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R-symmetry Charges Of Monopole Operators

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Abstract

M-theory is an attempt to unify the different 10-dimensional superstring theories in a single framework. In this 11-dimensional theory the strings become two-dimensional objects called M2-branes. The interactions of these branes are not very well understood at a fundamental level. At low energies, however, a three-dimensional superconformal field theory known as the ABJM theory has been conjectured to describe the world-volume dynamics of multiple M2-branes.

We introduce monopole operators in three-dimensional field theories and calculate the R-symmetry charges of such operators in $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory. This theory reduces to the ABJM theory in the IR, but our calculations are performed in the UV. Results for the ABJM case can be obtained by flowing to the IR, if the quantities involved are constant along the RG flow. Monopole operators with vanishing R-charges are needed in the ABJM theory, both for supersymmetry enhancement and for matching the spectrum with the dual gravity theory.

To describe the monopole operators we use the radial quantization method, allowing us to indirectly study the operators by looking at monopole states. We start by calculating the abelian R-charges carried by our monopole vacuum state. This is done by a normal ordering computation and proves that there exist monopoles with vanishing R-charge. Since the abelian charge can change along the RG flow, however, this does not prove anything for the ABJM theory. The non-abelian $\mathrm{SU}(2)_R$-charges are calculated by studying the collective coordinate parametrizing our monopole vacuum state. These charges are also found to be vanishing, and since non-abelian representations cannot change continuously the result is valid in the IR (ABJM) limit as well. As a part of our computations we also derive explicit expressions for the monopole spinor harmonics, defined as eigenspinors of the Dirac operator on a sphere around a magnetic monopole.

Keywords:
Monopole operator, ABJM, BLG, R-charge, superconformal field theory, Chern-Simons theory, M-theory

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“There is nothing more practical than a good theory”

-James C. Maxwell

Contents

1

Introduction

In the history of theoretical physics, unifications has always been of great importance. In unifying two different concepts under a single theoretical framework, a closer understanding of the laws of nature is obtained. Earlier examples of great unifying theories are electromagnetism and Einstein's theory of special relativity. The task that now lies ahead for physicists is to unify the strong and electroweak interactions, currently best described by the standard model of particle physics, with gravity, which is described by a completely different theory; general relativity.

In trying to describe the fundamental laws of nature, the main guiding principle is that of symmetries. A theory is symmetric under a specific set of transformations if these transformations leave the theory, often expressed by its action, invariant. The continuous symmetries of a theory can be divided into two classes; spacetime symmetries and internal symmetries. The former are symmetries affecting spacetime, such as translations, rotations and Lorentz boosts. In this class of symmetries we also find conformal symmetry and supersymmetry, which will be introduced below. The internal symmetries, on the other hand, reflect a redundance in the way we describe the theory mathematically. The so-called gauge symmetries, which in field theories transform different fields into each other, are extremely common in modern physics. Often, most of the properties of a theory follow directly from its symmetries.

The standard model is formulated in the framework of quantum field theory (QFT), where elementary particles are described as quantum excitations of fields. Quantum field theories are enormously widespread in fundamental physics and show up in many other contexts as well. For an introductory text on QFT, see [19].

Despite the big success of the standard model to describe the strong and electroweak interactions, it suffers from several problems. First, there are some "minor" ones concerning fine-tuning and the so-called hierarchy problem. These can be solved by introducing the concept of supersymmetry (SUSY); a symmetry between bosons and fermions. Most physicists believe that SUSY exists in nature, even though no experimental evidence has been found so far. For an introduction on the subject, see [3].

The major problem of the standard model, however, is that it does not include gravity. Supersymmetrical quantum field theories of gravity (supergravity) have been studied since the 1970:s and was for a while considered as potential candidates for a "theory of everything". These theories met a number of problems though, excluding them as alternatives for a fundamental theory. Instead, the 1980:s saw the rise of string theory as the main candidate. In string theory, the fundamental objects are not point particles, but vibrating one-dimensional strings. Interestingly, only string theories with supersymmetry (superstring theories) seem to be without inconsistencies. After a while, five separate superstring theories emerged, each one being 10-dimensional. Later, in the mid 1990:s, it was suggested by Edward Witten that these theories are different limits of an underlying more fundamental theory called M-theory [25]. In M-theory, the world is 11-dimensional and the strings in superstring theory become two-dimensional membranes called M2-branes. It was also found that 11-dimensional supergravity can be seen as a low-energy limit of M-theory, which revived the interest in this "dead" field.

To this day, M-theory is still poorly understood. Many leading physicists believe that completely new mathematics is required to correctly describe the theory at a fundamental level. Nevertheless, certain aspects and limits of the theory are possible to study via superstring- and supergravity theories. Also, a special class of quantum field theories known as conformal field theories (CFT) can describe the world-volume dynamics of strings and branes in certain cases.

In the late 1990:s, the so-called AdS/CFT-correspondence was suggested by Juan Maldacena [16]. This turned out to be an important breakthrough in theoretical physics and offered a completely new way of looking at things. Maldacena conjectured that superstring- and supergravity theories defined on Anti-de Sitter (AdS) spaces are actually dual (equivalent) to supersymmetric and conformal (superconformal) field theories on the boundaries of these spaces. Since then, many dualities of this kind has been found and new aspects of string- and M-theory can be studied via conformal field theories. Needless to say, this has increased the importance of CFT in fundamental physics. The AdS/CFT-correspondance can also be used the other way around. Many systems in condensed matter theory, for instance, that are described by CFT can now be studied using methods from string theory.

The concern of this thesis is a superconformal field theory known as the ABJM theory, which is proposed to describe the world-volume dynamics of multiple M2-branes. The ABJM theory is manifestly $\mathcal{N} = 6$ supersymmetric. For certain choices of parameters, however, it is conjectured to describe M2-brane configurations with $\mathcal{N} = 8$ supersymmetry. Thus, if the ABJM theory is a correct world-volume theory of M2-branes, the supersymmetry must be enhanced for these choices of parameters. That this really is the case has been explicitly proven using *monopole operators*. These must exist inherently in the theory and have certain properties for the SUSY enhancement mechanism to work out. The main subject of this thesis is to prove the existence of monopole operators with the desired properties in the ABJM theory. Most of the content in the thesis is based on the arguments and calculations made in [7], and no new results are presented.

1.1 Outline

In chapter 2, we introduce superconformal field theories and give the details of the BLG and the ABJM theory. We also describe a mechanism of supersymmetry enhancement in the ABJM theory and briefly explain the importance of monopole operators in this context.

In chapter 3, we introduce $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory as a UV completion to ABJM and rewrite it in the radial quantization formulation. We also define monopole operators in gauge theories and proceed to find a classical BPS monopole background solution to our theory.

In the first part of chapter 4, we consider a special case of the monopole background and calculate its $\mathrm{U}(1)_R$ charge. This is done by normal ordering the charge operator derived from the R-symmetry Noether current. In the second part of chapter 4, we instead look at the general monopole background and calculate the $\mathrm{SU}(2)_R$ charge. This is done by quantizing the collective coordinate that parametrizes the background.

In chapter 5, we summarize what we have done and comment on our results.

2

Superconformal Field Theories

The spacetime symmetry group of the standard model is the Poincaré group, containing translations and Lorentz transformations. As mentioned in the previous chapter, other important spacetime symmetries are supersymmetry and conformal symmetry. A field theory possessing both these kinds of symmetries, in addition to the Poincaré symmetries, is called a *superconformal* field theory. In fact, it can be shown that the superconformal algebra is the largest possible spacetime symmetry algebra of a quantum field theory [12]. With supersymmetry also comes R-symmetry, which is a symmetry rotating the different supersymmetry generators (supercharges) into one another. R-symmetry is a central concept in this thesis.

Conformal transformations are defined as transformations preserving angles. They can be divided into scaling transformations (dilatations) and special conformal transformations.

Conformal symmetry is not consistent with the existence of massive particles, which is why a theory like the standard model cannot be conformally invariant. Nevertheless, as hinted in the previous section and as explained below, CFT is very important in modern physics. In condensed matter physics, Euclidian two-dimensional CFT:s are used to describe critical point phenomena. In addition, a special class of three-dimensional gauge theories known as Chern-Simons theories are important in the description of phenomena that has to do with topological order. In a Chern-Simons theory, the gauge field dynamics is given by the Chern-Simons form

$$\frac{k}{4\pi}\int \mathrm{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right) \tag{2.1}$$

where the integer parameter k is called the Chern-Simons level.

In string theory, two-dimensional conformal field theories are known to arise on the world-sheet of strings. In superstring theory, the world-sheet dynamics is described by superconformal field theory.

Recently, three-dimensional superconformal Chern-Simons theories were suggested to describe the world-volume dynamics of coincident M2-branes. These specific theories are known as the BLG and the ABJM theory and will be described in the subsequent sections. Most of this thesis is concerned with properties of the ABJM theory.

2.1 BLG

In 2007, a world-volume theory for stacks of multiple M2-branes was found by Bagger and Lambert [4,5] and separately by Gustavsson [10] (BLG). The BLG theory is an $\mathcal{N} = 8$ superconformal Chern-Simons theory based on an algebraic structure called a three-algebra, with a basis T^a and a totally antisymmetric triple product:

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d. \tag{2.2}$$

There is also a symmetric trace-form allowing us to raise and lower three-algebra indices. Analogously to the Jacobi identity for ordinary Lie algebras, the three-algebra generators obey a fundamental identity. Expressed in the structure constants this identity reads

$$f^{abc}{}_g f^{efg}{}_d = 3 f^{ef[a}{}_g f^{bc]g}{}_d, \tag{2.3}$$

which equivalently can be written as

$$f^{[abc}{}_g f^{e]fg}{}_d = 0 \tag{2.4}$$

or

$$f^{ab[c}{}_g f^{d]efg} = -f^{cd[a}{}_g f^{b]efg}. \tag{2.5}$$

Furthermore, one can show that the structure constants are totally antisymmetric:

$$f^{abcd} = f^{[abcd]}. \tag{2.6}$$

For a specific realization of the three-algebra, related to the Lie algebra SO(4), it was later shown in [24] that it is possible to rewrite the theory as an ordinary SU(2)×SU(2) gauge theory, without any reference to the three-algebra structure constants. This realization, however, seems to be the only finite-dimensional one, which means that the BLG theory can only describe stacks of two M2-branes. As a solution to this problem, Aharony, Bergman, Jafferies and Maldacena (ABJM) were led to formulate another world-volume theory for stacks of M2-branes. The ABJM theory is described in section 2.2.

2.1.1 Field Content and Lagrangian

The original BLG theory consists of two dynamical fields, the scalar $X^i{}_a$ and the spinor Ψ_a, and an auxiliary gauge field $\tilde{A}_\mu{}^a{}_b$. Here, μ, ν,... are flat indices on the $2+1$-dimensional world-volume, i, j,... are SO(8) R-symmetry vector indices and a, b,... are three-algebra indices. The SO(8) spinor indices and the world-volume spinor indices are not explicitly written out. To construct a Lagrangian one also needs a basic gauge field $A_{\mu ab}$, which is related to the auxiliary gauge field by

$$\tilde{A}_\mu{}^a{}_b = A_{\mu cd} f^{cda}{}_b. \tag{2.7}$$

The BLG Lagrangian is

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}(D_\mu X^{ia})(D^\mu X^i{}_a) + \frac{i}{2}\bar{\Psi}^a \gamma^\mu D_\mu \Psi_a - \frac{i}{4}\bar{\Psi}_b \Gamma^{ij} X^i{}_c X^j{}_d \Psi_a f^{abcd} \\ -V + \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(A_{\mu ab}\partial_\nu \tilde{A}_\lambda{}^{ab} + \frac{2}{3} A_\mu{}^a{}_b \tilde{A}_\nu{}^b{}_c \tilde{A}_\lambda{}^c{}_a\right)\end{aligned} \tag{2.8}$$

with the potential

$$V = \frac{1}{12} f^{abcd} f^{efg}{}_d X^i{}_a X^j{}_b X^k{}_c X^i{}_e X^j{}_f X^k{}_g. \tag{2.9}$$

The covariant derivative is defined by

$$D_\mu X^{ia} = \partial_\mu X^{ia} + \tilde{A}_\mu{}^a{}_b X^{ib}. \tag{2.10}$$

SUSY transformation rules

The Lagranian (2.8) is invariant under the following SUSY transformations:

$$\delta X^{ia} = i\bar{\epsilon}\Gamma^i \Psi^a \tag{2.11}$$

$$\delta \Psi^a = D_\mu X^{ia} \gamma^\mu \Gamma^i \epsilon + \frac{1}{6} X^i{}_b X^j{}_c X^k{}_d \Gamma^{ijk} \epsilon f^{bcda} \tag{2.12}$$

$$\delta \tilde{A}_\mu{}^a{}_b = i\bar{\epsilon} X^i{}_c \Gamma^i \gamma_\mu \Psi_d f^{cda}{}_b. \tag{2.13}$$

An explicit verification of this invariance is carried out in Appendix A.1.

2.2 ABJM

As mentioned in section 2.1, the underlying algebraic structure of the BLG theory is so restrictive that it has only one realization. This realization is related to the gauge group SO(4) = SU(2) × SU(2), which limits the BLG theory to describe stacks of two M2-branes. We also mentioned that the BLG theory can be rewritten as an ordinary gauge theory with gauge group SU(2) × SU(2), without any reference to the three-algebra. As a generalization to this, Aharony, Bergman, Jafferis and Maldacena (ABJM) constructed a theory based on gauge group U(N) × U(N), describing the world-volume dynamics of N M2-branes [2]. For this to work out, however, they had to reduce the number of manifest supersymmetries from $\mathcal{N} = 8$ to $\mathcal{N} = 6$. Later, Bagger and Lambert proved that it is possible to reformulate the ABJM theory in terms of a three-algebra with a less restrictive structure than their original one [6]. Yet another formulation of the theory was given in [18], relating it to Jordan triple systems.

The original ABJM theory contains two Chern-Simons terms with opposite levels k and $-k$. Thus, $1/k$ is the coupling parameter of the theory. In fact, for level k, the theory is conjectured to describe M2 branes in an $\mathbb{R}^8/\mathbb{Z}_k$ background. A stack of M2 branes on $\mathbb{R}^8$ or $\mathbb{R}^8/\mathbb{Z}_2$ has $\mathcal{N}=8$ supersymmetry which means that the number of supersymmetries is expected be enhanced in ABJM theory with $k=1,2$. For the special case of gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$, it was shown in [22] that one actually gets enhanced supersymmetry. In fact, the resulting theory is identical to the BLG theory. For the ABJM theory to correctly describe multiple M2-branes of any number, however, the supersymmetry must be enhanced for all permissible choices of gauge groups. That this really is the case was explicitly proven in [11]. The proof relies heavily on the existence of certain monopole operators in the theory. A brief explanation of the role these operators play is given in section 2.2.2.

2.2.1 Details of the ABJM Theory

Below, we give the details of the ABJM theory as formulated in [18]. The Chern-Simons level in this case is $k=1$, which means that the theory describes M2-branes in an $\mathbb{R}^8$ background.

The four-index structure constants are written as

$$f^{ab}{}_{cd} = f^{[ab]}{}_{cd} = f^{ab}{}_{[cd]}\,. \tag{2.14}$$

Fields with upper and lower gauge indices are treated as objects in different vector spaces and, correspondingly, there is no metric to raise and lower indices. Furthermore, the structure constants obey the fundamental identity

$$f^{a[b}{}_{dc}f^{e]d}{}_{gh} = f^{be}{}_{d[g}f^{ad}{}_{h]c} \tag{2.15}$$

and

$$(f^{ab}{}_{cd})^* = f^{cd}{}_{ab} \tag{2.16}$$

under complex conjugation.

The dynamical fields in the theory are the scalars $Z^A{}_a$, the spinors Ψ_{Aa} and their complex conjugates $\bar{Z}_A{}^a$ and Ψ^{Aa}. Here, upper(lower) capital indices are (anti-)fundamental SU(4) R-symmetry indices. The basic gauge field $A_\mu{}^a{}_b$ and the auxiliary gauge field $\tilde{A}_\mu{}^a{}_b$ are related by

$$\tilde{A}_\mu{}^a{}_b = f^{ac}{}_{bd}A_\mu{}^d{}_c\,. \tag{2.17}$$

The ABJM Lagrangian is

$$\begin{aligned}
\mathcal{L} = \quad & -(D_\mu Z^A{}_a)(D^\mu \bar{Z}_A{}^a) - i\bar{\Psi}^{Aa}\gamma^\mu D_\mu \Psi_{Aa} \\
& -if^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Aa}Z^B{}_b\bar{Z}_B{}^c + 2if^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Ba}Z^B{}_b\bar{Z}_A{}^c \\
& -\frac{i}{2}\epsilon_{ABCD}f^{ab}{}_{cd}\bar{\Psi}^{Ac}\Psi^{Bd}Z^C{}_aZ^D{}_b - \frac{i}{2}\epsilon^{ABCD}f^{cd}{}_{ab}\bar{\Psi}_{Ac}\Psi_{Bd}\bar{Z}_C{}^a\bar{Z}_D{}^b \\
& -V + \frac{1}{2}\epsilon^{\mu\nu\lambda}\left(f^{ab}{}_{cd}A_\mu{}^d{}_b\partial_\nu A_\lambda{}^c{}_a + \frac{2}{3}f^{bd}{}_{gc}f^{gf}{}_{ae}A_\mu{}^a{}_bA_\nu{}^c{}_dA_\lambda{}^e{}_f\right)
\end{aligned} \tag{2.18}$$

where the potential is given by

$$V = \frac{2}{3}\Upsilon^{CD}{}_{Bd}\bar{\Upsilon}_{CD}{}^{Bd}, \tag{2.19}$$

$$\Upsilon^{CD}{}_{Bd} = f^{ab}{}_{cd}Z^{C}{}_{a}Z^{D}{}_{b}\bar{Z}_{B}{}^{c} + f^{ab}{}_{cd}\delta^{[C}{}_{B}Z^{D]}{}_{a}Z^{E}{}_{b}\bar{Z}_{E}{}^{c}. \tag{2.20}$$

The covariant derivative is

$$D_\mu Z^A{}_a = \partial_\mu Z^A{}_a - Z^A{}_b\tilde{A}_\mu{}^b{}_a, \tag{2.21}$$

$$D_\mu \bar{Z}_A{}^a = \partial_\mu \bar{Z}_A{}^a + \tilde{A}_\mu{}^a{}_b\bar{Z}_A{}^b, \tag{2.22}$$

$$D_\mu \Psi_{Bd} = \partial_\mu \Psi_{Bd} - \Psi_{Bd}\tilde{A}_\mu{}^a{}_d. \tag{2.23}$$

SUSY transformation rules

The SUSY transformation rules for the scalar and spinor fields are:

$$\delta Z^A{}_a = i\bar{\epsilon}^{AB}\Psi_{Ba} \tag{2.24}$$

$$\delta\Psi_{Bd} = \gamma^\mu D_\mu Z^A{}_d\epsilon_{AB} + f^{ab}{}_{cd}Z^C{}_aZ^D{}_b\bar{Z}_B{}^c\epsilon_{CD} - f^{ab}{}_{cd}Z^A{}_aZ^C{}_b\bar{Z}_C{}^c\epsilon_{AB}, \tag{2.25}$$

where the transformation parameters ϵ^{AB} and ϵ_{AB} obey

$$\epsilon^{AB} = \frac{1}{2}\epsilon^{ABCD}\epsilon_{CD}, \tag{2.26}$$

$$\epsilon_{AB} = \frac{1}{2}\epsilon_{ABCD}\epsilon^{CD}, \tag{2.27}$$

and

$$\epsilon^{AB} = (\epsilon_{AB})^*. \tag{2.28}$$

In Appendix A.2.3 we explicitly show that the ABJM Lagrangian is invariant under (2.24)-(2.25).

2.2.2 SUSY Enhancement and Monopole Operators

As said above, the supersymmetry in the ABJM theory must be enhanced to $\mathcal{N} = 8$ for $k = 1, 2$. This was proven in [11], where the authors could write the action in an $\mathrm{SO}(8)_R$ invariant form, identify an extra $\mathcal{N} = 2$ supersymmetry and show that it closes with the original $\mathcal{N} = 6$ algebra. This was done by considering, in addition to the original ABJM fields, fields with a different index structure. As an example, recall from section 2.2.1 that the ABJM scalar field and its complex conjugate has index structure $Z^A{}_a$ and $\bar{Z}_A{}^a$ respectively. For the SUSY enhancement to work out, however, fields with index structure Z^{Aa} and Z_{Aa} must also exist in the theory. In other words, we need a way to raise or lower gauge indices without changing the R-symmetry properties. This is the role of the monopole operators. In [11] the authors use monopole operators W^{ab} with two gauge indices. By attaching these operators to the original ABJM fields, one obtains fields with the desired index structure. Of course, the monopole operators must transform trivially under R-symmetry. In addition, their scaling dimension must be zero, since otherwise scalar or spinor fields with different index structures would have different dimensions. The question of whether there really, inherently in the ABJM theory, exist monopole operators with these properties will be the topic for the rest of this thesis.

As a side note, monopole operators are also important in the verification of the conjectured AdS/CFT duality between ABJM theory and M-theory on $\mathrm{AdS}_4 \times \mathbb{S}^7/\mathbb{Z}_k$.

3

$\mathcal{N} = 3$ Chern-Simons Yang-Mills Theory

The goal of this chapter and the next is to prove the existence of monopole operators with certain properties in the ABJM theory. These operators should transform non-trivially under gauge transformations, be R-symmetry singlets and have vanishing scaling dimension. By the superconformal algebra, the scaling dimension of the operators is actually related to the R-symmetry charges. If the R-symmetry charges of the operators can be shown to vanish, this is also true for the scaling dimensions. Therefore, the most important issue in this thesis is to calculate the R-symmetry charges of the monopole operators.

The charges can, in a weakly coupled theory, be calculated using perturbation theory. The coupling constant of the ABJM theory is $1/k$, so for large Chern-Simons level perturbation theory can be used. The problem is, however, that we are interested in the case of small k-values ($k = 1, 2$ to be more specific), in which case the theory is strongly coupled. The solution to this problem is to add a Yang-Mills term to the action, introducing another coupling constant g. Adding the Yang-Mills term one must also add additional dynamical fields to the theory in order to preserve some amount of supersymmetry, and the result of all this is an $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory [7]. In the IR, where g is divergent, this theory reduces to the $\mathcal{N} = 6$ ABJM theory. In the UV, on the other hand, the theory is weakly coupled and it is possible to use perturbation theory. Thus, values of quantities that are unaffected by the renormalization flow can be computed in the UV and still be valid in the ABJM (IR) limit. Since a non-abelian representation cannot change continuously, the non-abelian R-symmetry charge is such a quantity.

3.1 Field Content and Lagrangian

The $\mathcal{N}=3$ Chern-Simons Yang-Mills theory contains two gauge fields A_μ and $\hat{A}_\mu$, corresponding to the two gauge groups. These fields form two gauge multiplets together with the scalar fields ϕ and $\hat{\phi}$ and the spinor fields χ_σ, χ_ϕ, $\hat{\chi}_\sigma$ and $\hat{\chi}_\phi$, all transforming in the adjoint representation of the two gauge groups. In addition, we have two types of chiral multiplets. The first one contains the scalars Z^A and the spinors ζ^A in the fundamental and anti-fundamental of the two gauge groups respectively. The other type of chiral multiplet is formed by the scalar W_A and the spinor ω_A, transforming in the anti-fundamental representation of the first gauge group and the fundamental representation of the second one. Here, $A, B, \ldots$ are $\mathrm{SU(N}_f)$ flavour indices.[1] The gauge indices are not explicitly written out.

To make the R-symmetry manifest, we arrange the above scalar and spinor fields into $\mathrm{SU}(2)_R$ multiplets. First, we have the doublets

$$X^{Aa} = \begin{pmatrix} Z^A \\ W^{\dagger A} \end{pmatrix}, \tag{3.1}$$

$$X^\dagger_{Aa} = \begin{pmatrix} Z^\dagger_A \\ W_A \end{pmatrix} \tag{3.2}$$

and[2]

$$\xi^{Aa} = \begin{pmatrix} \bar{\omega}^A \mathrm{e}^{i\pi/4} \\ \zeta^A \mathrm{e}^{-i\pi/4} \end{pmatrix}, \tag{3.3}$$

$$\xi^\dagger_{Aa} = \begin{pmatrix} \omega_A \mathrm{e}^{-i\pi/4} \\ \bar{\zeta}_A \mathrm{e}^{i\pi/4} \end{pmatrix}. \tag{3.4}$$

The adjoint scalars can be written as $\phi^a_b = \phi_i(\sigma_i)^{a}_{b}$[3] (and similar for $\hat{\phi}$), where i is an $\mathrm{SU}(2)_R$ vector index. Finally, the adjoint fermions are grouped as

$$\lambda^{ab} = \begin{pmatrix} \chi_\sigma \mathrm{e}^{-i\pi/4} & \bar{\chi}_\phi \mathrm{e}^{-i\pi/4} \\ \chi_\phi \mathrm{e}^{i\pi/4} & -\bar{\chi}_\sigma \mathrm{e}^{i\pi/4} \end{pmatrix}, \tag{3.5}$$

$$\hat{\lambda}^{ab} = \begin{pmatrix} \hat{\chi}_\sigma \mathrm{e}^{-i\pi/4} & -\bar{\hat{\chi}}_\phi \mathrm{e}^{-i\pi/4} \\ -\hat{\chi}_\phi \mathrm{e}^{i\pi/4} & -\bar{\hat{\chi}}_\sigma \mathrm{e}^{i\pi/4} \end{pmatrix}. \tag{3.6}$$

The complex conjugates of the adjoint fields are

$$(\phi^a_b)^* \;=\; \phi^b_a = \epsilon_{ac}\epsilon^{bd}\phi^c_d, \tag{3.7}$$

$$(\lambda^{ab})^* \;=\; -\lambda_{ab} = -\epsilon_{ac}\epsilon_{bd}\lambda^{cd}. \tag{3.8}$$

[1] We will keep the number of flavours N_f in the computations arbitrary. The ABJM theory has $N_f = 2$.

[2] The bars refer to the ordinary spacetime Dirac conjugate

[3] Here, $(\sigma_i)^a_b$ denotes either a Pauli matrix or a transposed Pauli matrix, depending on the context. The $\mathrm{SU}(2)_R$ indices of the fields are always placed as in (3.1)-(3.4), Pauli matrices are written as $(\sigma_i)_a{}^b$ and their transposes as $(\sigma_i)^a{}_b$. These conventions will, in every given context, make clear which set of matrices we are dealing with.

We can now write down the action for the $\mathcal{N}=3$ Chern-Simons Yang-Mills theory. The kinetic part is given by[4]

$$
\begin{aligned}
\mathcal{S}_{\text{kin}} \;=\; & \int d^3x \,\mathrm{tr}\Big[-\frac{1}{2g^2}F^{\mu\nu}F_{\mu\nu}+\kappa\epsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda+\frac{2i}{3}A_\mu A_\nu A_\lambda\right)-\\
& -\frac{1}{2g^2}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}-\kappa\epsilon^{\mu\nu\lambda}\left(\hat{A}_\mu\partial_\nu\hat{A}_\lambda+\frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right)\\
& -\mathcal{D}_\mu X^\dagger\mathcal{D}^\mu X+i\xi^\dagger\not{\mathcal{D}}\xi\\
& -\frac{1}{2g^2}\mathcal{D}_\mu\phi^a_b\mathcal{D}^\mu\phi^b_a-\frac{1}{2}\kappa^2g^2\phi^a_b\phi^b_a-\frac{1}{2g^2}\mathcal{D}_\mu\hat{\phi}^a_b\mathcal{D}^\mu\hat{\phi}^b_a-\frac{1}{2}\kappa^2g^2\hat{\phi}^a_b\hat{\phi}^b_a\\
& -\frac{i}{2g^2}\lambda^{ab}\not{\mathcal{D}}\lambda_{ab}-\frac{\kappa}{2}i\lambda^{ab}\lambda_{ba}-\frac{i}{2g^2}\hat{\lambda}^{ab}\not{\mathcal{D}}\hat{\lambda}_{ab}-\frac{\kappa}{2}i\hat{\lambda}^{ab}\hat{\lambda}_{ba}\Big]
\end{aligned}
\tag{3.9}
$$

and the interaction part by

$$
\begin{aligned}
\mathcal{S}_{\text{int}} \;=\; & \int d^3x\,\mathrm{tr}\Big[-\kappa g^2X^\dagger_a\phi^a_bX^b+\kappa g^2X^a\hat{\phi}^b_aX^\dagger_b-i\xi^\dagger_a\phi^a_b\xi^b-i\xi^a\hat{\phi}^b_a\xi^\dagger_b\\
& +\epsilon_{ac}\lambda^{cb}X^a\xi^\dagger_b-\epsilon^{ac}\lambda_{cb}\xi^bX^\dagger_a-\epsilon_{ac}\hat{\lambda}^{cb}\xi^\dagger_bX^a+\epsilon^{ac}X^\dagger_a\hat{\lambda}_{cb}\xi^b\\
& +\frac{\kappa}{6}\phi^a_b[\phi^b_c,\phi^c_a]+\frac{\kappa}{6}\hat{\phi}^a_b[\hat{\phi}^b_c,\hat{\phi}^c_a]-\frac{1}{2g^2}i\lambda_{ab}[\phi^b_c,\lambda^{ac}]+\frac{1}{2g^2}i\hat{\lambda}_{ab}[\hat{\phi}^b_c,\hat{\lambda}^{ac}]\\
& -\frac{g^2}{4}(X\sigma_iX^\dagger)^2-\frac{g^2}{4}(X^\dagger\sigma_iX)^2-\frac{1}{2}(XX^\dagger)\phi^a_b\phi^b_a-\frac{1}{2}(XX^\dagger)\hat{\phi}^a_b\hat{\phi}^b_a\\
& -X^\dagger_{Aa}\phi^b_cX^{Aa}\hat{\phi}^c_b+\frac{1}{8g^2}[\phi^a_b,\phi^c_d][\phi^b_a,\phi^d_c]+\frac{1}{8g^2}[\hat{\phi}^a_b,\hat{\phi}^c_d][\hat{\phi}^b_a,\hat{\phi}^d_c]\Big].
\end{aligned}
\tag{3.10}
$$

The covariant derivative is

$$
\mathcal{D}_\mu X=\partial_\mu X+iA_\mu X-iX\hat{A}_\mu. \tag{3.11}
$$

It can be verified that the action given by (3.9) and (3.10) is invariant under the following $\mathcal{N}=3$ SUSY transformations, with variation parameter ε_{ab}:

$$
\begin{aligned}
\delta A_\mu &= -\frac{i}{2}\varepsilon_{ab}\gamma_\mu\lambda^{ab}\\
\delta\lambda^{ab} &= \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}\gamma_\lambda\varepsilon^{ab}-i\not{\mathcal{D}}\phi^b_c\varepsilon^{ac}+\frac{i}{2}[\phi^b_c,\phi^c_d]\varepsilon^{ad}\\
&\quad+\kappa g^2i\phi^b_c\varepsilon^{ac}+g^2iX^aX^\dagger_c\varepsilon^{cb}-\frac{ig^2}{2}(XX^\dagger)\varepsilon^{ab}\\
\delta\phi^a_b &= -\varepsilon_{cb}\lambda^{ca}+\frac{1}{2}\delta^a_b\varepsilon_{cd}\lambda^{cd}\\
\delta\hat{A}_\mu &= -\frac{i}{2}\varepsilon_{ab}\gamma_\mu\hat{\lambda}^{ab}\\
\delta\hat{\lambda}^{ab} &= \frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}\gamma_\lambda\varepsilon^{ab}+i\not{\mathcal{D}}\hat{\phi}^b_c\varepsilon^{ac}+\frac{i}{2}[\hat{\phi}^b_c,\hat{\phi}^c_d]\varepsilon^{ad}\\
&\quad+\kappa g^2i\hat{\phi}^b_c\varepsilon^{ac}-g^2i\varepsilon^{bc}X^\dagger_cX^a+\frac{ig^2}{2}(X^\dagger X)\varepsilon^{ab}
\end{aligned}
$$

[4] $\kappa=k/4\pi$, where k is the Chern-Simons level

$$\begin{aligned}
\delta\hat{\phi}^a_b &= -\varepsilon_{cb}\hat{\lambda}^{ca} + \frac{1}{2}\delta^a_b\varepsilon_{cd}\hat{\lambda}^{cd} \\
\delta X^{Aa} &= -i\varepsilon^a_b\xi^{Ab} \\
\delta X^\dagger_{Aa} &= -i\xi^\dagger_{Ab}\varepsilon^b_a \\
\delta\xi^{Aa} &= \not{D}X^{Ab}\varepsilon^a_b + \phi^a_b\varepsilon^b_c X^{Ac} + X^{Ac}\varepsilon^b_c\hat{\phi}^a_b \\
\delta\xi^\dagger_{Aa} &= \not{D}X^\dagger_{Ab}\varepsilon^b_a + \hat{\phi}^b_a\varepsilon^c_b X^\dagger_{Ac}\varepsilon^c_b\phi^b_a.
\end{aligned} \tag{3.12}$$

In the IR, we see that the kinetic terms for all the adjoint fields vanish from the action. Thus, these fields become auxiliary and can be integrated out. The Yang-Mills terms vanish in the IR as well, and it can be shown that the theory reduces to the ordinary ABJM theory in the $N_f = 2$ case, if one regroups the fields in an appropriate way [7].

