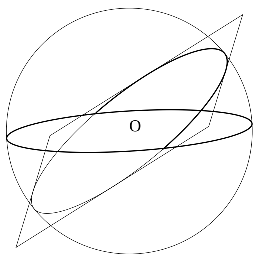
The Celestial vault

The vault as outside space

Standing on Earth you are aware of two things. A seemingly flat expanse (never mind if disturbed by high mountains and deep valleys), and a spherical vault that spans it. On Earth you can move around and in principle not only see things but also touch them and view them from different perspectives. But the objects of the sky are only accessible by sight, definitely not by touch. In ancient times two geometries developed. The flat Euclidean geometry with which we are most intimately acquainted and applies to our everyday world, and a spherical geometry which applies to the celestial world. Thus the Greeks in a sense were familiar with some non-Euclidean geometry, but prefered not to think of it in this way, for reasons that should become clear. For all what ancient people, including the Greeks, could know, the objects in the sky could be infinitely far away, thus truly inaccessible by touch, not part of the space in which we can in principle move. If so there would be an unbridgable separation between two worlds. This is science not superstition, although in retrospect it is tempting to think of it as such, because the science of astronomy rests on the assumption that the celestial objects are part of space and not infinitely far away. The most basic consequence of this is that we can talk about distances to celestial objects and thus meet the challenge of measuring them.

The vault as seen as a sphere inside space

The celestial sphere is something that we experience from the inside, but when we want to picture it we see it from the outside, and this is something quite different.



A sphere which is seen from the outside is something we want to touch. The great circles on it are clearly curved, they are circles in fact, but when seen from the center (O) they are seen as straight lines, because they are given by the intersection of planes through the center. In fact our field of vision is a sphere, the immobile eye has no means of judging distances, it can only perceive directions, and the visual sphere parametrizes all the directions there are, and gives us the first example of something bounded but yet without boundary.

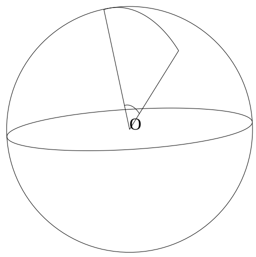
It is ironic that the geometry most accessible to our visual sense is the spherical, but of course what we think of as the remarkable global properties of non-Euclidean geometry, such as the failure of angles to add up to π is not as striking as our field of vision that allows scrutiny is quite limited[1]. The first

[1] Our field of vision is actually surprisingly extensive, but at its wide margins we can only detect movement as such, not the objects that move

step towards the spatial realization of celestial objects is to see them as actually placed on the inside of a sphere in space at whose center we are placed. One can then ask the question about the length of its radius, and more intriguingly what lies beyond it, because thinking of it as an actual sphere, it divides infinite Euclidean 3-dimensional space in a finite part and an unbounded infinite one, because for some reason we think that any ray emenating from the eye will travel indefinitely. Another possibility, which does not seem to have entered the minds of the ancients is that the ray would eventually return to the eye. That we would in fact live in a 3-dimensional sphere. If the celestial sphere would be a ballon, and we would expand it, the expansion would only go so far until it reached a maximum size, the equator of the universe, and then it would decrease in size and in the limit contract to a point, just as the latitudes starting at the northpole are expanding circles until they reach the equator (a great circle) and then contact until becoming a point at the southpole. The 3-dimensional sphere is no mystery, it can be constructed in 4-dimensional space as the loci of all points equidistant from the center, just as we define the circle in the plane, and the ordinary sphere in our three-dimensional space. However, it does not have to be constructed in that way, it does not require 4-dimensions to make sense, but it can be hard to imagine it intrinsically, just as it is very hard for us to imagine a circle unless we see it in the plane.

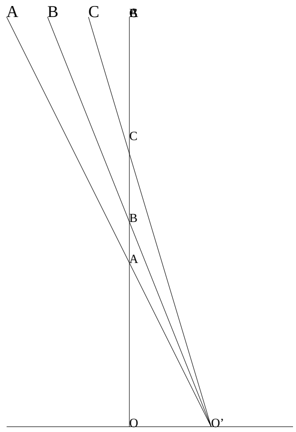
Depth and Parallax and angular Size

Now this is not fanciful, in fact it is hard to reject out of hand that the universe is in fact bounded without boundary. But this belongs to cosmology and we have to be concerned with more down-to-earth matters. In ordinary space we experience depth, for objects close to us this is part of stereoscopic vision, but in general we derive it from our movement in space. When we move we change perspective slightly, and the changes of those perspectives let us infer the depth of space, in a rather subtle but instinctive way. Stereoscopic vision depends on two eyes who are set apart and hence give slightly different points of view which the brain is able to integrate into one image. That the images are different you can easily convince yourself by alternating closing one eye and the other. The most basic phenomenon is that distant objects appear smaller than when close. This is useful when you 'know' the size of an object, but if you see an airplane in the sky when very young, you may think of it as a toy object quite close to you. But how big is an object, say the Moon in the sky? In one sense it is easier, or at least more direct, to measure objects on a sphere, in particular in your visual field. Due to the phenomenon of scaling in Euclidean space (meaning the existence of similar but non-congruent triangles), there is no canonical way of measuring distances, to do so you have to arbitrarily select a unit, and measure lengths in terms of that.



