *bosonic* physical**

operators ϕ; the fermionic operators can be included by changing the commutators and the anti-commutators appropriately.

Two physical states are considered equivalent if they are identical modulo Q-exact states, i.e. states of the form $Q|\lambda\rangle$, $|\lambda\rangle \in \mathcal{H}$ ($Q|\lambda\rangle$ is a "spurious state")

$$|\phi_i\rangle \sim |\phi_i\rangle + Q|\lambda\rangle \ , \quad |\phi_i\rangle \in \mathcal{H}_{phys} \tag{3.9}$$

or for operators

$$\phi_i \sim \phi_i + \{Q, \lambda\}. \tag{3.10}$$

This identification is reasonable, because operators of the form $\{Q, \lambda\}$ vanish in physical correlators, as we shall see below.

Thus Q generates a symmetry from which the physical states in $\mathcal{H}$ can be found, the physical states are the cohomology classes of $\mathcal{H}$

$$\mathcal{H}_{phys} = \frac{\ker Q}{\mathrm{Im}\ Q} \tag{3.11}$$

When extending the theory to the Hilbert space $\mathcal{H}$ the energy-momentum tensor in general does not vanish, but of course still does in the subspace $\mathcal{H}_{phys}$. To see this explicitly, extend the action to

$$S = S_{\text{topo}} + \{Q, R\} \tag{3.12}$$

— in $\mathcal{H}_{phys}$ we still have $S = S_{\text{topo}} + \{Q, R\} \sim S_{\text{topo}}$, hence the extension does not change anything in $\mathcal{H}_{phys}$. With the action (3.12) the energy-momentum tensor becomes Q-exact

$$\begin{aligned} T_{ab} &= \frac{1}{\sqrt{g}} \left(\frac{\delta S_{\text{topo}}}{\delta g^{ab}} + \{Q, \frac{\delta R}{\delta g^{ab}}\} \right) \\ &= \{Q, \frac{1}{\sqrt{g}} \frac{\delta R}{\delta g^{ab}}\} \\ &= \{Q, G_{ab}\} \ , \quad G_{ab} \equiv \frac{1}{\sqrt{g}} \frac{\delta R}{\delta g^{ab}} \end{aligned} \tag{3.13}$$

The quantity G_{ab} will play an important role in the following.

Q-exact Operators in Correlators

In this subsection I want to show that Q-exact operators, i.e. operators of the form

$$\{Q, \lambda\} \quad \text{— if } \lambda \text{ fermionic} \tag{3.14}$$

or

$$[Q, \sigma] \quad \text{— if } \sigma \text{ bosonic} \tag{3.15}$$

vanish in physical correlators

$$\langle\{Q,\lambda\}\phi_1\cdots\phi_s\rangle = 0, \quad \langle[Q,\sigma]\phi_1\cdots\phi_s\rangle = 0\ , \tag{3.16}$$

$\phi_1,\ldots,\phi_s$ physical operators. For the transformation

$$\tilde{\phi} = e^{\epsilon Q}\phi e^{-\epsilon Q} \tag{3.17}$$

with ϵ a small Grassmannian parameter, using the Lagrangian formulation, and using the invariance of the measure $\mathcal{D}\phi$ and the action $S[\phi]$ under the transformation (3.17), one finds the following for the bosonic operator $\sigma(\phi)$

$$\begin{aligned}
0 &= \int\mathcal{D}\tilde{\phi}e^{-S[\tilde{\phi}]}\sigma(\tilde{\phi}) - \int\mathcal{D}\phi e^{-S[\phi]}\sigma(\phi) \\
&= \int\mathcal{D}\phi e^{-S[\phi]}\left(e^{\epsilon Q}\sigma(\phi)e^{-\epsilon Q} - \sigma(\phi)\right) \\
&= \int\mathcal{D}\phi e^{-S[\phi]}[\epsilon Q,\sigma] \\
&= \langle[\epsilon Q,\sigma]\rangle \\
&= \epsilon\langle[Q,\sigma]\rangle
\end{aligned} \tag{3.18}$$

and in the same way one finds for the fermionic operator $\lambda(\phi)$

$$\langle[\epsilon Q,\lambda]\rangle = \epsilon\langle\{Q,\lambda\}\rangle = 0 \tag{3.19}$$

Thus when a Q-exact operator is inserted into a physical correlator, the correlator vanishes

$$\langle\{Q,\lambda\}\phi_1\cdots\phi_s\rangle = \langle\{Q,\lambda\phi_1\cdots\phi_s\}\rangle = 0 \tag{3.20}$$

with $\phi_1,\phi_2,\ldots,\phi_s$ physical operators and where I used the relation

$$\{Q,\lambda\phi\} = \{Q,\lambda\}\phi - \lambda\underbrace{[Q,\phi]}_{=0} \tag{3.21}$$

In particular, the energy-momentum tensor, which is Q-exact ($T_{ab} = \{Q,G_{ab}\}$) vanishes in physical correlators

$$\langle T_{ab}\phi_1\phi_2\cdots\phi_s\rangle = 0\ . \tag{3.22}$$

We can also check that it really does not matter which representative from a Q-cohomology class is used in the correlators

$$\begin{aligned}
\langle\phi_1\phi_2\cdots\phi_s\rangle &\stackrel{?}{=} \langle(\phi_1 + \{Q,\lambda\})\phi_2\cdots\phi_s\rangle \\
&= \langle\phi_1\phi_2\cdots\phi_s\rangle + \langle\{Q,\lambda\}\phi_2\cdots\phi_s\rangle \\
&= \langle\phi_1\phi_2\cdots\phi_s\rangle
\end{aligned} \tag{3.23}$$

thus the identification of the physical operators with the cohomology classes of Q, $\mathcal{H}_{phys} = \frac{\ker Q}{\operatorname{Im} Q}$, is valid.

$\mathcal{H}_{phys}$ Still Topological?

The introduction of a Q-cohomology makes the Hilbertspace $\mathcal{H}$ decompose into two orthogonal subspaces $\mathcal{H}_{phys}$ and $\mathcal{H}_{non-phys} = \mathcal{H}_{phys}^{\perp}$

$$\mathcal{H} = \mathcal{H}_{phys} \oplus \mathcal{H}_{phys}^{\perp} \; . \tag{3.24}$$

In the extended theory, $\mathcal{H}$, factorization properties similar to the factorizations (3.2, 3.3) arise when inserting the completeness relation $\mathbf{1} = \sum_{|\phi_i\rangle\in\mathcal{H}} |\phi_i\rangle\langle\phi_i|$ into the correlators. One should check that, after the extension of the theory, the subspace $\mathcal{H}_{phys}$ is still topological, i.e. that only *physical* operators contribute to the sum in the factorizations of *physical* correlators. Check this for the first factorization property (3.2):

$$\langle\phi_1\phi_2\cdots\phi_n\cdots\phi_s\rangle_g = \sum_{i,j}\langle\phi_1\phi_2\cdots\phi_n\phi_i\rangle_{g_1}\eta^{ij}\langle\phi_j\phi_{n+1}\cdots\phi_s\rangle_{g_2}$$

A state $|\phi_{n+1}\cdots\phi_s\rangle = \phi_{n+1}\cdots\phi_s|0\rangle$ with only physical operators $\phi_{n+1},\ldots,\phi_s \in \mathcal{H}_{phys}$ is itself a physical state (as might be expected ...)

$$Q|\phi_{n+1}\cdots\phi_s\rangle = Q\phi_{n+1}\cdots\phi_s|0\rangle = \phi_{n+1}\cdots\phi_s \underbrace{Q|0\rangle}_{=0} = 0 \tag{3.25}$$

thus all correlators of the form $\langle\psi_i\phi_{n+1}\cdots\phi_s\rangle = \langle\psi_i|\phi_{n+1}\cdots\phi_s\rangle$, with $|\psi_i\rangle \in \mathcal{H}_{phys}^{\perp}$ vanish. The sum in the second factorization property (3.3) also reduces to a sum over physical operators only, though the argument is a bit different.

3.1.2 Topological Conformal Field Theory

In this subsection I will make use of results from conformal field theory and $N = 2$ superconformal models. A short review of conformal field theory was given in Chapter 1, for a review of the $N = 2$ superconformal models, see below or the references there.

N=2 Superconformal Model

The $N = 2$ superconformal algebra consists, in addition to the energy-momentum tensor $T(z)$ from the (non-super-) conformal algebra, of a U(1) current $J(z)$ and two supercurrents $G^+(z)$ and $G^-(z)$ ($N = 2$ is the number of supercurrents in the theory), both with conformal dimension $\frac{3}{2}$. Superconformal transformations and the algebra of the superconformal model are found using techniques very similar to the non-super case described in Chapter 1, though one carefully has to keep track of two additional, Grassmannian, variables θ^+ and θ^-. The Grassmann variables live in superspace along with the usual bosonic variable z, superspace being described by the variable $Z \equiv (z, \theta^+, \theta^-)$, with a similar variable $\bar{Z} \equiv$

$(\bar{z}, \bar{\theta}^+, \bar{\theta}^-)$ for the right-moving part of the theory. Here I will give a short review of how the algebra of the $N=2$ superconformal model is obtained, mainly using [1].

The four quantities [1] $T(z)$, $J(z)$ and $G^\pm(z)$ can be encoded in a supercurrent $\tilde{J}(Z)$

$$\tilde{J}(Z) = J(z) + \tfrac{i}{\sqrt{2}}\theta^+ G^+(z) + \tfrac{i}{\sqrt{2}}\theta^- G^-(z) + \theta^-\theta^+ T(z) \ . \tag{3.26}$$

The supercurrent generates superconformal transformations of the superfield

$$\Phi(Z) = \phi(z) + \theta^+\psi^+ + \theta^-\psi^- + \theta^+\theta^- g(z) \ , \tag{3.27}$$

analogous with the conformal transformations of $\phi(z)$ generated by $T(z)$ in the non-super case (1.15)

$$\delta_E \Phi(Z) \equiv \left[\tilde{J}_E, \Phi(Z)\right] = \oint dZ' E(Z')\tilde{J}(Z')\Phi(Z) \tag{3.28}$$

with $dZ = \frac{dz}{2\pi i}d\theta^+ d\theta^-$ and with the contour of z' winding around z. $E(Z)$ is similar to $\epsilon(z)$ in the non-super-conformal case, and indeed $E(Z) \equiv \epsilon(z) - i\sqrt{2}\theta^+\beta^-(z) + i\sqrt{2}\theta^-\beta^+(z) + \theta^-\theta^+ t(z)$ where $\epsilon(z)$ generates the "usual" conformal transformations, $\beta^+(z)$ and $\beta^-(z)$ generate conformal supersymmetry transformations and $t(z)$ generates conformal U(1) transformations.

Again, quite similar to the non-super-conformal case, one finds operator product expansions etc. for $\tilde{J}(Z)$ and $\Phi(Z)$, and one finds the algebra [1]

$$\left[\tilde{J}_E, \tilde{J}_F\right] = \tilde{J}_G + \frac{c}{6}\oint dZ F(Z)[D^+, D^-]E'(Z) \tag{3.29}$$

where $G \equiv FE' - EF' + D^+FD^-E + D^-FD^+E$, with the covariant derivatives

$$D^\pm = \frac{\partial}{\partial\theta^\pm} + \tfrac{1}{2}\theta^\mp\frac{\partial}{\partial z} \ . \tag{3.30}$$

and with $E'(Z) = \frac{\partial}{\partial z}E(Z)$. In order to obtain the $N=2$ superconformal algebra, Laurent-expand the four supercurrents $T(z)$, $J(z)$ and $G^\pm(z)$ around $z=0$

$$T(z) = \sum_{n\in\mathsf{Z}} L_n z^{-n-2}, \quad J(z) = \sum_{n\in\mathsf{Z}} J_n z^{-n-1}$$

$$G^\pm(z) = \sum_r G_r^\pm z^{-r-\frac{3}{2}} \tag{3.31}$$

where the choice of summation $r \in \mathsf{Z} + \frac{1}{2}$ in the expansion of the supercurrents $G^\pm(z)$ brings us to the Neveu–Schwarz (NS) sector and the choice $r \in \mathsf{Z}$ to the

[1] I will call all four quantities "supercurrents" in the following, although strictly speaking $T(z)$ is not a current. Also $\tilde{J}(Z)$ will be called a supercurrent, because it is a superfield and a current.