3.1.1 Radial Quantization

In all three-dimensional conformal field theories, there is a one-to-one correspondence between local operators in the theory formulated on $\mathbb{R}^3$, and states in the theory formulated on $\mathbb{R}\times S^2$. This is called the operator-state correspondence and the formulation of the theory on $\mathbb{R}\times S^2$ is called the radial quantization picture [8, 9]. In this thesis, we are interested in the properties of certain operators in a conformal field theory on $\mathbb{R}^{1,2}$. If we transform the theory to $\mathbb{R}^3$, we can then use the operator-state correspondence and study states in the radial quantization picture to learn about the operators. The remaining part of this section will be devoted to transforming our action given by (3.9) and (3.10) from $\mathbb{R}^{1,2}$ to $\mathbb{R}\times S^2$.

The first step is to perform a Wick rotation by defining $x^3 = ix^0$, making the signature Euclidian. Next, we change to polar coordinates (r,θ,ϕ) and introduce a new dimensionless radial variable τ, by defining $r = \mathrm{e}^\tau$. These operations result in a theory on $\mathbb{R}\times S^2$ where the radial direction is considered to be the (Euclidian) time direction. Some details of the manifold $\mathbb{R}\times S^2$, as well as our Dirac matrix conventions, are given in Appendix B. In addition to the coordinate transformations we also perform Weyl rescalings of all the fields in the theory. These rescalings are defined by

$$B = \mathrm{e}^{-\dim(B)\tau}\tilde{B}, \tag{3.13}$$

where B is an old field on $\mathbb{R}^3$ with mass dimension $\dim(B)$ and $\tilde{B}$ is the new one on $\mathbb{R}\times S^2$. The coupling constant g turns into

$$\tilde{g} = \mathrm{e}^{\tau/2}g. \tag{3.14}$$

The above transformations will render all the coordinates and fields of the theory dimensionless. Moreover, the theory we started with is not conformally invariant, which means that the action will change under the Weyl rescalings. We are ultimately, however, only interested the IR limit of the theory, where it reduces to ABJM. In this limit, the theory is conformally invariant and the rescalings will not affect our results. Before the Weyl rescalings we will also redefine the adjoint fermions as $\lambda \to g\lambda$ (and similar for the hatted field) to put them in a form more suitable for perturbation theory. The resulting action after performing all these manipulations is given by[5]

$$\begin{aligned}
\mathcal{S}_{\text{kin}} &= \int d\tau d\Omega \,\mathrm{tr}\left[\frac{1}{2\tilde{g}^2}F^{mn}F_{mn} - i\kappa\epsilon^{mnk}\left(A_m\partial_n A_k + \frac{2i}{3}A_m A_n A_k\right) - \right.\\
&\quad +\frac{1}{2\tilde{g}^2}\hat{F}^{mn}\hat{F}_{mn} + i\kappa\epsilon^{mnk}\left(\hat{A}_m\partial_n\hat{A}_k + \frac{2i}{3}\hat{A}_m\hat{A}_n\hat{A}_k\right)\\
&\quad +\mathcal{D}_m X^\dagger\mathcal{D}^m X + \frac{1}{4}X^\dagger X - i\xi^\dagger \not{\mathcal{D}}\xi\\
&\quad +\frac{1}{2\tilde{g}^2}\mathcal{D}_m\phi^a_b\mathcal{D}^m\phi^b_a + \frac{1}{2}\kappa^2\tilde{g}^2\phi^a_b\phi^b_a + \frac{1}{2\tilde{g}^2}\mathcal{D}_m\hat{\phi}^a_b\mathcal{D}^m\hat{\phi}^b_a + \frac{1}{2}\kappa^2\tilde{g}^2\hat{\phi}^a_b\hat{\phi}^b_a\\
&\quad \left. +\frac{i}{2}\lambda^{ab}\not{\mathcal{D}}\lambda_{ab} + \frac{1}{2}\kappa^2\tilde{g}^2 i\lambda^{ab}\lambda_{ba} + \frac{i}{2}\hat{\lambda}^{ab}\not{\mathcal{D}}\hat{\lambda}_{ab} + \frac{1}{2}\kappa^2\tilde{g}^2 i\hat{\lambda}^{ab}\hat{\lambda}_{ba}\right]
\end{aligned} \tag{3.15}$$

for the kinetic part and

$$\begin{aligned}
\mathcal{S}_{\text{int}} &= \int d\tau d\Omega \,\mathrm{tr}\left[\kappa\tilde{g}^2 X^\dagger_a\phi^a_b X^b - \kappa\tilde{g}^2 X^a\hat{\phi}^b_a X^\dagger_b + i\xi^\dagger_a\phi^a_b\xi^b + i\xi^a\hat{\phi}^b_a\xi^\dagger_b\right.\\
&\quad -\tilde{g}\epsilon_{ac}\lambda^{cb}X^a\xi^\dagger_b + \tilde{g}\epsilon^{ac}\lambda_{cb}\xi^b X^\dagger_a + \tilde{g}\epsilon_{ac}\hat{\lambda}^{cb}\xi^\dagger_b X^a - \tilde{g}\epsilon^{ac}X^\dagger_a\hat{\lambda}_{cb}\xi^b\\
&\quad -\frac{\kappa}{6}\phi^a_b[\phi^b_c,\phi^c_a] - \frac{\kappa}{6}\hat{\phi}^a_b[\hat{\phi}^b_c,\hat{\phi}^c_a] + \frac{i}{2}\lambda_{ab}[\phi^b_c,\lambda^{ac}] - \frac{i}{2}\hat{\lambda}_{ab}[\hat{\phi}^b_c,\hat{\lambda}^{ac}]\\
&\quad +\frac{\tilde{g}^2}{4}(X\sigma_i X^\dagger)^2 + \frac{\tilde{g}^2}{4}(X^\dagger\sigma_i X)^2 + \frac{1}{2}(XX^\dagger)\phi^a_b\phi^b_a + \frac{1}{2}(XX^\dagger)\hat{\phi}^a_b\hat{\phi}^b_a\\
&\quad \left. +X^\dagger_{Aa}\phi^b_c X^{Aa}\hat{\phi}^c_b - \frac{1}{8\tilde{g}^2}[\phi^a_b,\phi^c_d][\phi^b_a,\phi^d_c] - \frac{1}{8\tilde{g}^2}[\hat{\phi}^a_b,\hat{\phi}^c_d][\hat{\phi}^b_a,\hat{\phi}^d_c]\right]
\end{aligned} \tag{3.16}$$

for the interactions. The derivative is of course, in addition to gauge covariant, also geometrically covariant:

$$\mathcal{D}_m X = \nabla_m X + iA_m X - iX\hat{A}_m, \tag{3.17}$$

where ∇_m for the manifold $\mathbb{R}\times S^2$ is calculated in Appendix B. Finally, the SUSY transformations (3.12) turn into:

$$\begin{aligned}
\delta A_m &= -\frac{i\tilde{g}}{2}\varepsilon_{ab}\gamma_m\lambda^{ab}\\
\delta\lambda^{ab} &= \frac{i}{2\tilde{g}}\epsilon^{mnk}F_{mn}\gamma_k\varepsilon^{ab} - \frac{i}{\tilde{g}}\not{\mathcal{D}}\phi^b_c\varepsilon^{ac} - \frac{2i}{3\tilde{g}}\phi^b_c\not{\nabla}\varepsilon^{ac} + \frac{i}{2\tilde{g}}[\phi^b_c,\phi^c_d]\varepsilon^a d\\
&\quad +\kappa\tilde{g}i\phi^b_c\varepsilon^{ac} + \tilde{g}iX^a X^\dagger_c\varepsilon^{cb} - \frac{i\tilde{g}}{2}(XX^\dagger)\varepsilon^{ab}\\
\delta\phi^a_b &= -\tilde{g}\varepsilon_{cb}\lambda^{ca} + \frac{\tilde{g}}{2}\delta^a_b\varepsilon_{cd}\lambda^{cd}\\
\delta\hat{A}_m &= -\frac{i\tilde{g}}{2}\varepsilon_{ab}\gamma_m\hat{\lambda}^{ab}
\end{aligned}$$

[5]We have dropped all tildes from the fields

$$\begin{aligned}
\delta\hat{\lambda}^{ab} &= \frac{i}{2\tilde{g}}\epsilon^{mnk}F_{mn}\gamma_k\varepsilon^{ab} + \frac{i}{\tilde{g}}\not{D}\hat{\phi}^b_c\varepsilon^{ac} + \frac{2i}{3\tilde{g}}\hat{\phi}^b_c\not{\nabla}\varepsilon^{ac} + \frac{i}{2\tilde{g}}[\hat{\phi}^b_c,\hat{\phi}^c_d]\varepsilon^a d \\
&\quad +\kappa\tilde{g}i\hat{\phi}^b_c\varepsilon^{ac} - \tilde{g}i\varepsilon^{bc}X^\dagger_c X^a + \frac{i\tilde{g}}{2}(X^\dagger X)\varepsilon^{ab} \\
\delta\hat{\phi}^a_b &= -\tilde{g}\varepsilon_{cb}\hat{\lambda}^{ca} + \frac{\tilde{g}}{2}\delta^a_b\varepsilon_{cd}\hat{\lambda}^{cd} \\
\delta X^{Aa} &= -i\varepsilon^a_b\xi^{Ab} \\
\delta X^\dagger_{Aa} &= -i\xi^\dagger_{Ab}\varepsilon^b_a \\
\delta\xi^{Aa} &= \not{D}X^{Ab}\varepsilon^a_b + \frac{1}{3}X^{Ab}\not{\nabla}\varepsilon^a_b + \phi^a_b\varepsilon^b_c X^{Ac} + X^{Ac}\varepsilon^b_c\hat{\phi}^a_b \\
\delta\xi^\dagger_{Aa} &= \not{D}X^\dagger_{Ab}\varepsilon^b_a + \frac{1}{3}X^\dagger_{Ab}\not{\nabla}\varepsilon^b_a + \hat{\phi}^b_a\varepsilon^c_b X^\dagger_{Ac} + X^\dagger_{Ac}\varepsilon^c_b\phi^b_a.
\end{aligned} \tag{3.18}$$

3.2 Monopole Operators

In this section, we define monopole operators in U(N) gauge theories and describe briefly how to find an appropriate classical monopole solution (i e field configuration) to our $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory. This classical solution will be the basis for our quantum mechanical computations in the next chapter.

In a U(1) gauge theory on $\mathbb{R}^3$, as is well known, a magnetic monopole is a gauge field configuration with a field strength singularity in a point in space, leading to a magnetic flux through a sphere surrounding that point. A monopole operator is defined as an operator that, if inserted at a specific point, creates such a singularity and flux. In addition, the monopole operator specifies the behaviour of the matter fields close to the insertion point. In the U(1) case, the gauge potential leading to a field strength singularity in the origin is given by

$$A_\phi = q(\pm 1 - \cos\theta), \tag{3.19}$$

where q is the magnetic charge and where the upper (lower) sign is for the northern (southern) hemisphere. The possible values of the magnetic charge are constrained by the Dirac quantization condition.

By the operator-state correspondence, a monopole operator with magnetic charge q is mapped to a state on $S^2\times\mathbb{R}$ with flux q through the sphere [9]. Also, the presence of a Chern-Simons term in the theory affects the effective charge of the monopole. By integrating the Chern-Simons term over the sphere, one obtains a state with charge kq, where k is the Chern-Simons level. Thus, the monopole operator is effectively described by a state on $S^2 \times \mathbb{R}$ with flux kq [14]. For us, this means that the properties we find for the monopole field configurations (which are our vacuum states) in the radially quantized theory can be considered properties of the monopole operators themselves.

As discussed in [13] and [14], the above definition of monopole operators can be generalized to U(N) gauge theories. In this case, the operator creates a singularity in a U(1) subgroup of U(N). To be more specific, one chooses a homomorphism U(1)→U(N) specifying the embedding of U(1) into U(N). The non-abelian monopole is defined as the image of the ordinary U(1) singularity under this homomorphism. The homomorphism will in general take an abelian Lie algebra element (like the magnetic charge) to the diagonal matrix $H = \text{diag}(q_1, q_2, ..., q_N)$, where q_i are integers. This matrix labels the non-abelian monopole operator and in fact, the integers q_i can be shown to define a highest weight representation of U(N). Thus, the choice of q_i specifying the embedding of U(1) into U(N) determines how the non-abelian monopole operator transforms under gauge transformations. By choosing different sets of integers one obtains monopole operators in different gauge representations. The non-abelian generalization of (3.19) is simply given by

$$A_\phi = H(\pm 1 - \cos\theta). \tag{3.20}$$

Also, analogously to the abelian case, a Chern-Simons term affects the effective charges and thus the possible representations. For Chern-Simons level k, the monopole operator transforms in the highest weight representation given by $kH = \text{diag}(kq_1, kq_2, ..., kq_N)$.

To summarize, a monopole operator inserted at a specific point in space creates a gauge field strength singularity and specifies the field configuration for the matter fields close to the insertion point. Properties of these fields configurations in the radial quantization picture, such as gauge- or R-symmetry representations, correspond to properties of the operators themselves.

In [14], monopole operators in U(N)×U(N) gauge theories were described, which of course is the case of interest in this thesis. The behaviour of the two gauge potentials A and $\hat{A}$ is specified by the two diagonal matrices H and $\hat{H}$, whose entries satisfy the constraint $\Sigma_i q_i = \Sigma_i \hat{q}_i$.

A special class of monopoles are the BPS[6] monopoles. A BPS monopole is defined as a monopole field configuration that saturates the so-called Bogomolny bound; a lower bound on the mass [23]. A BPS monopole also preserves the supersymmetry of the theory. Since our ultimate goal is to calculate R-symmetry charges of monopole operators, our monopole field configuration must be supersymmetry preserving. Thus, we will focus on BPS monopoles.

3.2.1 BPS Solution

A classical BPS monopole solution to our $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory was found in [7]. For some field configuration to be a valid BPS monopole, two conditions must be met. First, it has to satisfy the equations of motion for the theory; and second, the SUSY transformations (3.18) must still leave the action invariant. The starting point is the gauge field configuration (3.20) defined on $\mathbb{R}^3$, which will not change when going to $\mathbb{R} \times S^2$ since it is already dimensionless. By requiring the SUSY variations for the fermions to vanish one obtains the conditions $A_m = \hat{A}_m$ and $\hat{\phi}_i = -\phi_i = \eta H n_i(\tau)$, where $n^2 = 1$,

[6]Bogomolny-Prasad-Sommerfeld

and $\eta = \pm 1$ corresponds to BPS and anti-BPS monopoles respectively.[7] One also obtains expectation values for the bifundamental scalar fields and it can be shown that all these field configurations satisfy the equations of motion.

Let us now suppose that we want to do perturbation theory around these classical expectation values. To get the action in a form suitable for that, the gauge fields and the adjoint scalar fields must be rescaled by a factor of g. This means that the quantum fluctuations are of order g. The expectation values, on the other hand, are of order unity. Since all the computations will be carried out in the UV, where g is small, our expectation values for the gauge fields and adjoint scalars can be treated as a classical background. For the bifundamental scalar, no rescaling is needed to make the action suitable for perturbation theory. Thus, the classical solution is of the same order of magnitude as the quantum fluctuations and cannot be treated as a background in the UV.

To summarize, a classical BPS monopole solution to the $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory is given by the field configuration

$$A_\phi = \hat{A}_\phi = H(\pm 1 - \cos\theta) \tag{3.21}$$

$$\phi_i = -\hat{\phi}_i = -\eta H n_i(\tau), \tag{3.22}$$

where $\eta = 1$ corresponds to a BPS monopole and $\eta = -1$ to an anti-BPS monopole. In the next chapter, this field configuration will be treated as a classical background. We will examine the R-symmetry properties of the background since, by the operator-state correspondence, there must exist monopole operators with the same properties.

[7] An anti-BPS monopole is a BPS monopole with opposite magnetic charge.

4

R-symmetry Charges

In the previous chapter, we found that there is a classical BPS monopole solution to our $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory, where the expectation values of the gauge fields and adjoint scalars are given by (3.21) and (3.22). These expectation values are of a different order of magnitude than the quantum fluctuations and can in the following be treated as a classical background. The background preserves the supersymmetry and R-symmetry of the theory.

In this chapter, we will compute the R-symmetry charges of the background induced by fermionic fluctuations. In section 4.1 we consider a static background that breaks the R-symmetry from SU(2) to U(1). Of course, we can only calculate abelian R-symmetry charges in this way and these can change continuously under the renormalization flow. Therefore, we also need to consider the full $\mathrm{SU}(2)_R$-preserving background, allowing us to calculate non-abelian charges. This is done in section 4.2. Since we in the rest of the thesis will work with a theory on $\mathbb{R} \times S^2$, we have collected some geometrical considerations of this manifold in Appendix B.

4.1 $\mathrm{U(1)}_R$

In this section, we consider a special case of the BPS monopole background (3.21) and (3.22) given by

$$
\begin{aligned}
A_\phi &= \hat{A}_\phi = H(\pm 1 - \cos\theta) && (4.1)\\
\phi_i &= -\hat{\phi}_i = -\eta H \delta_{i3}, && (4.2)
\end{aligned}
$$

where we arbitrarily have selected the $\mathrm{SU}(2)_R$ direction to be $i = 3$. Our task is now to calculate the $\mathrm{U}(1)_R$ charge of this background. The idea is to consider all the non-background fields in the theory and compute the total charge operator. This operator will consist of a vacuum term and the field fluctuations, where the vacuum term corresponds to the charge of the monopole background and thus of the monopole operator that inserts the background. To begin with, we will make the calculations for an abelian toy model, containing only one fermion. The results of this calculation can then easily be generalized and applied to the $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory.

4.1.1 Toy Model

Let us start with an abelian gauge theory on $\mathbb{R} \times S^2$ containing one fermion $\psi(\tau, \Omega)$ with equation of motion

$$\not{D}\psi + \eta q \psi = 0. \tag{4.3}$$

The Dirac operator is given by

$$\not{D} = \not{\nabla} + i\not{A}, \tag{4.4}$$

where A is a monopole gauge field configuration of the form (4.1) with $H = q$. The mass term in the Dirac equation will later be shown to emerge from the coupling to a background scalar of the form (4.2) with $H = q$.

Let us now proceed to calculate the R-symmetry charges in this theory. Assume that the charge operator is given by

$$Q_1 = \int d\Omega\, \psi^\dagger \psi. \tag{4.5}$$

To avoid divergences that might occur at $\tau = 0$ we will use point-splitting. Thus, we set:

$$Q_1(\beta) = \int d\Omega\, \psi^\dagger \left(\tau + \frac{\beta}{2}, \Omega\right) \psi \left(\tau - \frac{\beta}{2}, \Omega\right), \tag{4.6}$$

where $\beta > 0$. After the calculations, we will obtain our result in the limit $\beta \to 0$.

Instead of (4.5) we could equally well have defined our R-symmetry charge operator as

$$Q_2 = -\int d\Omega\, \psi \psi^\dagger. \tag{4.7}$$

In most cases in quantum field theory, the charge operators consist of fluctuation terms and an infinite normal ordering constant. This normal ordering constant is simply dropped, and in this way one sets the charge of the vacuum to zero. In our case, however, the vacuum charge is precisely the quantity we are interested in since it corresponds to the charge of the monopole background and, thus, we expect to see a finite normal ordering constant. Q_1 and Q_2 will in general give different values for the normal ordering term and, therefore, we redefine the charge operator as their average.[1] The point-splitted version of our charge operator is now given by

$$Q(\beta) = \frac{1}{2} \int d\Omega \left[\psi^\dagger \left(\tau + \frac{\beta}{2}, \Omega\right) \psi \left(\tau - \frac{\beta}{2}, \Omega\right) - \psi \left(\tau + \frac{\beta}{2}, \Omega\right) \psi^\dagger \left(\tau - \frac{\beta}{2}, \Omega\right)\right]. \tag{4.8}$$

[1] See [8] for a discussion about this.

Our next step is to solve the Dirac equation (4.3) and insert the solution into the expression for the charge (4.8). To solve the equation, we first expand $\psi(\tau, \Omega)$ in monopole spinor harmonics. These are defined as eigenspinors of the Dirac operator on a sphere around a magnetic monopole, and in Appendix C we derive the explicit expressions for these spinors, as well as some of their properties. As seen from (C.84), the monopole spinor harmonics form a complete set of spinors on S^2, which means that we can use them to expand a general spinor. We write the expansion as follows

$$\psi(\tau, \Omega) = \sum_{m} \psi_m(\tau) \Upsilon^0_{qm}(\Omega) + \sum_{jm\varepsilon} \psi^\varepsilon_{jm}(\tau) \Upsilon^\varepsilon_{qjm}(\Omega), \tag{4.9}$$

where the τ-dependent ψ-functions are operator coefficients. Next, we insert this expression into the Dirac equation (4.3):

$$\begin{aligned} 0 &= (\not{D} + \eta q)\psi(\tau, \Omega) \\ &= (\not{D}_S + \gamma^\tau \partial_\tau + \eta q)\psi(\tau, \Omega) \\ &= \sum_m \gamma^\tau \dot{\psi}_m(\tau) \Upsilon^0_{qm} + \sum_{jm\varepsilon} \left(i\Delta^\varepsilon_{jq} \psi^\varepsilon_{jm}(\tau) \Upsilon^\varepsilon_{qjm} + \gamma^\tau \dot{\psi}^\varepsilon_{jm}(\tau) \Upsilon^\varepsilon_{qjm} \right) \\ &\quad + \eta q \sum_m \psi_m(\tau) \Upsilon^0_{qm} + \eta q \sum_{jm\varepsilon} \psi^\varepsilon_{jm}(\tau) \Upsilon^\varepsilon_{qjm}. \end{aligned} \tag{4.10}$$

Here, a dot denotes a derivative with respect to τ. Now, using (C.81), we have the following equation

$$\operatorname{sign}(q) \dot{\psi}_m(\tau) + \eta q \psi_m(\tau) = 0, \tag{4.11}$$

with solution

$$\psi_m(\tau) = c_m \mathrm{e}^{-\eta |q| \tau} \tag{4.12}$$

for some operator c_m. In the same way, we use (C.79) to obtain

$$i\Delta^\varepsilon_{jq} \psi^\varepsilon_{jm}(\tau) + \dot{\psi}^{-\varepsilon}_{jm} + \eta q \psi^\varepsilon_{jm}(\tau) = 0, \tag{4.13}$$

which we also can write as

$$\begin{pmatrix} \dot{\psi}^+_{jm}(\tau) \\ \dot{\psi}^-_{jm}(\tau) \end{pmatrix} = \begin{pmatrix} 0 & -(i\Delta^-_{jq} + \eta q) \\ -(i\Delta^+_{jq} + \eta q) & 0 \end{pmatrix} \begin{pmatrix} \psi^+_{jm}(\tau) \\ \psi^-_{jm}(\tau) \end{pmatrix}. \tag{4.14}$$

This system of coupled first order ODE:s can be rewritten as the following uncoupled second order ODE:s

$$\ddot{\psi}^\varepsilon_{jm} + E_j^2 \psi^\varepsilon_{jm} = 0, \tag{4.15}$$

where we have defined

$$\begin{aligned} E_j &= \sqrt{(i\Delta^+_{jq} + \eta q)(i\Delta^-_{jq} + \eta q)} \\ &= \sqrt{-\Delta^+_{jq}\Delta^-_{jq} + i\eta q(\Delta^+_{jq} + \Delta^-_{jq}) + q^2} \\ &= \sqrt{\frac{1}{4}((2j+1)^2 - 4q^2) + q^2} \\ &= j + \frac{1}{2}. \end{aligned} \tag{4.16}$$

We write the solution to (4.15) for $\varepsilon = +$ as

$$\psi^+_{jm}(\tau) = \frac{1}{\sqrt{2}} c_{jm} \mathrm{e}^{-E_j\tau} + \frac{1}{\sqrt{2}} d^\dagger_{jm} \mathrm{e}^{E_j\tau} \tag{4.17}$$

for some operators c_{jm} and $d^\dagger_{jm}$. Taking the τ-derivative of this expression and using (4.14) now yields

$$\begin{aligned}
\psi^-_{jm}(\tau) =& -\frac{1}{i\Delta^-_{jq} + \eta q}\dot{\psi}^+_{jm}(\tau) \\
=& -\frac{i\Delta^+_{jq} + \eta q}{E_j^2}\frac{E_j}{\sqrt{2}}\left(-c_{jm}\mathrm{e}^{-E_j\tau} + d^\dagger_{jm}\mathrm{e}^{E_j\tau}\right) \\
=& \frac{i\sqrt{(2j+1)^2 - 4q^2} + 2\eta q}{2j+1}\frac{1}{\sqrt{2}}\left(c_{jm}\mathrm{e}^{-E_j\tau} - d^\dagger_{jm}\mathrm{e}^{E_j\tau}\right) \\
=& \frac{1}{\sqrt{2}}\left(\frac{2\eta q}{2j+1} + i\sqrt{1 - \left(\frac{2q}{2j+1}\right)^2}\right)\left(c_{jm}\mathrm{e}^{-E_j\tau} - d^\dagger_{jm}\mathrm{e}^{E_j\tau}\right).
\end{aligned} \tag{4.18}$$

Before we insert (4.18) and (4.17) into the expansion we streamline the notation a bit by defining

$$u^+_j = v^+_j = \frac{1}{\sqrt{2}}, \tag{4.19}$$

$$u^-_j = -v^-_j = \frac{1}{\sqrt{2}}\left(\frac{2\eta q}{2j+1} + i\sqrt{1 - \left(\frac{2q}{2j+1}\right)^2}\right). \tag{4.20}$$

The expansion (4.9) can now be written

$$\psi(\tau, \Omega) = \sum_m c_m \mathrm{e}^{-\eta|q|\tau}\Upsilon^0_m + \sum_{jm\varepsilon}\left[c_{jm}u^\varepsilon_j \mathrm{e}^{-E_j\tau} + d^\dagger_{jm}v^\varepsilon_j\mathrm{e}^{E_j\tau}\right]\Upsilon^\varepsilon_{jm}. \tag{4.21}$$

Since τ is related to ordinary time by $\tau = it$, this is an ordinary oscillator expansion with positive and negative frequency modes. For the non-zero modes, we interpret E_j as the energy, c_{jm} and d_{jm} as annihilation operators and $c^\dagger_{jm}$ and $d^\dagger_{jm}$ as creation operators. For the zero modes we have to be a bit more careful since there is a difference between the BPS and the anti-BPS case. In the BPS case ($\eta = 1$), the zero mode is a positive frequency mode and c_m is an annihilation operator. In the anti-BPS case ($\eta = -1$), we must instead interpret it as a creation operator. The operators satisfy (for an appropriate normalization of ψ) the following anticommutation relations

$$\{c_m, c^\dagger_{m'}\} = \delta_{mm'} \tag{4.22}$$

$$\{c_{jm}, c^\dagger_{j'm'}\} = \delta_{jj'}\delta_{mm'} \tag{4.23}$$

$$\{d_{jm}, d^\dagger_{j'm'}\} = \delta_{jj'}\delta_{mm'}. \tag{4.24}$$

Furthermore, when we take the Hermitian conjugate of (4.21) we must also change the sign of τ. This follows because, if we Wick rotate back to ordinary time, there is a factor of i in the exponentials. Thus, we have

$$\psi^\dagger(\tau,\Omega) = \sum_m c_m^\dagger \mathrm{e}^{\eta|q|\tau} \Upsilon_m^{\dagger 0} + \sum_{jm\varepsilon} \left[c_{jm}^\dagger u_j^{*\varepsilon} \mathrm{e}^{E_j\tau} + d_{jm} v_j^{*\varepsilon} \mathrm{e}^{-E_j\tau} \right] \Upsilon_{jm}^{\dagger\varepsilon}. \tag{4.25}$$

We are now finally ready to calculate $Q(\beta)$ by inserting (4.21) and (4.25) into (4.8). We use the orthogonality of the monopole spinor harmonics (C.82) and (C.83) to perform the integral over Ω, simply trading it for deltas that turn the double sums into simple sums. This results in the following expression:

$$\begin{aligned} Q(\beta) \quad = \quad & \frac{1}{2}\sum_m c_m^\dagger c_m \mathrm{e}^{\eta|q|\beta} + \frac{1}{2}\sum_{jm\varepsilon} \Big[c_{jm}^\dagger c_{jm} |u_j^\varepsilon|^2 \mathrm{e}^{E_j\beta} + d_{jm} d_{jm}^\dagger |v_j^\varepsilon|^2 \mathrm{e}^{-E_j\beta} \\ & + c_{jm}^\dagger d_{jm}^\dagger u_j^{*\varepsilon} v_j^\varepsilon \mathrm{e}^{2E_j\tau} + d_{jm} c_{jm} u_j^\varepsilon v_j^{*\varepsilon} \mathrm{e}^{-2E_j\tau} \Big] \\ & - \frac{1}{2}\sum_m c_m c_m^\dagger \mathrm{e}^{-\eta|q|\beta} - \frac{1}{2}\sum_{jm\varepsilon} \Big[c_{jm} c_{jm}^\dagger |u_j^\varepsilon|^2 \mathrm{e}^{-E_j\beta} + d_{jm}^\dagger d_{jm} |v_j^\varepsilon|^2 \mathrm{e}^{E_j\beta} \\ & + c_{jm} d_{jm} u_j^\varepsilon v_j^{*\varepsilon} \mathrm{e}^{-2E_j\tau} + d_{jm}^\dagger c_{jm}^\dagger u_j^{*\varepsilon} v_j^\varepsilon \mathrm{e}^{2E_j\tau} \Big]. \end{aligned} \tag{4.26}$$

The first step in simplifying this expression is to evaluate the ε-sums. We easily see that $\sum_\varepsilon |u_j^\varepsilon|^2 = |v_j^\varepsilon|^2 = 1$ and $\sum_\varepsilon u_j^\varepsilon v_j^{*\varepsilon} = u_j^{*\varepsilon} v_j^\varepsilon = 0$. Then, we set $\beta = 0$ and arrive at

$$Q = \frac{1}{2}\sum_m (c_m^\dagger c_m - c_m c_m^\dagger) + \frac{1}{2}\sum_{jm} \left[c_{jm}^\dagger c_{jm} - c_{jm} c_{jm}^\dagger - d_{jm}^\dagger d_{jm} + d_{jm} d_{jm}^\dagger \right]. \tag{4.27}$$

We can now normal order all the terms and calculate the vacuum charge. For the non-zero modes we have the normal ordered part

$$Q_1 = \sum_{jm} \left[c_{jm}^\dagger c_{jm} - d_{jm}^\dagger d_{jm} \right]. \tag{4.28}$$

Because of the sign difference between the terms, the normal ordering constants cancel and we get no contribution to the vacuum charge from these terms. Turning to the zero mode terms, we have two different cases. In the BPS case, the normal ordered term is

$$Q_1 = \sum_m c_m^\dagger c_m \tag{4.29}$$

and the normal ordering constant

$$Q_0 = -\sum_m 1 = -\frac{1}{2}(2j+1) = -|q|. \tag{4.30}$$

In the anti-BPS case, the roles of c_m and $c_m^\dagger$ are switched and the sign of the charge is reversed. Thus, the final expression for the vacuum charge is

$$Q_0 = -\eta|q|. \tag{4.31}$$

Our result (4.31) is the monopole background charge induced by a fermion in our abelian toy model. Before we apply this result to our Chern-Simons Yang-Mills theory, we should of course also examine whether a scalar gives rise to a similar vacuum charge. However, it was shown in [8] that this is not the case due to their symmetric spectrum. As shown in Appendix C, the non-zero mode eigenspinors of the Dirac operator are paired; the two eigenspinors for each j and m have the same eigenvalue with opposite sign. This is the reason why the non-zero mode contributions to the normal ordering constant cancel. The zero mode state, however, is unpaired and no cancellation takes place. The corresponding spectrum for a scalar is symmetric and has no such unpaired states, which means that all its contributions to the vacuum charge cancel.