But the natural measure on the sphere is the angle a segment extends as seen from the center, thus the angular extension in your field of vision. Angles you can define intrinsically, as the notion of a right angle can be communicated, but not an arbitrary unit which refers to a particular object. It is no coincidence that the meter was defined as the equator to be 40 million meters. In practice it is hard to measure the circumference of the equator so it was replaced by a physical unit residing in Paris.
Now what you call a right angle, whether $\pi/2$ as mathematicians do or 90^o as is commonly done, an old tradition to which the astronomers still adhere is of less importance, you only need a currency exchange. Mathematically the size of angles are identified by the length of their corresponding arcs on a circle of radius one, the so called unit circle, and this has a great advantage when you want to express the trigonomteric functions in a series, especially sine and cosine. In particular sine and tangent for small angles are close to the numerical values of the angles. In particular for small angles, or equivalently distant objects, distance is more or less proportional to apparent size. This is not true for close distances and objects occupying a large part of your visual field where the relation is more subtle, but which most of us automatically deal with through our stereoscopic vision.

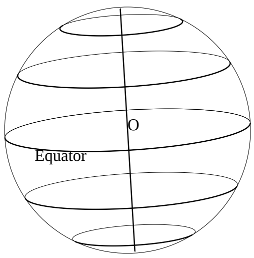
A tower seen from a distance equal to its height has an apparent extension of 45^0 which is very large and hard to take in at one time and thus atypical of the objects we look at in the sky (The Milky Way of course is the biggest object we can see in the sky, but we cannot take it all in at a glance, apart from the fact that it extends beyond any half-sphere and thus is never simultaneously visible in its entirety). The Moon and the Sun are approximately of the same apparent size meaning about half a degree (or $30'$ (minutes) or mathematically $\sim 1/120$) which is also about the apparent size of Boeing 747 at cruising altitude overhead. The bright star Mizar of the Double Dipper, has a fainter star Alcor separated by twelve minutes of arc, providing a modest challenge. Normal human eyesight detects more than seven stars in the Pleiades, involving significantly less separation. The naked human eye cannot, however, separate anything closer than half a minute of an arc, around thirty arc seconds. Jupiter at its closest extends about that angular distance, but still cannot really be seen as a disc. Thus the planets appear as point sources just as do the stars, but they shine with a steadiness not to be seen by the latter. As far as I know, no disc of any star has been resolved. The closest star is about a quarter million as distant as is the sun, and will extend about 0.01 seconds of arc one tenth of the apparent diameter of Pluto. Such minute resolutions cannot be obtained by telescopes on Earth due to the turbulent effects of the atmosphere.



When traveling by train you notice that objects close to the train appear very briefly in your visual field, while the further away they are the less they change their positions. The phenomenon is referred to as parallax. As you move, the direction to an object, seen as a point in your visual field will change, the change actually being equal to the apparent length of your travel as seen from the object. If you travel 100 meters an object 10 km away will have moved roughly half a degree, the apparent diameter of the Moon. Of course if the train changes its direction, so will those of the objects, but that is something different. If you rotate everything in your vision, regardless of distance, will change.

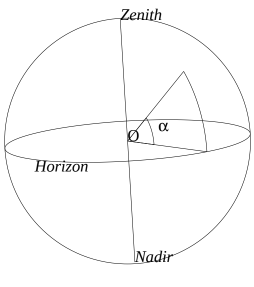
We all have our private fields of vision, which will change whenever we turn our heads or move our bodies. But there is also a common field of vision, given by the distant objects in the sky. The configurations of the stars in the sky will be the same for all observers, yet depending on your position on Earth and the time of day, the part of the sphere you will see will be different.

Spherical co-ordinates



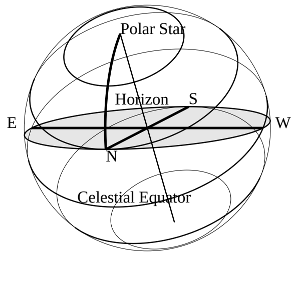
To mathematically describe the positions of objects on a sphere it is convenient to use spherical co-ordinates. This presupposes an axis as well as a fixed meridan. The axis define a pair of anti-podal points (the poles) and a great-circle the equator equidistant from both. Circles cut out by planes perpendicular to the axis will be referred to as the latitudes (or in astronomy - declinations), while great circles cut out by planes containing the axis, will be referred to as longitudes (or in astronomy acension).

Looking at the vault above you there is a center, the point just above you, referred to as zenith. It has a corresponding anti-pode - the nadir, which you cannot see.



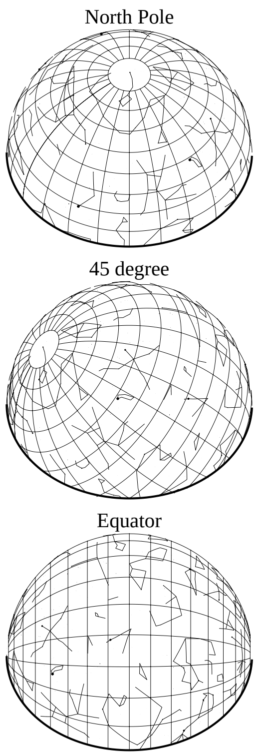
Anyway it gives a natural axis, and the equator will be given by the horizon, which we out on the ocean would experience as a straight, although it curves around as a circle. Thus any object seen on the sky will be given an altitude, namely its angular height above the horizon. However, there is no natural point on the horizon, which may serve as the zero meridan (just as there is none on Earth, the Greenwhich one won out, being a matter of politics or social conventions).

It is important to pin-point positions in the sky, all visual astronomy hinges on it, but those positions will change, so they are inconvenient to use in giving the positions of stars.