Ramond (R) sector. Choosing the following transformations one by one, one arrives at the modes of the supercurrents $T(z)$, $J(z)$ and $G^{\pm}(z)$ from $\tilde{J}_E$

$$\begin{aligned}
E &= \epsilon(z) = \epsilon z^{n+1} && \Rightarrow \ \tilde{J}_E = \epsilon L_n \\
E &= -i\sqrt{2}\theta^+\beta^-(z) = -i\sqrt{2}\theta^+\beta^- z^{r+\frac{1}{2}} && \Rightarrow \ \tilde{J}_E = \beta^- G_r^- \\
E &= i\sqrt{2}\theta^-\beta^+(z) = i\sqrt{2}\theta^-\beta^+ z^{r+\frac{1}{2}} && \Rightarrow \ \tilde{J}_E = \beta^+ G_r^+ \\
E &= \theta^-\theta^+ t(z) = \theta^-\theta^+ t z^n && \Rightarrow \ \tilde{J}_E = t J_n
\end{aligned} \tag{3.32}$$

with ϵ and t bosonic parameters and with β^+ and β^- Grassmannian parameters. E.g. for $E = \epsilon z^{n+1}$ we can symbolically write $\tilde{J}_E$ (understood within a commutator)

$$\begin{aligned}
\tilde{J}_E &= \oint_0 dZ E(Z)\tilde{J}(Z) \\
&= \oint_0 \frac{dz}{2\pi i} d\theta^+ d\theta^- \epsilon z^{n+1}\{\sum_m J_m z^{-m-1} + \tfrac{i}{\sqrt{2}}\theta^+ \sum_r G_r^+ z^{-r-\frac{3}{2}} \\
&\qquad + \tfrac{i}{\sqrt{2}}\theta^- \sum_r G_r^- z^{-r-\frac{3}{2}} + \theta^-\theta^+ \sum_m L_m z^{-m-2}\} \\
&= \epsilon \oint_0 \frac{dz}{2\pi i} \sum_m L_m z^{n+1-m-2} \\
&= \epsilon L_n
\end{aligned} \tag{3.33}$$

and for $E = -i\sqrt{2}\theta^+\beta^- z^{r+\frac{1}{2}}$

$$\begin{aligned}
\tilde{J}_E &= \oint_0 dZ E(Z)\tilde{J}(Z) \\
&= \oint_0 \frac{dz}{2\pi i} d\theta^+ d\theta^- (-i\sqrt{2})\theta^+\beta^- z^{r+\frac{1}{2}}\{\sum_m J_m z^{-m-1} + \tfrac{i}{\sqrt{2}}\theta^+ \sum_s G_s^+ z^{-s-\frac{3}{2}} \\
&\qquad + \tfrac{i}{\sqrt{2}}\theta^- \sum_s G_s^- z^{-s-\frac{3}{2}} + \theta^-\theta^+ \sum_m L_m z^{-m-2}\} \\
&= i\sqrt{2}\beta^- \oint_0 \frac{dz}{2\pi i} d\theta^+ d\theta^- \theta^+ z^{r+\frac{1}{2}} \tfrac{i}{\sqrt{2}}\theta^- \sum_s G_s^- z^{-s-\frac{3}{2}} \\
&= \beta^- \oint_0 \frac{dz}{2\pi i} \sum_s G_s^- z^{r+\frac{1}{2}-s-\frac{3}{2}} \\
&= \beta^- G_r^-
\end{aligned} \tag{3.34}$$

With these transformations, using the algebra (3.29), the $N = 2$ algebra is found to be [1, 2]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

$$
\begin{aligned}
[J_m, J_n] &= \frac{c}{3}\, m\, \delta_{n+m,0} & [L_m, G_r^\pm] &= (\frac{m}{2} - r) G^\pm_{m+r} \\
[J_m, G_r^\pm] &= \pm G^\pm_{m+r} & [L_m, J_n] &= -n\, J_{m+n}
\end{aligned}
$$

$$
\{G_r^-, G_s^+\} = 2L_{r+s} - (r-s) J_{r+s} + \frac{c}{3}(r^2 - \tfrac{1}{4})\delta_{r+s,0} \tag{3.35}
$$

For example, find the Virasoro algebra of (3.35): with $E = z^{m+1}$ and $F = z^{n+1}$ (here I left out the ϵ-parameters) one has

$$
\begin{aligned}
&E'(Z) = (m+1) z^m, \quad F'(Z) = (n+1) z^n \\
&D^+ E(Z) = (\frac{\partial}{\partial \theta^+} + \tfrac{1}{2}\theta^- \frac{\partial}{\partial z}) z^{m+1} = \tfrac{1}{2}(m+1)\theta^- z^m \\
&D^- E(Z) = \tfrac{1}{2}(m+1)\theta^+ z^m \\
&D^+ F(Z) = \tfrac{1}{2}(n+1)\theta^- z^n \\
&D^- F(Z) = \tfrac{1}{2}(n+1)\theta^+ z^n
\end{aligned} \tag{3.36}
$$

hence

$$
\begin{aligned}
G &= FE' - EF' + D^+ F D^- E + D^- F D^+ E \\
&= (m+1) z^{m+n} - (n+1) z^{m+n} + \tfrac{1}{4}(m+1)(n+1)\theta^- \theta^+ z^{m+n} \\
&\qquad + \tfrac{1}{4}(m+1)(n+1)\theta^+ \theta^- z^{m+n} \\
&= (m-n) z^{m+n}
\end{aligned} \tag{3.37}
$$

which, from the transformations (3.32), gives the mode

$$
\tilde{J}_G = (m-n) L_{m+n} \tag{3.38}
$$

as the first term of the algebra (3.29). To evaluate the second term of (3.29) we need

$$
\begin{aligned}
D^- E'(Z) &= \tfrac{1}{2}(m+1) m \theta^+ z^{m-1} \\
D^+ D^- E'(Z) &= \tfrac{1}{2}(m+1) m (\tfrac{\partial}{\partial \theta^+} + \tfrac{1}{2}\theta^- \tfrac{\partial}{\partial z})\theta^+ z^{m-1} \\
&= \tfrac{1}{2}(m+1) m (z^{m-1} + \tfrac{1}{2}(m-1)\theta^- \theta^+ z^{m-2}) \\
D^- D^+ E'(Z) &= \tfrac{1}{2}(m+1) m (z^{m-1} + \tfrac{1}{2}(m-1)\theta^+ \theta^- z^{m-2}) \\
[D^+, D^-] E'(Z) &= \tfrac{1}{2}(m+1) m (m-1)\theta^- \theta^+ z^{m-2}
\end{aligned} \tag{3.39}
$$

such that the second term of (3.29) becomes

$$
\begin{aligned}
&\frac{c}{6}\oint dZ F(Z) [D^+, D^-] E'(Z) \\
&\qquad = \frac{c}{6}\oint_0 \frac{dz}{2\pi i} d\theta^+ d\theta^- z^{n+1} \tfrac{1}{2} m (m^2 - 1)\theta^- \theta^+ z^{m-2} \\
&\qquad = \frac{c}{12} m (m^2 - 1)\delta_{m+n,0}
\end{aligned} \tag{3.40}
$$

in accordance with the Virasoro algebra in (3.35). The other (anti-) commutators in the algebra are found by choosing other transformations E and F according to the table (3.32).

The above structures have their right-moving analogues in the four supercurrents $\bar{T}(\bar{z})$, $\bar{J}(\bar{z})$ and $\bar{G}^{\pm}(\bar{z})$, and their mode-expansions give rise to an algebra similar to (3.35), with the *same* central charge c. In the following I will again suppress the right-moving parts, and I will be interested in *unitary* representations of the algebra.

The U(1)-charge of a state $|\phi\rangle$ of the theory is q

$$J_0|\phi\rangle = q|\phi\rangle \tag{3.41}$$

and the conformal dimension h

$$L_0|\phi\rangle = h|\phi\rangle \ . \tag{3.42}$$

Restricting the discussion to the NS sector for a while, the indices of the modes $G_r^{\pm}$ are half-integer. *Chiral* states are states $|\phi\rangle$ that satisfy

$$G^{+}_{-\frac{1}{2}}|\phi\rangle = 0 \tag{3.43}$$

and *primary* chiral states also satisfy

$$G_r^{\pm}|\phi\rangle = 0 \qquad\qquad \text{for} \quad r \geq \tfrac{1}{2} \tag{3.44}$$

from which follows the conditions

$$L_n|\phi\rangle = J_n|\phi\rangle = 0 \qquad\qquad \text{for} \quad n \geq 1 \ , \tag{3.45}$$

the first of which is familiar from the definition of primary states in non-superconformal theory.

In a unitary theory there are bounds on the values of the U(1)-charge q and the conformal dimension h. With the hermiticity condition $(G^{\pm}_{-r})^{\dagger} = G^{\mp}_{r}$ one finds, using (3.35)

$$\langle\phi|\{G^{-}_{\frac{1}{2}}, G^{+}_{-\frac{1}{2}}\}|\phi\rangle = \langle\phi|2L_0 - J_0|\phi\rangle = (2h-q)\langle\phi|\phi\rangle \tag{3.46}$$

and

$$\begin{aligned}\langle\phi|\{G^{-}_{\frac{1}{2}}, G^{+}_{-\frac{1}{2}}\}|\phi\rangle &= \langle\phi|G^{-}_{\frac{1}{2}}G^{+}_{-\frac{1}{2}}|\phi\rangle + \langle\phi|G^{+}_{-\frac{1}{2}}G^{-}_{\frac{1}{2}}|\phi\rangle \\ &= |G^{+}_{-\frac{1}{2}}|\phi\rangle|^2 + |G^{-}_{\frac{1}{2}}|\phi\rangle|^2 \geq 0 \end{aligned} \tag{3.47}$$

thus $h \geq \frac{1}{2}q$, also

$$\langle\phi|\{G^{-}_{-\frac{1}{2}}, G^{+}_{\frac{1}{2}}\}|\phi\rangle = \langle\phi|2L_0 + J_0|\phi\rangle = (2h+q)\langle\phi|\phi\rangle \tag{3.48}$$

and

$$\begin{aligned}\langle\phi|\{G^-_{-\frac{1}{2}},G^+_{\frac{1}{2}}\}|\phi\rangle &= \langle\phi|G^-_{-\frac{1}{2}}G^+_{\frac{1}{2}}|\phi\rangle+\langle\phi|G^+_{\frac{1}{2}}G^-_{-\frac{1}{2}}|\phi\rangle \\ &= |G^+_{\frac{1}{2}}|\phi\rangle|^2+|G^-_{-\frac{1}{2}}|\phi\rangle|^2\geq 0 \end{aligned}\tag{3.49}$$

hence the bound is $h \geq \frac{1}{2}|q|$ for any unitary state. The primary chiral states are precisely the states for which $h = \frac{1}{2}q$ (the anti-chiral states, defined by $G^-_{\frac{1}{2}}|\phi\rangle = 0$ and (3.44) are the states for which $h = -\frac{1}{2}q$); that a state with $h = \frac{1}{2}q$ is chiral primary can be shown as follows. First, suppose the state $|\phi\rangle$ satisfies $h = \frac{1}{2}q$. Then from

$$0 = (2h-q)\langle\phi|\phi\rangle = |G^+_{-\frac{1}{2}}|\phi\rangle|^2 + |G^-_{\frac{1}{2}}|\phi\rangle|^2 \tag{3.50}$$

one finds that $|\phi\rangle$ is chiral

$$G^+_{-\frac{1}{2}}|\phi\rangle = 0 \tag{3.51}$$

and that one of the conditions for primary states is satisfied

$$G^-_{\frac{1}{2}}|\phi\rangle = 0\ . \tag{3.52}$$

To see that the rest of the conditions in (3.44) are satisfied note that acting with J_n, $n \geq 1$, on a state lowers the conformal dimension by n but does not change the U(1)-charge q

$$L_0(J_n|\phi\rangle) = [L_0,J_n]|\phi\rangle + J_nL_0|\phi\rangle = (-n+h)J_n|\phi\rangle$$

$$J_0(J_n|\phi\rangle) = [J_0,J_n]|\phi\rangle + J_nJ_0|\phi\rangle = qJ_n|\phi\rangle \tag{3.53}$$

so, since the bound $h \geq \frac{1}{2}|q|$ must be satisfied also after acting with a number of J_n's on $|\phi\rangle$ we must demand that J_n, $n \geq 1$ vanishes on $|\phi\rangle$. Then the conditions (3.44) are satisfied by $|\phi\rangle$:

$$G^\pm_{n\mp\frac{1}{2}}|\phi\rangle = \pm[J_n,G^\pm_{\mp\frac{1}{2}}]|\phi\rangle = \pm J_n\underbrace{G^\pm_{\mp\frac{1}{2}}|\phi\rangle}_{=0}\mp G^\pm_{\mp\frac{1}{2}}\underbrace{J_n|\phi\rangle}_{=0} = 0 \tag{3.54}$$

hence a state with $h = \frac{1}{2}q$ is a chiral primary state.