4.1.2 Applications to $\mathcal{N} = 3$ Chern-Simons Yang-Mills Theory

Let us now apply the results of the previous section to the Chern-Simons Yang-Mills theory we are interested in. With the background scalar field configuration (3.22), the theory is invariant under $\mathrm{SU}(2)_R$ transformations. We can write these transformations as

$$\delta B^a = i\varepsilon^a_b B^b \tag{4.32}$$
$$\delta B_a = -i\varepsilon^b_a B_b. \tag{4.33}$$

The fermion part of the associated Noether current J^m is given by

$$\varepsilon^a_b (J^m)^b_a = \varepsilon^a_b \mathrm{tr}\left[\xi^\dagger_{Aa}\gamma^m\xi^{Ab} - \frac{1}{2}\lambda_{ca}\gamma^m\lambda^{cb} - \frac{1}{2}\lambda_{ac}\gamma^m\lambda^{bc} - \frac{1}{2}\hat\lambda_{ca}\gamma^m\hat\lambda^{cb} - \frac{1}{2}\hat\lambda_{ac}\gamma^m\hat\lambda^{bc}\right]. \tag{4.34}$$

When we consider the static background (4.2), the symmetry is broken to $\mathrm{U}(1)_R$. Since we have chosen the $i = 3$ $\mathrm{SU}(2)_R$ direction, the $\mathrm{U}(1)_R$ current is given by setting $\varepsilon^a_b = (\sigma_3)^a_b$ in (4.34). Changing back to the original fields, the $\mathrm{U}(1)_R$ current can now be written

$$\begin{aligned}
J^m =&(\sigma_3)^a_b \mathrm{tr}\left[\xi^\dagger_{Aa}\gamma^m\xi^{Ab} - \frac{1}{2}\lambda_{ca}\gamma^m\lambda^{cb}\right.\\
&\left.-\frac{1}{2}\lambda_{ac}\gamma^m\lambda^{bc} - \frac{1}{2}\hat\lambda_{ca}\gamma^m\hat\lambda^{cb} - \frac{1}{2}\hat\lambda_{ac}\gamma^m\hat\lambda^{bc}\right]\\
=&\mathrm{tr}\left[\xi^\dagger_{A1}\gamma^m\xi^{A1} - \xi^\dagger_{A2}\gamma^m\xi^{A2} - \lambda_{11}\gamma^m\lambda^{11}\right.\\
&\left.+\lambda_{22}\gamma^m\lambda^{22} - \hat\lambda_{11}\gamma^m\hat\lambda^{11} + \hat\lambda_{22}\gamma^m\hat\lambda^{22}\right]\\
=&\mathrm{tr}\left[\omega_A\gamma^m\bar\omega^A - \bar\zeta_A\gamma^m\zeta^A + \bar\chi_\sigma\gamma^m\chi_\sigma - \chi_\sigma\gamma^m\bar\chi_\sigma + \bar{\hat\chi}_\sigma\gamma^m\hat\chi_\sigma - \hat\chi_\sigma\gamma^m\bar{\hat\chi}_\sigma\right]\\
=&\mathrm{tr}\left[-\bar\omega_A\gamma^m\omega^A - \bar\zeta_A\gamma^m\zeta^A + 2\bar\chi_\sigma\gamma^m\chi_\sigma + 2\bar{\hat\chi}_\sigma\gamma^m\hat\chi_\sigma\right].
\end{aligned} \tag{4.35}$$

In the last step, we have flipped three of the fermion bilinears. The conserved U(1)$_R$ charge is now given by

$$\begin{aligned} Q &= \int d\Omega\, J^\tau \\ &= \int d\Omega \,\mathrm{tr}\left[-\bar{\omega}_A \gamma^\tau \omega^A - \bar{\zeta}_A \gamma^\tau \zeta^A + 2\bar{\chi}_\sigma \gamma^\tau \chi_\sigma + 2\bar{\hat{\chi}}_\sigma \gamma^\tau \hat{\chi}_\sigma\right] \\ &= \int d\Omega \,\mathrm{tr}\left[-\omega_A^\dagger \omega^A - \zeta_A^\dagger \zeta^A + 2\chi_\sigma^\dagger \chi_\sigma + 2\hat{\chi}_\sigma^\dagger \hat{\chi}_\sigma\right]. \end{aligned} \tag{4.36}$$

Let us now turn to the equations of motion. For the background given by (4.1) and (4.2), and when $\tilde{g} \to 0$, the Euler-Lagrange equations for the fermions yield

$$\not{D}\xi^{Aa} - \eta[H(\sigma_3)^a_b, \xi^{Ab}] = 0 \tag{4.37}$$
$$\not{D}\xi^\dagger_{Aa} + \eta[H(\sigma_3)^b_a, \xi^\dagger_{Ab}] = 0 \tag{4.38}$$
$$\not{D}\lambda^{ab} + \eta[H(\sigma_3)^b_c, \lambda^{ac}] = 0 \tag{4.39}$$
$$\not{D}\hat{\lambda}^{ab} + \eta[H(\sigma_3)^b_c, \hat{\lambda}^{ac}] = 0, \tag{4.40}$$

implying

$$\not{D}\omega_A + \eta[H, \omega_A] = 0 \tag{4.41}$$
$$\not{D}\zeta^A + \eta[H, \zeta^A] = 0 \tag{4.42}$$
$$\not{D}\chi_\sigma + \eta[H, \chi_\sigma] = 0 \tag{4.43}$$
$$\not{D}\hat{\chi}_\sigma + \eta[H, \hat{\chi}_\sigma] = 0. \tag{4.44}$$

Our next step is to relate these equations to the Dirac equation in our toy model. Writing out the gauge indices explicitly, the gauge group generator $H = \mathrm{diag}(q_1, ..., q_N)$ can be written as $H_{rs} = q_r \delta_{rs}$. Since there is a well defined commutator between H and the fermion fields, these must carry two gauge indices as well. For a generic fermion ψ, we now have

$$[H, \psi]_{rs} = q_r \delta_{rt} \psi_{ts} - \psi_{rt} q_t \delta_{ts} = (q_r - q_s)\psi_{rs}. \tag{4.45}$$

In the monopole background given by (3.21), the Dirac operator corresponding to the covariant derivative (3.17) is

$$\begin{aligned} \not{D}\psi &= \not{\nabla}\psi + i\gamma^\phi H(\pm 1 - \cos\theta)\psi - i\gamma^\phi \psi H(\pm 1 - \cos\theta) \\ &= \not{\nabla}\psi + i\gamma^\phi(\pm 1 - \cos\theta)[H, \psi]. \end{aligned} \tag{4.46}$$

Using (4.45), we see that each fermion matrix element ψ_{rs} separately satisfies the equation of motion

$$\not{\nabla}\psi_{rs} + i\gamma^\phi(\pm 1 - \cos\theta)(q_r - q_s)\psi_{rs} + \eta(q_r - q_s)\psi_{rs} = 0, \tag{4.47}$$

which is of the same form as the Dirac equation (4.3) in our abelian toy model with $q = q_r - q_s$.

Let us now compare our charge (4.36) to the charge operator (4.5) that was the starting point in our abelian model. Inside the trace, there are $4-2N_f$ terms of the same form as the one in (4.5). Moreover, taking the trace amounts to summing over all fermion matrix elements ψ_{rs}. Since, in addition, each such matrix element satisfies the Dirac equation (4.47), we can directly apply our toy model result (4.31) to the $\mathcal{N}=3$ Chern-Simons Yang-Mills theory. Thus, our final result for the $\mathrm{U}(1)_R$ charge of the BPS monopole background is given by

$$Q = -\eta(4-2N_f)\sum_{r,s=1}^{N}|q_r - q_s|. \tag{4.48}$$

In the above computation, we did not include the (non-background) bosons of the theory. As briefly commented at the end of the previous section, their contribution to the vacuum charge vanish due to the symmetric spectrum.

We see that there are two ways to make (4.48) identically vanish. One is to set $q_r = q_s$ for all r and s, but this strongly limits the possible gauge representations of the monopole (remember that the q:s define a highest weight representation of the gauge group U(N)). The other way is to set the number of flavours, N_f, equal to two. The ABJM theory, where our ultimate interest lies, has precisely this number of flavours. It is important to remember, however, that the above computations were carried out in the UV. Flowing to the IR (ABJM) limit could in principle change the value of the $\mathrm{U}(1)_R$ charge. Thus, we cannot be completely sure that the ABJM theory really allows for a monopole background with vanishing R-symmetry charge and arbitrary gauge representations. To be able to draw this conclusion, we must calculate non-abelian R-symmetry charges. This is the topic of the next section.

4.2 SU(2)$_R$

In the previous section we considered the background field configuration specified by (4.1) and (4.2). Here, the non-zero vacuum expectation values had one degree of freedom; the $\mathrm{U}(1)_R$ phase. This phase is a simple example of a *collective coordinate*. In general, a collective coordinate is a parameter describing the zero modes of a system. For instance, let us suppose we want to find static quantum fluctuations around a classical (zero-mode) solution of some system. If the system possesses a symmetry in a given coordinate x, the zero-mode states will be "spread out" in the corresponding direction. Therefore, to really find quantum states that are localized around the classical solution, one has to separate out the x-dependence from the problem. This results in a new problem involving all the other coordinates, and by solving this problem the true localized quantum states can be found. The only role of x is to parameterize the space of zero-mode states. For a more thorough discussion about collective coordinates, see [20].

In this thesis, however, we are interested in finding properties of the classical background solution itself. For this purpose, we can use the fact that the collective coordinate fully parameterizes the background; the properties of the collective coordinate corresponds to the properties of the background itself. As noted above, the background considered in the previous section was described by a U(1)$_R$ collective coordinate. In this section, we instead consider the full SU(2)$_R$-conserving background given by (3.21) and (3.22). The collective coordinate parametrizing this background is the unit vector $n_i(\tau)$ in the SU(2)$_R$ moduli space. Our task is now to determine the possible SU(2)$_R$ charges of $n_i(\tau)$.

A simple interpretation of our collective coordinate is that of a particle moving on a unit sphere. In this interpretation, the SU(2)$_R$ charge of $n_i(\tau)$ corresponds to the angular momentum of the particle. If there were no interactions between the background scalar and the other fields in the Chern-Simons Yang-Mills theory, the collective coordinate would behave as a free particle on a sphere. Since any angular momentum representation is possible for a free particle, we could in this case easily find a background with vanishing SU(2)$_R$ charge. What makes things more complicated is that there are interactions between the background scalar and the other fields. Thus, the collective coordinate is described by a particle subject to constraints that could possibly change the allowed SU(2)$_R$ representations. To find out the details of this, we will compute the effective action $\Gamma(n)$, describing the effects of the interaction terms on the collective coordinate. This action can then be used to draw conclusions about the SU(2)$_R$ properties of $n_i(\tau)$.

In the UV, where $\tilde{g} \to 0$, the surviving interaction terms in (3.16) containing $n_i(\tau)$ are

$$-in_i\xi^\dagger_{Aa}(\sigma_i)^a{}_b[H,\xi^{Ab}] - in_i\lambda_{ab}(\sigma_i)^b{}_c[H,\lambda^{ac}] - in_i\hat{\lambda}_{ab}(\sigma_i)^b{}_c[H,\hat{\lambda}^{ac}]. \quad (4.49)$$

Since these terms contain the fermions of the theory, we also need to include the fermion kinetic terms. Thus, the part of the action (3.15) and (3.16) we will consider is

$$\begin{aligned}\mathcal{S} = \int d\tau d\Omega\,\mathrm{tr}\Big[&-i\xi^\dagger_{Aa}\not{D}\xi^{Aa} - in_i\xi^\dagger_{Aa}(\sigma_i)^a{}_b[H,\xi^{Ab}] \\ &+\frac{i}{2}\lambda_{ab}\not{D}\lambda^{ab} - \frac{i}{2}n_i\lambda_{ab}(\sigma_i)^b{}_c[H,\lambda^{ac}] \\ &+\frac{i}{2}\hat{\lambda}_{ab}\not{D}\hat{\lambda}^{ab} - \frac{i}{2}n_i\hat{\lambda}_{ab}(\sigma_i)^b{}_c[H,\hat{\lambda}^{ac}]\Big]. \end{aligned} \quad (4.50)$$

By integrating out the fermions one obtains the effective action $\Gamma(n)$ for the collective coordinate:

$$\mathrm{e}^{-\Gamma(n)} = \int [d\xi^\dagger][d\xi][d\lambda][d\hat{\lambda}]\,\mathrm{e}^{-\mathcal{S}}. \quad (4.51)$$

Before we do this explicitly, we will consider a toy model action that we easily can generalize to (4.50), much like we did in the U(1)$_R$ case.

4.2.1 Toy Model

Let us first study the following action, containing the fermion $\mathrm{SU}(2)_R$ doublet $\psi^a(\tau)$:

$$\mathcal{S} = \int d\tau \left[-i\psi_a^\dagger \partial_\tau \psi^a - iqn_i(\tau)\psi_a^\dagger (\sigma_i)^a{}_b \psi^b \right] . \tag{4.52}$$

In this simple example, the fermion has no spatial dependence. We will show later, however, that the results obtained for (4.52) can easily be generalized to a case with spatial dependence by using the properties of the monopole spinor harmonics. The effective action is given by the following expression

$$\Gamma(n) = -\ln\det\left(i\partial_\tau - iqn_i(\tau)\sigma_i\right), \tag{4.53}$$

where $\det\left(i\partial_\tau - iqn_i(\tau)\sigma_i\right)$ is a functional determinant of the operator acting on the fermion.[2] In the definition of a functional determinant of an operator, the operator itself is treated as a matrix. In the general case, this matrix might have a lot of different indices in different "spaces". Also, if the operator is a function of some continuous variable, this variable is treated as an additional (diagonal) index. To calculate the determinant one can use the following matrix identity

$$\ln\det M = \mathrm{tr}\ln M. \tag{4.54}$$

Using (4.54) on a functional determinant, the trace operation corresponds, of course, to taking the ordinary trace of all the indices and integrating over all the continuous variables. Applied to our toy model, (4.54) yields

$$\Gamma(n) = -\,\mathrm{tr}\ln\left(i\partial_\tau - iqn_i(\tau)\sigma_i\right). \tag{4.55}$$

Here, the trace operation amounts to taking the ordinary $\mathrm{SU}(2)_R$ trace and integrating over τ. Before we do that, however, some further manipulations are in order.

We will not compute the action (4.55) exactly; rather, we will write it in a way that tells us something about the $\mathrm{SU}(2)_R$ representations of the collective coordinate. To proceed we take $n_i(\tau)$ to be quasi-static, which enables us to expand the action in $\dot{n}_i$. The general form of such an expansion is:

$$\Gamma(n) = \int d\tau \left[-V_{\text{eff}}(n) + i\dot{n}_i A_i(n) + \frac{1}{2}\dot{n}_i\dot{n}_j B_{ij}(n) + ... \right], \tag{4.56}$$

where $V_{\text{eff}}(n)$ is the effective potential and $A_i(n)$ and $B_{ij}(n)$ are arbitrary functions. In our particle-on-a-sphere picture of the collective coordinate, the first order term represents a coupling to an electromagnetic field with vector potential $A_i(n)$. The presence of such a term can affect the possible values of the angular momentum of the particle. What we will do next is to expand (4.55) and compare it to (4.56). If we find $A_i(n)$ to be non-zero, the collective

[2] Note that the relative sign between the terms in the determinant has changed. This is because we use Pauli matrices in (4.53) and transposed Pauli matrices in (4.52).

coordinate is properly described by a particle in an electromagnetic field. The form of $A_i(n)$ will then determine the possible angular momentum values of the particle and thus the allowed $\mathrm{SU}(2)_R$ representations for the collective coordinate.

Since the action (4.55) does not contain $\dot{n}_i(\tau)$ explicitly, some additional manipulations are in order before we can carry out the expansion. We begin by writing the collective coordinate in a form that makes the quasi-staticity more explicit:

$$n_i(\tau) = \mathring{n}_i + \tilde{n}_i(\tau). \tag{4.57}$$

Here, the constant part $\mathring{n}_i$ obeys $\mathring{n}^2 = 1$ and the τ-dependent part $\tilde{n}_i$ is a small fluctuation. This allows us to expand our expressions in the fluctuation, around $\mathring{n}_i$. Separately expanding each term of (4.56) in this way, we have to second order in the fluctuations

$$\begin{aligned}\Gamma(n) = \int d\tau \Big[& -V_{\text{eff}}(\mathring{n}) - \tilde{n}_i \partial_i V_{\text{eff}}(\mathring{n}) - \frac{1}{2}\tilde{n}_i \tilde{n}_j \partial_i \partial_j V_{\text{eff}}(\mathring{n}) \\ & + i\dot{\tilde{n}}_i A_i(\mathring{n}) + i\dot{\tilde{n}}_i \tilde{n}_j \partial_j A_i(\mathring{n}) + \frac{1}{2}\dot{\tilde{n}}_i \dot{\tilde{n}}_j B_{ij}(\mathring{n}) + \ldots \Big].\end{aligned} \tag{4.58}$$

When we expand the logarithm in (4.55), the result will be of the form (4.58). Remembering that our task is to determine $A_i(n)$, we will concentrate on finding the unique second order term where one of the two $\tilde{n}_i$-factors is a τ-derivative. Since the coefficient of the corresponding term in the general expansion is $\partial_j A_i(\mathring{n})$, this will enable us to say something about $A_i(n)$. Before we carry out the expansion, let us streamline the expressions a bit by defining $\mathring{m}_i = q\mathring{n}_i$, $\tilde{m}_i = q\tilde{n}_i$ and $\not{m} = m_i \sigma_i$. The action (4.55) can now be written[3]

$$\Gamma(n) = -\operatorname{tr}\ln\left(i\partial_\tau - i\mathring{\not{m}} - i\tilde{\not{m}}\right) = -\operatorname{tr}\ln\left(i\partial_\tau - i\mathring{\not{m}}\right) - \operatorname{tr}\ln\left(\mathbb{1} - \frac{1}{\partial_\tau - \mathring{\not{m}}}\tilde{\not{m}}\right). \tag{4.59}$$

Here, all the τ-dependece is in the second term. Expanding the logarithm in its Taylor series, we can pick out the second order term:

$$\Gamma_{(2)}(n) = \frac{1}{2}\operatorname{tr}\left[\frac{1}{\partial_\tau - \mathring{\not{m}}}\tilde{\not{m}}\frac{1}{\partial_\tau - \mathring{\not{m}}}\tilde{\not{m}}\right] = \frac{1}{2}\operatorname{tr}\left[\frac{\partial_\tau + \mathring{\not{m}}}{\partial_\tau^2 - \mathring{m}^2}\tilde{\not{m}}\frac{\partial_\tau + \mathring{\not{m}}}{\partial_\tau^2 - \mathring{m}^2}\tilde{\not{m}}\right]. \tag{4.60}$$

To be able to find the terms in (4.60) with exactly one derivative of $\tilde{\not{m}}$, we will use the following identity

$$\frac{1}{\partial^2 - \mathring{m}^2}\phi = \sum_{k=0}^{\infty}(-1)^k \underbrace{[\partial^2, [\partial^2, \ldots, [\partial^2}_{k}, \phi]\ldots]]\frac{1}{(\partial^2 - \mathring{m}^2)^{k+1}}. \tag{4.61}$$

[3]The identity $\ln AB = \ln A + \ln B$ holds only when the operators A and B commute. Otherwise, the extra terms can be calculated using the Baker-Hausdorff formula. The operators in (4.59) do not commute, but since everything is inside the trace the identity holds anyway. The trace of a commutator is always zero, which can be proven directly by using the linearity and cyclicity.

It is also important to remember that the ordering of the slashed m:s matters, since they contain Pauli matrices. We start by using (4.61) with $\phi = \not{\tilde{m}}$. Since we are interested in the terms with one first order derivative only, we can immediately throw away the terms with a second order derivative or with two first order derivatives. The only surviving terms are[4]

$$\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2} - [\partial_\tau^2, \not{\tilde{m}}]\frac{1}{(\partial_\tau^2 - \mathring{m}^2)^2} = \not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2} - 2\dot{\not{\tilde{m}}}\partial_\tau\frac{1}{(\partial_\tau^2 - \mathring{m}^2)^2}. \tag{4.62}$$

Applying $\partial_\tau + \not{\mathring{m}}$ to this expression now gives us

$$\begin{aligned}\frac{\partial_\tau + \not{\mathring{m}}}{\partial_\tau^2 - \mathring{m}^2}\not{\tilde{m}} =& \dot{\not{\tilde{m}}}\frac{1}{\partial_\tau^2 - \mathring{m}^2} + \not{\tilde{m}}\frac{\partial_\tau}{\partial_\tau^2 - \mathring{m}^2} + \not{\mathring{m}}\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2} \\ &- 2\dot{\not{\tilde{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2 - \mathring{m}^2)^2} - 2\not{\mathring{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2)^2}.\end{aligned} \tag{4.63}$$

Next, we calculate the square of (4.63), again throwing away terms of the wrong order:

$$\begin{aligned}\left(\frac{\partial_\tau + \not{\mathring{m}}}{\partial_\tau^2 - \mathring{m}^2}\not{\tilde{m}}\right)^2 &= \dot{\not{\tilde{m}}}\frac{1}{\partial_\tau^2 - \mathring{m}^2}\left[\not{\tilde{m}}\frac{\partial_\tau}{\partial_\tau^2 - \mathring{m}^2} + \not{\mathring{m}}\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2}\right] + \\ &+\not{\tilde{m}}\frac{\partial_\tau}{\partial_\tau^2 - \mathring{m}^2}\left[\dot{\not{\tilde{m}}}\frac{1}{\partial_\tau^2 - \mathring{m}^2} + \not{\tilde{m}}\frac{\partial_\tau}{\partial_\tau^2 - \mathring{m}^2} + \not{\mathring{m}}\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2}\right. \\ &\left.-2\dot{\not{\tilde{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2 - \mathring{m}^2)^2} - 2\not{\mathring{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2)^2}\right] + \\ &+\not{\mathring{m}}\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2}\left[\dot{\not{\tilde{m}}}\frac{1}{\partial_\tau^2 - \mathring{m}^2} + \not{\tilde{m}}\frac{\partial_\tau}{\partial_\tau^2 - \mathring{m}^2} + \not{\mathring{m}}\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2}\right. \\ &\left.-2\dot{\not{\tilde{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2 - \mathring{m}^2)^2} - 2\not{\mathring{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2)^2}\right] + \\ &-2\dot{\not{\tilde{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2 - \mathring{m}^2)^2}\left[\not{\tilde{m}}\frac{\partial_\tau}{\partial_\tau^2 - \mathring{m}^2} + \not{\mathring{m}}\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2}\right] - \\ &-2\not{\mathring{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2)^2}\left[\not{\tilde{m}}\frac{\partial_\tau}{\partial_\tau^2 - \mathring{m}^2} + \not{\mathring{m}}\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2}\right] \\ &= \dot{\not{\tilde{m}}}\not{\tilde{m}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2)^2} + \dot{\not{\tilde{m}}}\not{\mathring{m}}\not{\tilde{m}}\frac{1}{(\partial_\tau^2 - \mathring{m}^2)^2} \\ &+\not{\tilde{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2)^2} - 2\not{\tilde{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau^3}{(\partial_\tau^2 - \mathring{m}^2)^3} \\ &+\not{\tilde{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2)^2} - 2\not{\tilde{m}}\not{\mathring{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2 - \mathring{m}^2)^3} \\ &+\not{\tilde{m}}\not{\mathring{m}}\dot{\not{\tilde{m}}}\frac{1}{(\partial_\tau^2 - \mathring{m}^2)^2} - 2\not{\tilde{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau^3}{(\partial_\tau^2 - \mathring{m}^2)^3}\end{aligned}$$

[4]In the following, we will use the equality signs in a rather sloppy way. It is to be understood that we throw away the uninteresting terms.

$$
\begin{aligned}
&-2\mathring{\not{m}}\tilde{\not{m}}\dot{\tilde{\not{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2-\mathring{m}^2)^3}+\mathring{\not{m}}\tilde{\not{m}}\dot{\tilde{\not{m}}}\frac{1}{(\partial_\tau^2-\mathring{m}^2)^2}\\
&-2\mathring{\not{m}}\tilde{\not{m}}\dot{\tilde{\not{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2-\mathring{m}^2)^3}-2\mathring{\not{m}}\tilde{\not{m}}\mathring{\not{m}}\dot{\tilde{\not{m}}}\frac{\partial_\tau}{(\partial_\tau^2-\mathring{m}^2)^3}\\
&-2\mathring{\not{m}}\tilde{\not{m}}\dot{\tilde{\not{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2-\mathring{m}^2)^3}-2\mathring{\not{m}}\tilde{\not{m}}\mathring{\not{m}}\dot{\tilde{\not{m}}}\frac{\partial_\tau}{(\partial_\tau^2-\mathring{m}^2)^3}\\
&-2\dot{\tilde{\not{m}}}\tilde{\not{m}}\frac{\partial_\tau^3}{(\partial_\tau^2-\mathring{m}^2)^3}-2\dot{\tilde{\not{m}}}\mathring{\not{m}}\tilde{\not{m}}\frac{\partial_\tau^2}{(\partial_\tau^2-\mathring{m}^2)^3}\\
&-2\mathring{\not{m}}\dot{\tilde{\not{m}}}\tilde{\not{m}}\frac{\partial_\tau^2}{(\partial_\tau^2-\mathring{m}^2)^3}-2\mathring{\not{m}}\dot{\tilde{\not{m}}}\mathring{\not{m}}\tilde{\not{m}}\frac{\partial_\tau}{(\partial_\tau^2-\mathring{m}^2)^3}.
\end{aligned}
\tag{4.64}
$$

Collecting all these terms yields

$$
\begin{aligned}
\left(\frac{\partial_\tau+\mathring{\not{m}}}{\partial_\tau^2-\mathring{m}^2}\tilde{\not{m}}\right)^2 =& \dot{\tilde{m}}_i\tilde{m}_j(\sigma_i\sigma_j+2\sigma_j\sigma_i)\left(\frac{\partial_\tau}{(\partial_\tau^2-\mathring{m}^2)^2}-2\frac{\partial_\tau^3}{(\partial_\tau^2-\mathring{m}^2)^3}\right)\\
&+\dot{\tilde{m}}_i\tilde{m}_j\mathring{m}_k(\sigma_i\sigma_k\sigma_j+\sigma_j\sigma_k\sigma_i+\sigma_k\sigma_j\sigma_i)\frac{1}{(\partial_\tau^2-\mathring{m}^2)^2}\\
&-2\dot{\tilde{m}}_i\tilde{m}_j\mathring{m}_k\,(2\sigma_j\sigma_k\sigma_i+2\sigma_k\sigma_j\sigma_i\\
&+\sigma_i\sigma_k\sigma_j+\sigma_k\sigma_i\sigma_j)\,\frac{\partial_\tau^2}{(\partial_\tau^2-\mathring{m}^2)^3}\\
&-2\mathring{m}_i\dot{\tilde{m}}_j\mathring{m}_k\tilde{m}_l(\sigma_i\sigma_j\sigma_k\sigma_l+2\sigma_i\sigma_l\sigma_k\sigma_j)\frac{\partial_\tau}{(\partial_\tau^2-\mathring{m}^2)^3}.
\end{aligned}
\tag{4.65}
$$

Let us now take the $\mathrm{SU}(2)_R$ trace of this expression, making use of the following Pauli matrix identities:

$$
\begin{aligned}
\operatorname{tr}\sigma_i\sigma_j &= 2\delta_{ij} &(4.66)\\
\operatorname{tr}\sigma_i\sigma_j\sigma_k &= 2i\epsilon_{ijk} &(4.67)\\
\operatorname{tr}\sigma_i\sigma_j\sigma_k\sigma_l &= 2(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}). &(4.68)
\end{aligned}
$$

Inserting the result into (4.60), we obtain our expression for the second order terms in the expansion with exactly one derivative:

$$
\begin{aligned}
\Gamma_{(2,1)}(n) \quad=\quad & -3\operatorname{tr}\dot{\tilde{m}}_i\tilde{m}_i\frac{\partial_\tau^3+\mathring{m}^2\partial_\tau}{(\partial_\tau^2-\mathring{m}^2)^3}-i\operatorname{tr}\epsilon_{ijk}\dot{\tilde{m}}_i\tilde{m}_j\mathring{m}_k\frac{1}{(\partial_\tau^2-\mathring{m}^2)^2}\\
&-6\operatorname{tr}(2\dot{\tilde{m}}_i\tilde{m}_j\mathring{m}_i\mathring{m}_j-\dot{\tilde{m}}_i\tilde{m}_i\mathring{m}^2)\frac{\partial_\tau}{(\partial_\tau^2-\mathring{m}^2)^3}.
\end{aligned}
\tag{4.69}
$$

At this stage, the only part of the trace still to be performed is the τ-integration. We see that, in all the terms in (4.69), the τ-dependent part is separated from the differential operators. These two parts can be integrated separately. We also note that the first and third terms are actually total derivatives that will not survive the integration. Thus, we are left with the second term only. To integrate the operator part, we insert two complete sets of energy states. To see these steps more clearly we switch to bra-ket notation, and view $\frac{1}{(\partial_\tau^2 - \mathring{m}^2)^2}$ as the τ-representation of the operator A. We rewrite the integral as

$$\begin{aligned}
\int d\tau \frac{1}{(\partial_\tau^2 - \mathring{m}^2)^2} &= \int d\tau \, \langle \tau | A | \tau \rangle \\
&= \int d\tau \int d\omega d\omega' \, \langle \tau | \omega \rangle \, \langle \omega | \, A \, | \omega' \rangle \, \langle \omega' | \tau \rangle \\
&= \int \frac{d\omega d\omega'}{2\pi} \, \langle \omega | \, A \, | \omega' \rangle \, \delta(\omega' - \omega) \\
&= \int \frac{d\omega}{2\pi} \, \langle \omega | A | \omega \rangle \, ,
\end{aligned} \tag{4.70}$$

where we in the second step have used

$$\int d\tau \, \langle \omega' | \tau \rangle \, \langle \tau | \omega \rangle = \langle \omega' | \omega \rangle = \frac{1}{2\pi} \delta(\omega' - \omega). \tag{4.71}$$

To explicitly write the operator A in the ω-basis, we recognize $i\partial_\tau$ to be the energy operator. Therefore, $\langle \omega | A | \omega \rangle$ is obtained by simply setting $\partial_\tau = -i\omega$. Now that we know how to handle the integration of the differential operator, let us return to (4.69). Writing out the trace integrals, we arrive at

$$\begin{aligned}
\Gamma_{(2,1)}(n) &= -i \int d\tau \, \epsilon_{ijk} \dot{\tilde{m}}_i \tilde{m}_j \mathring{m}_k \int \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \mathring{m}^2)^2} \\
&= -\frac{i}{4} \int d\tau \, \epsilon_{ijk} \dot{\tilde{m}}_i \tilde{m}_j \frac{\mathring{m}_k}{|\mathring{m}|^3} \\
&= -\frac{i}{4} \operatorname{sign} q \int d\tau \, \epsilon_{ijk} \dot{\tilde{n}}_i \tilde{n}_j \frac{\mathring{n}_k}{|\mathring{n}|^3}.
\end{aligned} \tag{4.72}$$

Let us now return to the general expansion (4.58). The term we are interested in can be written as

$$\begin{aligned}
\Gamma_{(2,1)}(n) &= i \int d\tau \, \dot{\tilde{n}}_i \tilde{n}_j \partial_j A_i(\mathring{n}) = \frac{i}{2} \int d\tau \, \left(\dot{\tilde{n}}_i \tilde{n}_j \partial_j A_i(\mathring{n}) + \dot{\tilde{n}}_j \tilde{n}_i \partial_i A_j(\mathring{n}) \right) \\
&= -\frac{i}{2} \int d\tau \, \dot{\tilde{n}}_i \tilde{n}_j \left(\partial_i A_j(\mathring{n}) - \partial_j A_i(\mathring{n}) \right),
\end{aligned} \tag{4.73}$$

where we in the last step have integrated one of the terms by parts, obtaining a minus sign. Comparing (4.73) and (4.72) we can now conclude:

$$\partial_i A_j(n) - \partial_j A_i(n) = \frac{\operatorname{sign} q}{2} \epsilon_{ijk} \frac{n_k}{|n|^3}. \tag{4.74}$$

This equation is nothing less than the field strength for a magnetic monopole of charge $\operatorname{sign} q/2$, with $A_i(n)$ as the corresponding gauge potential. Thus, the above analysis shows that the toy model collective coordinate can be properly described by a particle on a sphere around a magnetic monopole. It is important to realize that this monopole is not related to our original spacetime monopole. The fact that there exists a monopole also in the SU(2)$_R$ moduli space is a pure coincidence. We can, however, use what we have already learned about magnetic monopoles to draw an important conclusion for the collective coordinate. In Appendix C we studied a fermion in a monopole background and learned that the angular momentum eigenvalues are bounded from below by the monopole charge (see (C.42)). Applying this to our particle analogy of the collective coordinate, we draw the conclusion that the possible SU(2)$_R$ representations are bounded from below.

The above analysis was done for the simple toy model action (4.52), that has no spatial dependence. Before we can apply our results to the $\mathcal{N}=3$ Chern-Simons Yang-Mills theory, we need to generalize this toy model a bit. Instead of (4.52), let us consider the following action

$$\mathcal{S}=\int d\tau d\Omega\left[-i\psi_a^\dagger \not{\mathcal{D}}\psi^a - iqn_i(\tau)\psi_a^\dagger(\sigma_i)^a{}_b\psi^b\right], \tag{4.75}$$

where the fermion is space dependent. We can expand the fermion in monopole spinor harmonics like we did in the previous section (see eq. (4.9)). Inserting the expansion into (4.75) and using (C.79), (C.81), (C.82) and (C.83) we arrive at[5]

$$\begin{aligned}\mathcal{S} &= \sum_m\int d\tau\left[-i\psi_m^\dagger\partial_\tau\psi_m - i\,\mathrm{sign}\,(q)qn_i\psi_m^\dagger\sigma_i\psi_m\right]\\ &\quad+\sum_{jm\varepsilon}\int d\tau\left[-i\psi_{jm}^{\varepsilon\dagger}\partial_\tau\psi_{jm}^\varepsilon+\Delta_{jq}^\varepsilon\psi_{jm}^{-\varepsilon\dagger}\psi_{jm}^\varepsilon - iqn_i\psi_{jm}^{-\varepsilon\dagger}\sigma_i\psi_{jm}^\varepsilon\right]\end{aligned} \tag{4.76}$$

Starting with the zero mode terms it is apparent that, for each m, we have essentially the previously considered case (4.52). The only difference is the $\mathrm{sign}\,q$-factor in the second term, which simply can be absorbed into n_i. Thus, the right hand side of our previous result (4.74) is multiplied by this factor, removing the previous $\mathrm{sign}\,q$-dependence. With the sum over m giving a factor of $2j+1=2|q|$, the total contribution from the zero modes is

$$\partial_iA_j(n)-\partial_jA_i(n)=|q|\epsilon_{ijk}\frac{n_k}{|n|^3}. \tag{4.77}$$

Next, we turn to the non-zero modes. In this case, we cannot immediately use our results from the simpler model, since the action is of a different form. Let us start with the effective action written as a functional determinant. Since different jm-modes do not mix, we can consider each one of these modes separately. For fixed values of j and m we have

$$\begin{aligned}\Gamma(n) &= -\ln\det\begin{pmatrix} i\partial_\tau & -\Delta^- - i\not{n}\\ -\Delta^+ - i\not{n} & i\partial_\tau\end{pmatrix}\\ &= -\,\mathrm{tr}\ln\begin{pmatrix} i\partial_\tau & -\Delta^- - i\not{n}\\ -\Delta^+ - i\not{n} & i\partial_\tau\end{pmatrix}.\end{aligned} \tag{4.78}$$

[5]We have suppressed the SU(2)$_R$ doublet indices to avoid cluttered notation.