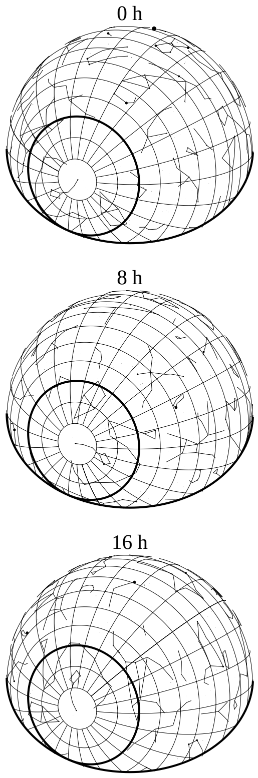
First the celestial sphere with its stars seems to rotate, this being an effect of the rotating earth. As you know it was an old question whether the Earth rotates or the sky, our senses tell us the former, it takes some effort of the imagination to fix the firmament and feel the ground rotate. The rotation of the celestial sphere goes through a fixed axis. This axis will coincide with the natural zenith-nadir ones, if and only if we are at one of the poles. Now at our present time we are lucky enough to have a bright star close to the axis on the Northern hemisphere, namely the Polar star.

With some simplification we can say that the sky rotates around the Polar star. At the Northpole, the Polar star will be at zenith, at the Southpole at nadir, and hence invisible.



At the poles we can see exactly half of the sky, nothing more, nothing left, and when the stars rotate, they keep their altitudes. Outside the poles, the Polar star is not on the natural axis, and will with that define a plane that cuts the sphere in a longitudal great circle (given two distinct points, not anti-podal, they will determine a unique great circle). The part on the same side as the Polar star will determine a point on the horizon, which will be due north. Thus only due to the rotating sky are we able to give the directions of north, and its opposite - the south. This does not work on the poles though. Any direction from the Northpole is south, any direction from the Southpole is north. The altitude of the Polar star gives the latitude of our position. When it is close to Zenith we are close to the North pole, when it is close to the horizon we are close to the equator. The altitude of a star is at highest when it is due south, at its lowest when it is due north (on the Northern hemisphere). When a celestial object is at its highest it is said to culminate. Given the axis of rotation, each star will have a declination given by its distance from the Polar star. Unlike the altitude of an observed star the declination is fixed.

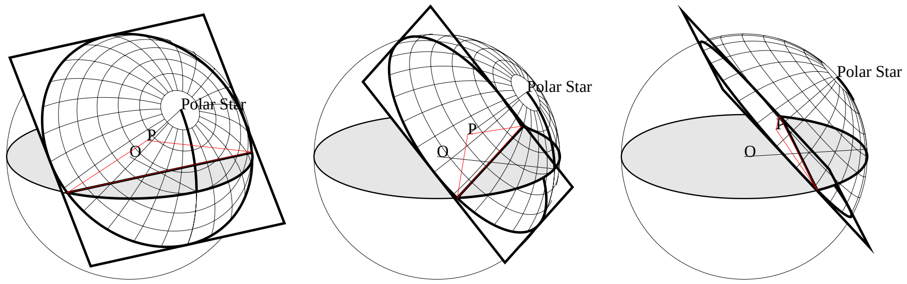
Positions of declination zero make up the celestial equator, those with positive declination make up the northern hemisphere, the one containing the Polar star, while those with negative declination will make up the southern hemisphere. In astronomy it is common to use the suffixes North and South, just as in geography, but in mathematics it is more convenient to use positive and negative. What is positive and negative is just a pure convention, but the point of conventions is to stick to them, otherwise there will be unnecessary confusion.

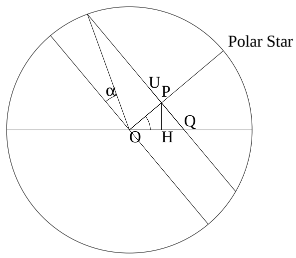


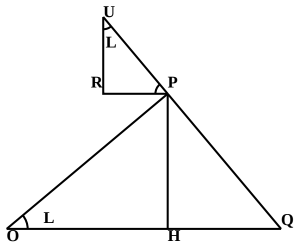
If the altitude of the Polar star is L (i.e. the latitude of the geographical position) and the declination of the star is D then the altitude of the star when due south will be $L - D + 90$ and due north $L + D - 90$. There will be three kinds of stars. If $L + D - 90 \geq 0$ or equivalently $D \geq 90 - L$ then the star will always be above the horizon (but it might be invisble for other reasons such as the sun being above the horizon as well, to which we will return). Such stars are called circumpolar stars. We may also have that $L - D + 90 \geq 180$ or equivalently $D \leq L - 90$ which means that the star is never above the horizion, and thus circumpolar at the antipode. Then for declinations between, the star is sometimes above, sometimes below the horizon. The objects on the celestial equator will be above horizon half of the time, and below the horizon the other half. Stars with positive declinations will on the Northern hemisphere be above the horizon more than half the time, and below less than half the time. The opposite will be true for stars with negative declination. At the equator there are no circumpolar stars, at the poles there are no intermediate stars. At the equator all the stars will be above the horizion half the time and below half the time as the circles traced by the stars will always be perpendicular to the horizon.

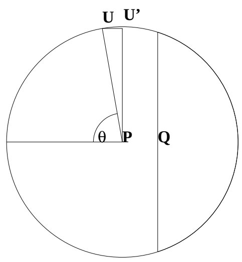
The Sky as a Clock

The rotation of the celestial sphere provides a clock. By a clock is meant a manifestation of time in space. A natural unit of time would be one revolution, meaning the time between two successive culminations of a given star. Then the time (of night) can be read off by the position of the star in its circle. This can be measured to some accuracy by determining its altitude. In ancient times there were no more accurate clocks.