Another important relation for the primary chiral states is that $h \leq c/6$:

$$0 \leq \langle\phi|\{G^-_{\frac{3}{2}},G^+_{-\frac{3}{2}}\}|\phi\rangle = \langle\phi|2L_0 - 3J_0 + 2\frac{c}{3}|\phi\rangle = (-4h+2\frac{c}{3})\langle\phi|\phi\rangle \tag{3.55}$$

where I used $h = \frac{1}{2}q$. Therefore, because $h \geq 0$ in a unitary theory (by similar arguments using $[L_1,L_{-1}]$) a non-degenerate conformal field theory has only a finite number of chiral states (the values of h are discrete). The chiral primary states form a closed ringstructure, and any chiral state $|\phi\rangle$ can be decomposed into

$$|\phi\rangle = |\tilde{\phi}\rangle + G^+_{-\frac{1}{2}}|\hat{\phi}\rangle \tag{3.56}$$

Figure 3.5: Primary chiral states in the Neveu–Schwarz sector, and ground states in the Ramond sector. The flow with $\theta = \frac{1}{2}$ interpolates between the two sectors, in particular primary chiral NS states flow to ground states in R.

where $|\tilde{\phi}\rangle$ is a chiral primary state [5].

Turning now to the Ramond sector one finds

$$\langle\phi|\{G_0^-, G_0^+\}|\phi\rangle = \langle\phi|2L_0 - \frac{c}{12}|\phi\rangle = (2h - \frac{c}{12})\langle\phi|\phi\rangle \tag{3.57}$$

and

$$\begin{aligned}\langle\phi|\{G_0^-, G_0^+\}|\phi\rangle &= \langle\phi|G_0^- G_0^+|\phi\rangle + \langle\phi|G_0^+ G_0^-|\phi\rangle \\ &= |G_0^+|\phi\rangle|^2 + |G_0^-|\phi\rangle|^2 \geq 0\end{aligned} \tag{3.58}$$

so that in the Ramond sector any state satisfies $h \geq c/24$. The ground states of the Ramond sector are the states annihilated by *both* G_0^+ and G_0^-, thus the ground states are exactly the states with $h = c/24$, see Figure 3.5.

The Neveu-Schwarz and the Ramond sector can be connected by a twisting of the modes of the four supercurrents and by letting the states in one sector flow to states in the other sector. Consider the flow given by the parameter θ (an "ordinary" real parameter, not to be confused with the Grassmannian variables θ^+ and θ^-) and the flow operator $\mathcal{U}_\theta$:

$$\begin{array}{ll} L_n^\theta \equiv \mathcal{U}_\theta L_n \mathcal{U}_\theta^{-1} = L_n + \theta J_n + \frac{c}{6}\theta^2\delta_{n,0} & J_n^\theta \equiv \mathcal{U}_\theta J_n \mathcal{U}_\theta^{-1} = J_n + \frac{c}{3}\theta\delta_{n,0} \\ G_r^{+,\theta} \equiv \mathcal{U}_\theta G_r^+ \mathcal{U}_\theta^{-1} = G_{r+\theta}^+ & G_r^{-,\theta} \equiv \mathcal{U}_\theta G_r^- \mathcal{U}_\theta^{-1} = G_{r-\theta}^- \end{array} \tag{3.59}$$

The flow Neveu–Schwarz $\rightarrow$ Ramond sector is given by $\theta = \frac{1}{2}$,

$$\begin{array}{ll} L_n^{\mathrm{R}} = L_n^{\mathrm{NS}} + \frac{1}{2}J_n^{\mathrm{NS}} + \frac{c}{24}\delta_{n,0} & J_n^{\mathrm{R}} = J_n^{\mathrm{NS}} + \frac{c}{6}\delta_{n,0} \\ G_n^{+,\mathrm{R}} = G_{n+\frac{1}{2}}^{+,\mathrm{NS}} & G_n^{-,\mathrm{R}} = G_{n-\frac{1}{2}}^{-,\mathrm{NS}} \end{array} \tag{3.60}$$

and the condition $G^{+,\mathrm{NS}}_{-\frac{1}{2}}|\phi\rangle_{\mathrm{NS}} = 0$ becomes

$$\mathcal{U}_{\frac{1}{2}} G^{+}_{-\frac{1}{2}} \mathcal{U}^{-1}_{\frac{1}{2}} \mathcal{U}_{\frac{1}{2}} |\phi\rangle_{\mathrm{NS}} = G^{+}_{0} |\phi\rangle_{\mathrm{R}} = 0 \tag{3.61}$$

and also

$$\mathcal{U}_{\frac{1}{2}} G^{-}_{\frac{1}{2}} \mathcal{U}^{-1}_{\frac{1}{2}} \mathcal{U}_{\frac{1}{2}} |\phi\rangle_{\mathrm{NS}} = G^{-}_{0} |\phi\rangle_{\mathrm{R}} = 0 \tag{3.62}$$

with $|\phi\rangle_{\mathrm{NS}}$ a state in the Neveu–Schwarz sector, and $|\phi\rangle_{\mathrm{R}} = \mathcal{U}_{\frac{1}{2}}|\phi\rangle_{\mathrm{NS}}$ the corresponding state in the Ramond sector. We see that chiral primary states in the NS sector flow to the ground states in the Ramond sector.

Topological Conformal Field Theory

The topological conformal field theories (TCFT) are special cases of the topological quantum field theories (TQFT). In a conformal field theory the trace of the energy-momentum tensor T_{ab} vanishes. This is indeed the case within the physical subspace $\mathcal{H}_{phys}$ in a TQFT, where not only the trace, but the whole energy-momentum tensor vanishes (understood within the Q-cohomology). However, outside $\mathcal{H}_{phys}$ there is no such restriction on T_{ab}; T_{ab} is Q-exact, but does not vanish. In a topological *conformal* field theory one imposes the condition that the theory shall be conformally invariant; the trace of the energy-momentum tensor must vanish in the *whole* Hilbertspace $\mathcal{H}$, not just in $\mathcal{H}_{phys}$. More exactly, the condition is stronger — one requires the vanishing of the trace of the quantity G_{ab}, defined in (3.13)

$$\sum_a G_a{}^a = 0 \tag{3.63}$$

which implies, from $T_{ab} = \{Q, G_{ab}\}$, the vanishing of the trace of T_{ab}

$$\sum_a T_a{}^a = \sum_a \{Q, G_a{}^a\} = 0 \ . \tag{3.64}$$

The conformal invariance of the theory (in a two-dimensional theory) decomposes the symmetries of the theory into left- and right-moving parts,

$$Q = Q_{\mathrm{L}} + Q_{\mathrm{R}} \tag{3.65}$$

which in many respects can be treated separately. I will consider mainly the left-movers, and often use the notation Q for Q_{L}. Define for the energy-momentum tensor and its Q-partner G_{ab} the holomorphic fields $T(z) \equiv T_{zz}$ and $G(z) \equiv G_{zz}$ (and for the right-movers the anti-holomorphic fields $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$ and $\bar{G}(\bar{z}) \equiv G_{\bar{z}\bar{z}}$). The relation between $T(z)$ and $G(z)$ is then

$$\boxed{T(z) = \{Q, G(z)\}} \tag{3.66}$$

$T(z)$ has conformal dimension 2 (see Chapter 1), and Q is fermionic and conformally invariant. Hence $G(z)$ is fermionic with conformal dimension 2. The BRST-current $Q(z)$ is introduced by

$$Q = \oint_0 \frac{dz}{2\pi i} Q(z) \tag{3.67}$$

and is therefore fermionic with conformal dimension 1. $Q(z)$ is Q-exact

$$\boxed{Q(z) = -[Q, J(z)]} \tag{3.68}$$

$J(z)$ has conformal dimension 1. The mode-expansions of the four "supercurrents"[2] the energy-momentum tensor $T(z)$, its Q-partner $G(z)$, the supercurrent $Q(z)$ and its Q-partner $J(z)$ are

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad & G(z) &= \sum_{n\in\mathbb{Z}} G_n z^{-n-2} \\ Q(z) &= \sum_{n\in\mathbb{Z}} Q_n z^{-n-1}, \qquad & J(z) &= \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \end{aligned} \tag{3.69}$$

and with techniques similar to the ones used in the previous section to find the $N = 2$ algebra (3.35), the mode-expansions give the closed TCFT algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} \qquad & [J_m, J_n] &= d\, m\, \delta_{n+m,0} \\ [L_m, G_n] &= (m-n)G_{m+n} \qquad & [J_m, G_n] &= -G_{m+n} \\ [L_m, Q_n] &= -n\, Q_{m+n} \qquad & [J_m, Q_n] &= Q_{m+n} \end{aligned}$$

$$\begin{aligned} \{G_m, Q_n\} &= L_{m+n} + nJ_{m+n} + \tfrac{1}{2} d\, m(m+1)\delta_{n+m,0} \\ [L_m, J_n] &= -n\, J_{m+n} - \tfrac{1}{2} d\, m(m+1)\delta_{m+n,0} \end{aligned} \tag{3.70}$$

(if more than the above four fields are introduced, then the above forms a subalgebra of a larger algebra with the modes of all the fields). Note that the Virasoro algebra in (3.70) has no central charge.

The algebra (3.70) shows similarities to the $N = 2$ superconformal algebra (3.35), and indeed, a crucial point [24] is now that by twisting the energy-momentum tensor $T^{N=2}(z)$ of the $N = 2$ superconformal model

$$T^{N=2}(z) \to T^{\mathrm{TCFT}}(z) \equiv T^{N=2}(z) + \tfrac{1}{2}\partial J(z) \tag{3.71}$$

the two algebras become equivalent! Using

$$T(z) + \tfrac{1}{2}\partial J(z) = \sum_{n\in\mathbb{Z}} \left(L_n z^{-n-2} + \tfrac{1}{2}(-n-1) J_n z^{-n-2} \right) \tag{3.72}$$

[2] Again, this is what I will call the four quantities $T(z)$, $G(z)$, $Q(z)$ and $J(z)$

the twist (3.71) becomes

$$\boxed{L_n^{N=2} \to L_n^{\mathrm{TCFT}} = L_n^{N=2} - \tfrac{1}{2}(n+1)J_n} \tag{3.73}$$

and the modes $Q_n = \frac{1}{2}G^+_{n-\frac{1}{2}}$ and $G_n = G^-_{n+\frac{1}{2}}$ (with $G^+_{n-\frac{1}{2}}$ and $G^-_{n+\frac{1}{2}}$ in the Neveu–Schwarz sector) are identified.