The matrix in this expression mixes the $\varepsilon = \pm$ components in (4.76). In addition to what we considered in the simpler model, the functional determinant should now also be taken with respect to this matrix. Defining $\Delta = \Delta^+ = -\Delta^-$ we have

$$\begin{aligned}
\Gamma(n) &= -\operatorname{tr}\ln\left[\begin{pmatrix} i\partial_\tau & \Delta - i\not{\mathring{m}} \\ -\Delta - i\not{\mathring{m}} & i\partial_\tau \end{pmatrix} + \begin{pmatrix} 0 & -i\not{\tilde{m}} \\ -i\not{\tilde{m}} & 0 \end{pmatrix}\right] \\
&= -\operatorname{tr}\ln\begin{pmatrix} i\partial_\tau & \Delta - i\not{\mathring{m}} \\ -\Delta - i\not{\mathring{m}} & i\partial_\tau \end{pmatrix} \\
&\quad -\operatorname{tr}\ln\left[\mathbb{1} + \begin{pmatrix} i\partial_\tau & \Delta - i\not{\mathring{m}} \\ -\Delta - i\not{\mathring{m}} & i\partial_\tau \end{pmatrix}^{-1} \begin{pmatrix} 0 & -i\not{\tilde{m}} \\ -i\not{\tilde{m}} & 0 \end{pmatrix}\right] \\
&= -\operatorname{tr}\ln\begin{pmatrix} i\partial_\tau & \Delta - i\not{\mathring{m}} \\ -\Delta - i\not{\mathring{m}} & i\partial_\tau \end{pmatrix} \\
&\quad -\operatorname{tr}\ln\left[\mathbb{1} - \frac{\begin{pmatrix} -\partial_\tau & -i\Delta - \not{\mathring{m}} \\ i\Delta - \not{\mathring{m}} & -\partial_\tau \end{pmatrix}}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} \begin{pmatrix} 0 & \not{\tilde{m}} \\ \not{\tilde{m}} & 0 \end{pmatrix}\right].
\end{aligned}$$

The second one of these logarithms can be expanded, just like in the previous case. The following calculations will be very similar to those for the simpler model, the only real difference being the extra matrix structure that mixes the $\varepsilon = \pm$ components. In each step, we throw away the uninteresting terms. The second order term in the expansion is

$$\begin{aligned}
\Gamma_{(2)}(n) &= \frac{1}{2}\operatorname{tr}\left[\frac{\begin{pmatrix} -\partial_\tau & -i\Delta - \not{\mathring{m}} \\ i\Delta - \not{\mathring{m}} & -\partial_\tau \end{pmatrix}}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} \begin{pmatrix} 0 & \not{\tilde{m}} \\ \not{\tilde{m}} & 0 \end{pmatrix}\right]^2 \\
&= \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} (-i\Delta - \not{\mathring{m}})\left(\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} - 2\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2}\right) & -(\dot{\not{\tilde{m}}} + \not{\tilde{m}}\partial_\tau)\frac{1}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} - 2\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2} \\ -(\dot{\not{\tilde{m}}} + \not{\tilde{m}}\partial_\tau)\frac{1}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} - 2\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2} & (i\Delta - \not{\mathring{m}})\left(\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} - 2\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2}\right) \end{pmatrix}^2\right].
\end{aligned} \tag{4.79}$$

When we square the matrix in (4.79) only the diagonal elements of the resulting matrix M need to be computed, since the trace is the sum of these elements. The first element is

$$\begin{aligned}
M_{11} &= \left[(-i\Delta - \not{\mathring{m}})\left(\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} - 2\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2}\right)\right]^2 \\
&\quad + \left[(\dot{\not{\tilde{m}}} + \not{\tilde{m}}\partial_\tau)\frac{1}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} + 2\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2}\right]^2
\end{aligned} \tag{4.80}$$

and the second one

$$\begin{aligned}
M_{22} &= \left[(i\Delta - \not{\mathring{m}})\left(\not{\tilde{m}}\frac{1}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} - 2\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2}\right)\right]^2 \\
&\quad + \left[(\dot{\not{\tilde{m}}} + \not{\tilde{m}}\partial_\tau)\frac{1}{\partial_\tau^2 - \mathring{m}^2 - \Delta^2} + 2\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2}\right]^2 . \qquad (4.81)
\end{aligned}$$

The only difference between these elements is the sign of Δ. Thus, we immediately see that after computing the squares, all terms with Δ and $\not{\mathring{m}}$ mixed will cancel. The surviving terms are

$$\begin{aligned}
M_{11} + M_{22} &= -2\Delta^2\left[-4\not{\mathring{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^3} - 2\dot{\not{\tilde{m}}}\not{\mathring{m}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^3}\right] \\
&\quad -8\not{\mathring{m}}\not{\tilde{m}}\not{\mathring{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^3} - 4\not{\mathring{m}}\dot{\not{\tilde{m}}}\not{\mathring{m}}\not{\tilde{m}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^3} \\
&\quad 2\left[\dot{\not{\tilde{m}}}\not{\tilde{m}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2} + \not{\tilde{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2}\right. \\
&\quad -2\not{\tilde{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^3} - \not{\tilde{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2} \\
&\quad \left. +2\not{\tilde{m}}\dot{\not{\tilde{m}}}\frac{\partial_\tau^2}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^3} + 2\dot{\not{\tilde{m}}}\not{\tilde{m}}\frac{\partial_\tau^2}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^3}\right] . \qquad (4.82)
\end{aligned}$$

Our next step is to take the SU(2)$_R$ trace of (4.82). Using (4.66) and (4.68) we arrive at

$$\begin{aligned}
\Gamma_{(2,1)}(n) &= \frac{1}{2}\,\mathrm{tr}\left[24\tilde{m}_i\dot{\tilde{m}}_i\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^3}\right. \\
&\quad -24\mathring{m}_i\dot{\tilde{m}}_j\mathring{m}_k\tilde{m}_l(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
&\quad \left. + 4\tilde{m}_i\dot{\tilde{m}}_i\left(\frac{\partial_\tau}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^2} + 2\frac{\partial_\tau^2}{(\partial_\tau^2 - \mathring{m}^2 - \Delta^2)^3}\right)\right] \qquad (4.83)
\end{aligned}$$

All these terms are total derivatives that do not survive the τ-integration, and this means that the non-zero modes give no contribution to the final result. To understand the mechanism behind this, we take a look at (4.80) and (4.81) again. The terms with three m:s are the only ones that survive the τ-integration. These terms do, however, cancel because of the mentioned sign difference between Δ^+ and Δ^-, which in turn is due to the pairing of the Dirac operator eigenvalues. We found that exactly the same mechanism is at work in the U(1)$_R$ case (see (4.28)).

To sum things up, we have found that the toy model collective coordinate $n_i(\tau)$ with action (4.75) can be described by a particle on a sphere around a monopole with charge $|q|$.

4.2.2 Applications to $\mathcal{N} = 3$ Chern-Simons Yang-Mills Theory

In the previous section, we found that the collective coordinate $n_i(\tau)$ in the toy model (4.75) could be properly described by a particle on a sphere around a magnetic monopole with charge $|q|$. Let us now apply this result to our $\mathcal{N}=3$ Chern-Simons Yang-Mills theory. Returning to (4.50), we note that we can handle the gauge indices in the same way as we did in the $\mathrm{U}(1)_R$ case (see (4.45)). Thus, starting with the ξ^A-part, we see that each fermion matrix element ξ_{rs} is of the same form as the toy model (4.75) with $q=q_r-q_s$. Summing over the gauge- and flavour indices, the total contribution to the monopole charge from these terms becomes $N_f\sum_{r,s=1}^N |q_r-q_s|$.

Next, we consider the λ^{ab}-terms. Summing over a, this part of the action becomes

$$\begin{aligned}\mathcal{S}=&\int d\tau d\Omega\,\mathrm{tr}\left[+\frac{i}{2}\lambda_{1b}\not{\mathcal{D}}\lambda^{1b}-\frac{i}{2}n_i\lambda_{1b}(\sigma_i)^b{}_c[H,\lambda^{1c}]\right.\\&\left.+\frac{i}{2}\lambda_{2b}\not{\mathcal{D}}\lambda^{2b}-\frac{i}{2}n_i\lambda_{2b}(\sigma_i)^b{}_c[H,\lambda^{2c}]\right]\\=&\int d\tau d\Omega\,\mathrm{tr}\left[-\frac{i}{2}(\lambda^{1b})^*\not{\mathcal{D}}\lambda^{1b}+\frac{i}{2}n_i(\lambda^{1b})^*(\sigma_i)^b{}_c[H,\lambda^{1c}]\right.\\&\left.-\frac{i}{2}(\lambda^{2b})^*\not{\mathcal{D}}\lambda^{2b}+\frac{i}{2}n_i(\lambda^{2b})^*(\sigma_i)^b{}_c[H,\lambda^{2c}]\right],\end{aligned}\tag{4.84}$$

where we have used $\lambda_{ab}=-(\lambda^{ab})^*$ (see (3.8)). These terms are almost of the same form as the ξ^A-part of (4.50), the only difference being the sign of the term containing the commutator, which is simply absorbed into the monopole charge. Since we have two expressions of this form, both with a factor of $\frac{1}{2}$ in front, the total contribution from the λ^{ab}-terms to the monopole charge is $-\sum_{r,s=1}^N|q_r-q_s|$. Performing the same analysis for the $\hat{\lambda}^{ab}$-terms, we get exactly the same result. Thus, the total contrubution to the monopole charge in the $\mathrm{SU}(2)_R$ moduli space, from all the terms in (4.50) is

$$Q_{\mathrm{mon}}=(N_f-2)\sum_{r,s=1}^N|q_r-q_s|.\tag{4.85}$$

Now, let us recall that the possible $\mathrm{SU}(2)_R$ representations of the collective coordinate $n_i(\tau)$ in our $\mathcal{N}=3$ Chern-Simons Yang-Mills theory is bounded from below by Q_{mon}. Since $n_i(\tau)$ parametrizes the (spacetime) monopole background, we can draw the conclusion that we must set $Q_{\mathrm{mon}}=0$ to allow for BPS monopole configurations with vanishing $\mathrm{SU}(2)_R$ charge. This puts us in exactly the same situation as in the $\mathrm{U}(1)_R$ case. We could either set $q_r-q_s=0$ for all r and s (which strongly limits the possible gauge representations of the monopole) or set $N_f=2$. As said before, the number of flavours in the ABJM theory is two. In the $\mathrm{U}(1)_R$ case we could not draw any conclusions about the ABJM theory, since all the computations were performed in the UV and the abelian charge can change in the RG flow to the ABJM limit. Now that we have found monopoles with vanishing $\mathrm{SU}(2)_R$ charge in the UV, however, we can really be sure that these also exist in the ABJM theory, since the RG flow cannot change a non-abelian representation.

5

Conclusions and Comments

Before we discuss our results, let us summarize what we have done in this thesis. The ultimate purpose of the calculations was to prove the existence of monopole operators with certain properties in the ABJM theory. These operators should transform non-trivially under gauge transformations and be R-symmetry singlets. The coupling parameter of the ABJM theory is the Chern-Simons level k, and since we were specifically interested in the cases of $k = 1, 2$ we had a strongly coupled theory, making it difficult to perform the calculations of the R-symmetry charges. The way to deal with this problem was to consider an extended theory that reduces to ABJM in the IR limit. Another coupling parameter g was introduced by adding a Yang-Mills term to the action. This enabled us to perform the calculations in the UV, where g is small, and then flow to the IR limit. To be completely sure that our results were valid in the IR, however, we had to calculate a non-abelian charge, which is protected under the RG flow. Because of this, our extended theory had to preserve at least $\mathcal{N} = 3$ supersymmetry.

To study monopole operators in our $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory, we used the radial-quantization method and transformed our theory from $\mathbb{R}^{1,2}$ to $\mathbb{R} \times S^2$. This allowed us to study properties of monopole field configurations to learn about the operators themselves. The next step was to find a classical monopole solution to our theory. In order to preserve the supersymmetry allowing us to calculate non-abelian R-charges, this had to be a BPS monopole everywhere along the RG flow. For this to work out, we had to give background expectation values to the adjoint scalars, in addition to those assigned to the gauge fields.

Having found a classical BPS monopole background to our Chern-Simons Yang-Mills theory, we could go to the far UV and perform our quantum mechanical computations there, with the goal of finding expressions for the R-symmetry charges of the monopole background. First, we considered a special case of our BPS monopole, with the R-symmetry broken from SU(2) to U(1).

Calculating the R-charge operator and normal ordering the terms, we found that the charge of the vacuum itself (corresponding to the monopole background field configuration) is given by (4.48). This result told us that the R-charge is vanishing when the number of flavours equals two, but not in the general case. Thus, we found that the ABJM theory has precisely the right number of flavours for the charge to vanish, but we could not be sure that our result was correct in the IR (ABJM) limit, since we had calculated an abelian charge.

Next, we considered the full SUSY-preserving monopole background and computed the corresponding $SU(2)_R$ charges. This was done in a completely different manner than in the $U(1)_R$ case, by calculating the effective action for the collective coordinate parameterizing the background. We found that the collective coordinate, in the $SU(2)_R$ moduli space, could be described by a particle on a sphere around a magnetic monopole whose charge is given by (4.85). Since the angular momentum values of a particle in a monopole background is bounded from below by the monopole charge, we could by our particle analogy conclude that the R-charge of the collective coordinate, and thus of the monopole background, also is bounded by (4.85) from below. Thus, we arrived at the same conclusion as in the $U(1)_R$ case; there are monopoles with vanishing R-charge in the ABJM theory, but not in the general case.

In both the abelian and the non-abelian case, we found that the R-charge of the monopole background is induced by the fermions in the theory. Without the coupling between the fermions and the background scalar, the R-charge of the monopole would vanish identically. The adjoint fermions give a significant contribution to the charge, but in the IR limit they are non-dynamical and can be integrated out. This arises the question of how one would obtain the same result directly by computations in the ABJM theory.

The whole argument in this thesis was based on the fact that we could find a UV completion to the original theory that preserves enough supersymmetry for the R-symmetry to be non-abelian. This method works for several other theories as well, but what if the UV completion theory has less than three supersymmetries, making it impossible to calculate a non-abelian R-charge? One could in that case, of course, calculate the $U(1)_R$ charge. Unless one finds a way to make sure that the abelian R-charge is constant along the RG flow, however, it would be difficult to determine it in the IR theory.

5.1 Towards a Deeper Understanding of the Monopole Operators

Monopole operators in three-dimensional conformal field theories have been extensively studied the last decade, mostly in connection to AdS/CFT dualities. These operators are, however, not very well understood at a basic level. In this thesis, and in most other contexts, the monopole operators are described in the radial quantization picture, using the operator-state correspondence. To understand the operators better, and thus make it easier to prove the conjectured dualities, one would instead like to find explicit expressions for them directly in the $\mathbb{R}^3$-theory. However, monopole operators create topological quantum numbers and cannot be expressed as polynomials in the fundamental fields. To solve this problem, it was suggested in [8] that a paper by Mandelstam [17] can serve as a model. In this paper, the author constructs solition creating operators in the two-dimensional sine-Gordon model, expressing these in the fundamental fields of the theory. To do something similar in three-dimensional CFT:s, however, is much more difficult.

Some steps towards an explicit description of the operators have been taken. In [15], the authors studied the SUSY enhancement mechanism in an ABJM-like theory and could from this derive a non-trivial condition for the monopole operators. This was taken one step further in [21], where the authors once again studied monopole operators through their role in SUSY enhancement, this time in an $\mathcal{N} = 6$ superspace formulation of ABJM. First, they assumed that the operators are covariantly constant, and proved that this can only be true in the $U(2) \times U(2)$- (or $SU(2) \times SU(2)$)-case. In this case, they really found an explicit expression. This expression had not the properties of a "proper" monopole operator though, from which they concluded that the SUSY in the $N = 2$ case is "kinematically" enhanced rather than topologically enhanced.[1] Relaxing the covariant consistency condition, they could derive a system of constraints satisfied by monopole operators in the general $U(N) \times U(N)$-case. It is unclear whether these conditions fully specify the operators.

[1] This should come as no surprise, since N=2 essentially is the BLG case, where the $\mathcal{N} = 8$ SUSY is already manifest

A

Proofs of SUSY Invariance

A.1 BLG Lagrangian

In this section we explicitly show that the BLG Lagrangian (2.8) is invariant under SUSY transformations (2.11)-(2.13). Starting from (2.11) and (2.12) only, we will be able to derive (2.13).

In section A.1.1, we give some details about the Dirac matrices in the theory and derive the Fierz identities. The variation of the Lagrangian and the cancellation of terms is carried out in sections A.1.2 and A.1.3.

A.1.1 Dirac Matrices

The BLG theory contains spacetime SO(2,1) Dirac matrices as well as SO(8) R-symmetry Dirac matrices. Below, we give details about these matrices and derive some important identities.

SO(2,1)

We use the following gamma matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

These matrices obey the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{A.2}$$

with the metric $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1)$. Our choice of charge conjugation matrix C must satisfy

$$C^{-1}\gamma^\mu C = (-\gamma^\mu)^\dagger. \tag{A.3}$$

It is easily checked that $C = -\gamma^0$ satisfies (A.3). The spinors in the theory obey the Majorana condition $\bar{\Psi} = \Psi^T C$.

The matrices

$$\gamma^0 C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma^1 C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^2 C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{A.4}$$

are all symmetric and together with the antisymmetric matrix C they span the space of real 2×2-matrices. We can use the symmetry of these matrices to flip fermion bilinears, for example:

$$\begin{aligned} \bar{\Psi}_1 \gamma^\mu \Psi_2 &= {\Psi_1}^T C \gamma^\mu \Psi_2 \\ &= ({\Psi_1}^T C \gamma^\mu \Psi_2)^T \\ &= -{\Psi_2}^T (C\gamma^\mu)^T \Psi_1 \\ &= -{\Psi_2}^T (C\gamma^\mu) \Psi_1 \\ &= -\bar{\Psi}_2 \gamma^\mu \Psi_1. \end{aligned} \tag{A.5}$$

The extra minus sign in the second step comes from interchanging the fermions. In the same way we can also prove $\bar{\Psi}_1 \gamma^\mu \gamma^\nu \Psi_2 = \bar{\Psi}_2 \gamma^\nu \gamma^\mu \Psi_1$ and similar identities for other bilinears. Finally, for future reference, we note that

$$\gamma^{\mu\nu} = \varepsilon^{\mu\nu\lambda} \gamma_\lambda, \tag{A.6}$$

where $\gamma^{\mu\nu}$ is the antisymmetrized product of gamma matrices.

SO(8)

For the SO(8) Dirac matrices we will not give explicit realizations, since it suffices to make some statements about their (anti-)symmetry properties. The SO(8)-matrices obey the Dirac algebra:

$$\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}. \tag{A.7}$$

Since the metric is just a delta we do not have to care about the position of the SO(8) vector indices, and we will write them upstairs all the time. If Γ^i is chosen to be antisymmetric we see that $C = \delta$ satisfies the condition

$$C^{-1} \Gamma^i C = (-\Gamma^i)^\dagger. \tag{A.8}$$

The antisymmetry of Γ^i also implies (anti-)symmetries for the different antisymmetrized products of gamma matrices. A list of these, together with C and Γ^i, is displayed below. The number of independent matrices of each kind is shown to the right.

$C = \delta$	Symmetric	1
Γ^i	Antisymmetric	8
Γ^{ij}	Antisymmetric	28
Γ^{ijk}	Symmetric	56
Γ^{ijkl}	Symmetric	70
Γ^{ijklm}	Antisymmetric	56
Γ^{ijklmn}	Antisymmetric	28
$\Gamma^{ijklmnp}$	Symmetric	8
Γ^9	Symmetric	1

Here, we have defined $\Gamma^9 \equiv \Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\Gamma^6\Gamma^7\Gamma^8$. The (anti-)symmetry of these matrices has to be accounted for when we flip fermion bilinears like we did in the previous section.

In total, there are 136 symmetric and 120 antisymmetric independent matrices. This equals the number of independent symmetric and antisymmetric 16×16-matrices, which means that our set of antisymmetrized gamma matrix products spans the set of real 16×16-matrices. Thus, our gamma matrices must be of size 16×16. We can, however, effectively view them as 8×8 blocks, carrying dotted or undotted indices depending on their position in the 16×16 matrix. The first 8 row- and column indices are undotted, and the last 8 dotted. If we define C to have two indices of the same kind and Γ^i to have one index of each kind, the index structure of the other matrices will follow. More about this when we derive the Fierz identities.

To compare different products of gamma matrices, it is often necessary to rewrite them as a sum of terms that are symmetric and antisymmetric in the different vector indices. Below, we derive a couple of such identities that are needed later.

First, consider $\Gamma^i\Gamma^{jk}$. Of course, each term in the expansion has to be antisymmetric in j and k, but i will be symmetric to j and k in one of the terms and antisymmetric in the other. Thus, we have:

$$\Gamma^i\Gamma^{jk} = a\Gamma^{ijk} + b\delta^{i[j}\Gamma^{k]} \tag{A.9}$$

for some coefficients a and b. Setting $(i, j, k) = (1, 1, 2)$ gives

$$\begin{aligned} \Gamma^1\Gamma^1\Gamma^2 &= b\frac{1}{2}(\delta^{11}\Gamma^2 - \delta^{12}\Gamma^1) \\ \Rightarrow \Gamma^2 &= \frac{1}{2}b\Gamma^2 \Rightarrow b = 2. \end{aligned} \tag{A.10}$$

In a similar manner, we can set $(i, j, k) = (1, 2, 3)$ to obtain $a = 1$. Thus, we conclude:

$$\Gamma^i\Gamma^{jk} = \Gamma^{ijk} + 2\delta^{i[j}\Gamma^{k]}. \tag{A.11}$$

Next, consider $\Gamma^{klm}\Gamma^{ij}$. The expansion can be written:

$$\Gamma^{klm}\Gamma^{ij} = a\Gamma^{klmij} + b\delta^{[i}{}_{[k}\Gamma^{j]}{}_{lm]} + c\delta^{ij}_{[kl}\Gamma_{m]}. \tag{A.12}$$

Setting $(k, l, m, i, j) = (1, 2, 3, 4, 5)$ yields $a = 1$. To determine b, we set $(k, l, m, i, j) = (3, 2, 1, 1, 4)$, which gives

$$\begin{aligned} \Gamma^3\Gamma^2\Gamma^4 &= \frac{b}{2}(\delta^1_{[3}\Gamma^4{}_{21]} - \delta^4_{[3}\Gamma^1{}_{21]}) \\ &= \frac{b}{12}(\delta^1_1\Gamma^4{}_{32]} - \delta^1_1\Gamma^4{}_{23}) \\ \Rightarrow b &= 6. \end{aligned} \tag{A.13}$$

Finally, we set $(k,l,m,i,j)=(3,2,1,1,2)$ to obtain

$$\begin{aligned}\Gamma^3 &= c\delta^{12}_{[32}\Gamma_{1]} = \frac{c}{2}(\delta^1_{[3}\delta^2_2\Gamma_{1]} - \delta^2_{[3}\delta^1_2\Gamma_{1]}) \\ &= \frac{c}{12}(-\delta^1_1\delta^2_2\Gamma_{3]} - \delta^2_2\delta^1_1\Gamma_3) \\ \Rightarrow c &= -6. \end{aligned} \tag{A.14}$$

Inserting these coefficients gives the expansion

$$\Gamma^{klm}\Gamma^{ij} = \Gamma^{klmij} + 6\delta^{[i}_{[k}\Gamma^{j]}{}_{lm]} - 6\delta^{ij}_{[kl}\Gamma_{m]}. \tag{A.15}$$

Fierz identities

Since the products of γ- and Γ-matrices span the space of 2×2-matrices and 16×16-matrices respectively, we can use tensor products of these matrices as basis elements in an expansion of an arbitrary matrix in the theory.

To expand a general matrix, one would have to use all possible combinations of basis elements for the 2×2-matrices and the 16×16-matrices. For our purpuses, however, it is sufficient to expand a matrix of the form $\Psi^{Aa}_\alpha\Psi^{Bb}_\beta$, where A, B are (undotted) SO(8) matrix indices and α, β are SO(2,1) matrix indices. If there is an imposed antisymmetry in a and b, the number of terms in the expansion will be reduced. Only terms that are symmetric in both AB and $\alpha\beta$, or antisymmetric in both these pairs of indices, will survive.

Also, in this special case, we expand a matrix with two undotted SO(8) indices, which means that the expansion will contain no matrix with dotted and undotted indices mixed (a matrix in the top right or bottom left block of the 16×16-matrix). Now, we are left with all matrices in the basis that have two indices of the same kind, but we only need half of these to expand a matrix in the top left block (which corresponds to both indices undotted). We can obtain these matrices by using the projection operator $\frac{1}{2}(1+\Gamma^9)$. The table in the previous section strongly hints that the different sets of matrices (except Γ^{ijkl}) can be grouped pairwise. For example, Γ^{ij} and Γ^{ijklmn} would be similar sets of matrices, projected into different blocks of the 16×16-matrix. As it turns out, the projection operators $\frac{1}{2}(1+\Gamma^9)$ and $\frac{1}{2}(1-\Gamma^9)$ project out Γ^{ij} and Γ^{ijklmn} respectively . When it comes to Γ^{ijkl}, half of the matrices are projected into the top left block, and we define

$$\Gamma^{+ijkl} = \frac{1}{2}(1+\Gamma^9)\Gamma^{ijkl}. \tag{A.16}$$

In total, we now have 28 antisymmetric matrices Γ^{ij} and 35+1=36 symmetric matrices (Γ^{+ijkl} and C). Thus, this set spans the set of real 8×8-matrices, as expected.

Taking all the above circumstances into account, we can now write down the expansion:

$$\begin{aligned}\Psi^{Aa}_\alpha\Psi^{Bb}_\beta =& a^{ij}(C^{-1})_{\alpha\beta}(\Gamma^{ij})^{AB} + b_\mu(\gamma^\mu C^{-1})_{\alpha\beta}\tilde{C}^{AB} \\ &+ d^{ijkl}_\mu(\gamma^\mu C^{-1})_{\alpha\beta}(\Gamma^{+ijkl})^{AB}, \end{aligned} \tag{A.17}$$

where we have put a tilde on the SO(8) charge conjugation matrix to distinguish it from the SO(2,1) one. Let us now determine the coefficients. First, we multiply (A.17) by $C^{\alpha\beta}(\Gamma^{kl})_{AB}$. The left hand side is:

$$\Psi_\alpha^{Aa}\Psi_\beta^{Bb}C^{\alpha\beta}(\Gamma^{kl})_{AB} \quad = \quad \bar{\Psi}^a\Gamma^{kl}\Psi^b \tag{A.18}$$

and the right hand side

$$\begin{aligned} a^{ij}(C^{-1})_{\alpha\beta}(\Gamma^{ij})^{AB}C^{\alpha\beta}(\Gamma^{kl})_{AB} &= a^{ij}\mathrm{tr}\left[C(C^{-1})^T\right]\mathrm{tr}\left[(\Gamma^{ij})(\Gamma^{kl})^T\right] \\ &= a^{ij}(-2)(-\mathrm{tr}\left[(\Gamma^{ij})(\Gamma^{kl})\right]) \\ &= a^{ij}(-2)(2\delta^{ij}_{kl}\mathrm{tr}\left[\mathbb{1}_{8\times 8}\right]] \\ &= -32a^{ij}\delta^{ij}_{kl} \\ &= -32a^{kl}. \end{aligned} \tag{A.19}$$

This means that we have $a^{ij} = -\frac{1}{32}\bar{\Psi}^a\Gamma^{ij}\Psi^b$. Multiplying by $(C\gamma^\nu)^{\alpha\beta}\tilde{C}_{AB}$ gives on the left hand side

$$\Psi_\alpha^{Aa}\Psi_\beta^{Bb}(C\gamma^\nu)^{\alpha\beta}\tilde{C}_{AB} = \bar{\Psi}^a\gamma^\nu\Psi^b \tag{A.20}$$

and on the right hand side

$$\begin{aligned} b_\mu(\gamma^\mu C^{-1})_{\alpha\beta}\tilde{C}^{AB}(C\gamma^\nu)^{\alpha\beta}\tilde{C}_{AB} &= b_\mu\mathrm{tr}\left[\gamma^\mu C^{-1}C\gamma^\nu\right]\mathrm{tr}\left[\tilde{C}\right] \\ &= 8b_\mu\mathrm{tr}\left[\gamma^\mu\gamma^\nu\right] \\ &= 8b_\mu(\eta^{\mu\nu}\mathrm{tr}\left[\mathbb{1}_{2\times 2}\right]) \\ &= 16b^\nu, \end{aligned} \tag{A.21}$$

which implies $b_\mu = \frac{1}{16}\bar{\Psi}^a\gamma_\mu\Psi^b$.