Given any object (star) on the sky we can determine the fraction of time it is above the horizon. This depends on its declination as well as the latitude of the observer. If the declination is α the star traces a circle of radius $\cos\alpha$ on the celestial sphere, whose radius is set to one. At time θ (note that we think of time as angle and thus as position on its circle, once we have decided on a zero position (conveniently at due south on the northern hemispher) and a direction of rotation.) its height will be given by $\sin(L)\sin(\alpha) + \cos(\theta)\cos(\alpha)\cos(L)$ where L is the latitude. To see this, note the following. First the point P is the center of the latitudinal cirle with radius $\cos(\alpha)$. The length $|OP|$ of the segment OP is given by $\sin(\alpha)$, hence the height of P above the horizon will be given by $|PH| = |OP|\sin(L) = \sin(\alpha)\sin(L)$ (Note that if α is negative, this height will be negative, i.e. P will be below the horizon). We also remark that the segment UP has the same length as the segment UU' which is given by $\cos(\theta)\cos(\alpha)$. We now need to compute the length of UR. This will be given by $|UP|\cos(L) = \cos(\theta)\cos(\alpha)\cos(L)$. The total height of U above the horizon will then be the formula above. Recall that if $L = 0$ i.e. on the equator, each star is above the horizon exactly half the time namely when $90 \leq \theta \leq 270$. Otherwise only stars with declination zero will have that property.

From this we can easily compute the critical angle (θ_0) when the object is at the horizon. It is determined by

$$\cos(\theta_0) = -\tan(L)\tan(\alpha)$$

`Example` The latitude L of Gothenburg is about 58^o and Sirius, the brightest star on the sky has declination -17^o giving $\theta_0 = 60.7$.

Thus Sirius is above the horizon about the third of the time. At summer the sun will also be above the horizon when that happens, so Sirius will be visible much less than that, and in fact only around winter. We will return to that.

We can also use the formula to determine the angle θ when measuring the altitude of a star.

`Example` An observer is situated at Mumbai with latitude 19^o. A star A culminates at an altitude of 40^o, while a star B with known declination of 60^o and known right acsension (longitude) has an altitude of 20^o. Determine the declination of A and the difference in longitude between the stars A and B. (Thus if the longitude of B is known we can work out the longtitude of A). To work out the declination of A is easy, but there are two solutions, depending on whether A is close to the Polar start or not. In the first case it distance to the Polar star is $40 - 19 = 21$ and thus the declination is $90 - 21 = 69$, in the second case the distance is $180 - 40 - 19 = 121$ and the declination is $90 - 121 = 31$. In the first case the culmination is in the northern part of the sky, in the second the southern. In the first case we observe it facing the Polar start, in the second with our backs to it.

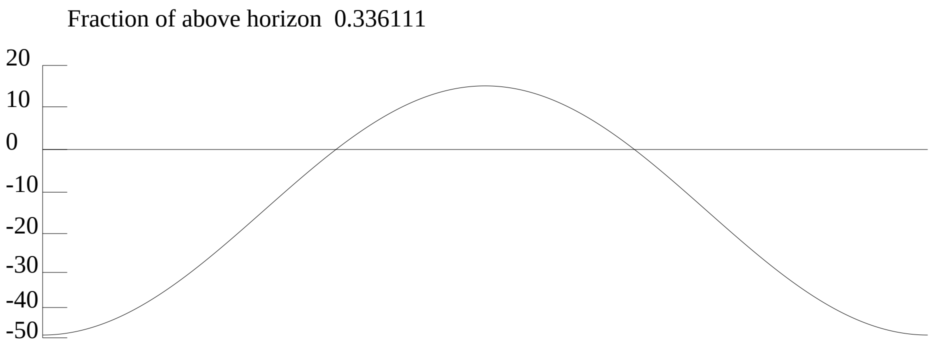
To use the star B as a clock, we work out its height above the horizon as $\sin(19)\sin(60)+\cos(\theta)\cos(19)\cos(60)$ thus $0.282+0.473\cos(theta)$. If the altitude is 20^o we solve for $0.282+0.473\cos(theta) = \sin(20) = 0.342$ thus $\cos(\theta) = 0.127$ with solutions $\theta = \pm 82.6941$ the sign depending on which side of the Polar star it is on. If on the east it is behind, if on the west ahead.

There are also other ways of making the computations.

`Matrix multiplication:`

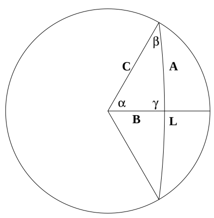
As you all know a rotation of angle θ is given by $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ this can be extended to 3-dimension by the matrix $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which fixes the $z-$axis. Let us denote it by Z_θ. In a similar way we can write down the matrices that give rotations around the $x-$ and $y-$ axi. Explicitly $X_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$ and $Y_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$. They are all orthogonal matrices, in fact they give examples of so called 1-parameters subgroups and for suitable $\theta_1, \theta_2, \theta_3$ any orthogonal matrix can be written in the form of a product $Z_{\theta_1} Y_{\theta_2} X_{\theta_3}$, as you know there is a 3-dimensional family of 3×3 orthogonal matrices. If you just look at the 2-dimensional family $Z_{\theta_1} Y_{\theta_2}$ you can move any point P except the poles $(0, 0, \pm 1)$ to any other point on the sphere. If we fix $\theta_2 = \pi/2 - L$ we have a 1-parameter subgroup of rotations around the axis through the point $(0,0,1)Y_{\theta_2}$. The orbits of points will be the orbits of stars on the sky at latitude L. Thus if you have a star of declination α you chose the point $P = (-\cos\alpha, 0, \sin\alpha)$ and look at the $z-$ coordinates of $PZ_{\theta_1}Y_{\pi/2-L}$. This can easily be computed getting $\sin L \sin\alpha + \cos L \cos\alpha \cos\theta$ recapturing the formula we obtained geometrically.

