DAMTP–97-133
FERMILAB–Conf-97/391-A
hep-th/9711195

Towards a String Formulation of Vortex Dynamics*

Elsebeth Schröder‡
Physikalisches Institut der Universität Bayreuth
D-95440 Bayreuth, Germany

Ola Törnkvist§
NASA/Fermilab Astrophysics Center
MS-209, P.O. Box 500, Batavia, IL 60510-0500, U.S.A.
and
Department of Applied Mathematics and Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, United Kingdom¶

19 November 1997

Abstract

We derive an exact equation of motion for a non-relativistic vortex in two- and three-dimensional models with a complex field. The velocity is given in terms of gradients of the complex field at the vortex position. We discuss the problem of reducing the field dynamics to a closed dynamical system with non-locally interacting strings as the fundamental degrees of freedom.

---

*Talk presented by O.T. at the International Workshop on Solitons, their Properties, Dynamics, Interactions, and Applications in Kingston, Ontario, Canada, 20-26 July 1997. To appear in the proceedings.

‡Electronic address: `schroder@nbi.dk`

§Electronic address: `O.Tornkvist@damtp.cam.ac.uk`

¶Present address.

1 Introduction

Vortices are extended solitonic objects that are (locally) stable solutions in many classical field theory models involving a complex field $\Phi = |\Phi|e^{iS}$. In three dimensions they resemble strings, but have a core of finite width where the fields deviate appreciably from their asymptotic values. The complex field Φ is zero on a one-dimensional string located inside the core. Along this string the phase S is multivalued, and singlevaluedness of Φ implies that the line integral $\oint dS$ along a contour enclosing the string is $2\pi n$ where n is the integer winding number of the vortex. Stable vortices with $n > 1$ will generally have n distinct strings of zeros located within the core. One can therefore regard such a multivortex as a bound state of unit-, or elementary, vortices with $n = 1$. For the special case of a single, stationary, straight, infinitely long, isolated, and therefore cylindrically symmetric vortex solution, it is well-known that $\Phi \sim r^{|n|}e^{in\varphi} \equiv z^n \ (z^{*(-n)})$, where r is the perpendicular distance to the string.

Non-relativistic models with vortex solutions include the Ginzburg-Pitaevski-Gross model for superfluid ^{4}He, with global U(1) symmetry, and the time-dependent Ginzburg-Landau model of a superconductor, with local U(1) symmetry. These are non-dissipative models whose field equations can be derived from Lagrangians. Their relativistic generalizations are known as the Goldstone model and abelian Higgs model, respectively, whose vortex solutions constitute global and gauged cosmic strings.

Non-relativistic vortices are solutions to equations with first-order time derivatives, which are typically of the form

$$\frac{d\Phi}{dt} = b\,\nabla^2\Phi + P(\Phi, \Phi^*)\Phi \ , \tag{1}$$

where $b \in \mathbf{C}$ and P is a polynomial. When $ib \notin \mathbf{R}$, the system is dissipative and no Lagrangian description exists. Vortex solutions in such a system have the distinctive feature that the lines of constant phase S are spirals rather than radial rays. Correspondingly, in three dimensions the sheets of constant phase are rolled up as a scroll around the central string. Such spiral vortex solutions occur in a variety of physical systems, such as chemical reaction-diffusion experiments, thermal convection, the growth of slime mold, non-linear laser optics, and even in the human heart.

2 String Formulation

The idea behind this work is to treat the string of zeros of Φ as a fundamental object, and to obtain an exact equation of motion for the string in terms of the fields that surround it. This is possible because the string is a feature of a *local* field theory, and therefore its motion can be determined from the behaviour of fields in an infinitesimal neighbourhood of the string. All the necessary information is encapsuled in the underlying field theory.

Because the strings have no thickness, the equation of motion for the string is exactly valid for an arbitrarily small intervortex distance as well as for arbitrarily large (but finite) vortex curvature. This is of considerable interest in the study of situations when vortex cores overlap, such as vortex-vortex scattering, bound multivortex states, cusp formation, the intersection and reconnection of vortex segments, collapsing string loops, and the question of vortex rigidity. In the string formulation, the strings interact non-locally through fields that are not confined to worldsheets but live in four-dimensional space-time.