Finally, we multiply the expansion by $(C\gamma^\nu)^{\alpha\beta}(\Gamma^{+mnpq})_{AB}$. In the same way as before, we get $\bar{\Psi}^a\gamma^\nu\Gamma^{+mnpq}\Psi^b$ on the left hand side. The right hand side is

$$\begin{aligned} & d_\mu^{ijkl}(\gamma^\mu C^{-1})_{\alpha\beta}(\Gamma^{+ijkl})^{AB}(C\gamma^\nu)^{\alpha\beta}(\Gamma^{+mnpq})_{AB} \\ = \;& d_\mu^{ijkl}\mathrm{tr}\left[\gamma^\mu\gamma^\nu\right]\mathrm{tr}\left[\Gamma^{+ijkl}(\Gamma^{+mnpq})^T\right] \\ = \;& 2d^{\nu ijkl}\mathrm{tr}\left[\frac{1}{2}(1+\Gamma^9)\Gamma^{ijkl}\Gamma^{mnpq}\right] \\ = \;& d^{\nu ijkl}\mathrm{tr}\left[(1+\Gamma^9)(4!\delta^{ijkl}_{mnpq}+\Gamma^9\epsilon^{ijklmnpq})\right] \\ = \;& d^{\nu ijkl}(4!\delta^{ijkl}_{mnpq}+\epsilon^{ijklmnpq})\mathrm{tr}\left[\mathbb{1}_{16\times 16}\right] \\ = \;& 32\cdot 4!d^{\nu mnpq}, \end{aligned} \tag{A.22}$$

which gives $d_\mu^{ijkl} = \frac{1}{32\cdot 4!}\bar{\Psi}^a\gamma_\mu\Gamma^{+ijkl}\Psi^b$. The expansion (A.17) can now be written:

$$\begin{aligned}\Psi^{Aa}_{\alpha}\Psi^{Bb}_{\beta} &= -\frac{1}{32}(\bar{\Psi}^a\Gamma^{ij}\Psi^b)(C^{-1})_{\alpha\beta}(\Gamma^{ij})^{AB} \\ &\quad +\frac{1}{16}(\bar{\Psi}^a\gamma_\mu\Psi^b)(\gamma^\mu C^{-1})_{\alpha\beta}\tilde{C}^{AB} \\ &\quad +\frac{1}{32\cdot 4!}(\bar{\Psi}^a\gamma_\mu\Gamma^{+ijkl}\Psi^b)(\gamma^\mu C^{-1})_{\alpha\beta}(\Gamma^{+ijkl})^{AB}. \end{aligned} \tag{A.23}$$

A.1.2 Variation of the Lagrangian

In this section we vary the BLG Lagrangian (2.8) with respect to the SUSY variations

$$\delta X^{ia} = i\bar{\epsilon}\Gamma^i\Psi^a, \tag{A.24}$$

$$\delta\Psi^a = D_\mu X^{ia}\gamma^\mu\Gamma^i\epsilon + \frac{1}{6}X^i{}_bX^j{}_cX^k{}_d\Gamma^{ijk}\epsilon f^{bcda}. \tag{A.25}$$

First, we note that taking the conjugate of (A.25) gives us

$$\begin{aligned}\delta\bar{\Psi}^a &= (\delta\Psi^a)^TC \\ &= \left(D_\mu X^{ia}\gamma^\mu\Gamma^i\epsilon + \frac{1}{6}X^i{}_bX^j{}_cX^k{}_d\Gamma^{ijk}\epsilon f^{bcda}\right)^T C \\ &= D_\mu X^{ia}\epsilon^T(\Gamma^i)^T(\gamma^\mu)^TC + \frac{1}{6}X^i{}_bX^j{}_cX^k{}_d\bar{\epsilon}(\Gamma^{ijk})^Tf^{bcda} \\ &= D_\mu X^{ia}\epsilon^T(-\Gamma^i)(-C\gamma^\mu) + \frac{1}{6}X^i{}_bX^j{}_cX^k{}_d\bar{\epsilon}\Gamma^{ijk}f^{bcda} \\ &= D_\mu X^{ia}\bar{\epsilon}\Gamma^i\gamma^\mu + \frac{1}{6}X^i{}_bX^j{}_cX^k{}_d\bar{\epsilon}\Gamma^{ijk}f^{bcda}. \end{aligned} \tag{A.26}$$

Furthermore, the transformation of the covariant derivative is

$$\begin{aligned}\delta(D_\mu X^{ia}) &= D_\mu(\delta X^{ia}) + (\delta\tilde{A}_\mu{}^a{}_b)X^{ib} \\ &= i\bar{\epsilon}\Gamma^iD_\mu\Psi^a + (\delta\tilde{A}_\mu{}^a{}_b)X^{ib} \end{aligned} \tag{A.27}$$

and

$$\begin{aligned}\delta(D_\mu\Psi^a) &= D_\mu(\delta\Psi^a) + (\delta\tilde{A}_\mu{}^a{}_b)\Psi^b \qquad (A.28) \\ &= D_\mu\left(D_\nu X^{ia}\gamma^\nu\Gamma^i\epsilon + \frac{1}{6}X^i{}_bX^j{}_cX^k{}_d\Gamma^{ijk}\epsilon f^{bcda}\right) + (\delta\tilde{A}_\mu{}^a{}_b)\Psi^b. \end{aligned}$$

Kinetic terms

The kinetic term for the scalar field is

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}(D_\mu X^{ia})(D^\mu X^i{}_a). \tag{A.29}$$

Varying this term we get

$$\begin{aligned}\delta\mathcal{L}_{\text{scalar}} &= -(D^\mu X^{ia})\delta(D_\mu X^i{}_a) \\ &= -(D^\mu X^i{}_a)\left(i\bar\epsilon\Gamma^i D_\mu\Psi^a + (\delta\tilde A_\mu{}^a{}_b)X^{ib}\right) \\ &= i(D^\mu D_\mu X^i{}_a)\bar\epsilon\Gamma^i\Psi^a - (D^\mu X^i{}_a)(\delta\tilde A_\mu{}^a{}_b)X^{ib}, \qquad \text{(A.30)}\end{aligned}$$

where we in the second step we have integrated by parts, obtaining an additional minus sign in the first term.
The Dirac term is given by:

$$\mathcal{L}_{\text{Dirac}} = \frac{i}{2}\bar\Psi^a\gamma^\mu D_\mu\Psi_a \qquad \text{(A.31)}$$

and its variation by

$$\begin{aligned}\delta\mathcal{L}_{\text{Dirac}} &= \frac{i}{2}(\delta\bar\Psi^a)\gamma^\mu D_\mu\Psi_a + \frac{i}{2}\bar\Psi^a\gamma^\mu\delta(D_\mu\Psi_a) \\ &= \frac{i}{2}(D_\nu X^{ia})\bar\epsilon\gamma^\nu\Gamma^i\gamma^\mu D_\mu\Psi_a + \\ &\quad +\frac{i}{12}X^i{}_bX^j{}_cX^k{}_d\bar\epsilon\Gamma^{ijk}f^{bcda}\gamma^\mu D_\mu\Psi_a \\ &\quad +\frac{i}{2}\bar\Psi^a\gamma^\mu D_\mu\left(D_\nu X^i{}_a\gamma^\nu\Gamma^i\epsilon + \frac{1}{6}X^i{}_bX^j{}_cX^k{}_d\Gamma^{ijk}\epsilon f^{bcd}{}_a\right) \\ &\quad +\frac{i}{2}\bar\Psi^a\gamma^\mu(\delta\tilde A_{\mu ab})\Psi^b \\ &= -\frac{i}{2}(D_\mu D_\nu X^{ia})\bar\epsilon\gamma^\nu\Gamma^i\gamma^\mu\Psi_a \\ &\quad +\frac{i}{12}X^i{}_bX^j{}_cX^k{}_d\bar\epsilon\Gamma^{ijk}f^{bcda}\gamma^\mu D_\mu\Psi_a \\ &\quad -\frac{i}{2}(D_\mu D_\nu X^{ia})\bar\epsilon\gamma^\nu\Gamma^i\gamma^\mu\Psi_a \\ &\quad +\frac{i}{12}X^i{}_bX^j{}_cX^k{}_d\bar\epsilon\Gamma^{ijk}f^{bcda}\gamma^\mu D_\mu\Psi_a \\ &\quad +\frac{i}{2}\bar\Psi^a\gamma^\mu(\delta\tilde A_\mu{}^a{}_b)\Psi^b \\ &= -i(D_\mu D_\nu X^{ia})\bar\epsilon\gamma^\nu\Gamma^i\gamma^\mu\Psi_a \\ &\quad +\frac{i}{6}X^i{}_bX^j{}_cX^k{}_d\bar\epsilon\Gamma^{ijk}f^{bcda}\gamma^\mu D_\mu\Psi_a \\ &\quad +\frac{i}{2}\bar\Psi^a\gamma^\mu(\delta\tilde A_{\mu ab})\Psi^b \\ &= -i(D^\mu D_\mu X^{ia})\bar\epsilon\Gamma^i\Psi_a - i(D_\mu D_\nu X^{ia})\bar\epsilon\gamma^{\nu\mu}\Gamma^i\Psi_a \qquad \text{(A.32)} \\ &\quad +\frac{i}{6}X^i{}_bX^j{}_cX^k{}_d\bar\epsilon\Gamma^{ijk}f^{bcda}\gamma^\mu D_\mu\Psi_a \qquad \text{(A.33)} \\ &\quad +\frac{i}{2}\bar\Psi^a\gamma^\mu(\delta\tilde A_{\mu ab})\Psi^b. \qquad \text{(A.34)}\end{aligned}$$

In the second step, we have integrated by parts in the first term and flipped the third term. In the last step we have used $\gamma^\nu\gamma^\mu = \eta^{\mu\nu} + \gamma^{\nu\mu}$.

Adding $\delta\mathcal{L}_{\text{scalar}}$ and $\delta\mathcal{L}_{\text{Dirac}}$ together we see that the terms with contracted derivatives cancel, and we are left with

$$\begin{aligned}\delta\mathcal{L}_{\text{scalar}} + \delta\mathcal{L}_{\text{Dirac}} &= -i(D_\mu D_\nu X^{ia})\bar{\epsilon}\gamma^{\nu\mu}\Gamma^i\Psi_a + &\text{(A.35)}\\ &+\frac{i}{6}X^i{}_bX^j{}_cX^k{}_d\bar{\epsilon}\Gamma^{ijk}f^{bcda}\gamma^\mu D_\mu\Psi_a + &\text{(A.36)}\\ &+\frac{i}{2}\bar{\Psi}^a\gamma^\mu(\delta\tilde{A}_{\mu ab})\Psi^b - &\text{(A.37)}\\ &-(D^\mu X^{ia})(\delta\tilde{A}_{\mu ab})X^{ib}. &\text{(A.38)}\end{aligned}$$

Chern-Simons term

The Chern-Simons term is given by

$$\mathcal{L}_{\text{CS}} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\right). \quad \text{(A.39)}$$

It is easiest to vary the two parts of the CS-term separately. Thus we have:

$$\begin{aligned}\delta\mathcal{L}^1_{CS} &= \delta\left(\frac{1}{2}\varepsilon^{\mu\nu\lambda}f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd}\right)\\ &= \frac{1}{2}f^{abcd}\varepsilon^{\mu\nu\lambda}\left((\delta A_{\mu ab})\partial_\nu A_{\lambda cd} + A_{\mu ab}\partial_\nu(\delta A_{\lambda cd})\right)\\ &= f^{abcd}\varepsilon^{\mu\nu\lambda}(\delta A_{\lambda ab})\partial_\mu A_{\nu cd} \qquad \text{(A.40)}\end{aligned}$$

and

$$\begin{aligned}\delta\mathcal{L}^2_{\text{CS}} =&\delta\left(\frac{1}{3}\varepsilon^{\mu\nu\lambda}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\right)\\ =&\frac{1}{3}\varepsilon^{\mu\nu\lambda}f^{cda}{}_g f^{efgb}\Big((\delta A_{\mu ab})A_{\nu cd}A_{\lambda ef}\\ &+ A_{\mu ab}(\delta A_{\nu cd})A_{\lambda ef} + A_{\mu ab}A_{\nu cd}(\delta A_{\lambda ef})\Big)\\ =&\frac{1}{3}\varepsilon^{\mu\nu\lambda}(\delta A_{\lambda ab})A_{\mu cd}A_{\nu ef}\left(f^{cda}{}_g f^{efgb} + f^{abe}{}_g f^{cdgf} + f^{efc}{}_g f^{abgd}\right). \qquad \text{(A.41)}\end{aligned}$$

Yukawa term

The Yukawa term is

$$\mathcal{L}_{\text{Yukawa}} = -\frac{i}{4}\bar{\Psi}_b\Gamma^{ij}X^i{}_cX^j{}_d\Psi_af^{abcd}. \qquad \text{(A.42)}$$

Varying this gives

$$\begin{aligned}
\delta\mathcal{L}_{\text{Yukawa}} =& -\frac{i}{4}(\delta\bar{\Psi}_b)\Gamma^{ij}X^i{}_cX^j{}_d\Psi_a f^{abcd} \\
&-\frac{i}{4}\bar{\Psi}_b\Gamma^{ij}X^i{}_cX^j{}_d(\delta\Psi_a)f^{abcd} \\
&-\frac{i}{4}\bar{\Psi}_b\Gamma^{ij}(\delta X^i{}_c)X^j{}_d\Psi_a f^{abcd} \\
&-\frac{i}{4}\bar{\Psi}_b\Gamma^{ij}X^i{}_c(\delta X^j{}_d)\Psi_a f^{abcd} \\
=& -\frac{i}{2}(\delta\bar{\Psi}_b)\Gamma^{ij}X^i{}_cX^j{}_d\Psi_a f^{abcd} \\
&-\frac{i}{2}\bar{\Psi}_b\Gamma^{ij}(\delta X^i{}_c)X^j{}_d\Psi_a f^{abcd} \\
=& -\frac{i}{2}\left(D_\mu X^k{}_b\bar{\epsilon}\Gamma^k\gamma^\mu + \frac{1}{6}X^k{}_eX^l{}_fX^m{}_g\bar{\epsilon}\Gamma^{klm}f^{efg}{}_b\right)\Gamma^{ij}X^i{}_cX^j{}_d\Psi_a f^{abcd} \\
&-\frac{i}{2}\bar{\Psi}_b\Gamma^{ij}(i\bar{\epsilon}\Gamma^i\Psi_c)X^j{}_d\Psi_a f^{abcd} \\
=& -\frac{i}{2}\bar{\epsilon}\Gamma^k\Gamma^{ij}\gamma^\mu\Psi_aX^i{}_cX^j{}_dD_\mu X^k{}_bf^{abcd} && \text{(A.43)} \\
&-\frac{i}{12}X^i{}_cX^j{}_dX^k{}_eX^l{}_fX^m{}_g\bar{\epsilon}\Gamma^{klm}\Gamma^{ij}\Psi_af^{efg}{}_bf^{abcd} && \text{(A.44)} \\
&+\frac{1}{2}(\bar{\epsilon}\Gamma^i\Psi_c)(\bar{\Psi}_b\Gamma^{ij}\Psi_a)X^j{}_df^{abcd}. && \text{(A.45)}
\end{aligned}$$

Potential term

We have

$$V = \frac{1}{12}f^{abcd}f^{efg}{}_dX^i{}_aX^j{}_bX^k{}_cX^i{}_eX^j{}_fX^k{}_g. \tag{A.46}$$

Since the product of scalar fields is symmetric under $a \leftrightarrow e$, $b \leftrightarrow f$ and $c \leftrightarrow g$ the variation is

$$\begin{aligned}
\delta V =& \frac{1}{6}f^{abcd}f^{efg}{}_d\Big((\delta X^i{}_a)X^j{}_bX^k{}_cX^i{}_eX^j{}_fX^k{}_g+ \\
&+X^i{}_a(\delta X^j{}_b)X^k{}_cX^i{}_eX^j{}_fX^k{}_g + X^i{}_aX^j{}_b(\delta X^k{}_c)X^i{}_eX^j{}_fX^k{}_g\Big). && \text{(A.47)}
\end{aligned}$$

Also, the product of structure constants $f^{abcd}f^{efg}{}_d$ is symmetric under $a, e \leftrightarrow b, f$ and $a, e \leftrightarrow c, g$. Thus, we can simplify (A.47) to

$$\begin{aligned}
\delta V &= \frac{1}{2}(\delta X^i{}_a)X^j{}_bX^k{}_cX^i{}_eX^j{}_fX^k{}_gf^{abcd}f^{efg}{}_d \\
&= \frac{i}{2}\bar{\epsilon}\Gamma^iX^j{}_bX^k{}_cX^i{}_eX^j{}_fX^k{}_g\Psi_af^{abcd}f^{efg}{}_d. && \text{(A.48)}
\end{aligned}$$

A.1.3 Cancellation of Terms

In this section, we show that all the terms from the variation in the previous section cancel.

Two-derivative terms

The two derivative part (A.35) of the kinetic terms can be rewritten:

$$\begin{aligned}
& -i(D_\mu D_\nu X^{ia})\bar{\epsilon}\gamma^{\nu\mu}\Gamma^i\Psi_a \\
= \;& i(D_\mu D_\nu X^{ia})\bar{\epsilon}\gamma^{\mu\nu}\Gamma^i\Psi_a \\
= \;& i\left(\partial_\mu\partial_\nu X^{ia} + \tilde{A}_\mu{}^a{}_b\partial_\nu X^{ib} + \partial_\mu(\tilde{A}_\nu{}^a{}_b X^{ib}) + \tilde{A}_\mu{}^a{}_c\tilde{A}_\nu{}^c{}_b X^{ib}\right)\bar{\Psi}_a\gamma^{\mu\nu}\Gamma^i\epsilon \\
= \;& i\left(\tilde{A}_\mu{}^a{}_b\partial_\nu X^{ib} + (\partial_\mu\tilde{A}_\nu{}^a{}_b)X^{ib} + \tilde{A}_\nu{}^a{}_b\partial_\mu X^{ib} + \tilde{A}_\mu{}^a{}_c\tilde{A}_\nu{}^c{}_b X^{ib}\right)\bar{\Psi}_a\gamma^{\mu\nu}\Gamma^i\epsilon \\
= \;& i\left(\partial_\mu\tilde{A}_\nu{}^a{}_b + \tilde{A}_\mu{}^a{}_c\tilde{A}_\nu{}^c{}_b\right)X^{ib}\bar{\Psi}_a\gamma^{\mu\nu}\Gamma^i\epsilon \\
= \;& i\varepsilon^{\mu\nu\lambda}\left(\partial_\mu\tilde{A}_\nu{}^a{}_b + \tilde{A}_\mu{}^a{}_c\tilde{A}_\nu{}^c{}_b\right)X^{ib}\bar{\Psi}_a\gamma_\lambda\Gamma^i\epsilon \\
= \;& i\varepsilon^{\mu\nu\lambda}\left(\partial_\mu A_{\nu cd}f^{cdab} + A_{\mu de}A_{\nu fg}f^{dea}{}_c f^{fgcb}\right)X^i{}_b\bar{\Psi}_a\gamma_\lambda\Gamma^i\epsilon, \qquad \text{(A.49)}
\end{aligned}$$

where we in the last step used (A.6).

Let us now compare (A.49) to the variation of the Chern-Simons term. It is easily seen that (A.40) will cancel against the first term in (A.49) if we set

$$\delta A_{\lambda ab} = -iX^i{}_b\bar{\Psi}_a\gamma_\lambda\Gamma^i\epsilon. \qquad \text{(A.50)}$$

Using the fundamental identity for the structure constants, we can also show that (A.41) cancels against the second term in (A.49) for this choice of $\delta A_{\lambda ab}$. We can rewrite (A.50) as

$$\delta\tilde{A}_\mu{}^a{}_b = i\bar{\epsilon}X^i{}_c\Gamma^i\gamma_\mu\Psi_d f^{cda}{}_b. \qquad \text{(A.51)}$$

Equation (A.51) is the transformation rule of the gauge field required for SUSY invariance of the Lagrangian.

One-derivative terms

From the kinetic terms we have

$$\begin{aligned}
\frac{i}{6}X^i{}_bX^j{}_cX^k{}_d\bar{\epsilon}\Gamma^{ijk}f^{bcda}\gamma^\mu D_\mu\Psi_a =& -\frac{i}{6}D_\mu(X^i{}_bX^j{}_cX^k{}_d)\bar{\epsilon}\Gamma^{ijk}\gamma^\mu\Psi_a f^{bcda} \\
=& -\frac{i}{2}(D_\mu X^i{}_b)X^j{}_cX^k{}_d\bar{\epsilon}\Gamma^{ijk}\gamma^\mu\Psi_a f^{bcda} \\
=& \frac{i}{2}(D_\mu X^i{}_b)X^j{}_cX^k{}_d\bar{\epsilon}\Gamma^{ijk}\gamma^\mu\Psi_a f^{abcd} \qquad \text{(A.52)}
\end{aligned}$$

and

$$\begin{aligned}
-(D^\mu X^i{}_a)(\delta\tilde{A}_\mu{}^a{}_b)X^{ib} =& -(D^\mu X^i{}_a)(i\bar{\epsilon}X^k{}_c\Gamma^k\gamma_\mu\Psi_d f^{cda}{}_b)X^{ib} \\
=& -i(D_\mu X^i{}_a)X^i{}_bX^k{}_c\bar{\epsilon}\Gamma^k\gamma^\mu\Psi_d f^{cdab} \\
=& -i(D_\mu X^i{}_a)X^i{}_bX^k{}_c\bar{\epsilon}\Gamma^k\gamma^\mu\Psi_d f^{abcd} \\
=& \; i(D_\mu X^i{}_b)X^j{}_cX^k{}_d\bar{\epsilon}\Gamma^k\delta^{ij}\gamma^\mu\Psi_a f^{abcd}. \qquad \text{(A.53)}
\end{aligned}$$

The contribution from the Yukawa term is

$$\begin{aligned} & -\frac{i}{2}\bar{\epsilon}\Gamma^k\Gamma^{ij}\gamma^\mu\Psi_a X^i{}_c X^j{}_d D_\mu X^k{}_b f^{abcd} \\ = & -\frac{i}{2}(D_\mu X^i{}_b) X^j{}_c X^k{}_d \bar{\epsilon}\Gamma^i\Gamma^{jk}\gamma^\mu\Psi_a f^{abcd}. \end{aligned} \tag{A.54}$$

Adding all these terms together, we arrive at

$$i(D_\mu X^i{}_b) X^j{}_c X^k{}_d \bar{\epsilon}\gamma^\mu \left(\Gamma^k\delta^{ij} + \frac{1}{2}\Gamma^{ijk} - \frac{1}{2}\Gamma^i\Gamma^{jk}\right)\Psi_a f^{abcd}. \tag{A.55}$$

The expression inside the parenthesis can be rewritten using (A.11):

$$\begin{aligned} \Gamma^k\delta^{ij} + \frac{1}{2}\Gamma^{ijk} - \frac{1}{2}\Gamma^i\Gamma^{jk} &= \Gamma^k\delta^{ij} + \frac{1}{2}\Gamma^{ijk} - \frac{1}{2}(\Gamma^{ijk} + 2\delta^{i[j}\Gamma^{k]}) \\ &= \Gamma^k\delta^{ij} - \delta^{i[j}\Gamma^{k]}. \end{aligned} \tag{A.56}$$

Since we have an imposed antisymmetry in j and k from the scalar fields outside the parenthesis, these two terms cancel.

Terms without derivatives

From the kinetic terms we have

$$\begin{aligned} \frac{i}{2}\bar{\Psi}_a\gamma^\mu(\delta\tilde{A}_\mu{}^a{}_b)\Psi^b &= \frac{i}{2}\bar{\Psi}_a\gamma^\mu(i\bar{\epsilon}X^i{}_c\Gamma^i\gamma_\mu\Psi_d f^{cda}{}_b)\Psi^b \\ &= -\frac{1}{2}X^i{}_c(\bar{\Psi}_a\gamma^\mu\Psi^b)(\bar{\epsilon}\Gamma^i\gamma_\mu\Psi_d) f^{cda}{}_b \end{aligned} \tag{A.57}$$

and from the potential

$$\begin{aligned} -\delta V &= -\frac{i}{2}\bar{\epsilon}\Gamma^i X^j{}_b X^k{}_c X^i{}_e X^j{}_f X^k{}_g \Psi_a f^{abcd} f^{efg}{}_d \\ &= -\frac{i}{2} X^j{}_b X^k{}_c X^i{}_e X^j{}_f X^k{}_g \bar{\epsilon}\Gamma^i\Psi_a f^{abcd} f^{efg}{}_d \\ &= -\frac{i}{2} X^i{}_b X^j{}_c X^k{}_e X^l{}_f X^m{}_g \bar{\epsilon}\Gamma^k\delta^{il}\delta^{jm}\Psi_a f^{abcd} f^{efg}{}_d. \end{aligned} \tag{A.58}$$

The contributions from the Yukawa term are

$$\begin{aligned} & -\frac{i}{12}X^i{}_c X^j{}_d X^k{}_e X^l{}_f X^m{}_g \bar{\epsilon}\Gamma^{klm}\Gamma^{ij}\Psi_a f^{efg}{}_b f^{abcd} \\ = & -\frac{i}{12}X^i{}_b X^j{}_c X^k{}_e X^l{}_f X^m{}_g \bar{\epsilon}\Gamma^{klm}\Gamma^{ij}\Psi_a f^{abcd} f^{efg}{}_d \end{aligned} \tag{A.59}$$

and

$$\frac{1}{2}(\bar{\epsilon}\Gamma_i\Psi_c)(\bar{\Psi}_b\Gamma^{ij}\Psi_a)X^j{}_d f^{abcd}. \tag{A.60}$$

Here, we can identify two types of terms. (A.57) and (A.60) are products of fermion bilinears and we will have to use the Fierz identities to make them cancel. (A.58) and (A.59) are more straightforward, so let us start with these. The sum of the two terms is

$$-\frac{i}{2}X^i{}_bX^j{}_cX^k{}_eX^l{}_fX^m{}_g\bar{\epsilon}\left(\Gamma^k\delta^{il}\delta^{jm}+\frac{1}{6}\Gamma^{klm}\Gamma^{ij}\right)\Psi_af^{abcd}f^{efg}{}_d. \quad (A.61)$$

Using (A.15) and the imposed antisymmetries we can rewrite the expression inside the parenthesis as

$$\delta^{ij}_{[lm}\Gamma_{k]}+\frac{1}{6}\Gamma^{klmij}+\delta^{[i}_{[k}\Gamma^{j]}{}_{lm]}-\delta^{ij}_{[kl}\Gamma_{m]} \quad = \quad \frac{1}{6}\Gamma^{klmij}+\delta^{[i}_{[k}\Gamma^{j]}{}_{lm]}. \quad (A.62)$$

Let us first concentrate on the Γ^{klmij}-term. Since we have a total antisymmetry in all the vector indices, we have a total antisymmetry also in $bcefg$. Thus, the product of structure constants in (A.61) vanishes by the fundamental identity (2.4). Next, we turn to the $\delta^{[i}_{[k}\Gamma^{j]}{}_{lm]}$-term, which because of the imposed antisymmetry in the corresponding indices can be written $\delta^{ik}\Gamma^{jlm}$, implying a symmetry in be and an antisymmetry in cfg. Now, using the fundamental identity (2.3), we can show this term to vanish.

Fierzing the remaining terms

Let us finally turn to the terms containing products of fermion bilinears. They are:

$$\begin{aligned}&-\frac{1}{2}X^i{}_c(\bar{\Psi}_a\gamma^\mu\Psi^b)(\bar{\epsilon}\Gamma^i\gamma_\mu\Psi_d)f^{cda}{}_b+\frac{1}{2}(\bar{\epsilon}\Gamma_i\Psi_c)(\bar{\Psi}_b\Gamma^{ij}\Psi_a)X^j{}_df^{abcd}\\ =\ &\frac{1}{2}X^j{}_c\Big((\bar{\Psi}_a\Gamma^{ij}\Psi_b)(\bar{\epsilon}\Gamma^i\Psi_d)-(\bar{\Psi}_a\gamma^\mu\Psi_b)(\bar{\epsilon}\gamma_\mu\Gamma^j\Psi_d)\Big)f^{abcd}. \quad (A.63)\end{aligned}$$

If we write out the SO(8) $(A, B, ...)$ and SO(2,1) $(\alpha, \beta, ...)$ matrix indices explicitly, the first term in the parenthesis is

$$\Psi^A_{\alpha a}C^{\alpha\beta}(\Gamma^{ij})_{AB}\Psi^B_{\beta b}\epsilon^C_{\gamma c}C^{\gamma\delta}(\Gamma^i)_{CD}\Psi^D_{\delta d}. \quad (A.64)$$

Let us now expand $\Psi^A_{\alpha a}\Psi^D_{\delta d}$ using (A.23). Inserting the expansion into the above expression yields

$$\begin{aligned}&\Big(-\frac{1}{32}(\bar{\Psi}_a\Gamma^{kl}\Psi_d)(C^{-1})_{\alpha\delta}(\Gamma^{kl})^{AD}+\frac{1}{16}(\bar{\Psi}_a\gamma_\mu\Psi_d)(\gamma^\mu C^{-1})_{\alpha\delta}\tilde{C}^{AD}\\ &+\frac{1}{32\cdot 4!}(\bar{\Psi}_a\gamma_\mu\Gamma^{+klmn}\Psi_d)(\gamma^\mu C^{-1})_{\alpha\delta}(\Gamma^{+klmn})^{AD}\Big)C^{\alpha\beta}(\Gamma^{ij})_{AB}\Psi^B_{\beta b}\epsilon^C_\gamma C^{\gamma\delta}(\Gamma^i)_{CD}\\ =&-\frac{1}{32}(\bar{\Psi}_a\Gamma^{kl}\Psi_d)\epsilon^C_\gamma C^{\gamma\delta}C_{\delta\alpha}C^{\alpha\beta}(\Gamma^i)_{CD}(-\Gamma^{kl})^{DA}(\Gamma^{ij})_{AB}\Psi^B_{\beta b}\\ &+\frac{1}{16}(\bar{\Psi}_a\gamma_\mu\Psi_d)\epsilon^C_\gamma C^{\gamma\delta}(\gamma^\mu C^{-1})_{\delta\alpha}C^{\alpha\beta}(\Gamma^i)_{CD}\tilde{C}^{DA}(\Gamma^{ij})_{AB}\Psi^B_{\beta b}\\ &+\frac{1}{32\cdot 4!}(\bar{\Psi}_a\gamma_\mu\Gamma^{+klmn}\Psi_d)\epsilon^C_{\gamma c}C^{\gamma\delta}(\gamma^\mu C^{-1})_{\delta\alpha}C^{\alpha\beta}(\Gamma^i)_{CD}(\Gamma^{+klmn})^{DA}(\Gamma^{ij})_{AB}\Psi^B_{\beta b}\\ =&-\frac{1}{32}(\bar{\Psi}_a\Gamma^{kl}\Psi_d)(\bar{\epsilon}\Gamma^i\Gamma^{kl}\Gamma^{ij}\Psi_b)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{16}(\bar{\Psi}_a\gamma_\mu\Psi_d)(\bar{\epsilon}\gamma^\mu\Gamma^i\Gamma^{ij}\Psi_b) \\
&+ \frac{1}{32\cdot 4!}(\bar{\Psi}_a\gamma_\mu\Gamma^{+klmn}\Psi_d)(\bar{\epsilon}\gamma^\mu\Gamma^i\Gamma^{+klmn}\Gamma^{ij}\Psi_b).
\end{aligned} \tag{A.65}
$$

In the same way, the second term in the parenthesis of (A.63) can be written

$$
\Psi^A_{\alpha a}(C\gamma^\mu)^{\alpha\beta}\tilde{C}_{AB}\Psi^B_{\beta b}\epsilon^C_\gamma(C\gamma_\mu)^{\gamma\delta}(\Gamma^j)_{CD}\Psi^D_{\delta d} \tag{A.66}
$$

and by similar calculations we can rewrite this as

$$
\begin{aligned}
- &\quad \frac{1}{32}(\bar{\Psi}_a\Gamma^{kl}\Psi_d)(\bar{\epsilon}\gamma_\mu\gamma^\mu\Gamma^j\Gamma^{kl}\Psi_b) \\
+ &\quad \frac{1}{16}(\bar{\Psi}_a\gamma_\nu\Psi_d)(\bar{\epsilon}\gamma_\mu\gamma^\nu\gamma^\mu\Gamma^j\Psi_b) \\
+ &\quad \frac{1}{32\cdot 4!}(\bar{\Psi}_a\gamma_\nu\Gamma^{+klmn}\Psi_d)(\bar{\epsilon}\gamma_\mu\gamma^\nu\gamma^\mu\Gamma^j\Gamma^{+klmn}\Psi_b).
\end{aligned} \tag{A.67}
$$

Adding all these terms together we arrive at

$$
\begin{aligned}
& (\bar{\Psi}_a\Gamma^{ij}\Psi_b)(\bar{\epsilon}\Gamma^i\Psi_d) && \text{(A.68)} \\
- \;& (\bar{\Psi}_a\gamma^\mu\Psi_b)(\bar{\epsilon}\gamma_\mu\Gamma^j\Psi_d) && \text{(A.69)} \\
= \;& -\frac{1}{32}(\bar{\Psi}_a\Gamma^{kl}\Psi_d)(\bar{\epsilon}\Gamma^i\Gamma^{kl}\Gamma^{ij}\Psi_b) && \text{(A.70)} \\
& +\frac{1}{16}(\bar{\Psi}_a\gamma_\nu\Psi_d)(\bar{\epsilon}\gamma^\nu\Gamma^i\Gamma^{ij}\Psi_b) && \text{(A.71)} \\
& +\frac{1}{32\cdot 4!}(\bar{\Psi}_a\gamma_\mu\Gamma^{+klmn}\Psi_d)(\bar{\epsilon}\gamma^\mu\Gamma^i\Gamma^{+klmn}\Gamma^{ij}\Psi_b) && \text{(A.72)} \\
& +\frac{1}{32}(\bar{\Psi}_a\Gamma^{kl}\Psi_d)(\bar{\epsilon}\gamma_\mu\gamma^\mu\Gamma^j\Gamma^{kl}\Psi_b) && \text{(A.73)} \\
& -\frac{1}{16}(\bar{\Psi}_a\gamma_\nu\Psi_d)(\bar{\epsilon}\gamma_\mu\gamma^\nu\gamma^\mu\Gamma^j\Psi_b) && \text{(A.74)} \\
& -\frac{1}{32\cdot 4!}(\bar{\Psi}_a\gamma_\nu\Gamma^{+klmn}\Psi_d)(\bar{\epsilon}\gamma_\mu\gamma^\nu\gamma^\mu\Gamma^j\Gamma^{+klmn}\Psi_b). && \text{(A.75)}
\end{aligned}
$$

To simplify this expression, we first need to calculate some additional identities for products of gamma matrices. First, we note that $\gamma_\mu\gamma^\mu = 3$ and $\gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu(2\eta^{\mu\nu} - \gamma^\mu\gamma^\nu) = 2\gamma^\nu - 3\gamma^\nu = -\gamma^\nu$. Next, we insert $i = j$ into equation (A.11) to obtain $\Gamma^i\Gamma^{ik} = 7\Gamma^k$. Using these identities, the terms (A.71) and (A.74) above can now be rewritten

$$
\begin{aligned}
& \frac{1}{16}(\bar{\Psi}_a\gamma_\nu\Psi_d)(\bar{\epsilon}\gamma^\nu\Gamma^i\Gamma^{ij}\Psi_b) \\
& -\frac{1}{16}(\bar{\Psi}_a\gamma_\nu\Psi_d)(\bar{\epsilon}\gamma_\mu\gamma^\nu\gamma^\mu\Gamma^j\Psi_b \\
= \;& \frac{7}{16}(\bar{\Psi}_a\gamma_\nu\Psi_d)(\bar{\epsilon}\gamma^\nu\Gamma^j\Psi_b) \\
& +\frac{1}{16}(\bar{\Psi}_a\gamma_\nu\Psi_d)(\bar{\epsilon}\gamma^\nu\Gamma^j\Psi_b) \\
= \;& \frac{1}{2}(\bar{\Psi}_a\gamma_\nu\Psi_d)(\bar{\epsilon}\gamma^\nu\Gamma^j\Psi_b).
\end{aligned} \tag{A.76}
$$

Now consider $\Gamma^i\Gamma^{kl}\Gamma^{ij}$, which we can rewrite as

$$\begin{aligned}\Gamma^i\Gamma^{kl}\Gamma^{ij} &= \Gamma^i\Gamma^{kl}(\Gamma^i\Gamma^j - \delta^{ij})\\ &= (6\Gamma^{kl} - 2\Gamma^{kl})\Gamma^j - \Gamma^i\Gamma^{kl}\delta^{ij}\\ &= 4\Gamma^{kl}\Gamma^j - \Gamma^j\Gamma^{kl}.\end{aligned}\tag{A.77}$$

Using this, the terms (A.70) and (A.73) can be rewritten as

$$\begin{aligned}&-\frac{1}{32}(\bar\Psi_a\Gamma^{kl}\Psi_d)(\bar\epsilon\Gamma^i\Gamma^{kl}\Gamma^{ij}\Psi_b) + \frac{1}{32}(\bar\Psi_a\Gamma^{kl}\Psi_d)(\bar\epsilon\gamma_\mu\gamma^\mu\Gamma^j\Gamma^{kl}\Psi_b)\\ =\ &-\frac{1}{32}(\bar\Psi_a\Gamma^{kl}\Psi_d)(\bar\epsilon(4\Gamma^{kl}\Gamma^j - \Gamma^j\Gamma^{kl})\Psi_b) + \frac{3}{32}(\bar\Psi_a\Gamma^{kl}\Psi_d)(\bar\epsilon\Gamma^j\Gamma^{kl}\Psi_b)\\ =\ &\frac{4}{32}(\bar\Psi_a\Gamma^{kl}\Psi_d)(\bar\epsilon(\Gamma^j\Gamma^{kl} - \Gamma^{kl}\Gamma^j)\Psi_b)\\ =\ &\frac{4}{32}(\bar\Psi_a\Gamma^{kl}\Psi_d)(\bar\epsilon(\Gamma^{jkl} + 2\delta^{j[k}\Gamma^{l]} - \Gamma^{klj} + 2\delta^{j[k}\Gamma^{l]})\Psi_b)\\ =\ &\frac{16}{32}(\bar\Psi_a\Gamma^{kl}\Psi_d)(\bar\epsilon(\delta^{jk}\Gamma^l)\Psi_b)\\ =\ &-\frac{1}{2}(\bar\Psi_a\Gamma^{lj}\Psi_d)(\bar\epsilon\Gamma^l\Psi_b).\end{aligned}\tag{A.78}$$

Finally, consider terms (A.72) and (A.75). We can write

$$\begin{aligned}\Gamma^i\Gamma^{+klmn}\Gamma^{ij} &= \frac{1}{2}(1+\Gamma^9)\Gamma^i\Gamma^{klmn}(\Gamma^i\Gamma^j - \delta^{ij})\\ &= -\frac{1}{2}\Gamma^j\Gamma^{klmn} - \frac{1}{2}\Gamma^j\Gamma^9\Gamma^{klmn}\\ &= -\Gamma^j\Gamma^{+klmn},\end{aligned}\tag{A.79}$$

which gives us

$$\begin{aligned}&\frac{1}{32\cdot 4!}(\bar\Psi_a\gamma_\mu\Gamma^{+klmn}\Psi_d)(\bar\epsilon\gamma^\mu\Gamma^i\Gamma^{+klmn}\Gamma^{ij}\Psi_b)\\ &-\frac{1}{32\cdot 4!}(\bar\Psi_a\gamma_\nu\Gamma^{+klmn}\Psi_d)(\bar\epsilon\gamma_\mu\gamma^\nu\gamma^\mu\Gamma^j\Gamma^{+klmn}\Psi_b)\\ =\ &-\frac{1}{32\cdot 4!}(\bar\Psi_a\gamma_\mu\Gamma^{+klmn}\Psi_d)(\bar\epsilon\gamma^\mu\Gamma^j\Gamma^{+klmn}\Psi_b)\\ &+\frac{1}{32\cdot 4!}(\bar\Psi_a\gamma_\nu\Gamma^{+klmn}\Psi_d)(\bar\epsilon\gamma^\nu\Gamma^j\Gamma^{+klmn}\Psi_b)\\ =\ &0.\end{aligned}\tag{A.80}$$

Thus, we have the result

$$(\bar\Psi_a\Gamma^{ij}\Psi_b)(\bar\epsilon\Gamma^i\Psi_d) - (\bar\Psi_a\gamma^\mu\Psi_b)(\bar\epsilon\gamma_\mu\Gamma^j\Psi_d)\tag{A.81}$$

$$= \frac{1}{2}(\bar\Psi_a\gamma_\nu\Psi_d)(\bar\epsilon\gamma^\nu\Gamma^j\Psi_b) - \frac{1}{2}(\bar\Psi_a\Gamma^{lj}\Psi_d)(\bar\epsilon\Gamma^l\Psi_b)\tag{A.82}$$

which implies

$$(\bar\Psi_a\Gamma^{ij}\Psi_b)(\bar\epsilon\Gamma^i\Psi_d) - (\bar\Psi_a\gamma^\mu\Psi_b)(\bar\epsilon\gamma_\mu\Gamma^j\Psi_d) = 0\tag{A.83}$$

This completes our proof that the BLG Lagrangian is invariant under transformations (2.11)-(2.13).