One of the crucial features about this twist is it is constructed such that although the $N = 2$ algebra does have a central charge c, the central charge does not survive the twist (as a central charge, i.e. in the Virasoro algebra), but it re-appears as a "background charge" $d = c/3$. For example, the Virasoro algebra of (3.70), using the twist and omitting the superscript $N = 2$, is obtained by

$$\begin{aligned}
[L_m^{\mathrm{TCFT}}, L_n^{\mathrm{TCFT}}] &= [L_m - \tfrac{1}{2}(m+1)J_m, L_n - \tfrac{1}{2}(n+1)J_n] \\
&= [L_m, L_n] - \tfrac{1}{2}(n+1)[L_m, J_n] - \tfrac{1}{2}(m+1)[J_m, L_n] \\
&\qquad + \tfrac{1}{4}(n+1)(m+1)[J_m, J_n] \\
&= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} - \tfrac{1}{2}(m+1)mJ_{m+n} \\
&\qquad + \tfrac{1}{2}(n+1)nJ_{m+n} + \tfrac{1}{4}(n+1)(m+1)\frac{c}{3}m\delta_{m+n,0} \\
&= (m-n)(L_{m+n} - \tfrac{1}{2}(n+m+1)J_{m+n}) \\
&= (m-n)L_{m+n}^{\mathrm{TCFT}}
\end{aligned} \tag{3.74}$$

— no central charge. Also

$$\begin{aligned}
[L_m^{\mathrm{TCFT}}, J_n^{\mathrm{TCFT}}] &= [L_m, J_n] - \tfrac{1}{2}(m+1)[J_m, J_n] \\
&= -nJ_{m+n} - \tfrac{1}{2}(m+1)\frac{c}{3}m\delta_{m+n,0} \\
&= -nJ_{m+n}^{\mathrm{TCFT}} - \frac{d}{2}(m+1)m\delta_{m+n,0}
\end{aligned} \tag{3.75}$$

A conjecture has been that in conformal field theories the vacuum $|0\rangle$ was the only state annihilated by the zero-mode of the energy-momentum tensor, L_0, and that a theory with vanishing central charge was trivial. This is indeed true for many conformal field theories, including the $N = 2$ superconformal theories. But, as we shall see, even though the TCFT's have no central charge, they do have a non-trivial structure.

The way the modes of the two supercurrents $G(z)$ and $Q(z)$ are obtained by twisting, or flowing, from the supercurrents $G^+(z)$ and $G^-(z)$ in the $N = 2$ algebra

$$G^+_{n-\frac{1}{2}} \to 2Q_n \qquad\qquad G^-_{n+\frac{1}{2}} \to G_n \tag{3.76}$$

is remarkably close to the flow of the two supercurrents from the Neveu–Schwarz to the Ramond sector, though the flows are not identical

$$G^{+,\mathrm{NS}}_{n+\frac{1}{2}} \to G^{+,\mathrm{R}}_{n} \qquad\qquad G^{-,\mathrm{NS}}_{n-\frac{1}{2}} \to G^{-,\mathrm{R}}_{n} \tag{3.77}$$

Most of the treatment in the previous section of the chiral and the chiral primary states only depended on the modes of the two currents $G^{\pm}(z)$, in particular given the flow (3.77) a chiral primary state in the NS sector would flow to a ground state in the Ramond sector no matter how the flow acts upon the modes J_n and L_n — as long as this second part of the flow does not involve G^+_r and G^-_r. This is (almost — except a slight difference between (3.76) and (3.77)) the case for the flow from the $N=2$ (Neveu–Schwarz) algebra to the TCFT algebra, and one would therefore expect to find structures like the Ramond ground states in the TCFT model corresponding to chiral primary states in the $N=2$ model.

The $N=2 \to$ TCFT flow associates a chiral state $|\phi\rangle$ in the $N=2$ (Neveu–Schwarz) model

$$G^+_{-\frac{1}{2}}|\phi_{\mathrm{chiral}}\rangle_{\mathrm{NS}} = 0\ , \qquad |\phi_{\mathrm{chiral}}\rangle_{\mathrm{NS}} = |\tilde{\phi}_{\mathrm{chiral\ prim}}\rangle_{\mathrm{NS}} + G^+_{-\frac{1}{2}}|\hat{\phi}\rangle_{\mathrm{NS}} \tag{3.78}$$

with a representative of a physical state (using $G^+_{-\frac{1}{2}} \to 2Q_0 = 2Q$)

$$Q|\phi_{\mathrm{phys}}\rangle = 0\ , \qquad |\phi_{\mathrm{phys}}\rangle = |\tilde{\phi}_{\mathrm{phys}}\rangle + Q|\hat{\phi}\rangle \tag{3.79}$$

where the representative $|\tilde{\phi}_{\mathrm{phys}}\rangle$ is physical and in addition is annihilated by the modes G_n and Q_n, $n \geq 0$; this is (one of the) "Ramond ground states" one would expect to find, using the analogy of the two flows. Thus there exists a representative $|\tilde{\phi}_{\mathrm{phys}}\rangle$ in each class of physical operators that satisfies

$$L_0|\tilde{\phi}_{\mathrm{phys}}\rangle = \{G_0, Q\}|\tilde{\phi}_{\mathrm{phys}}\rangle = 0 \tag{3.80}$$

i.e. $h^{\mathrm{TCFT}} = 0$ even when $|\tilde{\phi}_{\mathrm{phys}}\rangle$ is *not* the vacuum.

Hence (a representative in each class of) the physical states correspond to the (non-trivial!) ring of chiral primary states, yet the representative has conformal dimension zero from (3.80) and lives within a model with no central charge. This shows that

non-trivial models with zero central charge do *exist!*

The bound $0 \leq h \leq c/6$, or $0 \leq q \leq c/3$, on primary chiral states in the NS sector carries over into the TCFT

$$J_0|\phi\rangle = q|\phi\rangle \quad \text{with} \quad 0 \leq q \leq d (= c/3) \tag{3.81}$$

because $J(z)$ is not changed by the $N=2 \to$ TCFT flow.

The analogy between the two flows does not hold much further than this; the two flows of the modes of $J(z)$ and $T(z)$ are quite different

$$\begin{array}{llll} L_n^{\mathrm{R}} = L_n^{\mathrm{NS}} + \frac{1}{2} J_n^{\mathrm{NS}} + \dfrac{c}{24}\delta_{n,0} & ; & J_n^{\mathrm{R}} = J_n^{\mathrm{NS}} + \dfrac{c}{6}\delta_{n,0} \\ L_n^{\mathrm{TCFT}} = L_n^{N=2} - \frac{1}{2}(n+1) J_n^{N=2} & ; & J_n^{\mathrm{TCFT}} = J_n^{N=2} \end{array} \qquad (3.82)$$

or

$$\begin{array}{llll} T^{\mathrm{R}}(z) = T^{\mathrm{NS}}(z) + \dfrac{J^{\mathrm{NS}}(z)}{2z} + \dfrac{c}{24z^2} & ; & J^{\mathrm{R}}(z) = J^{\mathrm{NS}}(z) + \dfrac{c}{6z} \\ T^{\mathrm{TCFT}}(z) = T^{N=2}(z) + \partial_z J^{N=2}(z) & ; & J^{\mathrm{TCFT}}(z) = J^{N=2}(z) \end{array} \qquad (3.83)$$

Everything of the above can be carried out in the right-moving sector as well.

3.2 Perturbing Away from TCFT

The $N = 2$ superconformal models have been extensively by now, and are well-known [5]. Via the twist (3.71) any $N = 2$ superconformal model can be identified with a TCFT, and therefore, as we just saw, one can take advantage of the knowledge from the $N = 2$ superconformal models in describing the TCFT's, which is a big step forward. But the TCFT's are, as previously mentioned, only special cases of the general topological field theories. When using the topological field theories, one would like to be able to treat also TQFT's that are not necessarily conformally invariant, both in string theory (e.g. in a non-perturbative description) and in other fields of physics, where the concept of topological invariance might prove itself useful. Then why (does the literature) treat the TCFT's in such detail? We shall see shortly that with the outset in a TCFT, one might *perturb* the theory in such a way that correlators in the perturbed, non-conformal TQFT can be found directly from correlators in the TCFT.

Turning to a super-field formulation of TQFT's, additional non-local physical fields[3] are introduced from the existing local physical fields. These additional fields, when coupled to a parameter each (coupling constants), admit a perturbation of the theory away from the original theory, with correlators in the perturbed theory obtainable from correlators in the un-perturbed theory. Specifically one can perturb away from a TCFT, with easily obtainable correlators (via the identification with a $N = 2$ model), to a *non-conformal* TQFT, in which the correlators can now be obtained from the TCFT ones. Thus at least some TQFT's can be described from what is known for the TCFT's (there is no guarantee that *any* TQFT can be reached by perturbing a TCFT this way).

[3]Here I mix the terminology a bit — what was previously referred to as "operators" (e.g. physical operators ϕ_i) will here be called "fields".

I will now introduce the superfield formulation, first for a D-dimensional manifold, before specializing to $D = 2$ [25, 27]. First define the conserved charges related to $T_{\alpha\beta}$ and $G_{\alpha\beta}$ (with $\alpha, \beta = 0, 1, \ldots, D-1$ and $x = (x^0, x^1, \ldots, x^{D-1})$) by

$$P_\alpha \equiv \int d^{D-1}x\, T_{\alpha 0}\ , \qquad G_\alpha \equiv \int d^{D-1}x\, G_{\alpha 0} \tag{3.84}$$

with fixed x^0, and from (3.13)

$$P_\alpha = \{Q, G_\alpha\}\ . \tag{3.85}$$

The superfield $\Phi(x,\theta)$ is introduced from a local physical field $\phi(x)$ by

$$\Phi(x,\theta) = e^G \phi(x) e^{-G} \tag{3.86}$$

where $\theta = (\theta^0, \theta^1, \ldots, \theta^{D-1})$ is a collection of Grassmann variables, and $G \equiv \theta^\alpha G_\alpha$. Here, and in the rest of this subsection, repeated indices are summed over. The superfield can be expanded in an exact D-dimensional Taylor expansion in the various Grassmann variables, the result being[4]

$$\Phi(x,\theta) = \phi^{(0)}(x) + \phi^{(1)}_\alpha(x)\theta^\alpha + \phi^{(2)}_{\alpha_1\alpha_2}(x)\theta^{\alpha_1}\theta^{\alpha_2} + \ldots + \phi^{(D)}_{\alpha_1\ldots\alpha_D}(x)\theta^{\alpha_1}\cdots\theta^{\alpha_D} \tag{3.87}$$

If the zero-form field $\phi^{(0)}(x) = \phi(x)$ is a physical field,

$$[Q, \phi^{(0)}(x)] = 0 \tag{3.88}$$

then the n-form fields satisfy the important *descent equations*

$$\boxed{d\phi^{(n)}(x) = -[Q, \phi^{(n+1)}(x)]} \tag{3.89}$$

where I used the notation $\phi^{(n)}(x) \equiv \phi^{(n)}_{\alpha_1\ldots\alpha_n}(x)\theta^{\alpha_1}\cdots\theta^{\alpha_n}$. I will now prove the descent equations.

The quantity $P_\alpha dx^\alpha$ is invariant; θ^α has the same conformal dimension as dx^α (namely, -1), so that the θ^α's can be used to keep track of the invariance under coordinate changes of $P_\alpha dx^\alpha$ via $P \equiv \theta^\alpha P_\alpha$. The relation between P and G is, from $T_{\alpha\beta} = \{Q, G_{\alpha\beta}\}$,

$$P = [G, Q\,]. \tag{3.90}$$

Expanding the expression (3.86) for the superfield $\Phi(x,\theta)$

$$\begin{aligned}\Phi(x,\theta) &= e^G\phi(x)e^{-G} \\ &= \phi(x) + [G,\phi(x)] + \tfrac{1}{2}[G,[G,\phi(x)]] + \ldots \\ &\qquad + \frac{1}{n!}\underbrace{[G,[G,\ldots[G}_{n\ G\text{'s}},\phi(x)]\ldots]] + \ldots\end{aligned} \tag{3.91}$$

[4]Notation $\phi^{(n)}$ for the n-form field should not be confused with the conformal dimension appearing as a superscript in Chapter 1.