3 The String Equation of Motion

The principal steps in deriving the string equation of motion are most clearly demonstrated in the case of non-relativistic vortices [1] that satisfy a field equation of the form (1). The equation for relativistic vortices has been worked out by Ben-Ya'acov [2].

Inserting $\Phi \equiv Re^{iS}$ into eq. (1) with $b = b_R + ib_I$, one obtains the amplitude and phase equations

$$\frac{d}{dt}\ln R = \mathrm{Re}(P) + b_R\left(\frac{1}{R}\nabla^2 R - (\nabla S)^2\right) - \frac{b_I}{R^2}\nabla\cdot(R^2\nabla S) \tag{2}$$

$$\frac{d}{dt}S = \mathrm{Im}(P) + b_I\left(\frac{1}{R}\nabla^2 R - (\nabla S)^2\right) + \frac{b_R}{R^2}\nabla\cdot(R^2\nabla S)\ . \tag{3}$$

Local curvilinear coordinates are then introduced in a neighbourhood of the string by writing $\boldsymbol{x} = \boldsymbol{X}(s,t) + x\boldsymbol{N}(s,t) + y\boldsymbol{B}(s,t)$ where s is the arc-length along the string, $\boldsymbol{X}(s,t)$ is the position of the string at a given time t, and $\boldsymbol{N}, \boldsymbol{B}$ are the normal and binormal unit vectors, respectively. An arbitrary position $\boldsymbol{x}$ near the string is thus specified at any time t by the coordinates s, x, and y. Local polar coordinates are defined by $x = r\cos\varphi$, $y = r\sin\varphi$.

One can now work out the gradient and Laplacian operators in the curvilinear coordinates. They depend on the curvature κ and torsion τ of the string. The time derivative in eqs. (2) and (3) is expressed as

$$\frac{d}{dt} = -(\dot{\boldsymbol{X}} + x\dot{\boldsymbol{N}} + y\dot{\boldsymbol{B}})\cdot\nabla + \frac{\partial}{\partial t}\ , \tag{4}$$

where $\dot{\boldsymbol{X}} = \partial\boldsymbol{X}/\partial t$ etc., and $\partial/\partial t$ indicates the time derivative in a frame following the local segment of the string such that s, x, y are constant.

The equation of motion for the string is derived by identifying the parts of eqs. (2) and (3) that are singular as $r \to 0$ and demanding that these singularities cancel order by order. To this end, let us separate out the non-differentiable parts of R and S, writing $\ln R = \ln R_{\text{sing.}} + \ln w$ and $S = S_{\text{sing.}} + \theta$, where $\ln w$ and θ are everywhere differentiable.

The calculations are simplified if we choose a "gauge" where the singular parts are given by $R_{\text{sing.}} = r^{|n|}$ and $S_{\text{sing.}} = n\varphi$ as for the straight, isolated vortex solution. The final result is independent of this restriction [1].

The singular terms of order r^{-2} are easily shown to cancel in both equations. Terms of order r^{-1} on the left-hand side of the equations contain the string velocity $\dot{\boldsymbol{X}}$, while those on the right-hand side do not. In this way an equation for $\dot{\boldsymbol{X}}$ is obtained. For example, dS/dt contains the term $-\dot{\boldsymbol{X}} \cdot \nabla S_{\text{sing.}} = -(n/r)\,\hat{\boldsymbol{\varphi}} \cdot \dot{\boldsymbol{X}}$, and $(\nabla S)^2$ contains the term $\nabla S_{\text{sing.}} \cdot \nabla\theta = (n/r)\,\hat{\boldsymbol{\varphi}} \cdot \nabla\theta$. Collecting all terms of order r^{-1} in both equations one obtains the following expression for the string velocity:

$$\begin{aligned} \dot{\boldsymbol{X}} &= b_{\scriptscriptstyle I}\left(\kappa\frac{n}{|n|}\boldsymbol{B} + 2\nabla_\perp\theta - 2\frac{n}{|n|}\boldsymbol{T}\times\nabla\ln w\right) \\ &+ b_{\scriptscriptstyle R}\left(\kappa\boldsymbol{N} - 2\nabla_\perp\ln w - 2\frac{n}{|n|}\boldsymbol{T}\times\nabla\theta\right)\,, \end{aligned} \qquad (5)$$

where $\boldsymbol{T} = \partial\boldsymbol{X}/\partial s$ is the string tangent vector and $\nabla_\perp = -\boldsymbol{T}\times(\boldsymbol{T}\times\nabla)$ is the gradient projection perpendicular to the string. Gradients are to be evaluated at the position of the string. The two-dimensional result is obtained as $\kappa \to 0$.