A.2 ABJM Lagrangian

In this section, we prove that the ABJM Lagrangian (2.18) is invariant under the SUSY transformations (2.24)-(2.25).

A.2.1 Dirac Matrices

The only Dirac matrices in the theory are the SO(2,1) ones. For these, we use the same conventions as in the BLG theory (see (A.1)).

A.2.2 Variation of the Lagrangian

Let us now vary (2.18) with respect to

$$\delta Z^A{}_a = i\bar{\epsilon}^{AB}\Psi_{Ba}, \tag{A.84}$$

$$\delta\Psi_{Bd} = \gamma^\mu D_\mu Z^A{}_d \epsilon_{AB} + f^{ab}{}_{cd} Z^C{}_a Z^D{}_b \bar{Z}_B{}^c \epsilon_{CD} - f^{ab}{}_{cd} Z^A{}_a Z^C{}_b \bar{Z}_C{}^c \epsilon_{AB}. \tag{A.85}$$

First, we write down the transformation rules for the complex conjugates $\bar{Z}_A{}^a$ and Ψ^{Bd}:

$$\delta\bar{Z}_A{}^a = (\delta Z^A{}_a)^* = i\bar{\epsilon}_{AB}\Psi^{Ba} \tag{A.86}$$

$$\begin{aligned}\delta\Psi^{Bd} = (\delta\Psi_{Bd})^* = \gamma^\mu D_\mu \bar{Z}_A{}^d \epsilon^{AB} + f^{cd}{}_{ab} \bar{Z}_C{}^a \bar{Z}_D{}^b Z^B{}_c \epsilon^{CD} \\ - f^{cd}{}_{ab} \bar{Z}_A{}^a \bar{Z}_C{}^b Z^C{}_c \epsilon^{AB}.\end{aligned} \tag{A.87}$$

The covariant derivatives (2.21)-(2.23) transform as

$$\delta(D_\mu Z^A{}_a) = D_\mu(\delta Z^A{}_a) - Z^A{}_b(\delta\tilde{A}_\mu{}^b{}_a) \tag{A.88}$$

$$\delta(D_\mu \bar{Z}_A{}^a) = D_\mu(\delta\bar{Z}_A{}^a) + (\delta\tilde{A}_\mu{}^a{}_b)\bar{Z}_A{}^b \tag{A.89}$$

$$\delta(D_\mu\Psi_{Bd}) = D_\mu(\delta\Psi_{Bd}) - \Psi_{Ba}(\delta\tilde{A}_\mu{}^a{}_d). \tag{A.90}$$

Kinetic terms

The kinetic term for the scalar field is

$$\mathcal{L}_{\text{scalar}} = (D_\mu Z^A{}_a)(D^\mu \bar{Z}_A{}^a). \tag{A.91}$$

Varying this term yields

$$\begin{aligned}\delta\mathcal{L}_{\text{scalar}} =& - \delta(D_\mu Z^A{}_a)(D^\mu\bar{Z}_A{}^a) - (D_\mu Z^A{}_a)\delta(D^\mu\bar{Z}_A{}^a) \\ =& - i\bar{\epsilon}^{AB}(D_\mu\Psi_{Ba})(D^\mu\bar{Z}_A{}^a) + Z^A{}_b(\delta\tilde{A}_\mu{}^b{}_a)(D^\mu\bar{Z}_a{}^A) \\ & - (D^\mu Z^A{}_a)i\bar{\epsilon}_{AB}D_\mu\Psi^{Ba} - (D^\mu Z^A{}_a)(\delta\tilde{A}_\mu{}^a{}_b)\bar{Z}_A{}^b \\ =& i(D_\mu D^\mu\bar{Z}_A{}^a)\bar{\epsilon}^{AB}\Psi_{Ba} + i(D_\mu D^\mu Z^A{}_a)\bar{\epsilon}_{AB}\Psi^{Ba} \\ & + (\delta\tilde{A}_\mu{}^a{}_b)(Z^A{}_a D^\mu\bar{Z}_A{}^b - \bar{Z}_A{}^b D^\mu Z^A{}_a),\end{aligned} \tag{A.92}$$

where we have integrated by parts in the last step.
The Dirac term is

$$\mathcal{L}_{\text{Dirac}} = -i\bar{\Psi}^{Aa}\gamma^\mu D_\mu \Psi_{Aa} \tag{A.93}$$

and the variation

$$\begin{aligned}
\delta\mathcal{L}_{\text{Dirac}} =& - i\left(\delta\bar{\Psi}^{Bd}\right)\gamma^\mu D_\mu\Psi_{Bd} - i\bar{\Psi}^{Bd}\gamma^\mu\delta\left(D_\mu\Psi_{Bd}\right) \\
& + i\left(D_\mu\bar{Z}_A{}^d\right)\bar{\epsilon}^{AB}\gamma^\mu\gamma^\nu D_\nu\Psi_{Bd} - \\
& - if^{cd}{}_{ab}\left[\bar{Z}_C{}^a\bar{Z}_D{}^b Z^B{}_c\bar{\epsilon}^{CD} - \bar{Z}_A{}^a\bar{Z}_C{}^b Z^C{}_c\bar{\epsilon}^{AB}\right]\gamma^\mu D_\mu\Psi_{Bd} \\
& - i\bar{\Psi}^{Bd}\gamma^\mu\left[D_\mu\left(\gamma^\nu D_\nu Z^A{}_d\epsilon_{AB} + f^{ab}{}_{cd}Z^C{}_a Z^D{}_b\bar{Z}_B{}^c\epsilon_{CD}\right.\right. \\
& \left.\left. - f^{ab}{}_{cd}Z^A{}_a Z^C{}_b\bar{Z}_C{}^c\epsilon_{AB}\right) - \Psi_{Ba}\left(\delta\tilde{A}_\mu{}^a{}_d\right)\right] \\
=& - i\left(D_\nu D_\mu\bar{Z}_A{}^d\right)\bar{\epsilon}^{AB}\gamma^\mu\gamma^\nu\Psi_{Bd} - i\left(D_\mu D_\nu Z^A{}_d\right)\bar{\Psi}^{Bd}\gamma^\mu\gamma^\nu\epsilon_{AB} \\
& - if^{cd}{}_{ab}\left[\bar{Z}_C{}^a\bar{Z}_D{}^b Z^B{}_c\bar{\epsilon}^{CD} - \bar{Z}_A{}^a\bar{Z}_C{}^b Z^C{}_c\bar{\epsilon}^{AB}\right]\gamma^\mu D_\mu\Psi_{Bd} \\
& - i\bar{\Psi}^{Bd}\gamma^\mu f^{ab}{}_{cd}D_\mu\left[Z^C{}_a Z^D{}_b\bar{Z}_B{}^c\epsilon_{CD} - Z^A{}_a Z^C{}_b\bar{Z}_C{}^c\epsilon_{AB}\right] \\
& + i\bar{\Psi}^{Bd}\gamma^\mu\Psi_{Ba}\left(\delta\tilde{A}_\mu{}^a{}_d\right).
\end{aligned} \tag{A.94}$$

Interaction terms

The third term in the Lagrangian is

$$\mathcal{L}_3 = -if^{ab}{}_{cd}\bar{\Psi}^{Bd}\Psi_{Ba}Z^E{}_b\bar{Z}_E{}^c \tag{A.95}$$

Varying this term gives

$$\begin{aligned}
\delta\mathcal{L}_3 =& - if^{ab}{}_{cd}\left[(\delta\bar{\Psi}^{Bd})\Psi_{Ba}Z^E{}_b\bar{Z}_E{}^c + \bar{\Psi}^{Bd}(\delta\Psi_{Ba})Z^E{}_b\bar{Z}_E{}^c\right. \\
& \left. + \bar{\Psi}^{Bd}\Psi_{Ba}\left(\delta Z^E{}_b\right)\bar{Z}_E{}^c + \bar{\Psi}^{Bd}\Psi_{Ba}Z^E{}_b(\delta\bar{Z}_E{}^c)\right] \\
=& - if^{ab}{}_{cd}\left[(\delta\bar{\Psi}^{Bd})\Psi_{Ba}Z^E{}_b\bar{Z}_E{}^c + \bar{\Psi}^{Bd}\Psi_{Ba}\left(\delta Z^E{}_b\right)\bar{Z}_E{}^c\right] + c.c. \\
=& - if^{ab}{}_{cd}\left[-(D_\mu\bar{Z}_A{}^d)\bar{\epsilon}^{AB}\gamma^\mu + f^{ed}{}_{fg}\bar{Z}_C{}^f\bar{Z}_D{}^g Z^B{}_e\bar{\epsilon}^{CD}\right. \\
& \left. - f^{ed}{}_{fg}\bar{Z}_A{}^f\bar{Z}_C{}^g Z^C{}_e\bar{\epsilon}^{AB}\right]\Psi_{Ba}Z^E{}_b\bar{Z}_E{}^c \\
& + f^{ab}{}_{cd}(\bar{\Psi}^{Bd}\Psi_{Ba})(\bar{\epsilon}^{EF}\Psi_{Fb})\bar{Z}_E{}^c + c.c.,
\end{aligned} \tag{A.96}$$

where $c.c.$ denotes the complex conjugate.
The fourth term is

$$\mathcal{L}_4 = 2if^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Ba}Z^B{}_b\bar{Z}_A{}^c \tag{A.97}$$

and the variation

$$\begin{aligned}\delta\mathcal{L}_4 =&2if^{ab}{}_{cd}\left[(\delta\bar{\Psi}^{Ad})\Psi_{Ba}Z^B{}_b\bar{Z}_A{}^c + \bar{\Psi}^{Ad}\Psi_{Ba}(\delta Z^B{}_b)\bar{Z}_A{}^c\right] + c.c.\\ =&2if^{ab}{}_{cd}\left[-\left(D_\mu\bar{Z}_D{}^d\right)\bar{\epsilon}^{DA}\gamma^\mu + f^{ed}{}_{fg}\bar{Z}_C{}^f\bar{Z}_D{}^g Z^A{}_e\bar{\epsilon}^{CD}\right.\\ &\left.-f^{ed}{}_{fg}\bar{Z}_B{}^f\bar{Z}_C{}^g Z^C{}_e\bar{\epsilon}^{BA}\right]\Psi_{Ea}Z^E{}_b\bar{Z}_A{}^c\\ &-2f^{ab}{}_{cd}(\bar{\Psi}^{Ad}\Psi_{Ba})(\bar{\epsilon}^{BD}\Psi_{Db})\bar{Z}_A{}^c + c.c.. \end{aligned} \tag{A.98}$$

For the fifth and sixth terms, $\mathcal{L}_5$ and $\mathcal{L}_6$, we first note that $\mathcal{L}_6 = (\mathcal{L}_5)^*$. The fifth term is

$$\mathcal{L}_5 = -\frac{i}{2}\epsilon_{ABCD}f^{ab}{}_{cd}\bar{\Psi}^{Ac}\Psi^{Bd}Z^C{}_a Z^D{}_b \tag{A.99}$$

and its variation

$$\begin{aligned}\delta\mathcal{L}_5 =&-\frac{i}{2}\epsilon_{ABCD}f^{ab}{}_{cd}\left[(\delta\bar{\Psi}^{Ac})\Psi^{Bd}Z^C{}_a Z^D{}_b + \bar{\Psi}^{Ac}(\delta\Psi^{Bd})Z^C{}_a Z^D{}_b\right.\\ &\left.+\bar{\Psi}^{Ac}\Psi^{Bd}(\delta Z^C{}_a)Z^D{}_b + \bar{\Psi}^{Ac}\Psi^{Bd}Z^C{}_a(\delta Z^D{}_b)\right]\\ =&-i\epsilon_{ABCD}f^{ab}{}_{cd}\left[(\delta\bar{\Psi}^{Ac})\Psi^{Bd}Z^C{}_a Z^D{}_b + \bar{\Psi}^{Ac}\Psi^{Bd}(\delta Z^C{}_a)Z^D{}_b\right]\\ =&-i\epsilon_{ABCD}f^{ab}{}_{cd}\left[-D_\mu\bar{Z}_E{}^c\bar{\epsilon}^{EA}\gamma^\mu + f^{ec}{}_{fg}\bar{Z}_E{}^f\bar{Z}_F{}^g Z^A{}_e\bar{\epsilon}^{EF}\right.\\ &\left.-f^{ec}{}_{fg}\bar{Z}_E{}^f\bar{Z}_F{}^g Z^F{}_e\bar{\epsilon}^{EA}\right]\Psi^{Bd}Z^C{}_a Z^D{}_b\\ &+\epsilon_{ABCD}f^{ab}{}_{cd}(\bar{\Psi}^{Ac}\Psi^{Bd})(\bar{\epsilon}^{CE}\Psi_{Ea})Z^D{}_b. \end{aligned} \tag{A.100}$$

Chern-Simons term

The Chern-Simons term is

$$\mathcal{L}_{CS} = \frac{1}{2}\epsilon^{\mu\nu\lambda}\left[f^{ab}{}_{cd}A_\mu{}^d{}_b\partial_\nu A_\lambda{}^c{}_a + \frac{2}{3}f^{bd}{}_{gc}f^{gf}{}_{ae}A_\mu{}^a{}_b A_\nu{}^c{}_d A_\lambda{}^e{}_f\right]. \tag{A.101}$$

Varying the derivative part yields

$$\begin{aligned}\delta\mathcal{L}^1_{CS} =&\frac{1}{2}\epsilon^{\mu\nu\lambda}f^{ab}{}_{cd}\left[(\delta A_\mu{}^d{}_b)\partial_\nu A_\lambda{}^c{}_a + A_\mu{}^d{}_b\partial_\nu(\delta A_\lambda{}^c{}_a)\right]\\ =&\epsilon^{\mu\nu\lambda}f^{ab}{}_{cd}(\delta A_\mu{}^d{}_b)\partial_\nu A_\lambda{}^c{}_a, \end{aligned} \tag{A.102}$$

where we, to rewrite the second term, have integrated by parts and used the antisymmetry of the indices. Varying the non-derivative part gives us

$$\begin{aligned}\delta\mathcal{L}^2_{CS} =&\frac{1}{3}\epsilon^{\mu\nu\lambda}f^{bd}{}_{gc}f^{gf}{}_{ae}\left[(\delta A_\mu{}^a{}_b)A_\nu{}^c{}_d A_\lambda{}^e{}_f +\right.\\ &\left.A_\mu{}^a{}_b(\delta A_\nu{}^c{}_d)A_\lambda{}^e{}_f + A_\mu{}^a{}_b A_\nu{}^c{}_d(\delta A_\lambda{}^e{}_f)\right]. \end{aligned} \tag{A.103}$$

To be able to further rewrite (A.103), we first use (2.15) to write

$$\begin{aligned}
f^{bd}{}_{gc}f^{fg}{}_{ae} =& f^{bd}{}_{g[c}f^{fg}{}_{a]e} + f^{bd}{}_{g(c}f^{fg}{}_{a)e} \\
=& f^{f[b}{}_{ge}f^{d]g}{}_{ca} + f^{bd}{}_{g(c}f^{fg}{}_{a)e} \\
=& \frac{1}{2}\left[f^{fb}{}_{ge}f^{dg}{}_{ca} - f^{fd}{}_{ge}f^{bg}{}_{ca} + f^{bd}{}_{gc}f^{fg}{}_{ae} + f^{bd}{}_{ga}f^{fg}{}_{ce}\right] \\
=& \frac{1}{2}\left[f^{bf}{}_{ge}f^{dg}{}_{ac} + f^{fd}{}_{ge}f^{bg}{}_{ac} + f^{bd}{}_{gc}f^{fg}{}_{ae} + f^{bd}{}_{ga}f^{fg}{}_{ce}\right]. \quad \text{(A.104)}
\end{aligned}$$

Using this, we now have

$$\begin{aligned}
& \epsilon^{\mu\nu\lambda}f^{bd}{}_{gc}f^{fg}{}_{ae}\left[(\delta A_\mu{}^a{}_b)A_\nu{}^c{}_d A_\lambda{}^e{}_f\right] \\
=& \frac{1}{2}\epsilon^{\mu\nu\lambda}f^{bd}{}_{gc}f^{fg}{}_{ae}\left[(\delta A_\mu{}^a{}_b)A_\nu{}^e{}_f A_\lambda{}^c{}_d + (\delta A_\mu{}^a{}_f)A_\nu{}^e{}_d A_\lambda{}^c{}_b\right. \\
& \left. +(\delta A_\mu{}^a{}_b)A_\nu{}^c{}_d A_\lambda{}^e{}_f + (\delta A_\mu{}^c{}_b)A_\nu{}^a{}_d A_\lambda{}^e{}_f\right] \\
=& \frac{1}{2}\epsilon^{\mu\nu\lambda}f^{bd}{}_{gc}f^{fg}{}_{ae}\left[(\delta A_\mu{}^e{}_f)A_\nu{}^a{}_b A_\lambda{}^c{}_d - (\delta A_\mu{}^c{}_d)A_\nu{}^a{}_b A_\lambda{}^e{}_f\right] \\
=& \frac{1}{2}\epsilon^{\mu\nu\lambda}f^{bd}{}_{gc}f^{fg}{}_{ae}\left[A_\mu{}^a{}_b A_\nu{}^c{}_d(\delta A_\lambda{}^e{}_f) + A_\mu{}^a{}_b(\delta A_\nu{}^c{}_d)A_\lambda{}^e{}_f\right]. \quad \text{(A.105)}
\end{aligned}$$

Using (A.105) in (A.103) we arrive at

$$\delta\mathcal{L}^2_{CS} = \epsilon^{\mu\nu\lambda}f^{bd}{}_{gc}f^{gf}{}_{ae}(\delta A_\mu{}^a{}_b)A_\nu{}^c{}_d A_\lambda{}^e{}_f, \quad \text{(A.106)}$$

which means that the total variation of the Chern-Simons term is

$$\begin{aligned}
\delta\mathcal{L}_{CS} =& \epsilon^{\mu\nu\lambda}\left[f^{ab}{}_{cd}(\delta A_\mu{}^d{}_b)\partial_\nu A_\lambda{}^c{}_a + f^{bd}{}_{gc}f^{gf}{}_{ae}(\delta A_\mu{}^a{}_b)A_\nu{}^c{}_d A_\lambda{}^e{}_f\right] \\
=& \epsilon^{\mu\nu\lambda}(\delta A_\lambda{}^a{}_b)\left[\partial_\mu\tilde{A}_\nu{}^b{}_a + \tilde{A}_\mu{}^b{}_c\tilde{A}_\nu{}^c{}_a\right]. \quad \text{(A.107)}
\end{aligned}$$

Scalar potential

The scalar potential is

$$\begin{aligned}
V =& \frac{2}{3}\Upsilon^{CD}{}_{Bd}\bar{\Upsilon}_{CD}{}^{Bd} \\
=& \frac{2}{3}f^{ab}{}_{cd}f^{ed}{}_{fg}\left[Z^C{}_a Z^D{}_b Z^B{}_e \bar{Z}_B{}^c\bar{Z}_C{}^f\bar{Z}_D{}^g + Z^C{}_a Z^D{}_b Z^E{}_e\bar{Z}_B{}^c\bar{Z}_E{}^g\delta^B_{[C}\bar{Z}_{D]}{}^f\right. \\
& \left. +Z^E{}_b Z^B{}_e\delta^{[C}_B Z^{D]}{}_a\bar{Z}_E{}^c\bar{Z}_C{}^f\bar{Z}_D{}^g + Z^E{}_b Z^F{}_e\delta^{[C}_B Z^{D]}{}_a\bar{Z}_E{}^c\bar{Z}_F{}^g\delta^B_{[C}\bar{Z}_{D]}{}^f\right] \\
=& \frac{2}{3}f^{ab}{}_{cd}f^{ed}{}_{fg}\left[X^{CDB}_{abe}\bar{X}^{cfg}_{BCD} + X^{CDE}_{abe}\bar{X}^{cfg}_{CDE} + X^{ECD}_{bea}\bar{X}^{cfg}_{ECD} + \frac{3}{2}X^{EFD}_{bea}\bar{X}^{cfg}_{EDF}\right] \\
=& \frac{2}{3}f^{ab}{}_{cd}f^{ed}{}_{fg}\left[X^{BCA}_{abe}\bar{X}^{cfg}_{ABC} + \frac{1}{2}X^{CAB}_{abe}\bar{X}^{cfg}_{ABC}\right], \quad \text{(A.108)}
\end{aligned}$$

where we have defined $X^{ABC}_{abc} \equiv Z^A{}_a Z^B{}_b Z^C{}_c$ and $\bar{X}^{abc}_{ABC} \equiv \bar{Z}_A{}^a \bar{Z}_B{}^b \bar{Z}_C{}^c$. Varying (A.108) yields

$$\begin{aligned}
-\delta V &= -\frac{2}{3} f^{ab}{}_{cd} f^{ed}{}_{fg} \left[\delta \left(X^{BCA}_{abe} \right) \bar{X}^{cfg}_{ABC} + \frac{1}{2} \delta \left(X^{CAB}_{abe} \right) \bar{X}^{cfg}_{ABC} \right] + c.c. \\
&= -\frac{2}{3} f^{ab}{}_{cd} f^{ed}{}_{fg} \bar{X}^{cfg}_{ABC} \left[2 \left(\delta Z^B{}_a \right) Z^C{}_b Z^A{}_e + \left(\delta Z^A{}_e \right) Z^B{}_a Z^C{}_b \right. \\
&\left. + \frac{1}{2} \left(\delta Z^C{}_a \right) Z^A{}_b Z^B{}_e + \frac{1}{2} \left(\delta Z^A{}_b \right) Z^C{}_a Z^B{}_e + \frac{1}{2} \left(\delta Z^B{}_e \right) Z^C{}_a Z^A{}_b \right] + c.c. \\
&= -\frac{1}{3} \bar{X}^{cfg}_{ABC} \left(\delta Z^A{}_a \right) Z^B{}_b Z^C{}_e \left[4 f^{ab}{}_{gd} f^{ed}{}_{cf} + 2 f^{be}{}_{cd} f^{ad}{}_{fg} \right. \\
&\left. + f^{ab}{}_{fd} f^{ed}{}_{gc} + f^{ba}{}_{cd} f^{ed}{}_{gf} + f^{be}{}_{gd} f^{ad}{}_{cf} \right] + c.c.. \qquad \text{(A.109)}
\end{aligned}$$

Let us now rewrite the expression inside the brackets using (2.15). We also note that the part outside the brackets imposes a symmetry under the permutation $bf \leftrightarrow eg$. Thus, we can write

$$\begin{aligned}
& 4 f^{ba}{}_{dg} f^{ed}{}_{cf} - 2 f^{be}{}_{dc} f^{ad}{}_{fg} + f^{ba}{}_{df} f^{ed}{}_{gc} - f^{ba}{}_{dc} f^{ed}{}_{gf} - f^{be}{}_{dg} f^{ad}{}_{cf} \\
=& 2 f^{ba}{}_{d[g} f^{ed}{}_{c]f} + 3 f^{ba}{}_{dg} f^{ed}{}_{cf} + 2 f^{be}{}_{d[c} f^{ad}{}_{g]f} - f^{be}{}_{dc} f^{ad}{}_{fg} + f^{ba}{}_{df} f^{ed}{}_{gc} \\
=& 2 f^{e[b}{}_{df} f^{a]d}{}_{gc} + 3 f^{ba}{}_{dg} f^{ed}{}_{cf} + 2 f^{a[b}{}_{df} f^{e]d}{}_{cg} - f^{be}{}_{dc} f^{ad}{}_{fg} + f^{ba}{}_{df} f^{ed}{}_{gc} \\
=& f^{eb}{}_{df} f^{ad}{}_{gc} - f^{ea}{}_{df} f^{bd}{}_{gc} + 3 f^{ba}{}_{dg} f^{ed}{}_{cf} + f^{ab}{}_{df} f^{ed}{}_{cg} \\
& - f^{ae}{}_{df} f^{bd}{}_{cg} - f^{be}{}_{dc} f^{ad}{}_{fg} + f^{ba}{}_{df} f^{ed}{}_{gc} \\
=& 3 f^{ba}{}_{dg} f^{ed}{}_{cf} + 2 f^{ea}{}_{df} f^{bd}{}_{cg} + 2 f^{ab}{}_{df} f^{ed}{}_{cg} + f^{eb}{}_{df} f^{ad}{}_{gc} - f^{be}{}_{dc} f^{ad}{}_{fg} \\
=& 3 f^{ba}{}_{dg} f^{ed}{}_{cf} + 4 f^{a[b}{}_{df} f^{e]d}{}_{cg} + f^{eb}{}_{df} f^{ad}{}_{gc} - f^{be}{}_{dc} f^{ad}{}_{fg} \\
=& 3 f^{ba}{}_{dg} f^{ed}{}_{cf} + 2 f^{be}{}_{dc} f^{ad}{}_{gf} - 2 f^{be}{}_{dg} f^{ad}{}_{cf} + f^{eb}{}_{df} f^{ad}{}_{gc} - f^{be}{}_{dc} f^{ad}{}_{fg} \\
=& 3 f^{ba}{}_{dg} f^{ed}{}_{cf} + 3 f^{be}{}_{dc} f^{ad}{}_{gf} + 3 f^{eb}{}_{df} f^{ad}{}_{gc}. \qquad \text{(A.110)}
\end{aligned}$$

Inserting (A.110) into (A.109) we arrive at

$$-\delta V = -\bar{X}^{cfg}_{ABC} \left(\delta Z^A{}_a \right) Z^B{}_b Z^C{}_e \left[f^{ba}{}_{dg} f^{ed}{}_{cf} + f^{be}{}_{dc} f^{ad}{}_{gf} + f^{eb}{}_{df} f^{ad}{}_{gc} \right]. \qquad \text{(A.111)}$$

A.2.3 Cancellation of Terms

In this section, we systematically show that all the terms from the variation in the previous section cancel.

Two-derivative terms

From the scalar kinetic term and the Dirac term we have

$$
\begin{aligned}
& i(D_\mu D^\mu \bar{Z}_A{}^a)\bar{\epsilon}^{AB}\Psi_{Ba} + i(D_\mu D^\mu Z^A{}_a)\bar{\epsilon}_{AB}\Psi^{Ba} \\
& - i\left(D_\nu D_\mu \bar{Z}_A{}^d\right)\bar{\epsilon}^{AB}\gamma^\mu\gamma^\nu\Psi_{Bd} - i\left(D_\mu D_\nu Z^A{}_d\right)\bar{\Psi}^{Bd}\gamma^\mu\gamma^\nu\epsilon_{AB} \\
=& i(D_\mu D^\mu \bar{Z}_A{}^a)\bar{\epsilon}^{AB}\Psi_{Ba} + i(D_\mu D^\mu Z^A{}_a)\bar{\epsilon}_{AB}\Psi^{Ba} \\
& - i\left(D_\nu D_\mu \bar{Z}_A{}^d\right)\bar{\epsilon}^{AB}(\eta^{\mu\nu} + \gamma^{\mu\nu})\Psi_{Bd} - i\left(D_\mu D_\nu Z^A{}_d\right)\bar{\Psi}^{Bd}(\eta^{\mu\nu} + \gamma^{\mu\nu})\epsilon_{AB} \\
=& - i\left(D_\nu D_\mu \bar{Z}_A{}^d\right)\bar{\epsilon}^{AB}\gamma^{\mu\nu}\Psi_{Bd} - i\left(D_\mu D_\nu Z^A{}_d\right)\bar{\Psi}^{Bd}\gamma^{\mu\nu}\epsilon_{AB} \\
=& i\left(D_\mu D_\nu \bar{Z}_A{}^d\right)\bar{\epsilon}^{AB}\gamma^{\mu\nu}\Psi_{Bd} + i\left(D_\mu D_\nu Z^A{}_d\right)\bar{\epsilon}_{AB}\gamma^{\mu\nu}\Psi^{Bd} \\
=& i\epsilon^{\mu\nu\lambda}\left(D_\mu D_\nu \bar{Z}_A{}^d\right)\bar{\epsilon}^{AB}\gamma_\lambda\Psi_{Bd} + i\epsilon^{\mu\nu\lambda}\left(D_\mu D_\nu Z^A{}_d\right)\bar{\epsilon}_{AB}\gamma_\lambda\Psi^{Bd}.
\end{aligned}
\tag{A.112}
$$

To proceed, let us calculate

$$
\begin{aligned}
D_\mu D_\nu Z^A{}_d =& D_\mu\left(\partial_\nu Z^A{}_d - Z^A{}_b \tilde{A}_\nu{}^b{}_d\right) \\
=& \partial_\mu\partial_\nu Z^A{}_d - \partial_\mu\left(Z^A{}_b\tilde{A}_\nu{}^b{}_d\right) - \left(\partial_\nu Z^A{}_b\right)\tilde{A}_\mu{}^b{}_d + Z^A{}_b\tilde{A}_\nu{}^b{}_c\tilde{A}_\mu{}^c{}_d \\
=& \partial_\mu\partial_\nu Z^A{}_d - \left(\partial_\mu Z^A{}_b\right)\tilde{A}_\nu{}^b{}_d - Z^A{}_b\partial_\mu\tilde{A}_\nu{}^b{}_d \\
& - \left(\partial_\nu Z^A{}_b\right)\tilde{A}_\mu{}^b{}_d + Z^A{}_b\tilde{A}_\nu{}^b{}_c\tilde{A}_\mu{}^c{}_d,
\end{aligned}
\tag{A.113}
$$

which means that we have

$$
D_{[\mu}D_{\nu]}Z^A{}_d = Z^A{}_b\left(-\partial_\mu\tilde{A}_\nu{}^b{}_d + \tilde{A}_\nu{}^b{}_c\tilde{A}_\mu{}^c{}_d\right)
\tag{A.114}
$$

and similarly

$$
D_{[\mu}D_{\nu]}\bar{Z}_A{}^d = \left(\partial_\mu\tilde{A}_\nu{}^d{}_b + \tilde{A}_\mu{}^d{}_c\tilde{A}_\nu{}^c{}_b\right)\bar{Z}_A{}^b.
\tag{A.115}
$$

Using (A.114) and (A.115), our expression in (A.112) reduces to

$$
\begin{aligned}
& i\epsilon^{\mu\nu\lambda}\left(\partial_\mu\tilde{A}_\nu{}^d{}_b + \tilde{A}_\mu{}^d{}_c\tilde{A}_\nu{}^c{}_b\right)\bar{Z}_A{}^b\bar{\epsilon}^{AB}\gamma_\lambda\Psi_{Bd} \\
& + i\epsilon^{\mu\nu\lambda}Z^A{}_b\left(-\partial_\mu\tilde{A}_\nu{}^b{}_d + \tilde{A}_\nu{}^b{}_c\tilde{A}_\mu{}^c{}_d\right)\bar{\epsilon}_{AB}\gamma_\lambda\Psi^{Bd} \\
=& i\epsilon^{\mu\nu\lambda}\left(\partial_\mu\tilde{A}_\nu{}^b{}_a + \tilde{A}_\mu{}^b{}_c\tilde{A}_\nu{}^c{}_a\right)\left(-\bar{Z}_B{}^a\bar{\epsilon}^{AB}\gamma_\lambda\Psi_{Ab} + Z^B{}_b\bar{\epsilon}_{AB}\gamma_\lambda\Psi^{Aa}\right).
\end{aligned}
\tag{A.116}
$$

Comparing (A.116) to (A.107), we see that the two-derivative terms cancel provided that

$$
\delta A_\mu{}^a{}_b = i\bar{Z}_B{}^a\bar{\epsilon}^{AB}\gamma_\lambda\Psi_{Ab} - iZ^B{}_b\bar{\epsilon}_{AB}\gamma_\lambda\Psi^{Aa},
\tag{A.117}
$$

which is the transformation rule of the gauge field required for SUSY invariance.