Then, when comparing to the Taylor expansion (3.87) and keeping in mind that every $G = \theta^\alpha G_\alpha$ is a sum of terms, each involving one factor of θ, we see that

$$\begin{aligned}
\phi^{(0)}(x) &= \phi(x) \\
\phi^{(1)}(x) &\equiv \phi^{(1)}_\alpha(x)\theta^\alpha = [G, \phi(x)] \\
\phi^{(2)}(x) &\equiv \phi^{(2)}_{\alpha_1\alpha_2}(x)\theta^{\alpha_1}\theta^{\alpha_2} = \tfrac{1}{2}[G,[G,\phi(x)]] = \tfrac{1}{2}[G,\phi^{(1)}(x)] \\
&\vdots \\
\phi^{(n)}(x) &\equiv \phi^{(n)}_{\alpha_1\ldots\alpha_n}(x)\theta^{\alpha_1}\cdots\theta^{\alpha_n} = \frac{1}{n!}[G,[G,\ldots[G,\phi(x)]\ldots]] = \frac{1}{n}[G,\phi^{(n-1)}(x)] \\
&\vdots \\
\phi^{(D)}(x) &\equiv \phi^{(D)}_{\alpha_1\ldots\alpha_D}(x)\theta^{\alpha_1}\cdots\theta^{\alpha_D} = \frac{1}{D}[G,\phi^{(D-1)}(x)] \qquad (3.92)
\end{aligned}$$

The usual connection between the differential operator $d = \theta^\alpha \frac{\partial}{\partial x^\alpha}$ and the translational operator $P = \theta^\alpha P_\alpha$ is

$$d\,f = [P, f]\ . \qquad (3.93)$$

The descent equations (3.89) are shown by induction, for $\phi^{(0)}(x) = \phi(x)$ a physical field. First

$$\begin{aligned}
d\,\phi^{(n)}(x) &= [P, \phi^{(n)}(x)] \\
&= [[G, Q\,], \phi^{(n)}(x)] \\
&= [[G, \phi^{(n)}(x)], Q\,] - [[Q, \phi^{(n)}(x)], G\,] \\
&= [(n+1)\phi^{(n+1)}(x), Q\,] - [[Q, \phi^{(n)}(x)], G\,] \qquad (3.94)
\end{aligned}$$

using $[G, \phi^{(n)}(x)] = (n+1)\phi^{(n+1)}(x)$. For $n = 0$ the last term vanishes because $\phi^{(0)}(x)$ is physical, and the relation reads

$$d\,\phi^{(0)}(x) = -[Q, \phi^{(1)}(x)]\ . \qquad (3.95)$$

Assume that the descent equations have been shown for all $n-1$ and less -form fields,

$$d\,\phi^{(n-1)}(x) = -[Q, \phi^{(n)}(x)] \qquad (3.96)$$

then

$$\begin{aligned}
d\,\phi^{(n)}(x) &= [(n+1)\phi^{(n+1)}(x), Q\,] - [[Q, \phi^{(n)}(x)], G] \\
&= -(n+1)[Q, \phi^{(n+1)}(x)] - [d\,\phi^{(n-1)}(x), G\,] \\
&= -(n+1)[Q, \phi^{(n+1)}(x)] - n\,d\,\phi^{(n)}(x) \qquad (3.97)
\end{aligned}$$

where in the last line I used the relations (3.92) on the field $d\,\phi^{(n)}(x)$

$$d\,\phi^{(n)}(x) = \frac{1}{n}[G, d\,\phi^{(n-1)}(x)] \ . \tag{3.98}$$

Re-arranging the terms in (3.97) we find the descent equation (3.89) for the field $\phi^{(n)}(x)$. In particular

$$d\phi^{(D)}(x) = -[Q, \phi^{(D+1)}(x)] = 0 \tag{3.99}$$

because the top-form in the expansion (3.87) is $\phi^{(D)}(x)$.

Introduce the notation $\mathsf{D} = Q - d$, understood as $\mathsf{D}\cdot = -[Q, \cdot] - d\,\cdot$, then for $\phi^{(0)}(x)$ a physical field

$$\begin{aligned}
\mathsf{D}\Phi(x,\theta) &= \mathsf{D}\sum_{n=0}^{D} \phi^{(n)}(x) \\
&= \sum_{n=0}^{D} \Big(-[Q, \phi^{(n)}(x)] - d\,\phi^{(n)}(x)\Big) \\
&= -[Q, \phi^{(0)}(x)] + \sum_{n=1}^{D} d\,\phi^{(n-1)}(x) - \sum_{n=0}^{D} d\,\phi^{(n)}(x) \\
&= -[Q, \phi^{(0)}(x)] - d\,\phi^{(D)}(x) \\
&= 0
\end{aligned} \tag{3.100}$$

— the first term vanishes because $\phi^{(0)}(x)$ is physical, the second term from (3.99). The $\mathsf{D}\Phi$ condition is very similar to the chirality conditions in the $N = 2$ superconformal models [5], D is a linear and nilpotent operator and vanishes on Φ. As we just saw, there *is* a connection between chirality in the $N = 2$ models and physical fields in the topological models.

From the descent equations we see that all the n-forms, integrated over a (closed) n-dimensional manifold, are (non-local) physical fields

$$[Q, \int_{\text{closed}} d^n x\, \phi^{(n)}_{\alpha_1\ldots\alpha_n}(x)] = -\int_{\text{closed}} d^n x\, d\phi^{(n-1)}_{\alpha_1\ldots\alpha_n}(x) = 0 \tag{3.101}$$

The integration over the top-form $\phi^{(D)}(x)$ is over the whole D-dimensional volume, so that adding it, with coupling constants, to the action

$$S(t) = S(0) - \sum_m t_m \int \phi^{(D)}_m \tag{3.102}$$

amounts to a modification of the Lagrangian, by which the theory can be perturbed while preserving the defining topological properties of the theory, although loosing the conformal properties. The set of couplings $\{t_n\}$ parametrize the space of topological models.

In two dimensions the superfield is written

$$\Phi_i(z,\bar{z},\theta,\bar{\theta}) = \phi_i(z,\bar{z}) + \phi_i^{(1)}(z,\bar{z})\theta + \bar{\phi}_i^{(1)}(z,\bar{z})\bar{\theta} + \phi_i^{(2)}(z,\bar{z})\theta\bar{\theta} \ . \tag{3.103}$$

The top-form is the two-form $\phi_i^{(2)}(z,\bar{z})$, and it is the one used to perturb the theory into a parameter family of TQFT's

$$S(t) = S(0) - \sum_m t_m \int \phi_m^{(2)} \ . \tag{3.104}$$

I have left out the explicit z and $\bar{z}$-dependence in the integral here and in the following.

In the Lagrangian formulation the correlators are defined by

$$\int \mathcal{D}\phi e^{-S}\sigma = \langle\sigma\rangle \tag{3.105}$$

thus a correlator in the perturbed theory can be expressed by a correlator in the un-perturbed theory

$$\begin{aligned} \langle\phi_1\phi_2\cdots\phi_s\rangle\{t\} &= \int \mathcal{D}\phi e^{-S(0)+\sum_n t_n \int \phi_n^{(2)}} \phi_1\phi_2\cdots\phi_s \\ &= \int \mathcal{D}\phi e^{-S(0)} \exp\left(\sum_n t_n \int \phi_n^{(2)}\right) \phi_1\phi_2\cdots\phi_s \\ &= \langle\phi_1\phi_2\cdots\phi_s \exp\left(\sum_n t_n \int \phi_n^{(2)}\right)\rangle\{t=0\} \end{aligned} \tag{3.106}$$

where $\{t\}$ means perturbed (with $t = (t_0, t_1, \ldots)$) and $\{t = 0\}$ is un-perturbed. Hence, to determine a correlator in the perturbed theory there is no need for the explicit path-integral formulation!

Thus by differentiating a perturbed correlator (l.h.s. of above) with respect to some of the coupling constants, and thereafter setting the couplings to zero (go back to unperturbed theory) one obtains correlators in the unperturbed theory *with non-local fields*

$$\frac{\partial}{\partial t_i}\langle\phi_1\phi_2\cdots\phi_s\rangle_g\{t\}\bigg|_{t=0} = \langle\phi_1\phi_2\cdots\phi_s\int\phi_i^{(2)}\rangle_g\{t=0\} \tag{3.107}$$

3.2.1 Two- and Three-point Functions in TQFT

The two- and three-point functions will play a special role, therefore introduce the notation (with $\phi_0 = \mathbf{1}$)

$$c_{ijk} \ = \ \langle\phi_i\phi_j\phi_k\rangle_{g=0} \tag{3.108}$$

$$\eta_{ij} = \langle\phi_i\phi_j\rangle_{g=0} \ = \ \langle\phi_0\phi_i\phi_j\rangle_{g=0} = c_{0ij} \ . \tag{3.109}$$

Here, as in the rest of the presentation, I consider only bosonic operators. Everything is easily extended to include fermionic operators by changing commutators, anti-commutators and signs appropriately. Usually, in conformal field theory, the correlators depend on the positions of the operators, and an implicit time-ordering of the operators is understood ($\langle\phi_1(z_1)\phi_2(z_2)\rangle$, $|z_1| > |z_2|$, $|z|$ is the time-coordinate in radial coordinates). But since the correlators in a TQFT do not depend on the positions of the operators, the operators can be changed about within the correlator, and the correlators are symmetric in indices.

The three-point functions satisfy the following relation:

$$\sum_m c_{ij}{}^m c_{mkl} = \sum_m c_{ik}{}^m c_{mjl} \tag{3.110}$$

which intuitively seems clear from Figure 3.110 using the same way to illustrate the correlators as in Figures 3.2 – 3.4. The relation can be proved using the factorization properties:

$$\begin{aligned} \langle\phi_i\phi_j\phi_k\phi_l\rangle_{g=0} &= \sum_{m,n}\langle\phi_i\phi_j|\phi_n\rangle_{g=0}\eta^{nm}\langle\phi_m|\phi_k\phi_l\rangle_{g=0} \\ &= \sum_{m,n} c_{ijn}\eta^{nm}c_{mkl} \\ &= \sum_m c_{ij}{}^m c_{mkl} \end{aligned} \tag{3.111}$$

and, interchanging ϕ_j and ϕ_k, the same correlator can be written

$$\begin{aligned} \langle\phi_i\phi_j\phi_k\phi_l\rangle_{g=0} &= \langle\phi_i\phi_k\phi_j\phi_l\rangle_{g=0} \\ &= \sum_m c_{ik}{}^m c_{mjl} \end{aligned} \tag{3.112}$$

thereby showing relation (3.110).

Formally, define an *operator algebra* of the physical operators

$$\phi_i \times \phi_j = \sum_k c_{ij}{}^k \phi_k \tag{3.113}$$

where

$$c_{ij}{}^k = \sum_l c_{ijl}\eta^{lk}\ . \tag{3.114}$$

The algebra should be understood as an algebra that holds *within* correlators. The algebra is commutative and associative — it is clearly commutative because $c_{ij}{}^k = c_{ji}{}^k$, associativity follows from relation (3.110) and commutativity

$$(\phi_i \times \phi_j) \times \phi_k = \left(\sum_m c_{ij}{}^m \phi_m\right) \times \phi_k$$

Figure 3.6: Illustration to relation (3.110)

$$
\begin{aligned}
&= \sum_{m,l} c_{ij}{}^{m} c_{mk}{}^{l} \phi_l \\
&= \sum_{m,l} c_{jk}{}^{m} c_{im}{}^{l} \phi_l \\
&= \sum_{m} c_{jk}{}^{m} \phi_i \times \phi_m \\
&= \phi_i \times \left(\sum_{m} c_{jk}{}^{m} \phi_m \right) \\
&= \phi_i \times (\phi_j \times \phi_k)
\end{aligned} \tag{3.115}
$$

Now, define the three-point function in a *perturbed* theory as

$$c_{ijk}(t) = \langle \phi_i \phi_j \phi_k \rangle_{g=0}\{t\} = \langle \phi_i \phi_j \phi_k \exp\left(\sum_n t_n \int \phi_n^{(2)} \right) \rangle_{g=0}\{t=0\} \tag{3.116}$$

where the $\{t\}$ and $\{t=0\}$ mean correlator in resp. perturbed and un-perturbed theory. As (3.116) suggests, the $c_{ijk}(t)$ can be represented as an expectation value in the un-perturbed theory (the conformal point). These perturbed three-point functions can be used to define an associative, commutative algebra for *all* values of t, because the perturbed theory is also topological. One finds that all correlators with more than three local fields can be built up by two- and three-point functions of the local fields. For example the four-point function with one non-local field is decomposed into (see below)

$$\langle \phi_i \phi_j \phi_k \phi_l \int \phi_n^{(2)} \rangle_{g=0} = \sum_p \{ c_{ij}{}^{p}{}_{,n} c_{pkl} + c_{ij}{}^{p} c_{pkl,n} \} - \sum_{r,s} c_{ij}{}^{r} \eta_{rs,n} c^{s}{}_{kl} \tag{3.117}$$

with the definitions

$$
\begin{aligned}
c_{ijk,n} &\equiv \frac{\partial}{\partial t_n} \langle \phi_i \phi_j \phi_k \rangle_{g=0}\{t\} \Big|_{t=0} = \langle \phi_i \phi_j \phi_k \int \phi_n^{(2)} \rangle_{g=0} \\
\eta_{ij,n} &\equiv \langle \phi_i \phi_j \int \phi_n^{(2)} \rangle_{g=0}
\end{aligned} \tag{3.118}
$$

It will later be shown that the two-point functions are independent of the perturbations, such that $\eta_{ij,n}$ vanishes. Any correlator can be decomposed down to correlators of the form $\langle \phi_i \phi_j \phi_k \int \phi_n^{(2)} \cdots \rangle$, the fundamental amplitudes.