As an application we can graph the z-coordinate (note that this will be sine of the altitude) of Sirius at the horizon of Gothenburg.



`Spherical Trigonometry:`

This was developed already by the Greek, and was a subject astronomers had to learn until the 60's (at least). Nowadays the computations are done mechanically by computers (as above) and need not be so efficient and conceptual.



We have the data of a circle of radius C with center at P and a great circle L (line) cutting a radius orthogonally, getting a triangle to the left. We want to compute the angle α because that will give us the fraction of the circle 'above' the line L (given by the horizon). The radius C we can read off from the declination of the star, and B is given by the altitude of the Polar star.

We will now use two facts from spherical geometry. The first is the spherical form of Pythagoras theorem which is written

$$\cos C = \cos A \cos B$$

Furthermore we have the spherical form of the sine theorem

$$\frac{\sin\alpha}{\sin A} = \frac{\sin\beta}{\sin B} = \frac{\sin\gamma}{\sin C}$$

we get $\sin\alpha = \frac{\sin A}{\sin C}$ from the sine-theorem (recall that γ is right angle, and hence $\sin(\gamma) = 1$) and $\cos A = \frac{\cos C}{\cos B}$ from Pythagoras. This leads to

$$\sin\alpha = \frac{\sqrt{\cos^2 B - \cos^2 C}}{\cos B \sin C}$$

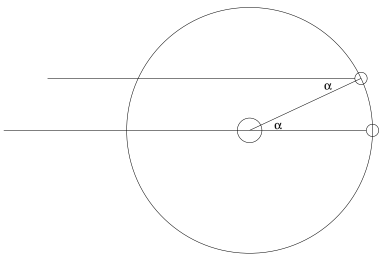
and with some straightforward trigonometric manipulations we end up with the elegant

$$\cos\alpha = \frac{\tan B}{\tan C}$$

Note that if we set $\alpha = \pi - \theta$ and $B = L$ and $C = \pi/2 - \alpha$ we retrieve our original formula.

The Movement of the Sun

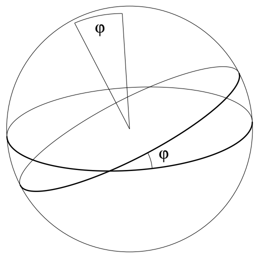
The celestial sphere makes a complete rotation in 23h 56m (using our units of time), which is 4 minutes short of the 24 hour period. The former period is also the exact period of the rotation of the Earth. Those four minutes are important. The year contains 365 days (and a little bit more) but during that time the earth has rotated 366 times around its axis (and a little bit more). If the Earth would face the same side to the Sun (as the Moon does to the Earth) the notion of a day would not make sense, or there would be zero days to the year, but with respect to the stars the Earth would make one (=0+1) complete revolution.



When the Earth has made one complete revolution with respect to the stars, it has also moved a certain angle (α) in its orbit around the Sun, and the Sun will no longer be in the same position, the discrepancy being given by the same angle. If the Earth rotates in the same direction as it orbits around the Sun (as it does, both rotations are counter-clockwise when seen from the Northern part of the hemisphere (from above)) then it is behind and has to rotate α more. If the directions are opposite it is ahead.
The angle α is approximately 1/365 of a complete revolution, and expressed in hours $24h/365 \sim 4m$ which has to be added to get the full 24 hour period of a complete rotation with respect to the Sun.

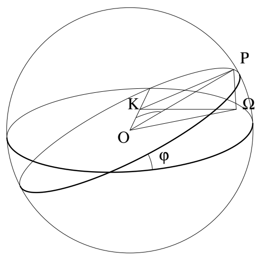
The position of the sun is not fixed visavi the stars, although its positions among the stars is not easy to directly observe because the brightness of the sun blots them all out except during truly exceptional circumstances such as given by a total eclipse. But this does not prevent us from measuring its position nevertheless which will be technically a bit more complicated. The sun will trace a great circle with respect to the fixed stars and complete one revolution in one year. As seen from the Northern hemisphere (i.e. if the Polar star is up) the rotation with respect to the fixed stars will be counter-clockwise. Mor succinctly the movement will be in the opposite direction of the apparent motion of the stars. Thus if the Earth would stop rotating, and thus the stars fixed on the sky, the sun would rise in the West and set in the East, making a complete revolution in one year. Roughly there are 360 days in a year, and 360^o in a complete revolution. This means that in one day it moves about one degree. Instead of using degrees we can divide the circle in 24 hours, one hour will then correspond to 15^o and thus each degree to 4 minutes. Those four minutes are the same as the four minutes the rotational period is short of 24 hours.

The Ecliptic

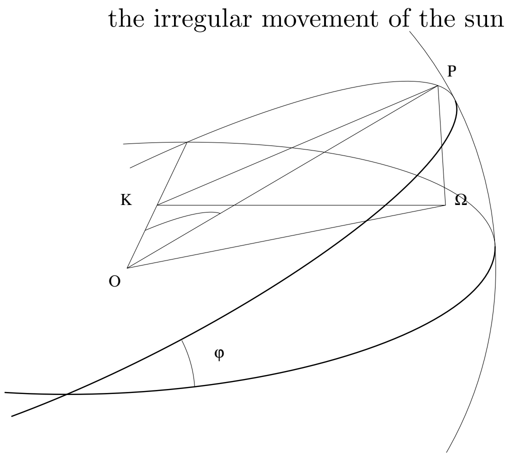


The great circle the sun transverses will make an angle (ψ) with the (celestial) equator and is called the ecliptic. The plane of the equator is normal to the axis of rotation, the plane of the ecliptic is the same as that of the orbit of the earth. The angle is about 23.5^o. When the ecliptic is on the upper part of the celestial sphere, the declination of the sun is positive. Its maximal value (23.5^o) is achieved at midsummer, its lowest at midwinter.