One-derivative terms

From the scalar kinetic term $\delta\mathcal{L}_{\text{scalar}}$ we have

$$\begin{aligned}
&\left(\delta\tilde{A}_\mu{}^a{}_b\right)\left(Z^A{}_a D^\mu \bar{Z}_A{}^b - \bar{Z}_A{}^b D^\mu Z^A{}_a\right) \\
=&\left(\delta\tilde{A}_\mu{}^a{}_b\right) Z^A{}_a D^\mu \bar{Z}_A{}^b + c.c. \\
=&if^{ac}{}_{bd}\left(\bar{Z}_C{}^d\bar{\epsilon}^{BC}\gamma^\mu\Psi_{Bc} - Z^C{}_c\bar{\epsilon}_{BC}\gamma^\mu\Psi^{Bd}\right) Z^A{}_a D_\mu \bar{Z}_A{}^b + c.c. \\
=&if^{ab}{}_{cd}\left(\bar{Z}_C{}^d\bar{\epsilon}^{BC}\gamma^\mu\Psi_{Bb} - Z^C{}_b\bar{\epsilon}_{BC}\gamma^\mu\Psi^{Bd}\right) Z^A{}_a D_\mu \bar{Z}_A{}^c + c.c.,
\end{aligned} \tag{A.118}$$

where we in the first step have used that $\delta\tilde{A}_\mu{}^a{}_b$ is imaginary, which is easily seen from (2.17) and (A.117). The contribution from the Dirac term is

$$\begin{aligned}
&-if^{cd}{}_{ab}\left(\bar{Z}_C{}^a\bar{Z}_D{}^b Z^B{}_c\bar{\epsilon}^{CD} - \bar{Z}_A{}^a\bar{Z}_C{}^b Z^C{}_c\bar{\epsilon}^{AB}\right)\gamma^\mu D_\mu\Psi_{Bd} \\
&-if^{ab}{}_{cd}\bar{\Psi}^{Bd}\gamma^\mu D_\mu\left(Z^C{}_a Z^D{}_b\bar{Z}_B{}^c\epsilon_{CD} - Z^A{}_a Z^C{}_b\bar{Z}_C{}^c\epsilon_{AB}\right) \\
=&if^{ab}{}_{cd}D_\mu\left(\bar{Z}_C{}^c\bar{Z}_D{}^d Z^B{}_a\bar{\epsilon}^{CD} - \bar{Z}_A{}^c\bar{Z}_C{}^d Z^C{}_a\bar{\epsilon}^{AB}\right)\gamma^\mu\Psi_{Bb} \\
&+if^{ab}{}_{cd}\bar{\epsilon}_{CD}\gamma^\mu D_\mu\left(Z^C{}_a Z^D{}_b\bar{Z}_B{}^c\right)\Psi^{Bd} \\
&-if^{ab}{}_{cd}\bar{\epsilon}_{AB}\gamma^\mu D_\mu\left(Z^A{}_a Z^C{}_b\bar{Z}_C{}^c\right)\Psi^{Bd} \\
=&if^{ab}{}_{cd}\Big[2\left(D_\mu\bar{Z}_C{}^c\right)\bar{Z}_D{}^d Z^B{}_a\bar{\epsilon}^{CD} + \bar{Z}_C{}^c\bar{Z}_D{}^d\left(D_\mu Z^B{}_a\right)\bar{\epsilon}^{CD} \\
&-\left(D_\mu\bar{Z}_A{}^c\right)\bar{Z}_C{}^d Z^C{}_a\bar{\epsilon}^{AB} - \bar{Z}_A{}^c\left(D_\mu\bar{Z}_C{}^d\right)Z^C{}_a\bar{\epsilon}^{AB} \\
&-\bar{Z}_A{}^c\bar{Z}_C{}^d\left(D_\mu Z^C{}_a\right)\bar{\epsilon}^{AB}\Big]\gamma^\mu\Psi_{Bb} \\
&+i\bar{\epsilon}_{CD}\gamma^\mu f^{ab}{}_{cd}\left[2\left(D_\mu Z^C{}_a\right)Z^D{}_b\bar{Z}_B{}^c + Z^C{}_a Z^D{}_b\left(D_\mu\bar{Z}_B{}^c\right)\right]\Psi^{Bd} \\
&-if^{ab}{}_{cd}\bar{\epsilon}_{AB}\gamma^\mu\Big[\left(D_\mu Z^A{}_a\right)Z^C{}_b\bar{Z}_C{}^c \\
&+Z^A{}_a\left(D_\mu Z^C{}_b\right)\bar{Z}_C{}^c + Z^A{}_a Z^C{}_b\left(D_\mu\bar{Z}_C{}^c\right)\Big]\Psi^{Bd} \\
=&if^{ab}{}_{cd}\left(D_\mu\bar{Z}_A{}^c\right)\left[2\bar{Z}_D{}^d Z^B{}_a\bar{\epsilon}^{AD} - \bar{Z}_C{}^d Z^C{}_a\bar{\epsilon}^{AB} + \bar{Z}_C{}^d Z^A{}_a\bar{\epsilon}^{CB}\right]\gamma^\mu\Psi_{Bb} \\
&+if^{ab}{}_{cd}\left(D_\mu\bar{Z}_A{}^c\right)\left[Z^C{}_a Z^D{}_b\bar{\epsilon}_{CD}\gamma^\mu\Psi^{Ad} - Z^C{}_a Z^A{}_b\bar{\epsilon}_{CB}\gamma^\mu\Psi^{Bd}\right] + c.c..
\end{aligned} \tag{A.119}$$

The contribution from $\delta\mathcal{L}_3$ is

$$\begin{aligned}
&if^{ab}{}_{cd}\left(D_\mu\bar{Z}_A{}^d\right)\bar{\epsilon}^{AB}\gamma^\mu\Psi_{Ba}Z^E{}_b\bar{Z}_E{}^c + c.c. \\
=&-if^{ab}{}_{cd}\left(D_\mu\bar{Z}_A{}^c\right)\bar{\epsilon}^{AB}\gamma^\mu\Psi_{Ba}Z^E{}_b\bar{Z}_E{}^d + c.c.
\end{aligned} \tag{A.120}$$

and from $\delta\mathcal{L}_4$

$$\begin{aligned}
&-2if^{ab}{}_{cd}\left(D_\mu\bar{Z}_D{}^d\right)\bar{\epsilon}^{DA}\gamma^\mu\Psi_{Ea}Z^E{}_b\bar{Z}_A{}^c + c.c. \\
=&2if^{ab}{}_{cd}\left(D_\mu\bar{Z}_A{}^c\right)\bar{\epsilon}^{AB}\gamma^\mu\Psi_{Ea}Z^E{}_b\bar{Z}_B{}^b + c.c..
\end{aligned} \tag{A.121}$$

From $\delta\mathcal{L}_5$ and $\delta\mathcal{L}_6$, the contribution is

$$\begin{aligned}&i\epsilon_{ABCD}f^{ab}{}_{cd}\left(D_\mu\bar{Z}_E{}^c\right)\bar{\epsilon}^{EA}\gamma^\mu\Psi^{Bd}Z^C{}_aZ^D{}_b+c.c.\\=&i\epsilon_{EBCD}f^{ab}{}_{cd}\left(D_\mu\bar{Z}_A{}^c\right)\bar{\epsilon}^{AE}\gamma^\mu\Psi^{Bd}Z^C{}_aZ^D{}_b+c.c..\end{aligned}\tag{A.122}$$

Putting all these contributions together yields

$$\begin{aligned}&if^{ab}{}_{cd}\left(D_\mu\bar{Z}_A{}^c\right)\Big[\bar{\epsilon}^{BC}\gamma^\mu\Psi_{Bb}\bar{Z}_C{}^d\bar{Z}_A{}^a-\bar{\epsilon}_{BC}\gamma^\mu\Psi_{Bd}Z^C{}_bZ^A{}_a\\&+2\bar{\epsilon}^{AD}\gamma^\mu\Psi_{Bb}\bar{Z}_D{}^dZ^B{}_a-\bar{\epsilon}^{AB}\gamma^\mu\Psi_{Bb}+\bar{\epsilon}^{CB}\gamma^\mu\Psi_{Bb}\bar{Z}_C{}^dZ^A{}_a\\&+\bar{\epsilon}_{CD}\gamma^\mu\Psi^{Ad}Z^C{}_aZ^D{}_b-\bar{\epsilon}_{CB}\gamma^\mu\Psi^{Bd}Z^C{}_aZ^A{}_b-\bar{\epsilon}^{AB}\gamma^\mu\Psi_{Ba}Z^E{}_b\bar{Z}_E{}^d\\&+2\bar{\epsilon}^{AB}\gamma^\mu\Psi_{Ea}Z^E{}_b\bar{Z}_B{}^d+\epsilon_{EBCD}\bar{\epsilon}^{AE}\gamma^\mu\Psi^{Bd}Z^C{}_aZ^D{}_b\Big]+c.c.\\=&if^{ab}{}_{cd}\left(D_\mu\bar{Z}_A{}^c\right)\Big[-2\bar{\epsilon}_{BC}\gamma^\mu\Psi^{Bd}Z^C{}_bZ^A{}_a+\bar{\epsilon}_{BC}\gamma^\mu\Psi^{Ad}Z^B{}_aZ^C{}_b\\&+\epsilon_{EBCD}\bar{\epsilon}^{AE}\gamma^\mu\Psi^{Bd}Z^C{}_aZ^D{}_b\Big]+c.c..\end{aligned}\tag{A.123}$$

To show that the last three terms cancel, let us use a trick. Since the (anti-) fundamental $SU(4)_R$ indices can take only four values, an antisymmetrized expression with five such indices will be identically zero. Since the last term in (A.123) has five fundamental (upper) indices, we can try to rewrite it by antisymmetrizing and explicitly writing out the different terms. Since we have an imposed antisymmetry in $EBCD$, coming from ϵ_{EBCD}, we only have to write out one term for each position of the A-index. The other terms are merely permutations of $EBCD$, giving us 4! copies of each term we write out. Thus, we have

$$\begin{aligned}0=&\epsilon_{EBCD}\bar{\epsilon}^{AE}\gamma^\mu\Psi^{Bd}Z^C{}_aZ^D{}_b-\epsilon_{EBCD}\bar{\epsilon}^{EA}\gamma^\mu\Psi^{Bd}Z^C{}_aZ^D{}_b\\&-\epsilon_{EBCD}\bar{\epsilon}^{BE}\gamma^\mu\Psi^{Ad}Z^C{}_aZ^D{}_b-\epsilon_{EBCD}\bar{\epsilon}^{CE}\gamma^\mu\Psi^{Bd}Z^A{}_aZ^D{}_b\\&-\epsilon_{EBCD}\bar{\epsilon}^{DE}\gamma^\mu\Psi^{Bd}Z^C{}_aZ^A{}_b\\=&2\epsilon_{EBCD}\bar{\epsilon}^{AE}\gamma^\mu\Psi^{Bd}Z^C{}_aZ^D{}_b+2\bar{\epsilon}_{CD}\gamma^\mu\Psi^{Ad}Z^C{}_aZ^D{}_b-4\bar{\epsilon}_{BC}\gamma^\mu\Psi^{Bd}Z^A{}_aZ^C{}_b,\end{aligned}\tag{A.124}$$

where we have used (2.27) and the fact that we have an imposed antisymmetry in a and b coming from the structure constant. Now, inserting (A.124) into (A.123), we easily see that the last one-derivative terms cancel.

Terms without derivatives

Let us start with the terms of the form $Z^5\epsilon\psi$. From $\mathcal{L}_3$ we have

$$\begin{aligned}&-if^{ab}{}_{cd}f^{ed}{}_{fg}\bar{X}^{fgc}_{CDE}Z^B{}_eZ^E{}_b\bar{\epsilon}^{CD}\Psi_{Ba}\\&+if^{ab}{}_{cd}f^{ed}{}_{fg}\bar{X}^{fgc}_{ACE}Z^C{}_eZ^E{}_b\bar{\epsilon}^{AB}\Psi_{Ba}+c.c.,\end{aligned}\tag{A.125}$$

from $\mathcal{L}_4$

$$\begin{aligned}&2if^{ab}{}_{cd}f^{ed}{}_{fg}\bar{X}^{fgc}_{CDA}Z^A{}_eZ^E{}_b\bar{\epsilon}^{CD}\Psi_{Ea}\\&-2if^{ab}{}_{cd}f^{ed}{}_{fg}\bar{X}^{fgc}_{BCA}Z^C{}_eZ^E{}_b\bar{\epsilon}^{BA}\Psi_{Ea}+c.c.,\end{aligned}\tag{A.126}$$

and from $\mathcal{L}_5$ and $\mathcal{L}_6$

$$\begin{aligned}&if^{fg}{}_{ec}f^{dc}{}_{ab}\epsilon^{ABCD}\bar{X}^{eab}_{ACD}Z^E{}_fZ^F{}_g\bar{\epsilon}_{EF}\Psi_{Bd}\\&-if^{fg}{}_{ec}f^{dc}{}_{ab}\epsilon^{ABCD}\bar{X}^{eab}_{FCD}Z^E{}_fZ^F{}_g\bar{\epsilon}_{EA}\Psi_{Bd}+c.c.\\=&if^{ab}{}_{cd}f^{ed}{}_{fg}\epsilon^{ABCD}Z^E{}_aZ^F{}_b\left[\bar{\epsilon}_{EF}\Psi_{Be}\bar{X}^{cfg}_{ACD}-\bar{\epsilon}_{EA}\Psi_{Be}\bar{X}^{cfg}_{FCD}\right]+c.c..\end{aligned}\tag{A.127}$$

To further rewrite (A.127), we will use the same technique of cycling indices as we did in (A.124). For simplicity we denote $\bar{\epsilon}_{EF}\Psi_{Be}\bar{X}^{cfg}_{ACD}$ by $EFBACD$. Now, we will try to rewrite $EFBACD - EABFCD$ to a form where all the terms have CD in the first two index places, since this will make it possible to use (2.27) in (A.127). Since five indices is enough to make the antisymmetrization identically zero, we can let one of the six indices remain fixed while cycling the others. Finally, before we perform the cycling, we also note that we can make use of several antisymmetries. These are in the first two indices, the last two indices and in $ABCD$. To begin with, we let B be fixed and write

$$\begin{aligned}0=&EFBACD-EABFCD-ECBAFD-EDBACF-AFBECD\\&-CFBAED-DFBACE+ACBEFD+CDBAEF+ADBECF\\=&EFBACD-2EABFCD-4ECBAFD+2ACBEFD+CDBAEF,\end{aligned}\tag{A.128}$$

which implies

$$\begin{aligned}EABFCD-EFBACD=&-EABFCD-4ECBAFD\\&+ACBEFD+CDBAEF.\end{aligned}\tag{A.129}$$

Next, we let F be fixed and write

$$\begin{aligned}0=&EABFCD-AEBFCD-BAEFCD-CABFED-DABFCE\\=&2EABFCD-BAEFCD-2CABFED,\end{aligned}\tag{A.130}$$

implying

$$EABFCD=\frac{1}{2}BAEFCD+CABFED.\tag{A.131}$$

Still keeping F fixed, we also write

$$0=2ECBAFD-BCEAFD-ACBEFD-DCBAFE,\tag{A.132}$$

which gives us

$$ECBAFD = \frac{1}{2}BCEAFD + \frac{1}{2}ACBEFD + \frac{1}{2}DCBAFE. \tag{A.133}$$

Now, inserting (A.131) and (A.133) into (A.129), we arrive at

$$\begin{aligned}
&EABFCD - EFBACD \\
=&-\frac{1}{2}BAEFCD - CABFED - 2BCEAFD \\
&-2ACBEFD - 2DCBAFE \\
&+2ACBEFD + CDBAEF \\
=&-2BCEAFD - CDABEF \\
&-CABFED - \frac{1}{2}BAEFCD \\
=&2CDEABF + CDABEF \\
&+CDBFEA + \frac{1}{2}CDEFAB.
\end{aligned} \tag{A.134}$$

This is our desired result of the cycling. Inserting (A.134) into (A.127) and using (2.27) yields

$$\begin{aligned}
&if^{ab}{}_{cd}f^{ed}{}_{fg}\epsilon^{ABCD}Z^E{}_a Z^F{}_b\left[\bar{\epsilon}_{EF}\Psi_{Be}\bar{X}^{cfg}_{ACD} - \bar{\epsilon}_{EA}\Psi_{Be}\bar{X}^{cfg}_{FCD}\right] + c.c. \\
=&-if^{ab}{}_{cd}f^{ed}{}_{fg}Z^E{}_a Z^F{}_b\bar{\epsilon}^{AB}\left[4\Psi_{Ee}\bar{X}^{cfg}_{ABF} + 2\Psi_{Ae}\bar{X}^{cfg}_{BEF}\right. \\
&\left.+2\Psi_{Be}\bar{X}^{cfg}_{FEA} + \Psi_{Ee}\bar{X}^{cfg}_{FAB}\right] + c.c..
\end{aligned} \tag{A.135}$$

The last terms of type $Z^5\epsilon\psi$ are the ones coming from the scalar potential. These are

$$-\bar{X}^{cfg}_{ABC}\left(\delta Z^A{}_a\right)Z^B{}_b Z^C{}_e\left[f^{ba}{}_{dg}f^{ed}{}_{cf} + f^{be}{}_{dc}f^{ad}{}_{gf} + f^{eb}{}_{df}f^{ad}{}_{gc}\right] + c.c.. \tag{A.136}$$

Now that we have collected all the $Z^5\epsilon\psi$-terms, we note that they are of two different types. The first type has its ψ-index contracted with one of the ϵ-indices, and thus contains a δZ. Collecting all these terms, we have

$$\begin{aligned}
&if^{ab}{}_{cd}f^{ed}{}_{fg}\left[\bar{X}^{fgc}_{ACE}Z^C{}_e Z^E{}_b\bar{\epsilon}^{AB}\Psi_{Ba}\right. \\
&\left.-2\bar{X}^{fgc}_{EFB}Z^E{}_a Z^F{}_b\bar{\epsilon}^{AB}\Psi_{Ae} - 2\bar{X}^{fgc}_{EAF}Z^E{}_a Z^F{}_b\bar{\epsilon}^{AB}\Psi_{Be}\right] \\
&-\bar{X}^{cfg}_{ABC}\left(\delta Z^A{}_a\right)Z^B{}_b Z^C{}_e\left[f^{ba}{}_{dg}f^{ed}{}_{cf} + f^{be}{}_{dc}f^{ad}{}_{gf} + f^{eb}{}_{df}f^{ad}{}_{gc}\right] + c.c. \\
=&\bar{X}^{cfg}_{ABC}\left(\delta Z^A{}_a\right)Z^B{}_b Z^C{}_e\left[f^{ab}{}_{fd}f^{ed}{}_{cg} + 2f^{be}{}_{cd}f^{ad}{}_{fg} - 2f^{be}{}_{gd}f^{ad}{}_{fc} - f^{ba}{}_{dg}f^{ed}{}_{cf}\right. \\
&\left.-f^{be}{}_{dc}f^{ad}{}_{gf} - f^{eb}{}_{df}f^{ad}{}_{gc}\right] + c.c.
\end{aligned} \tag{A.137}$$

The terms inside the brackets can be shown to cancel using (2.15) and the symmetry under $bf \leftrightarrow eg$ from outside the brackets. The calculation goes as follows:

$$\begin{aligned}
& f^{ab}{}_{fd} f^{ed}{}_{cg} + 2f^{be}{}_{cd} f^{ad}{}_{fg} - 2f^{be}{}_{gd} f^{ad}{}_{fc} - f^{ba}{}_{dg} f^{ed}{}_{cf} - f^{be}{}_{dc} f^{ad}{}_{gf} - f^{eb}{}_{df} f^{ad}{}_{gc} \\
=& f^{eb}{}_{dc} f^{ad}{}_{fg} + f^{be}{}_{dg} f^{ad}{}_{fc} + f^{ba}{}_{df} f^{ed}{}_{cg} - f^{ba}{}_{dg} f^{ed}{}_{cf} \\
=& 2f^{eb}{}_{d[g} f^{ad}{}_{c]f} + f^{ba}{}_{df} f^{ed}{}_{cg} - f^{ba}{}_{dg} f^{ed}{}_{cf} \\
=& 2f^{a[e}{}_{df} f^{b]d}{}_{gc} + f^{ba}{}_{df} f^{ed}{}_{cg} - f^{ba}{}_{dg} f^{ed}{}_{cf} \\
=& f^{ae}{}_{df} f^{bd}{}_{gc} - f^{ab}{}_{df} f^{ed}{}_{gc} + f^{ba}{}_{df} f^{ed}{}_{cg} - f^{ba}{}_{dg} f^{ed}{}_{cf} \\
=& 0.
\end{aligned} \tag{A.138}$$

Moving on to the remaining $Z^5\epsilon\psi$-terms, we have

$$\begin{aligned}
& i f^{ab}{}_{cd} f^{ed}{}_{fg} \Big[-\bar{X}^{cfg}_{ECD} Z^B{}_e Z^E{}_b \bar{\epsilon}^{CD} \Psi_{Ba} + 2\bar{X}^{cfg}_{ACD} Z^A{}_e Z^E{}_b \bar{\epsilon}^{CD} \Psi_{Ea} \\
& \quad + 2\bar{X}^{cfg}_{ABC} Z^C{}_e Z^E{}_b \bar{\epsilon}^{AB} \Psi_{Ea} - 4\bar{X}^{cfg}_{ABF} Z^E{}_a Z^F{}_b \bar{\epsilon}^{AB} \Psi_{Ee} - \bar{X}^{cfg}_{FAB} Z^E{}_a Z^F{}_b \bar{\epsilon}^{AB} \Psi_{Ee} \Big] \\
=& i\bar{X}^{cfg}_{EDC} Z^B{}_e Z^E{}_b \bar{\epsilon}^{CD} \Psi_{Ba} \Big[-f^{ab}{}_{cd} f^{ed}{}_{fg} + 2f^{ae}{}_{cd} f^{bd}{}_{fg} \\
& \quad + 2f^{ae}{}_{fd} f^{bd}{}_{gc} - 4f^{eb}{}_{fd} f^{ad}{}_{gc} - f^{eb}{}_{cd} f^{ad}{}_{fg} \Big].
\end{aligned} \tag{A.139}$$

Again, we will show that the terms inside the brackets cancel, using the imposed antisymmetry in $f \leftrightarrow g$:

$$\begin{aligned}
& 4f^{eb}{}_{df} f^{ad}{}_{gc} + 2f^{ea}{}_{dc} f^{bd}{}_{fg} + 2f^{ea}{}_{df} f^{bd}{}_{gc} + f^{eb}{}_{dc} f^{ad}{}_{fg} + f^{ab}{}_{dc} f^{ed}{}_{fg} \\
=& 4f^{eb}{}_{d[f} f^{ad}{}_{g]c} + 4f^{ae}{}_{d[c} f^{bd}{}_{g]f} + f^{eb}{}_{dc} f^{ad}{}_{fg} + f^{ab}{}_{dc} f^{ed}{}_{fg} \\
=& 4f^{a[e}{}_{dc} f^{b]d}{}_{fg} + 4f^{b[a}{}_{df} f^{e]d}{}_{cg} + f^{eb}{}_{dc} f^{ad}{}_{fg} + f^{ab}{}_{dc} f^{ed}{}_{fg} \\
=& 2f^{ae}{}_{dc} f^{bd}{}_{fg} - 2f^{ab}{}_{dc} f^{ed}{}_{fg} + 2f^{ba}{}_{df} f^{ed}{}_{cg} - 2f^{be}{}_{df} f^{ad}{}_{cg} + f^{eb}{}_{dc} f^{ad}{}_{fg} + f^{ab}{}_{dc} f^{ed}{}_{fg} \\
=& - f^{ab}{}_{dc} f^{ed}{}_{fg} + 4f^{b[a}{}_{df} f^{e]d}{}_{cg} + 2f^{ae}{}_{dc} f^{bd}{}_{fg} + f^{eb}{}_{dc} f^{ad}{}_{fg} \\
=& - f^{ab}{}_{dc} f^{ed}{}_{fg} + 2f^{ae}{}_{dc} f^{bd}{}_{gf} - 2f^{ae}{}_{dg} f^{bd}{}_{cf} + 2f^{ae}{}_{dc} f^{bd}{}_{fg} + f^{eb}{}_{dc} f^{ad}{}_{fg} \\
=& - f^{ab}{}_{dc} f^{ed}{}_{fg} - 2f^{ae}{}_{dg} f^{bd}{}_{cf} + f^{eb}{}_{dc} f^{ad}{}_{fg} \\
=& 2f^{b[a}{}_{dc} f^{e]d}{}_{fg} - 2f^{ae}{}_{dg} f^{bd}{}_{cf} \\
=& 2f^{ae}{}_{d[f} f^{bd}{}_{g]c} - 2f^{ae}{}_{dg} f^{bd}{}_{cf} \\
=& 0.
\end{aligned} \tag{A.140}$$

Having cancelled all the $Z^5\epsilon\psi$-terms, only the $\psi^3\epsilon Z$-terms remain. From the Dirac term we have

$$\begin{aligned}
& i\left(\bar{\Psi}^{Bd}\gamma^\mu \Psi_{Ba}\right)\left(\delta \tilde{A}_\mu{}^a{}_d\right) \\
=& - f^{ac}{}_{db}\left(\bar{\Psi}^{Dd}\gamma^\mu \Psi_{Da}\right)\left(\bar{\epsilon}^{AB}\gamma_\mu \Psi_{Ac}\right)\bar{Z}_B{}^b + c.c.
\end{aligned}$$

$$=f^{ab}{}_{cd}\left(\bar{\Psi}^{Dd}\gamma^{\mu}\Psi_{Da}\right)\left(\bar{\epsilon}^{AB}\gamma_{\mu}\Psi_{Ab}\right)\bar{Z}_B{}^c+c.c., \tag{A.141}$$

from $\mathcal{L}_3$

$$f^{ab}{}_{cd}\left(\bar{\Psi}^{Bd}\Psi_{Ba}\right)\left(\bar{\epsilon}^{EF}\Psi_{Fb}\right)\bar{Z}_E{}^c+c.c., \tag{A.142}$$

from $\mathcal{L}_4$

$$-2f^{ab}{}_{cd}\left(\bar{\Psi}^{Ad}\Psi_{Ba}\right)\left(\bar{\epsilon}^{BD}\Psi_{Db}\right)\bar{Z}_A{}^c+c.c., \tag{A.143}$$

and from $\mathcal{L}_5$ and $\mathcal{L}_6$

$$\begin{aligned}
&f^{cd}{}_{ab}\,\epsilon^{ABCD}\left(\bar{\Psi}_{Ac}\Psi_{Bd}\right)\left(\bar{\epsilon}_{CE}\bar{\Psi}_{Ea}\right)\bar{Z}_D{}^b+c.c.\\
&=f^{cd}{}_{ab}\,\epsilon^{ABCD}\left[-\left(\bar{\Psi}_{Ac}\Psi_{Ed}\right)\left(\bar{\epsilon}_{CD}\bar{\Psi}_{Ea}\right)\bar{Z}_B{}^b+\frac{1}{2}\left(\bar{\Psi}_{Ac}\Psi_{Bd}\right)\left(\bar{\epsilon}_{CD}\bar{\Psi}_{Ea}\right)\bar{Z}_E{}^b\right]+c.c.\\
&=-2f^{ab}{}_{cd}\left(\bar{\Psi}_{Aa}\Psi_{Eb}\right)\left(\bar{\epsilon}^{AB}\bar{\Psi}_{Ec}\right)\bar{Z}_B{}^d+f^{ab}{}_{cd}\left(\bar{\Psi}_{Aa}\Psi_{Bb}\right)\left(\bar{\epsilon}^{AB}\bar{\Psi}_{Ec}\right)\bar{Z}_E{}^d+c.c.,
\end{aligned} \tag{A.144}$$

where we in the first step have cycled anti-fundamental (lower) $\mathrm{SU}(4)_R$ indices. In total, the $\psi^3\epsilon Z$-terms are

$$\begin{aligned}
&f^{ab}{}_{cd}\bar{Z}_B{}^c\Big[\left(\bar{\epsilon}^{AB}\gamma^{\mu}\Psi_{Ab}\right)\left(\bar{\Psi}^{Ed}\gamma_{\mu}\Psi_{Ea}\right)\\
&-\left(\bar{\epsilon}^{AB}\Psi_{Ab}\right)\left(\bar{\Psi}^{Ed}\Psi_{Ea}\right)-2\left(\bar{\epsilon}^{AB}\Psi^{Ed}\right)\left(\bar{\Psi}_{Ab}\Psi_{Ea}\right)\Big]\\
&+f^{ab}{}_{cd}\bar{Z}_B{}^c\Big[-2\left(\bar{\epsilon}^{AE}\Psi_{Eb}\right)\left(\bar{\Psi}^{Bd}\Psi_{Aa}\right)-\left(\bar{\epsilon}^{AE}\Psi^{Bb}\right)\left(\bar{\Psi}_{Aa}\Psi_{Eb}\right)\Big]+c.c..
\end{aligned} \tag{A.145}$$

The only way to proceed from (A.145) is to perform Fierz expansions of the fermion bilinears. This will not be nearly as messy as in the BLG theory, since the only gamma matrices we have to deal with are the SO(2,1) ones. The general expansion of a product of two fermions χ^1 and χ^2 is

$$\chi^1_{\alpha}\chi^2_{\beta}=a\left(C^{-1}\right)_{\alpha\beta}+b_{\mu}\left(\gamma^{\mu}C^{-1}\right)_{\alpha\beta}, \tag{A.146}$$

where a and b_μ are coefficients to be determined. Multiplying (A.146) with $C^{\alpha\beta}$, the right hand side is

$$a\,\mathrm{tr}\left[C^{-1}C^{T}\right]+b_{\mu}\,\mathrm{tr}\left[\gamma^{\mu}C^{-1}C^{T}\right]=-2a \tag{A.147}$$

and the left hand side $-\bar{\chi}^1\chi^2$. This implies

$$a=\frac{1}{2}\bar{\chi}^1\chi^2. \tag{A.148}$$

If we instead multiply with $(C\gamma^{\nu})^{\alpha\beta}$ the right hand side is

$$a\,\mathrm{tr}\left[C^{T}C\gamma^{\nu}\right]+b_{\mu}\,\mathrm{tr}\left[\gamma^{\mu}C^{-1}C\gamma^{\nu}\right]=b_{\mu}\,\mathrm{tr}\left[\gamma^{\mu}\gamma^{\nu}\right]=b_{\mu}\,\mathrm{tr}\left[\eta^{\mu\nu}\right]=2b^{\nu} \tag{A.149}$$

and the left hand side $-\bar{\chi}^1\gamma^{\nu}\chi^2$, implying

$$b_{\mu}=-\frac{1}{2}\bar{\chi}^1\gamma_{\mu}\chi^2. \tag{A.150}$$

The general expansion (A.146) can now be written

$$\chi^1_\alpha \chi^2_\beta = \frac{1}{2}\left(\bar{\chi}^1\chi^2\right)\left(C^{-1}\right)_{\alpha\beta} - \frac{1}{2}\left(\bar{\chi}^1\gamma_\mu\chi^2\right)\left(\gamma^\mu C^{-1}\right)_{\alpha\beta}. \tag{A.151}$$

Let us now apply (A.151) to (A.145). Starting with the bracket containing two terms, we set $\chi^1 = \epsilon^{AE}$, $\chi^2 = \Psi_{Eb}$, $\chi^3 = \Psi^{Bd}$ and $\chi^4 = \Psi_{Aa}$. With these definitions, we see that any expression inside the brackets is antisymmetric under $\chi^2 \leftrightarrow \chi^4$. Next, we write

$$\left(\bar{\chi}^1\chi^2\right)\left(\bar{\chi}^3\chi^4\right) = \chi^1_\alpha\chi^2_\beta C^{\alpha\beta}\cdot\chi^3_\gamma\chi^4_\delta C^{\gamma\delta} \tag{A.152}$$

Using (A.151) we can expand $\chi^2_\beta\chi^4_\delta$, obtaining

$$\chi^2_\beta\chi^4_\delta = \frac{1}{2}\left(\bar{\chi}^2\chi^4\right)\left(C^{-1}\right)_{\beta\delta}, \tag{A.153}$$

where we have used the antisymmetry mentioned above to make the second term in the expansion vanish. Inserting (A.153) into (A.152) yields

$$\begin{aligned}\left(\bar{\chi}^1\chi^2\right)\left(\bar{\chi}^3\chi^4\right) =& \frac{1}{2}\left(\bar{\chi}^2\chi^4\right)\chi^1_\alpha\chi^3_\gamma C^{\alpha\beta}\left(C^{-1}\right)_{\beta\delta}\left(C^T\right)^{\delta\gamma}\\ =& -\frac{1}{2}\left(\bar{\chi}^2\chi^4\right)\left(\bar{\chi}^1\chi^3\right).\end{aligned} \tag{A.154}$$

Using (A.154) in the second bracket of (A.145), we can now easily see that the two terms cancel.