3.2.2 Decomposing a Four-point Function

In this section I will show how the four-point function (3.117) is built up by two- and three-point functions. Using the notation $c_{ijk,n} \equiv \langle \phi_i \phi_j \phi_k \int \phi_n^{(2)} \rangle_{g=0}$ and $\eta_{ij,n} \equiv \langle \phi_i \phi_j \int \phi_n^{(2)} \rangle_{g=0}$ one finds

$$\begin{aligned}
\frac{\partial}{\partial t_n} \langle \phi_i \phi_j \phi_k \phi_l \rangle_{g=0} \Big|_{\{t=0\}} &= \frac{\partial}{\partial t_n} \sum_{m,p} \langle \phi_i \phi_j \phi_m \rangle_{g=0} \eta^{mp} \langle \phi_p \phi_k \phi_l \rangle_{g=0} \Big|_{\{t=0\}} \\
&= \frac{\partial}{\partial t_n} \sum_{m,p} c_{ijm} \eta^{mp} c_{pkl} \Big|_{\{t=0\}} \\
&= \sum_{m,p} \{ c_{ijm,n} \eta^{mp} c_{pkl} \\
&\qquad + c_{ijm} (- \sum_{r,s} \eta^{mr} \eta_{rs,n} \eta^{sp}) c_{pkl} + c_{ijm} \eta^{mp} c_{pkl,n} \} \\
&= \sum_{p} \{ c_{ij}{}^{p}{}_{,n} c_{pkl} + c_{ij}{}^{p} c_{pkl,n} \} - \sum_{r,s} c_{ij}{}^{r} \eta_{rs,n} c^{s}{}_{kl} \qquad (3.119)
\end{aligned}$$

where, to evaluate $\frac{\partial}{\partial t_n} \eta^{mp} \Big|_{\{t=0\}}$ again I used

$$\begin{aligned}
1 = A A^{-1} \quad &\Rightarrow \quad 0 = (\partial A) A^{-1} + A \partial (A^{-1}) \\
&\Rightarrow \quad -A^{-1} (\partial A) A^{-1} = \partial (A^{-1})
\end{aligned}$$

with $\eta_{mp} = A$,

$$\frac{\partial}{\partial t_n} \eta^{mp} \Big|_{\{t=0\}} = - \sum_{r,s} \eta^{mr} \eta_{rs,n} \eta^{sp} \ . \qquad (3.120)$$

It will be shown later that the two-point function η_{ij} is independent of the couplings, hence $\eta_{ij,n} = 0$.

3.2.3 U(1) Selection Rule

The general correlators

$$\langle \phi_1 \cdots \phi_s \int \phi_{s+1}^{(2)} \cdots \int \phi_n^{(2)} \rangle_g \qquad (3.121)$$

obey a selection rule on the U(1)-charge,

$$\sum_{i=1}^{s} q_i + \sum_{j=s+1}^{n} (q_j - 1) = d(1-g) \qquad (3.122)$$

where $d = c/3$ is the parameter that appears in the algebra (3.70).

Figure 3.7: How to obtain the U(1)-selection rule for genus $g > 0$ from genus zero correlators.

To show this, first note from the proof of the descent equations (3.86) that the two-form of a physical operator can be written

$$
\begin{aligned}
\phi_i^{(2)}(z,\bar z)\theta\bar\theta &\sim \tfrac{1}{2}[\theta G_{-1} + \bar\theta \bar G_{-1}, [\theta G_{-1} + \bar\theta\bar G_{-1}, \phi_i^{(0)}(z,\bar z)]] \\
&= \tfrac{1}{2}[\theta G_{-1}, [\bar\theta \bar G_{-1}, \phi_i^{(0)}(z,\bar z)]] + \tfrac{1}{2}[\bar\theta\bar G_{-1}, [\theta G_{-1}, \phi_i^{(0)}(z,\bar z)]] \\
&= [\theta G_{-1}, [\bar\theta\bar G_{-1}, \phi_i^{(0)}(z,\bar z)]] - \tfrac{1}{2}[\phi_i^{(0)}(z,\bar z), \underbrace{[\bar\theta\bar G_{-1}, \theta G_{-1}]}_{=0}] \\
&= [\theta G_{-1}, [\bar\theta\bar G_{-1}, \phi_i^{(0)}(z,\bar z)]] \qquad (3.123)
\end{aligned}
$$

and since G_{-1} acting upon a state brings the U(1)-charge one unit down, the U(1)-charge of $\phi_i^{(2)}(z,\bar z)$ is $(q_i - 1, \bar q_i - 1)$ if the charge of $\phi_i^{(0)}(z,\bar z)$ is $(q_i, \bar q_i)$.

In the commutator

$$[L_m, J_n] = -n\, J_{m+n} - \tfrac{1}{2} d\, m(m+1)\delta_{m+n,0} \qquad (3.124)$$

of the TCFT algebra (3.70) the first term is what we would expect from $J(z)$ if treated as a conformal field with conformal dimension 1 (see (1.20)). The anomaly (the second term) appears after the twist from an $N = 2$ algebra, and it has an interpretation as a "background charge". First consider correlators of genus $g = 0$. Before twisting, U(1)-charge conservation gives us that a correlator of the form $\langle Q|q\rangle$ vanishes unless $Q = q$, where Q and q are the U(1)-charges of the states $|Q\rangle$ and $|q\rangle$. After twisting, the hermitean conjugate of J_0 becomes

$$J_0^\dagger = [J_1, L_{-1}]^\dagger = [L_1, J_{-1}] = J_0 - d \qquad (3.125)$$

such that

$$
\begin{aligned}
\langle Q|J_0^\dagger|q\rangle &= Q\langle Q|q\rangle \\
\langle Q|J_0^\dagger|q\rangle &= \langle Q|(J_0 - d)|q\rangle = (q-d)\langle Q|q\rangle \qquad (3.126)
\end{aligned}
$$

hence $\langle Q|q\rangle$ vanishes in the twisted theory unless $q = Q + d$, in particular a correlator $\langle 0|\sigma|0\rangle$ vanishes unless the U(1)-charge of σ is $q_\sigma = d$. For the correlators (3.121) on a sphere ($g = 0$) this gives the U(1) selection rule

$$\sum_{i=1}^{s} q_i + \sum_{j=s+1}^{n} (q_j - 1) = d \; . \tag{3.127}$$

To show the rule for correlators with $g > 0$ the factorization rules will be used, illustrated by Figure 3.7. A genus $g > 0$ correlator can be built up from $g = 0$ correlators. For each handle two extra (local) operators ϕ_{m_l} and ϕ_{n_l} are inserted into the genus zero correlator, and they are "glued" to the two-point function $\eta^{m_l n_l}$, i.e. the correlator can be factorized into

$$\begin{aligned} &\langle \phi_1 \cdots \phi_s \int \phi_{s+1}^{(2)} \cdots \int \phi_n^{(2)} \rangle_g = \\ &\qquad \sum \left(\prod_{l=1}^{g} \eta^{m_l n_l} \right) \langle \left(\prod_{l=1}^{g} \phi_{m_l} \phi_{n_l} \right) \phi_1 \cdots \phi_s \int \phi_{s+1}^{(2)} \cdots \int \phi_n^{(2)} \rangle_0 \end{aligned} \tag{3.128}$$

where the sum is over all physical operators (as usual). Thus, using the genus zero selection rule on each of the correlators in the sum one finds

$$\sum_{i=1}^{s} q_i + \sum_{j=s+1}^{n} (q_j - 1) + \sum_{l=1}^{g} (q_{m_l} + q_{n_l}) = d$$

$$q_{m_l} + q_{n_l} = d \tag{3.129}$$

which gives the general U(1) selection rule (3.122).

The U(1)-selection rule implies that non-trivial correlators of the form

$$\langle \phi_1 \cdots \phi_s \rangle \tag{3.130}$$

(i.e. without non-local fields) only exist on the sphere, because from (3.81) we have $q_i \geq 0$, therefore for higher genus

$$0 \leq \sum q_i = d(1 - g) \tag{3.131}$$

will not be satisfied.

3.2.4 Free Energy

When perturbing the TCFT, one generally does not arrive at a conformal theory. A necessary requirement, in order for the perturbed theory to be conformal, is that $\int \phi_j^{(2)}$ is conformally invariant, i.e. that $\phi_j^{(2)}$ has conformal dimension $(h_j, \bar{h}_j) = (1, 1)$. This is not a sufficient requirement, but if also the U(1)-charge conservation is kept when perturbing, then the perturbed theory is conformal. From the relation

$$\langle \phi_1 \cdots \phi_s \rangle_g \{t\} = \langle \phi_1 \cdots \phi_s \exp \left(\sum_j t_j \int \phi_j^{(2)} \right) \rangle_g \{t=0\} \tag{3.132}$$

between correlators in the perturbed ($\{t\}$) and the un-perturbed theory ($\{t=0\}$) U(1) charge conservation requires that $(q_j, \bar{q}_j) = (0,0)$. Generally the perturbed correlators satisfy the scaling law

$$\sum_j (q_j - 1) t_j \frac{\partial}{\partial t_j} \langle \phi_1 \cdots \phi_s \rangle_g \{t\} = \left(d(1-g) - \sum_{i=1}^{s} q_i \right) \langle \phi_1 \cdots \phi_s \rangle_g \{t\} \ . \tag{3.133}$$

Namely, the correlator can be rewritten, by expanding the exponentials

$$\langle \phi_1 \cdots \phi_s \rangle_g \{t\} = \sum_{m_1, \ldots, m_j, \ldots} \left(\prod_j \frac{1}{m_j!} \right) \langle \phi_1 \cdots \phi_s \left(\prod_j (t_j \int \phi_j^{(2)})^{m_j} \right) \rangle_g \{t=0\} \tag{3.134}$$

The U(1) selection rule (3.122) used on the r.h.s. correlators in the unperturbed (and thereby topological conformal) theory gives

$$\sum_{i=1}^{s} q_i + m_1(q_{m_1} - 1) + \ldots + m_j(q_{m_j} - 1) + \ldots = d(1-g) \tag{3.135}$$

On the other hand, acting with $t_j \frac{\partial}{\partial t_j}$ upon the correlator gives a factor of m_j, thus the factors m_j in the above sum can be replaced by the differential operator $t_j \frac{\partial}{\partial t_j}$ acting on the correlators; by re-arranging the terms one finds the scaling law (3.133).

In the previous subsections we saw that a correlator of the form

$$\langle \phi_1 \phi_2 \phi_3 \int \phi_4^{(2)} \cdots \int \phi_n^{(2)} \rangle_g \tag{3.136}$$

is symmetric in the indices $1,2,3$. On a sphere ($g=0$) one can show that the correlator is actually symmetric in *all* indices $1,2,\ldots,n$; I will sketch the proof below.