The velocity can also be expressed in terms of the original magnitude R and phase S. Because of an exact cancellation of singularities, this expression is identical to eq. (5) with the substitutions $\theta \to S$, $\ln w \to \ln R$. A beautifully compact expression is obtained by defining $\dot{Z} \equiv (\boldsymbol{N} + i\boldsymbol{B}) \cdot \dot{\boldsymbol{X}}$ and $z = x + iy$. Then one finds $\dot{Z} = b\,[-4\,\partial(\ln\Phi)/\partial z^* + \kappa]$ or $\dot{Z}^* = b\,[-4\,\partial(\ln\Phi)/\partial z + \kappa]$ for positive and negative n respectively, where the right-hand sides are evaluated on the string ($z = 0$). From the physical requirement of finite string velocity one then infers that the field Φ near the string is either holomorphic or anti-holomorphic.

The result shows that, apart from Biot-Savart terms proportional to curvature, the string velocity expression includes local gradients of the magnitude and phase of the complex field Φ. A similar result is obtained for the non-dissipative relativistic vortex, satisfying the equation [2]

$$X^{\mu;a}_{\ \ ;a} = -\frac{n}{|n|}\epsilon^{\mu}{}_{\nu\lambda\rho}X^{\lambda}_{,a}X^{\rho}_{,b}\epsilon^{ab}\sqrt{-\gamma}\,\partial^\nu\theta + 2(g^{\mu\nu} - \gamma^{ab}X^{\mu}_{,a}X^{\nu}_{,b})\,\partial_\nu\ln w\,, \qquad (6)$$

where σ^a ($a = 0, 1$) are coordinates on the worldsheet and γ^{ab} is the induced worldsheet metric. Eq. (6) reduces to the Nambu-Goto equation for a free relativistic string when the right-hand side is zero.

4 Conclusions and Outlook

We have shown that the motion of a vortex string, defined as the set of zeros of the complex field Φ, can be expressed exactly in terms of gradients of the magnitude and phase of Φ at the position of the string. This represents only partial progress towards a formulation of vortex dynamics in terms of strings, since Φ satisfies a non-linear partial differential equation whose solutions are not known. Further developments require approximations. Near the Bogomolnyi limit of 2D gauge theories one may make an adiabatic approximation in which slowly moving vortices at each instant take on the field configuration of known static solutions [3]. The field dynamics can then be reduced to a finite-dimensional system of differential equations for the vortex positions. Other approaches involve perturbations of the straight, isolated vortex solution, the small parameters being e.g. the curvature, b_R/b_I, the core radius [4]. Such approximations usually include only local contributions to the dynamics or make specific assumptions about the asymptotic field behavior. We emphasise that the equation derived here is exact and incorporates also long-range contributions from other vortices and remote segments of the same vortex.

Acknowledgments

One of us (O.T.) thanks the conference organisers for their splendid work. Support for O.T. was provided in part by the Swedish Natural Science Research Council (NFR) and by DOE and NASA under Grant NAG5-2788, in part by EPSRC under Grant GR/K50641. E.S. was supported by the Danish Natural Science Research Council.

References

[1] O. Törnkvist and E. Schröder, Phys. Rev. Lett. **78** (1997) 1908.

[2] U. Ben-Ya'acov, Nucl. Phys. **B382** (1992) 597.

[3] N. Manton, Ann. Phys. **256** (1997) 114.

[4] M. Gabbay, E. Ott and P.N. Guzdar, Phys. Rev. Lett. **78** (1997) 2012; I.S. Aranson and A.R. Bishop, *Stretching of vortex lines and generation of vorticity in the three-dimensional complex Ginzburg-Landau Equation*, preprint patt-sol/9705009.