The two points of crossings are refered to as the equinoxes. The declination being zero, means that day and night will be of equal length. One will be the spring equinox, the other the autumn. In general we can compute the declinationdepending on the time of the year. This is one particular aspect of computing the separation of two great-circles knowing the angle between them, which is clearly the angle between their normals.



In terms of the angle α given by KOP we want to compute the angle ($\delta(\alpha)$) given by $PO\Omega$. The length KP is easily computed as $\sin(\alpha)$ hence the length $P\Omega$ is given by $\sin(\alpha)\sin(\psi)$ where ψ is the angle $PK\Omega$ given by the angles of the two great circles, and hence the desired angle will satisfy $\sin(\delta(\alpha)) = \sin(\alpha)\sin(\psi)$. Note that the angle given by $KO\Omega$ will not be the same as KOP which will account for the irregular movement of the sun along its path.



`Example:` This might be worth a digression.

Denote the angle $KO\Omega$ with $\theta(\alpha)$ note that $\theta(\alpha) = \alpha$ for $\alpha = -\pi/2, 0, \pi/2$. Consulting the picture on the previous page we see that $|K\Omega| = \sin(\alpha)\cos(\psi)$ while $|OK| = \cos(\alpha)$ from which we conclude that

$$\tan(\theta(\alpha)) = \tan(\alpha)\cos(\psi)$$

As $\tan(t)$ is an increasing function for $-\pi/2 < t < \pi/2$ and $\cos(\psi) < 1$ we see that if $\alpha > 0$ then $\theta < \alpha$ and when ($\alpha < 0$ we have $\theta > \alpha$. If the Sun would move at a uniform speed along its path, its projection onto the celestial equator does not, and it is really movement along that, which amounts to longitudinal change, that keeps time. Let us now compute the maximal discrepancy $\theta(\alpha) - \alpha$. Differentiating we want to find when $\theta'(\alpha) - 1 = 0$. Using the identity above we get

$$(1 + \tan^2(\theta))\theta' = (1 + \tan^2(\alpha))\cos(\psi)$$

and thus

$$(1 + \tan^2(\theta)) = (1 + \tan^2(\alpha))\cos(\psi)$$

the left hand side can be replaced by $(1 + \tan^2(\alpha)\cos^2(\psi))$ allowing us to solve for $\tan(\alpha)$ and obtain

$$\tan(\alpha) = \frac{1}{\sqrt{\cos(\psi)}}$$

and hence

$$\tan(\theta) = \sqrt{\cos(\psi)}$$

The identity $\tan(x)\tan(\pi/2 - x)$ should be obvious from the basic $\cos(\pi/2 - x) = \sin(x)$ and vice versa, and can be rewritten as $\tan(\pi/4 + x)\tan(\pi/4 - x) = 1$ we can thus write $\theta = \pi/4 - x, \alpha = \pi/4 + x$ for some $x > 0$ and we are interested in $2x$. We can write down the first terms for the Taylor expansion

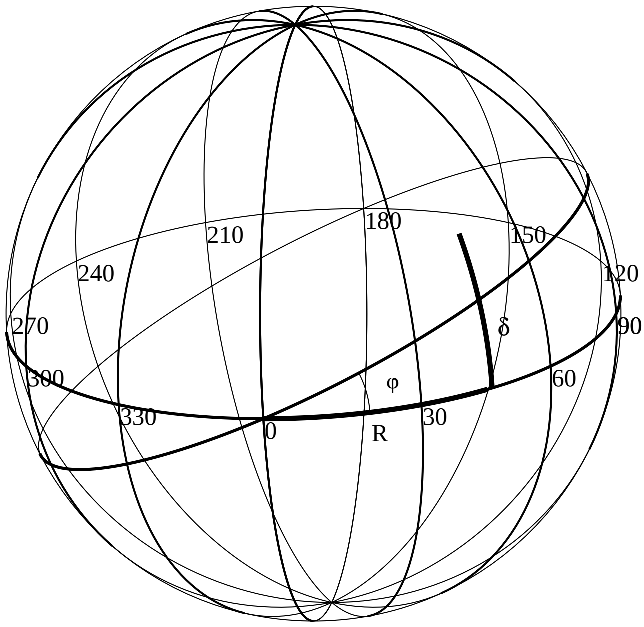
$$\tan(\pi/4 + x) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \ldots$$

while $\cos(\psi) = 0.91706..$ and $\sqrt{\cos(\psi)} = 0.957633..$ and hence $\frac{1}{\sqrt{\cos(\psi)}} = 1.04424..$ hence we can approximate x by $0.02212..$ (this will overshoot by an amount of $2 \times 5 \times 10^{-4} + .. \sim 10^-3$ which we will choose to ignore. Thus the discrepancy is about $0.044..$ which corresponds to about $2.52..^o$ or about ten minutes the sun will be off. This is the geometrical part of the so called 'equation of time' which measures how much the sun is of time, meaning that it does not always culminate at noon as it is supposed to do. The geometrical part is due to the inclination of the axis, the more the axis is inclined the larger the discrepancy. Then there is also a dynamic element, which is due to the elliptic orbit, and thus that the Sun does not travel along the Ecliptic at a uniform speed.