Consider a correlator with superfields

$$\langle \Phi_1(z_1, \bar{z}_1, \theta_1, \bar{\theta}_1) \cdots \Phi_s(z_s, \bar{z}_s, \theta_s, \bar{\theta}_s) \rangle_{g=0} \ , \tag{3.137}$$

with $\Phi(z,\bar{z},\theta,\bar{\theta}) = \phi^{(0)}(z,\bar{z}) + \phi^{(1)}(z,\bar{z})\theta + \bar{\phi}^{(1)}(z,\bar{z})\bar{\theta} + \phi^{(2)}(z,\bar{z})\theta\bar{\theta}$. The correlation function is invariant under the super-Möbius transformations [25, 27]

$$\begin{aligned} z &\rightarrow \tilde{z} = \frac{az+b}{cz+d} \ , \qquad ad - bc = 1 \\ \theta &\rightarrow \tilde{\theta} = \frac{\theta + \alpha z^2 + \beta z + \gamma}{(cz+d)^2} \ . \end{aligned} \tag{3.138}$$

When integrating the correlator (3.137) over superspace the invariance implies that for three of the operators the $(z,\bar{z})$ coordinates might be fixed and the $(\theta,\bar{\theta})$ be set to zero.

Then, from $\Phi(z,\bar{z},\theta,\bar{\theta})\Big|_{\theta,\bar{\theta}=0} = \phi^{(0)}(z,\bar{z})$ and

$$\int d^2z d^2\theta \Phi(z,\bar{z},\theta,\bar{\theta}) = \int d^2z \phi^{(2)}(z,\bar{z}) \tag{3.139}$$

the integrated form of the correlator (3.137) becomes

$$\begin{aligned}\int \prod_{i=1}^{s} d^2z_i d^2\theta_i \langle \Phi_1(z_1,\bar{z}_1,\theta_1,\bar{\theta}_1)\cdots\Phi_s(z_s,\bar{z}_s,\theta_s,\bar{\theta}_s)\rangle_{g=0} = \\ \langle \phi_1^{(0)}\phi_2^{(0)}\phi_3^{(0)} \int \phi_4^{(2)} \cdots \int \phi_s^{(2)}\rangle_{g=0} \ .\end{aligned} \tag{3.140}$$

Since (3.137) is symmetric in all indices and the Möbius invariance does not prescribe which three operators are to be fixed, the correlator (3.140) is also symmetric in *all* indices, in other words it does not matter which three operators are chosen to be local operators.

The three-point functions

$$c_{ijk}(t) = \langle \phi_i\phi_j\phi_k\rangle_0\{t\} = \langle \phi_i\phi_j\phi_k \exp\left(\sum_n t_n \int \phi_n^{(2)}\right)\rangle_0\{t=0\} \tag{3.141}$$

satisfy the integrability condition

$$\begin{aligned}\frac{\partial}{\partial t_l} c_{ijk}(t) &= \langle \phi_i\phi_j\phi_k \int \phi_l^{(2)} \exp\left(\sum_n t_n \int \phi_n^{(2)}\right)\rangle_0\{t=0\} \\ &= \langle \phi_i\phi_j\phi_l \int \phi_k^{(2)} \exp\left(\sum_n t_n \int \phi_n^{(2)}\right)\rangle_0\{t=0\} \\ &= \frac{\partial}{\partial t_k} c_{ijl}(t)\end{aligned} \tag{3.142}$$

where I used the symmetry of the indices at the conformal point to interchange k and l. In particular, since the (local) operator $\phi_0 = \mathbf{1}$ does not have a two-form ($\phi_0^{(2)} \sim [G_{-1},[\bar{G}_{-1},\phi_0]] = 0$) there is no coupling constant t_0 in the perturbed action, therefore

$$\frac{\partial}{\partial t_l}\eta_{ij}(t) = \frac{\partial}{\partial t_l} c_{ij0}(t) = \frac{\partial}{\partial t_0} c_{ijl}(t) = 0 \tag{3.143}$$

which shows that the two-point functions are independent of the perturbations. The decomposition of for example the genus zero four-point function (3.117) therefore reduces to

$$\langle \phi_i\phi_j\phi_k\phi_l \int \phi_n^{(2)}\rangle_{g=0} = \sum_p \{c_{ij}{}^p{}_{,n} c_{pkl} + c_{ij}{}^p c_{pkl,n}\} \ . \tag{3.144}$$

The integrability condition (3.142) makes it possible to (locally) integrate $c_{ijk}(t)$ and obtain a unique function $F_0(t)$ that satisfies

$$c_{ijk}(t) = \frac{\partial^3 F_0(t)}{\partial t_i \partial t_j \partial t_k} \tag{3.145}$$

and which can be written as a generating function

$$\begin{aligned} F_0(t) &= \langle \exp\left(\sum_n t_n \int \phi_n^{(2)}\right) \rangle_{g=0} \{t=0\} \\ &= \langle \exp\left(\sum_n t_n \int \Phi_n\right) \rangle_{g=0} \{t=0\} \ . \end{aligned} \tag{3.146}$$

$F_0(t)$ is the (genus zero part of the) *free energy* of the TCFT [25, 27]. For the function $F_0(t)$ the scaling law (3.133) becomes

$$\sum_j (q_j - 1) t_j \frac{\partial}{\partial t_j} F_0(t) = (d-3) F_0(t) \ . \tag{3.147}$$

$F_0(t)$ is the generator of all connected (genus zero) correlators, thereby acting as the partition function for the *connected* correlators. Thus a genus zero correlator can be found from

$$\langle \int \Phi_1 \cdots \int \Phi_s \rangle_0 = \frac{\partial^s F_0(t)}{\partial t_1 \cdots \partial t_s} \ . \tag{3.148}$$

3.3 Coupling to 2D Gravity

The topological conformal field theories described in the previous sections exhibit a Q-symmetry. When coupling these field theories as matter parts to two-dimensional quantum gravity, the quantum gravity must be modified such as to include the matter-part Q-symmetry, in order to get a topological theory.

This section is by no means meant as a *presentation* of topological gravity / topological string theory. This part of the theory did not take up any significant part of my work, and here I merely want to mention the basic ideas used when coupling the topological matter part to gravity. For introductions to the topic, see [25, 29].

There seems to be some confusion about the use of the terms "topological gravity" and "pure topological gravity"; here I will use the terms as used by the authors of [25, 27, 29], where "topological gravity" means topological matter coupled to gravity, in such a way that the nilpotent Q-symmetry introduced in the matter sector survives as a symmetry of the total theory. By "pure topological gravity" I mean a theory that has $d(= c/3) = 0$ in the matter sector, thus the matter sector is trivial — $c = 0$ in the corresponding $N = 2$ superconformal model — and the gravity sector can be treated on its own, though keeping the Q-symmetry that topological gravity was thus endowed with.

As in ordinary string theory, the matter sector is coupled to Liouville fields and ghost fields, after gauge fixing. In order to preserve the Q-symmetry arising from the TCFT matter sector (hereafter renamed $Q_{\rm S}$-symmetry) in the *topological* string theory, each of the Liouville and ghost fields will have a $Q_{\rm S}$-partner. Having introduced all these fields appropriately the string theory now has *two* nilpotent symmetries — the $Q_{\rm S}$- and the $Q_{\rm BRST}$-symmetry — and the theory has to be invariant under the sum of the two,

$$Q_{\rm total} = Q_{\rm S} + Q_{\rm BRST} \tag{3.149}$$

(similarly in the right-moving sector $\bar{Q}_{\rm total} = \bar{Q}_{\rm S} + \bar{Q}_{\rm BRST}$). The physical Hilbert-space is then defined as the cohomology classes[5]

$$\mathcal{H}_{phys} = \frac{\ker Q_{\rm total}}{{\rm Im}\; Q_{\rm total}} \tag{3.150}$$

and the physical operators are of the form

$$\mathcal{O}_{i,n} = \Phi_i \cdot \sigma_n \tag{3.151}$$

where σ_n is one of an infinite set of gravitational (super-) operators with charge $n \in \mathsf{N}_0$, and Φ_i is a superoperator of the matter sector with the zero-form ϕ_i a physical operator, as described in the previous sections.

I will now restrict the discussion to *pure* topological gravity, i.e. to $d = c/3 = 0$ in the matter sector. As previously discussed, $c = 0$ in the untwisted $N = 2$ superconformal model leaves us with model that only contains the vacuum state (whereas the twisted version, the TCFT, always has zero central charge, yet still has a non-trivial structure). Thus, from (3.151) the correlators of interest are of the form

$$\langle \sigma_{k_1} \cdots \sigma_{k_s} \rangle_g \tag{3.152}$$

with a charge conservation rule $\sum_i (k_i - 1) = 3(g-1)$, cf. [29]. The operator $\sigma_0(z, \bar{z})$ has an interpretation as a puncture operator (it marks a point at $(z, \bar{z})$).

The correlators show to satisfy the following recursion relation [27] (and [25, 29] with a different normalization)

$$\begin{aligned} \langle \sigma_{n+1} \prod_{k \in S} \sigma_k \rangle_g \;\; &= \;\; \sum_{k \in S} (2k+1) \langle \sigma_{n+k} \prod_{k' \neq k} \sigma_{k'} \rangle_g \\ &\quad + \sum_{j=1}^{n} \Bigg\{ \langle \sigma_{j-1} \sigma_{n-j} \prod_{k \in S} \sigma_k \rangle_{g-1} \end{aligned}$$

[5] With a slight restriction on the Hilbertspace $\mathcal{H}$ (namely that the momentum has to be the same in the left- and rightmoving sectors), which turns out to be important for the possibility of actually constructing physical operators [25].

$$+\tfrac{1}{2}\sum_{\substack{S=X\cup Y\\ g=g_1+g_2}}\langle\sigma_{j-1}\prod_{k\in X}\sigma_k\rangle_{g_1}\langle\sigma_{n-j}\prod_{k\in Y}\sigma_k\rangle_{g_2}\Bigg\} \quad (3.153)$$

with $S=\{k_1,\dots,k_s\}$. The summation over $S=X\cup Y$ in the last set of terms is over all partitions of S into the two disjunct subsets X and Y. Note that this relation is recursive in the genus number of the correlators, it indicates that all terms in a genus expansion are needed.

The operators σ_k are superoperators, and when integrated over, the Riemann surface part of the integral (over their two-forms) $\int d^2z\,\sigma^{(2)}_{k_1}(z,\bar z)\sigma^{(2)}_{k_2}(0,0)$ has a δ-function distribution, i.e. the contribution is only non-vanishing when the two operators collide

$$\int d^2z\,\sigma^{(2)}_{k_1}(z,\bar z)\sigma^{(2)}_{k_2}(0,0)=(2k_2+1)\sigma^{(2)}_{k_1+k_2-1}(0,0)\ . \quad (3.154)$$

Hence, in a splitting of (3.152) with an extra operator σ_{n+1}, into correlators with fewer operators and/or lower genus, one will only expect contributions from contact terms of the above kind and from factorizations of the Riemann surface (the definition of a topological theory), equations (3.2) and (3.3). The factorization terms are here of the form

$$\sum_{i+j=n-1}A^{(n)}_{ij}\langle\sigma_i\sigma_{k_1}\dots\sigma_{k_r}\rangle_{g'}\langle\sigma_j\sigma_{k_{r+1}}\dots\sigma_{k_s}\rangle_{g-g'} \quad (3.155)$$

and

$$\sum_{i+j=n-1}B^{(n)}_{ij}\langle\sigma_i\sigma_j\sigma_{k_1}\dots\sigma_{k_s}\rangle_{g-1} \quad (3.156)$$

The values of the indices i and j, which are also the charges of the respective operators, are restricted in order to have charge conservation. The physical correlators must be symmetric in all operators; using this and the contact (3.154) on correlators like

$$\langle\sigma_n\sigma_m\prod_{k\in S}\sigma_k\rangle=\langle\sigma_m\sigma_n\prod_{k\in S}\sigma_k\rangle \quad (3.157)$$

consistency shows [29] that $A^{(n)}_{ij}$ and $B^{(n)}_{ij}$ are independent of the indices i, j and n, and that their value is $\frac{1}{2}$ [27]. Because the physical correlators are symmetric in the operators, the summation in (3.156) gives double terms; changing the summation to $\sum_{j=1}^{n}$ gives a factor of two that cancels the factor $B^{(n)}_{ij}=\frac{1}{2}$, whereby the second sum in (3.153) is obtained. Likewise, the summation in (3.155) is changed, and when summing over all possible genera and over all possible partitions of the operators $\prod\sigma_k$, one finds the third set of terms in (3.153). The contact property (3.154) is reflected in the first sum of (3.153).