The inclination of the earth's axis is rather big, which will mean that there are marked seasonal variations for latitudes close to the poles. Inside the arctic circles the sun will be circumpolar for some time of the year. This clearly means that there is sunlight around the clock. The closer to the pole, the longer the period of extended lightness (or darkness) reaching a maximum of half a year at the poles.

The determination of the exact positions of the equinoxes is very important in all basic astronomy because they are well-defined positions on the celestial equator and define among other things the duration of a year. The longitude that passes through the vernal (spring) equinox is by definition set as the zero one, and then the meridans are counted from that. This system of spherical co-ordinates is the standard one to give the positions of celestial objects which will then be independent of time, which is of course a great advantage. Given the co-ordinates of a star, we can predict its position on the celestial sphere at any time and at any location on earth.

Thus the position of a star is given by its declination (latitude) (δ), i.e. its height above the celestial equator, and by its so called Right ascension (longitude) (R) which is counted from the vernal (spring) equinox in a counter clockwise direction.



The Right Ascension is normally given not in degrees but in hours, where 24 hours correspond to 360^o. Thus 1 h correspond to 15^o and 4 minutes to 1^o as noted before. The reason for this terminology is that objects on the celestial equator will be above the horizon half the time and below half the time. If an object on it with right ascension 0 will rise at a certain time, one with right ascension R will rise R hours later. More generally the time difference will apply to all objects when culminations are concerned. We will of course approximate the rotation time with 24 hours which involves an error of about 0.3%. The Right Ascension of the Sun will increase during the year. If we approximate the year with 360 days, and each month with 30 days, the position of the sun will increase two hours for each month (and 4 minutes for each day as we have remarked above). For astronomical reasons it would be convenient if the year started on March 1 with the vernal equinox, as was the case previously (**Sept**ember, **Oct**ober, **Nov**ember and **Dec**ember being the seventh, eighth, ninth and tenth month respectively). Now there is a shift, the Sun will be at the vernal equinox around March 20, while the autumnal equinox will be around September 23, the summer and winter solstices will be approximately around June and December 21 (see the table at the end of the section). This would correspond to $0, 6, 12$ and 18 hours but not quite. The durations of summer and winter (positive and negative declination of the Sun on the northern hemisphere) are not the same, something which was observed already in antiquity. The reason for this is that the earth travels in an elliptical orbit, in which the sun is not in the center, and the velocity varies. The closest approach is in January, i.e. during winter on the northern hemisphere, and Earth will spend less time in the (northern) winter part of the year than in the summer. The discrepancy is a couple of days.

More remarkably though was that the Greeks (Hipparchos) discovered that the position of the equinoxes moved. The movement is slight, only some $50''$ a year, and it takes some 26000 years for this to make a complete revolution. In fact the axis of the earth rotates around a line perpendicular to the orbital plane, thus the pole position traces out a circle on the sky. This has practical consequences. There is a convention to change the co-ordinates every 50 years, the last change being made in 2000 (before that we had 1950). One talks about different epochs. One may then have a different co-ordinate system, in which the ecliptic serves as the equator (with some, I believe, arbitrary convention as to the zero meridan) this is referred to as the ecliptic (as opposed to the equatorial) system. Given the ecliptic co-ordinates, the equatorial co-ordinates can easily be computed for each epoch.

The precession of the equinoxes presents a problem of what should be the definition of a year. One definition of a year would be the duration of a complete revolution of the year as regards the fixed stars. But that would mean that equinoxes, and hence mid-summer and mid-winter would drift over the years, so in 13'000 years, mid-summer would appear in December. Thus it makes more sense, as noted above, to measure the period between two vernal equinoxes. There will be a discrepancy of about 1/26000 of a year, which is roughly 1000 seconds, i.e. about a quarter of an hour.

Now there is no integral multiples of days in a year, which is an inconvenience when you make calendars. As the length of a year is roughly 365 and a quarter day, we have the convention of four years cycles, with one leap year. This is the Julian calender. Each year we get a quarter of a day behind, which is compensated each fourth year by adding an extra day to catch up. This is the well-known Julian calender. Now the discrepancy is not exactly a quarter. By compensating with a day each four years we overdo it. In fact in a 400 year period we overdo it by three days. In about 2000 years we have a discrepancy of about two weeks. This is not considered acceptable and in the 16th century the Gregorian calender was introduced making the extra stipulation that years divisible by 100 are only leap years if they are divisible by 400, thus 1900 was not a leap year, although 2000 was. Of course nothing is perfect the Gregorian calender will eventually have to be modified. More precisely The Gregorian year is 365.2425 days (as opposed to the 365.25 days of the Julian year) If we multiply this by 400 we get $400 \times 365 + 97$ thus we need 97 leap years in the period. But the mean tropical year is 365.24219. This means that in 400 years the Gregorian would overshoot by 0.124 days. This is close to 1/8, thus in every 3200 years we should lop off a single day. If we would not do this for a 100'000 years, the seasons would be off by a month. Hardly surprising that the astronomers stick to the Julian calendar.