By repeatedly applying the relations (3.153) one can reduce the genus and eliminate operators, and any correlator can be reduced to an expression in terms of $\langle\sigma_1\rangle_{g=1}=\frac{1}{8}$ and $\langle\sigma_0^3\rangle_{g=0}=1$ [27].

At this point I should mention that there seems to be no "standard" yet in the literature on the use of normalizations for the operators σ_k, even in papers from the same authors differing notation is used.

3.3.1 Perturbing and the Virasoro Constraints

Similar to the analysis of the perturbed theories in the previous sections, the superoperators σ_k integrated over superspace, or the two-forms integrated over the Riemann surface $\int d^2z d^2\theta \sigma_k(z,\bar{z},\theta,\bar{\theta}) = \int d^2z \sigma_k^{(2)}(z,\bar{z})$ might be added to the world sheet action with coupling constants t_k

$$S \to S - \sum_k t_k \int \sigma_k \ . \tag{3.158}$$

The perturbed theory has an interpretation of a topological string moving in a non-trivial background, parametrized by the (infinite set of) couplings t_k. The free energy, which is the generator of *connected* correlators, is for fixed genus g defined by

$$F_g(t) = \left\langle \exp \sum_n t_n \sigma_n \right\rangle_g \{t=0\} \tag{3.159}$$

(where by $\{t=0\}$ again I mean that the expectation value is to be taken in the unperturbed theory) and for general genus (formally) by

$$F(t) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) \tag{3.160}$$

where λ is the string coupling constant (remember that in the matrix model approach the quantity κ could be fixed such as to correspond to the string coupling constant of pure topological gravity). The string partition function is

$$Z(t) = e^{F(t)} = \exp \sum_{g=0}^{\infty} \lambda^{2g-2} \left\langle \exp \sum_n t_n \sigma_n \right\rangle_g \tag{3.161}$$

In a specific background, $\{t\} = \{\tilde{t}\}$, the correlators might be found from

$$\langle \sigma_{k_1} \cdots \sigma_{k_s} \rangle = \langle \sigma_{k_1} \cdots \sigma_{k_s} \rangle \{t=\tilde{t}\} = \left. \frac{\partial^s F(t)}{\partial t_{k_1} \cdots \partial t_{k_s}} \right|_{t=\tilde{t}} \tag{3.162}$$

when using the genus expansion (3.160) to find the correlator for a particular genus g.

An important point now is that the recursion relations for the correlators (3.153), in a perturbed theory leads to the following equation for the square root of the partition function, $\tau^2(t) = Z(t)$:

$$L_n \tau(t) = 0 \qquad n \geq -1 \tag{3.163}$$

where the L_n's are given by

$$
\begin{aligned}
L_{-1} &= \sum_{m=1}^{\infty}(m+\tfrac{1}{2})t_m\frac{\partial}{\partial t_{m-1}}+\tfrac{1}{8}\lambda^{-2}t_0^2 \\
L_0 &= \sum_{m=0}^{\infty}(m+\tfrac{1}{2})t_m\frac{\partial}{\partial t_m}+\tfrac{1}{16} \\
L_n &= \sum_{m=0}^{\infty}(m+\tfrac{1}{2})t_m\frac{\partial}{\partial t_{m+n}}+\tfrac{1}{2}\lambda^2\sum_{m=1}^{n}\frac{\partial^2}{\partial t_{m-1}\partial t_{n-m}}. \qquad (3.164)
\end{aligned}
$$

Thus the (square root of the) perturbed topological string theory partition function satisfies *the same* Virasoro constraints as the (square root of the) one-matrix model partition function does! And in both cases the connected surfaces are obtained from the logarithm of the partition functions. Therefore, if the Virasoro constraints above can be shown to be consistent with the recursion relation (3.153) this is strong evidence for the equivalence of the two approaches, at least for the simple system of pure gravity.

I want to show the consistency of the Virasoro constraints and the correlator recursion relations when the theory is "un-perturbed". "Un-perturbed" is in quotation marks because the recursion relation is actually obtained with the values $t_m = 0$ for $m \neq 1$, but $t_1 = -\frac{1}{3}$. For this, and other reasons, some authors prefer to shift the value of $t_1 \to t_1 + \frac{1}{3}$, so that $t_m = 0$ for all $m \geq 0$, but then an extra term has to be added [27] to the Virasoro operators (3.164). Let me keep t_1 at the unshifted value here. But first consider general values of the variables t_m. Then (3.163) yields

$$
\begin{aligned}
0 &= L_n\tau(t) \\
&= L_n e^{\frac{1}{2}F(t)} \\
&= \sum_{m=0}^{\infty}(m+\tfrac{1}{2})t_m\frac{\partial}{\partial t_{m+n}}e^{\frac{1}{2}F(t)}+\tfrac{1}{2}\lambda^2\sum_{m=1}^{n}\frac{\partial^2}{\partial t_{m-1}\partial t_{n-m}}e^{\frac{1}{2}F(t)} \\
&= \tfrac{1}{2}\tau\sum_{m=0}^{\infty}(m+\tfrac{1}{2})t_m\frac{\partial}{\partial t_{m+n}}F(t) \\
&\quad +\tfrac{1}{4}\tau\lambda^2\sum_{m=1}^{n}\frac{\partial^2}{\partial t_{m-1}\partial t_{n-m}}F(t)+\tfrac{1}{8}\tau\lambda^2\sum_{m=1}^{n}\frac{\partial F(t)}{\partial t_{m-1}}\frac{\partial F(t)}{\partial t_{n-m}} \qquad (3.165)
\end{aligned}
$$

Thus, using that the derivative $\frac{\partial}{\partial t_m}$ acting on the free energy $F_g(t)$ brings down an operator σ_m into the correlator, $\frac{\partial}{\partial t_m}F_g(t) = \langle\sigma_m\rangle_g$, where the correlator is taken in the perturbed point, we have

$$
0 = \frac{L_n\tau(t)}{\tau(t)}
$$

$$
\begin{aligned}
&= \tfrac{1}{2}\sum_{m=0}^{\infty}(m+\tfrac{1}{2})t_m \sum_{g=0}^{\infty}\lambda^{2g-2}\langle\sigma_{m+n}\rangle_g + \tfrac{1}{4}\sum_{m=1}^{n}\sum_{g=0}^{\infty}\lambda^{2g}\langle\sigma_{m-1}\sigma_{n-m}\rangle_g \\
&\quad + \tfrac{1}{8}\sum_{m=1}^{n}\sum_{g_1}\sum_{g_2}\lambda^{2(g_1+g_2)-2}\langle\sigma_{m-1}\rangle_{g_1}\langle\sigma_{n-m}\rangle_{g_2} \qquad (3.166)
\end{aligned}
$$

Now, for convenience define $D_s \equiv \frac{\partial^s}{\partial t_{k_1}\cdots\partial t_{k_s}}$, then when acting with D_s on the above we get

$$
\begin{aligned}
0 &= 2D_s\frac{L_n\tau(t)}{\tau(t)} \\
&= D_s\sum_{m=0}^{\infty}(m+\tfrac{1}{2})t_m\sum_{g=0}^{\infty}\lambda^{2g-2}\langle\sigma_{m+n}\rangle_g + \tfrac{1}{2}D_s\sum_{m=1}^{n}\sum_{g=0}^{\infty}\lambda^{2g}\langle\sigma_{m-1}\sigma_{n-m}\rangle_g \\
&\quad + \tfrac{1}{4}D_s\sum_{m=1}^{n}\sum_{g_1}\sum_{g_2}\lambda^{2(g_1+g_2)-2}\langle\sigma_{m-1}\rangle_{g_1}\langle\sigma_{n-m}\rangle_{g_2} \\
&= \sum_{m=0}^{\infty}(m+\tfrac{1}{2})t_m\sum_{g=0}^{\infty}\lambda^{2g-2}\langle\sigma_{m+n}\prod_{k\in S}\sigma_k\rangle_g \\
&\quad + \sum_{m=0}^{\infty}(m+\tfrac{1}{2})\sum_{k'\in S}\delta_{k',m}\sum_{g=0}^{\infty}\lambda^{2g-2}\langle\sigma_{m+n}\prod_{\substack{k\neq k'\\ k\in S}}\sigma_k\rangle_g \\
&\quad + \tfrac{1}{2}\sum_{m=1}^{n}\sum_{g=0}^{\infty}\lambda^{2g}\langle\sigma_{m-1}\sigma_{n-m}\prod_{k\in S}\sigma_k\rangle_g \\
&\quad + \tfrac{1}{4}\sum_{m=1}^{n}\sum_{g_1}\sum_{g_2}\lambda^{2(g_1+g_2)-2}\sum_{S=X\cup Y}\langle\sigma_{m-1}\prod_{k\in X}\sigma_k\rangle_{g_1}\langle\sigma_{n-m}\prod_{k\in Y}\sigma_k\rangle_{g_2} \qquad (3.167)
\end{aligned}
$$

— remember that the sum $S = X\cup Y$ is over all *disjunct* sets X and Y. Note that the sum is over all partitions in the "formal" sense that even if some of the s indices $k_1, k_2, \ldots, k_s$ are the same, e.g. $k_{13} = k_{17} = 5$, the indices should be distinguished in the partitions, e.g. the partition

$$X = \{\ldots, k_{13}(=5), \ldots\},\ Y = \{\ldots, k_{17}(=5), \ldots\}$$

is considered different from

$$X = \{\ldots, k_{17}(=5), \ldots\},\ Y = \{\ldots, k_{13}(=5), \ldots\}.$$

Turning now to particular values of the couplings $\{t_n\}$, selecting a *conformal* point (the $k = 1$ critical point, with $t_k = \frac{-1}{2k-1}$ [6]) namely with $t_1 = -\frac{1}{3}$ and $t_m = 0$ for $m \neq 1$, and selecting the coefficient of the $\lambda^{2\hat{g}-2}$-term, (3.167) reduces to

$$0 \;=\; -\langle\sigma_{n+1}\prod_{k\in S}\sigma_k\rangle_{\hat{g}}$$

$$+\sum_{k'\in S}(2k'+1)\langle\sigma_{k'+n}\prod_{\substack{k\neq k'\\ k\in S}}\sigma_k\rangle_{\hat{g}}+\sum_{m=1}^{n}\langle\sigma_{m-1}\sigma_{n-m}\prod_{k\in S}\sigma_k\rangle_{\hat{g}-1}$$

$$+\tfrac{1}{2}\sum_{m=1}^{n}\sum_{g_2}\sum_{S=X\cup Y}\langle\sigma_{m-1}\prod_{k\in X}\sigma_k\rangle_{\hat{g}-g_2}\langle\sigma_{n-m}\prod_{k\in Y}\sigma_k\rangle_{g_2} \qquad (3.168)$$

which is exactly the recursion relation (3.153).

Actually, in (3.165) I left out the two terms

$$\delta_{n,-1}\frac{1}{8}\lambda^{-2}t_0^2+\delta_{n,0}\frac{1}{16}$$

in L_n. However, the $\delta_{n,0}$-term vanishes whenever S is not the empty set, and the only contribution from the $\delta_{n,-1}$-term is when $S=\{0,0\}$, in that case

$$2D_s(\delta_{n,-1}\frac{1}{8}\lambda^{-2}t_0^2)=\frac{1}{2}\lambda^{-2}\delta_{n,-1}\ .$$

Since the only other term in (3.167) that survives when $n=-1$ and $S=\{0,0\}$ is

$$-\tfrac{1}{2}\sum_{g}\langle\sigma_0\sigma_0\sigma_0\rangle_g\lambda^{2g-2}$$

one finds that $\langle\sigma_0\sigma_0\sigma_0\rangle_0=1$, in accordance with what was claimed earlier in this section.

Lastly, I want to mention that the Virasoro constraints generalize to constraints with operators of the W-algebra. By coupling topological minimal matter to gravity one finds constraints of the form

$$W_n^{(s)}\cdot\tau=0 \qquad (3.169)$$

which is also the case for the multi matrix model approach.

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