Below is a table of the times for the equinoxes and the solstices, meaning when the Sun actually passes through those points on the Ecliptic

Year	Ver.	Equinox	Sum.	Solstice	Aut.	Equinox	Win.	Solstice
	March		June		Sept	ember	Dec	ember
2010	20	17:32	21	11:28	23	03:09	21	23:38
2011	20	23:21	21	17:16	23	09:04	22	05:30
2012	20	05:14	20	23:09	22	14:49	21	11:12
2013	20	11:02	21	05:04	22	20:44	21	17:11
2014	20	16:57	21	10:51	23	02:29	21	23:03
2015	20	22:45	21	16:38	23	08:21	22	04:48
2016	20	04:30	20	22:34	22	14:21	21	10:44
2017	20	10:28	21	04:24	22	20:02	21	16:28
2018	20	16:15	21	10:07	23	01:54	21	22:23
2019	20	21:58	21	15:54	23	07:50	22	04:19
2020	20	03:50	20	21:44	22	3:31	21	10:02

Note that we have an approximate shift of six hours every year, and after each leap-year there is back-wards shift of 24 hours which in most cases means that the date changes by one day. In particular the length of a year varies from year to year. Notice that there is a drift of 45 minutes every four years at the times of the vernal equinoxes. In 400 years this corresponds to 75 hours, which is roughly three days, hence the need for the Gregorian calendar.

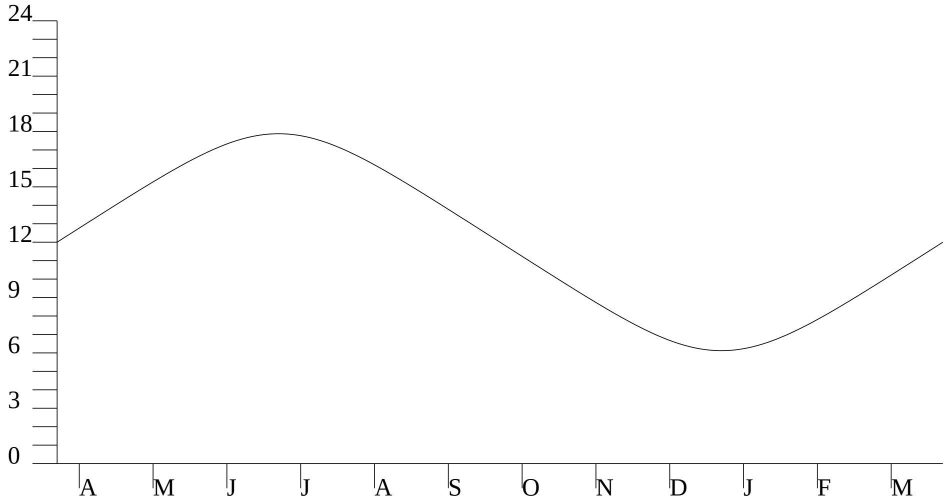
Length of Day-Light

The length of day-light depends on the time of the year and the latitude. In summers days are longer than nights, and in winters shorter than nights. At the equinoxes they are the same length. At the equator night and day are equally long throughout the year.

The length of light depends on the declination δ of the sun. In fact we have computed it as $\arccos(-\tan(L)\tan(\delta))/\pi$ where L is the latitude. We have also computed how δ varies with the time of the year. If t gives time counting from the vernal equinox (roughly 1^o a day) we have

$$\sin(\delta(t)) = \sin(t)\sin(\psi)$$

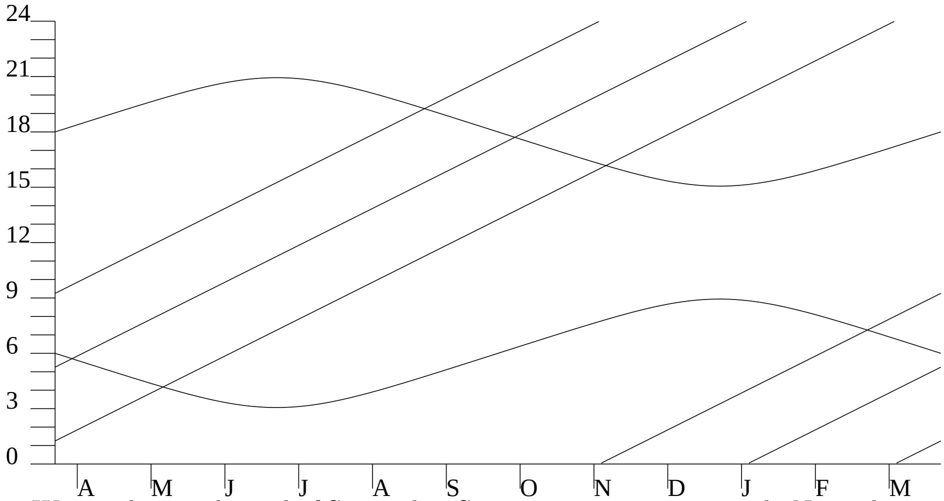
This gives $\cos(\delta(t)) = \sqrt{1 - \sin^2(t)\sin^2(\psi)}$ allowing us to compute $\tan(\delta(t))$. At the horizon of Gothenburg we can graph it as follows.



The Sun and Sirius

As noted at the horizon of Gothenburg Sirius will be above the horizon for about a third of the time, i.e. on the average of 8 hours a day. The sun is above the horizon on the average of 12 hours a day, regardless of latitude. To each position of the Sun there will be an antipodal position, and the day lights hours of one will be the complement of those of the hours, using the rotation days. Thus at Gothenburg Sirius will be visible on the average of 4 night hours during the day. It will not be observable in the summer though, as then the right ascension of the Sun is close to it, but quite visible in the winter when the Sun is on the opposite part of the sky. Circumpolar stars will below the Arctic circle be visible the year round, while those between will more or less depend on the time of the year to be seen. The Orion is a typical constellation of the winter.

We can now combine the graph above with the durations of Sirius being above the horizon to get the followign



We see that at the end of September Sirius sets at sunset, in early November it rises at sunset and the peak period of watching is from that time until mid February, when the Sun and Sirius are never simultaneously above the horizon and the star can be seen for a full 8 hours during the night.