Finally, let us turn to the bracket in (A.145) with three terms. Similarly to the previous case, we define $\chi^1 = \epsilon^{AB}$, $\chi^2 = \Psi_{Ab}$, $\chi^3 = \Psi^{Ed}$ and $\chi^4 = \Psi_{Ea}$ and compute

$$\begin{aligned}\left(\bar{\chi}^1\chi^3\right)\left(\bar{\chi}^2\chi^4\right) =& \chi^1_\alpha\chi^3_\beta C^{\alpha\beta}\chi^2_\gamma\chi^4_\delta C^{\gamma\delta}\\ =& -\frac{1}{2}\left(\bar{\chi}^1\chi^2\right)\chi^3_\beta\chi^4_\delta\left(C^T\right)^{\beta\alpha}C^{\gamma\delta}\left(C^{-1}\right)_{\alpha\gamma}\\ &+\frac{1}{2}\left(\bar{\chi}^1\gamma_\mu\chi^2\right)\chi^3_\beta\chi^\delta_4\left(C^T\right)^{\beta\alpha}C^{\gamma\delta}\left(\gamma^\mu C^{-1}\right)_{\alpha\gamma}\\ =& -\frac{1}{2}\left(\bar{\chi}^1\chi^2\right)\left(\bar{\chi}^3\chi^4\right)+\frac{1}{2}\left(\bar{\chi}^1\gamma_\mu\chi^2\right)\left(\bar{\chi}^3\gamma^\mu\chi^4\right).\end{aligned} \tag{A.155}$$

Using (A.155) in (A.145), we see that the terms in the first bracket cancel. This completes our proof that the ABJM Lagrangian (2.18) is invariant under the SUSY transformations specified by (2.24)-(2.25).

B

Details of $\mathbb{R} \times S^2$

B.1 Metric and Dirac Matrices

The metric on $S^2 \times \mathbb{R}$, with coordinates (τ, θ, ϕ), is given by

$$g_{mn} = \delta_{ab} e^a_m e^b_n = \mathrm{diag}(1, 1, \sin^2\theta) \tag{B.1}$$

and the diagonal dreibeins by

$$e^a_m = \mathrm{diag}(1, 1, \sin\theta). \tag{B.2}$$

In the tangent frame, we use the Dirac matrices $(\gamma^0, \gamma^1, \gamma^2) = (\sigma^3, \sigma^1, \sigma^2)$, which means that we have $(\gamma^\tau, \gamma^\theta, \gamma^\phi) = (\sigma^3, \sigma^1, \frac{\sigma^2}{\sin\theta})$ in the coordinate frame.

B.2 Covariant Derivative

The spin connection ω_{mab} can be calculated as follows. First, since the torsion vanishes, we have the condition

$$\partial_m e^a_n + \omega_m{}^a{}_b e^b_n = 0. \tag{B.3}$$

Multiplying this equation by $e^n_b e^m_c$ yields

$$e^n_b \partial_c e^a_n + \omega_c{}^a{}_b = 0. \tag{B.4}$$

By lowering the a-index and antisymmetrizing in b and c we arrive at

$$e_{[b}{}^n \partial_{c]} e_{na} = -\omega_{[b,c]a}, \tag{B.5}$$

where we also have used $\omega_{mab} = -\omega_{mba}$. By manipulating (B.5) we can now solve for $\omega_{b,ca}$. Let us start by adding some suitable terms with renamed indices to each side of the equation:

$$e_{[b}{}^n \partial_{c]} e_{na} - e_{[c}{}^n \partial_{a]} e_{nb} + e_{[a}{}^n \partial_{b]} e_{nc} = -\omega_{[b,c]a} + \omega_{[c,a]b} - \omega_{[a,b]c}. \tag{B.6}$$

The right hand side simplifies to

$$\begin{aligned} & -\omega_{[b,c]a} + \omega_{[c,a]b} - \omega_{[a,b]c} = \\ = \; & \frac{1}{2}(\omega_{b,ca} + \omega_{c,ba} + \omega_{c,ab} - \omega_{a,cb} - \omega_{a,bc} + \omega_{b,ac}) \\ = \; & -\omega_{b,ca}. \end{aligned} \tag{B.7}$$

Thus, we arive at the solution

$$\begin{aligned} \omega_{kca} =& e_k^b \omega_{b,ca} \\ =& -\frac{1}{2} e_k^b ({e_b}^n \partial_c e_{na} - {e_c}^n \partial_b e_{na} - {e_c}^n \partial_a e_{nb} \\ & + {e_a}^n \partial_c e_{nb} + {e_a}^n \partial_b e_{nc} - {e_b}^n \partial_a e_{nc}). \end{aligned} \tag{B.8}$$

By setting $k = \tau$ and $k = \theta$ it is easily seen that $\omega_{\tau ab}$ and $\omega_{\theta ab}$ vanish for all a and b. Setting $k = \phi$ gives

$$\begin{aligned} \omega_{\phi ca} &= -\frac{1}{2} e_\phi^2 (e_2^\phi \partial_c e_{\phi a} - e_c^\phi \partial_a e_{\phi 2} + e_a^\phi \partial_c e_{\phi 3} - e_3^\phi \partial_a e_{\phi c}) \\ &= -\frac{1}{2} e_\phi^2 (e_2^\phi e_c^m \partial_m e_{\phi a} - e_c^\phi e_a^m \partial_m e_{\phi 2} + e_a^\phi e_c^m \partial_m e_{\phi 2} - e_2^\phi e_a^m \partial_m e_{\phi c}). \end{aligned} \tag{B.9}$$

This expression is antisymmetric in ac, just as it should be. Setting $c = 0$ clearly gives zero, which means that the only nonzero components of the spin connection are:

$$\omega_{\phi 12} = -\omega_{\phi 21} = -\frac{1}{2}(\partial_\theta \sin\theta + \partial_\theta \sin\theta) = -\cos\theta. \tag{B.10}$$

The covariant derivative acting on a spinor ψ is

$$\nabla_m \psi = (\partial_m + \frac{1}{4}\omega_{mab}\gamma^{ab})\psi, \tag{B.11}$$

with $\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$.

C

Monopole Spinor Harmonics

In this Appendix, we derive explicit expressions for the monopole spinor harmonics. These are defined as eigenspinors of the Dirac operator on S^2, in a monopole background. We first prove that the spinors in question are also angular momentum eigenspinors, which allows us to use the SU(2) algebra to calculate them.

C.1 Dirac Operator

In an abelian gauge theory on $\mathbb{R} \times S^2$ with a magnetic monopole at the origin, the Dirac operator is given by $\not{D} = \not{\nabla} + i\not{A}$. Here, A is an abelian monopole background of the form

$$A_\phi = q(\pm 1 - \cos\theta), \tag{C.1}$$

where q is the magnetic charge and where the upper (lower) sign is for the northern (southern) hemisphere. The Dirac operator on S^2 is given by

$$\not{D}_S = \not{D} - \gamma^\tau \partial_\tau, \tag{C.2}$$

since the monopole gauge field has no component along the radial direction. Explicitly we have

$$\begin{aligned}
\not{D}_S &= \gamma^\theta \partial_\theta + \gamma^\phi \partial_\phi + \frac{1}{4}\gamma^\phi(\omega_{\phi 12}\gamma^{12} + \omega_{\phi 21}\gamma^{21}) + i\gamma^\phi A_\phi \\
&= \sigma^1 \partial_\theta + \frac{1}{\sin\theta}\sigma^2 \partial_\phi + \frac{1}{4\sin\theta}\sigma^2(-2i\cos\theta\sigma^3) + \frac{i}{\sin\theta}\sigma^2 A_\phi \\
&= \sigma^1\left(\partial_\theta + \frac{1}{2}\cot\theta\right) + \sigma^2\left(\frac{1}{\sin\theta}\partial_\phi + \frac{i}{\sin\theta}A_\phi\right).
\end{aligned} \tag{C.3}$$

For later use, let us also calculate the square of the Dirac operator:

$$\begin{aligned}
\not{\mathcal{D}}_S^2 &= \left(\partial_\theta + \frac{1}{2}\cot\theta\right)^2 + \frac{1}{\sin^2\theta}\left(\partial_\phi + iA_\phi\right)^2 \\
&\quad + i\sigma^3\left(\partial_\theta + \frac{1}{2}\cot\theta\right)\frac{1}{\sin\theta}\left(\partial_\phi + iA_\phi\right) \\
&\quad - i\sigma^3\frac{1}{\sin\theta}\left(\partial_\phi + iA_\phi\right)\left(\partial_\theta + \frac{1}{2}\cot\theta\right) \\
&= \partial_\theta^2 + \frac{1}{4}\cot^2\theta - \frac{1}{2\sin^2\theta} + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}(\partial_\phi + iA_\phi)^2 \\
&\quad + i\sigma^3\left(-\frac{\cos\theta}{\sin^2\theta}(\partial_\phi + iA_\phi) + \frac{i}{\sin\theta}\partial_\theta A_\phi\right) \\
&= \partial_\theta^2 + \cot\theta\partial_\theta - \frac{1}{4\sin\theta} - \frac{1}{4} + \frac{1}{\sin^2\theta}(\partial_\phi + iA_\phi)^2 \\
&\quad + i\sigma^3\left(-\frac{\cos\theta}{\sin^2\theta}(\partial_\phi + iA_\phi) + iq\right). \qquad \text{(C.4)}
\end{aligned}$$

C.2 Angular Momentum Operators

In flat $\mathbb{R}^3$, the orbital angular momentum of a fermion in the monopole background (C.1) is given by [26]:

$$\vec{L} = \vec{r} \times (\vec{p} + \vec{A}) + q\frac{\vec{r}}{r}. \qquad \text{(C.5)}$$

The Cartesian components of this vector, with $p_i = -i\partial_i$, are

$$\begin{aligned}
L_x &= -iy(\partial_z + A_z) + iz(\partial_y + A_y) + q\frac{x}{r} & \text{(C.6)} \\
L_y &= -iz(\partial_x + A_x) + ix(\partial_z + A_z) + q\frac{y}{r} & \text{(C.7)} \\
L_z &= -ix(\partial_y + A_y) + iy(\partial_x + A_x) + q\frac{z}{r}. & \text{(C.8)}
\end{aligned}$$

Expressing this in spherical coordinates yields

$$\begin{aligned}
L_x &= -i(-\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi) - \cot\theta\cos\phi A_\phi + q\sin\theta\cos\phi & \text{(C.9)} \\
L_y &= -i(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) - \cot\theta\sin\phi A_\phi + q\sin\theta\sin\phi & \text{(C.10)} \\
L_z &= -i\partial_\phi \pm q. & \text{(C.11)}
\end{aligned}$$

The total angular momentum operator $\vec{J}$ is given by adding the spin term $\frac{\vec{\sigma}}{2}$. Thus, we have

$$J_x = -i(-\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi) - \cot\theta\cos\phi A_\phi + q\sin\theta\cos\phi + \frac{\sigma_1}{2} \qquad \text{(C.12)}$$

$$J_y = -i(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) - \cot\theta\sin\phi A_\phi + q\sin\theta\sin\phi + \frac{\sigma_2}{2} \qquad \text{(C.13)}$$

$$J_z = -i\partial_\phi \pm q + \frac{\sigma_3}{2} \qquad \text{(C.14)}$$

and in the Cartan-Weyl basis [1]

$$\begin{aligned}
J_\pm =& J_x \pm iJ_y \\
=& -i(-\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi) - \cot\theta\cos\phi A_\phi + q\sin\theta\cos\phi \pm \\
& \pm(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) \pm i(-\cot\theta\sin\phi A_\phi + q\sin\theta\sin\phi) + \frac{\sigma_1 \pm i\sigma_2}{2} \\
=& (\pm\cos\phi + i\sin\phi)\partial_\theta + \cot\theta(\mp\sin\phi + i\cos\phi)\partial_\phi - \\
& -\cot\theta(\cos\phi \pm i\sin\phi)A_\phi + q\sin\theta(\cos\phi \pm i\sin\phi) + \frac{\sigma_1 \pm i\sigma_2}{2} \\
=& e^{\pm i\phi}\left(\pm\partial_\theta + \cot\theta(i\partial_\phi - A_\phi) + q\sin\theta\right) + \frac{\sigma_1 \pm i\sigma_2}{2} && \text{(C.15)} \\
J_z =& -i\partial_\phi \pm q + \frac{\sigma_3}{2}. && \text{(C.16)}
\end{aligned}$$

Our next step is to transform these operators from flat $\mathbb{R}^3$ to the curved manifold $S^2 \times \mathbb{R}$. The operators act on spinors and must therefore transform accordingly. The transformation of a spinor when going from $\mathbb{R}^3$ to $S^2 \times \mathbb{R}$ was derived in [1] and is given by

$$\psi \to V\psi, \tag{C.17}$$

where the unitary matrix V and its Hermitian conjugate is given by

$$V = \mathrm{e}^{\frac{i\sigma_2}{2}\theta}\mathrm{e}^{\frac{i\sigma_3}{2}\phi} = \begin{pmatrix} \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \tag{C.18}$$

$$V^\dagger = \mathrm{e}^{-\frac{i\sigma_3}{2}\phi}\mathrm{e}^{-\frac{i\sigma_2}{2}\theta} = \begin{pmatrix} \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}. \tag{C.19}$$

The corresponding transformation of an operator S acting on the spinors is:

$$S \to S' = VSV^\dagger. \tag{C.20}$$

Let us first apply this transformation to the three Pauli matrices:

$$\begin{aligned}
V\sigma_1 V^\dagger &= \begin{pmatrix} \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} & -\mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ \mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} \\ \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} & -\mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi + i\sin\phi \\ \cos\theta\cos\phi - i\sin\phi & -\sin\theta\cos\phi \end{pmatrix} \\
&= \cos\theta\cos\phi\sigma_1 - \sin\phi\sigma_2 + \sin\theta\cos\phi\sigma_3 && \text{(C.21)}
\end{aligned}$$

[1]Here, the $\pm$ that distinguishes the two step operators is not to be confused with the $\pm$ in J_z, which has to do with the different expressions for the monopole background in the northern and southern hemispheres.

$$\begin{aligned}
V\sigma_2 V^\dagger &= \begin{pmatrix} \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} & -\mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ \mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \\
&= i \begin{pmatrix} \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & -\mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} \\ \mathrm{e}^{\frac{-i\phi}{2}}\cos\frac{\theta}{2} & -\mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\sin\phi & \cos\theta\sin\phi - i\cos\phi \\ \cos\theta\sin\phi + i\cos\phi & -\sin\theta\sin\phi \end{pmatrix} \\
&= \cos\theta\sin\phi\sigma_1 + \cos\phi\sigma_2 + \sin\theta\sin\phi\sigma_3
\end{aligned} \tag{C.22}$$

$$\begin{aligned}
V\sigma_3 V^\dagger &= \begin{pmatrix} \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} & -\mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ \mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{\frac{i\phi}{2}}\cos\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} & -\mathrm{e}^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -\mathrm{e}^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & -\mathrm{e}^{+\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \\
&= -\sin\theta\sigma_1 + \cos\theta\sigma_3.
\end{aligned} \tag{C.23}$$

Also, it is easily seen that the partial derivatives of $V^\dagger$ are given by

$$\partial_\theta V^\dagger = V^\dagger(-\frac{i\sigma_2}{2}), \tag{C.24}$$

$$\partial_\phi V^\dagger = (-\frac{i\sigma_3}{2})V^\dagger. \tag{C.25}$$

Using (C.21)-(C.25), it is now an easy task to calculate the transformation of the angular momentum operators:

$$\begin{aligned}
J'_\pm = VJ_\pm V^\dagger &= V\left(e^{\pm i\phi}\left(\pm\partial_\theta + \cot\theta(i\partial_\phi - A_\phi) + q\sin\theta\right) + \frac{\sigma_1 \pm i\sigma_2}{2}\right)V^\dagger \\
&= e^{\pm i\phi}\left(\pm\partial_\theta + \cot\theta(i\partial_\phi - A_\phi) + q\sin\theta\right) \pm \\
&\quad \pm e^{\pm i\phi}\left(-\frac{i\sigma_2}{2}\right) + ie^{\pm i\phi}\cot\theta V\left(-\frac{i\sigma_3}{2}\right)V^\dagger \\
&= e^{\pm i\phi}\left(\pm\partial_\theta + \cot\theta(i\partial_\phi - A_\phi) + q\sin\theta\right) \mp \\
&\quad \mp\frac{i}{2}e^{\pm i\phi}\sigma_2 + \frac{1}{2}\cot\theta e^{\pm i\phi}[-\sin\theta\sigma_1 + \cos\theta\sigma_3] + \\
&\quad +\frac{1}{2}[\cos\theta\cos\phi\sigma_1 - \sin\phi\sigma_2 + \sin\theta\cos\phi\sigma_3] \pm \\
&\quad \pm\frac{i}{2}[\cos\theta\sin\phi\sigma_1 + \cos\phi\sigma_2 + \sin\theta\sin\phi\sigma_3] \\
&= e^{\pm i\phi}\left(\pm\partial_\theta + \cot\theta(i\partial_\phi - A_\phi) + q\sin\theta + \frac{\sigma_3}{2\sin\theta}\right),
\end{aligned} \tag{C.26}$$

$$\begin{aligned}
J'_z = VJ_zV^\dagger &= V\left(-i\partial_\phi \pm q + \frac{\sigma_3}{2}\right)V^\dagger = -i\partial_\phi \pm q + V\left(-\frac{\sigma_3}{2} + \frac{\sigma_3}{2}\right)V^\dagger \\
&= -i\partial_\phi \pm q.
\end{aligned} \tag{C.27}$$

The total angular momentum operators on $S^2 \times \mathbb{R}$ are thus given by (C.26) and (C.27). In the following, we drop the primes and denote the operators on $S^2 \times \mathbb{R}$ by $J_\pm$ and J_z. Let us now verify that these operators satisfy the SU(2) algebra:

$$\begin{aligned}
[J_+, J_-] &= \Big[\mathrm{e}^{i\phi}\left(\partial_\theta + \cot\theta(i\partial_\phi - A_\phi) + q\sin\theta + \frac{\sigma_3}{2\sin\theta}\right), \\
&\quad \mathrm{e}^{-\phi}\left(-\partial_\theta + \cot\theta(i\partial_\phi - A_\phi) + q\sin\theta + \frac{\sigma_3}{2\sin\theta}\right)\Big] \\
&= \left[\mathrm{e}^{i\phi}\partial_\theta, i\mathrm{e}^{-i\phi}\cot\theta\partial_\phi\right] + [\partial_\theta, -\cot\theta A_\phi + q\sin\theta] + \frac{\sigma_3}{2}\left[\partial_\theta, \frac{1}{\sin\theta}\right] + \\
&\quad + \left[i\cot\theta\mathrm{e}^{i\phi}\partial_\phi, -\mathrm{e}^{-i\phi}\partial_\theta\right] - \cot^2\theta\left[\mathrm{e}^{i\phi}\partial_\phi, \mathrm{e}^{-i\phi}\partial_\phi\right] \\
&\quad + i(-\cot^2\theta A_\phi + q\cos\theta)\left[\mathrm{e}^{i\phi}\partial_\phi, \mathrm{e}^{-i\phi}\right] + i\frac{\sigma_3}{2}\frac{\cos\theta}{\sin^2\theta}\left[\mathrm{e}^{i\phi}\partial_\phi, \mathrm{e}^{-i\phi}\right] \\
&\quad + [-\cot\theta A_\phi, -\partial_\theta] - i\cot^2\theta A_\phi\left[\mathrm{e}^{i\phi}, \mathrm{e}^{-i\phi}\partial_\phi\right] - [-q\sin\theta, -\partial_\theta] - \\
&\quad + iq\cos\theta\left[\mathrm{e}^{i\phi}, \mathrm{e}^{-i\phi}\partial_\phi\right] + \frac{\sigma_3}{2}\left[\frac{1}{\sin\theta}, -\partial_\theta\right] + i\frac{\sigma_3}{2}\frac{\cos\theta}{\sin^2\theta}\left[\mathrm{e}^{i\phi}, \mathrm{e}^{-i\phi}\partial_\phi\right] \\
&= -i\frac{1}{\sin^2\theta}\partial_\phi + \cot\theta\partial_\theta - i\frac{1}{\sin^2\theta}\partial_\phi - \cot\theta\partial_\theta - 2q\frac{\cos\theta}{\sin^2\theta} \pm 2q\frac{1}{\sin^2\theta} - \\
&\quad - \sigma_3\frac{\cos\theta}{\sin^2\theta} + 2i\cot^2\theta\partial_\phi \mp 2q\cot^2\theta + 2q\frac{\cos^3\theta}{\sin^2\theta} + 2q\cos\theta + \sigma_3\frac{\cos\theta}{\sin^2\theta} \\
&= 2i\left(-\frac{1}{\sin^2\theta} + \cot^2\theta\right)\partial_\phi \\
&\quad - 2q\left(\frac{\cos\theta}{\sin^2\theta} \mp \frac{1}{\sin^2\theta} \pm \cot\theta - \frac{\cos^3\theta}{\sin^2\theta} - \cos\theta\right) \\
&= -2i\partial_\phi \pm 2q = 2J_z, \qquad \text{(C.28)} \\
[J_z, J_+] &= [-i\partial_\phi, J_+] = J_+, \qquad \text{(C.29)} \\
[J_z, J_-] &= [-i\partial_\phi, J_-] = -J_-. \qquad \text{(C.30)}
\end{aligned}$$

Next, we evaluate the commutators with the Dirac operator (C.3):

$$\begin{aligned}
[\not{D}_S, J_\pm] =& i\sigma_1\mathrm{e}^{\pm i\phi}[\partial_\theta, \cot\theta] + \sigma_1\mathrm{e}^{\pm i\phi}[\partial_\theta, -\cot\theta A_\phi + q\sin\theta] + \\
& \frac{1}{2}\mathrm{e}^{\pm i\phi}\left[\sigma_1\partial_\theta, \frac{1}{\sin\theta}\sigma_3\right] \mp \frac{1}{2}\mathrm{e}^{\pm i\phi}\sigma_1[\partial_\theta, \cot\theta] + \frac{1}{4}\mathrm{e}^{\pm i\phi}\frac{\cos\theta}{\sin^2\theta}[\sigma_1, \sigma_3] \\
& \pm\sigma_2\left[\frac{1}{\sin\theta}\partial_\phi, \mathrm{e}^{\pm i\phi}\partial_\theta\right] + +i\sigma_2\frac{\cos\theta}{\sin^2\theta}\left[\partial_\phi, \mathrm{e}^{\pm i\phi}\partial_\phi\right] \\
& + \frac{\sigma_2}{\sin\theta}(-\cot\theta A_\phi + q\sin\theta)\left[\partial_\phi, \mathrm{e}^{\pm i\phi}\right] + \frac{1}{2\sin^2\theta}\left[\sigma_2\partial_\phi, \sigma_3\mathrm{e}^{\pm i\phi}\right] \\
& + i\sigma_2\mathrm{e}^{\pm i\phi}\left[\partial_\theta, \frac{-A_\phi}{\sin\theta}\right] + \frac{iA_\phi}{2\sin^2\theta}\mathrm{e}^{\pm i\phi}[\sigma_2, \sigma_3] \\
=& \sigma_1\mathrm{e}^{\pm i\phi}\left[-\frac{i}{\sin^2\theta}\partial_\phi - q\left(\mp\frac{1}{\sin^2\theta} + \frac{\cos\theta}{\sin^2\theta}\right) \pm \frac{1}{2\sin^2\theta}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{i}{2\sin^2\theta}(\pm i+2\partial_\phi)-\frac{q}{\sin^2\theta}(\pm 1-\cos\theta)\Big]+\\
&+\sigma_2 \mathrm{e}^{\pm i\phi}\left[+\frac{i\cos\theta}{2\sin^2\theta}-\frac{i}{\sin\theta}\partial_\phi-\frac{i\cos\theta}{2\sin^2\theta}+\frac{i}{\sin\theta}\partial_\theta\pm\frac{\cos\theta}{\sin^2\theta}\partial_\phi\mp\right.\\
&\left.\mp\frac{\cos\theta}{\sin^2\theta}\mp iq\left(\pm\frac{\cos\theta}{\sin^2\theta}-\frac{1}{\sin^2\theta}\right)\mp iq\left(\mp\frac{\cos\theta}{\sin^2\theta}+\frac{1}{\sin^2\theta}\right)\right]\\
=&0
\end{aligned}
\tag{C.31}
$$

and trivially

$$[\not{D}_S, J_z] = 0. \tag{C.32}$$

Since these commutators are zero, the angular momentum operators and the Dirac operator on S^2 are simultaneously diagonalizable. Thus, the monopole spinor harmonics are also angular momentum eigenspinors, and we can use all the machinery of the SU(2) algebra to calculate them.
Finally, we calculate the Casimir operator J^2:

$$
\begin{aligned}
J^2 =&J_z^2+\frac{1}{2}\{J_+,J_-\}\\
=&-\partial_\phi^2+q^2\mp 2iq\partial_\phi+\frac{1}{2}\left\{\mathrm{e}^{i\phi}\left(\partial_\theta+\cot\theta(i\partial_\phi-A_\phi)+q\sin\theta+\frac{\sigma_3}{2\sin\theta}\right),\right.\\
&\left.\mathrm{e}^{-i\phi}\left(-\partial_\theta+\cot\theta(i\partial_\phi-A_\phi)+q\sin\theta+\frac{\sigma_3}{2\sin\theta}\right)\right\}\\
=&-\partial_\phi^2+q^2\mp 2iq\partial_\phi-\partial_\theta^2+\frac{1}{2}\left\{\mathrm{e}^{i\phi}\partial_\theta,\mathrm{e}^{-i\phi}\cot\theta(i\partial_\phi-A_\phi)\right\}+\\
&+\frac{1}{2}\left\{\partial_\theta,q\sin\theta\right\}+\frac{1}{2}\left\{\partial_\theta,\frac{\sigma_3}{2\sin\theta}\right\}\\
&+\frac{1}{2}\left\{\mathrm{e}^{i\phi}\cot\theta(i\partial_\phi-A_\phi),-\mathrm{e}^{-i\phi}\partial_\theta\right\}+\\
&+\frac{1}{2}\left\{\mathrm{e}^{i\phi}\cot\theta(i\partial_\phi-A_\phi),\mathrm{e}^{-i\phi}\cot\theta(i\partial_\phi-A_\phi)\right\}\\
&+\frac{1}{2}\left\{\mathrm{e}^{i\phi}\cot\theta(i\partial_\phi-A_\phi),\mathrm{e}^{-i\phi}q\sin\theta\right\}\\
&+\frac{1}{2}\left\{\mathrm{e}^{i\phi}\cot\theta(i\partial_\phi-A_\phi),\mathrm{e}^{-i\phi}\frac{\sigma_3}{2\sin\theta}\right\}+\\
&+\frac{1}{2}\left\{q\sin\theta,-\partial_\theta\right\}+\frac{1}{2}\left\{\mathrm{e}^{i\phi}q\sin\theta,\mathrm{e}^{-i\phi}\cot\theta(i\partial_\phi-A_\phi)\right\}+\\
&+\frac{1}{2}\left\{q\sin\theta,q\sin\theta\right\}+\frac{1}{2}\left\{q\sin\theta,\frac{\sigma_3}{2\sin\theta}\right\}\\
&+\frac{1}{2}\left\{\frac{\sigma_3}{2\sin\theta}\mathrm{e}^{i\phi},\mathrm{e}^{-i\phi}\cot\theta(i\partial_\phi-A_\phi)\right\}+\\
&+\frac{1}{2}\left\{\frac{\sigma_3}{2\sin\theta},-\partial_\theta\right\}+\frac{1}{2}\left\{\frac{\sigma_3}{2\sin\theta},q\sin\theta\right\}+\frac{1}{2}\left\{\frac{\sigma_3}{2\sin\theta},\frac{\sigma_3}{2\sin\theta}\right\}\\
=&-\partial_\phi^2+q^2\mp 2iq\partial_\phi-\partial_\theta^2-\cot\theta\partial_\theta+\cot^2\theta(i\partial_\phi-A_\phi)^2\\
&+2q\cos\theta(i\partial_\phi-A_\phi)+\sigma_3\left(\frac{\cos\theta}{\sin^2\theta}(i\partial_\phi-A_\phi)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ q^2 \sin^2\theta + q\sigma_3 - \frac{1}{4\sin^2\theta} \\
=& - \partial_\theta^2 - \cot\theta\partial_\theta - \frac{1}{4\sin^2\theta} + q^2 + \sigma_3 \left(\frac{\cos\theta}{\sin^2\theta}(i\partial_\phi - A_\phi) + q \right) + \\
&+ \cot^2\theta \left(-\partial_\phi^2 + q^2(\pm 1 - \cos\theta)^2 - 2iq(\pm 1 - \cos\theta)\partial_\phi \right) \\
&- \partial_\phi^2 \mp 2iq\partial_\phi + 2q\cos\theta(i\partial_\phi \mp q + q\cos\theta) + q^2\sin^2\theta \\
=& - \partial_\theta^2 - \cot\theta\partial_\theta - \frac{1}{4\sin^2\theta} + q^2 + \sigma_3 \left(\frac{\cos\theta}{\sin^2\theta}(i\partial_\phi - A_\phi) + q \right) + \\
&+ (\cot^2\theta + 1)\,(i\partial_\phi - A_\phi)^2 \\
=& - \partial_\theta^2 - \cot\theta\partial_\theta - \frac{1}{4\sin^2\theta} + q^2 + i\sigma_3 \left(\frac{\cos\theta}{\sin^2\theta}(\partial_\phi + iA_\phi) - iq \right) - \\
&- \frac{1}{\sin^2\theta}(\partial_\phi + iA_\phi)^2.
\end{aligned}
\tag{C.33}
$$

Comparing this expression to (C.4) yields:

$$
J^2 = -\not{D}_S^2 - \frac{1}{4} + q^2. \tag{C.34}
$$

C.3 Eigenvalue Equations

Since the monopole spinor harmonics are angular momentum eigenspinors they are, in addition to the monopole charge q, labelled by the SU(2) quantum numbers j and m. We denote these spinors by Υ_{qjm}. The eigenvalue equations are

$$
J^2\Upsilon_{qjm} = j(j+1)\Upsilon_{qjm} \tag{C.35}
$$

$$
J_z\Upsilon_{qjm} = m\Upsilon_{qjm} \tag{C.36}
$$

and

$$
-i\not{D}_S\Upsilon_{qjm} = \lambda\Upsilon_{qjm}, \tag{C.37}
$$

which implies

$$
\not{D}_S^2\Upsilon_{qjm} = -\lambda^2\Upsilon_{qjm}. \tag{C.38}
$$

Due to (C.34), the eigenvalues λ and j are related in the following way:

$$
\left(-J^2 - \not{D}_S^2 - \frac{1}{4} + q^2 \right) \Upsilon_{qjm} = \left(-j(j+1) + \lambda^2 - \frac{1}{4} + q^2 \right) \Upsilon_{qjm} = 0, \tag{C.39}
$$

which yields

$$
\lambda = \pm\frac{1}{2}\sqrt{(2j+1)^2 - 4q^2}. \tag{C.40}
$$

This relation tells us that, for every value of q and j, there are two different eigenvalues of the Dirac operator. From now on, we denote these eigenvalues by

$$\Delta^{\pm}_{qj} = \pm\frac{1}{2}\sqrt{(2j+1)^2 - 4q^2} \tag{C.41}$$

and their corresponding eigenspinors by $\Upsilon^{\pm}_{qjm}$.[2] The relation (C.41) gives us a lower bound on j (and therefore on the size of the SU(2) representation), namely:

$$j \geq |q| - \frac{1}{2}. \tag{C.42}$$

The zero mode eigenspinor of the Dirac operator has angular momentum eigenvalue $j = |q| - \frac{1}{2}$ and does not exist for $q = 0$. Clearly, there is only one independent zero mode eigenspinor.

C.4 Finding the Eigenspinors

Let us now go on to find the explicit expressions for the monopole spinor harmonics. Using (C.36) and (C.27) we have

$$\partial_\phi \Upsilon_{qjm} = i(m \mp q)\Upsilon_{qjm}, \tag{C.43}$$

which means that we can separate out the ϕ-dependence in Υ_{qjm}. Thus, we write

$$\Upsilon_{qjm}(\theta, \phi) = \mathrm{e}^{i(m\mp q)\phi} \begin{pmatrix} \alpha(\theta) \\ \beta(\theta) \end{pmatrix}, \tag{C.44}$$

where we have suppressed the q-, j-, and m-labels on the θ-dependent two component spinor and where $\alpha(\theta)$ and $\beta(\theta)$ are two arbitrary functions.

C.4.1 Lowest Weight Eigenspinors

Our next step is to find the eigenspinors corresponding to the lowest SU(2) weight $m = -j$. Applying the angular momentum step operators to this spinor will then give us the complete set of eigenspinors. The lowest weight eigenspinors obey the equation

$$J_- \Upsilon_{q,-m,m}(\theta, \phi) = 0, \tag{C.45}$$

which is the same as

$$\mathrm{e}^{-i\phi}\left(-\partial_\theta + \cot\theta(i\partial_\phi - A_\phi) + q\sin\theta + \frac{\sigma_3}{2\sin\theta}\right)\Upsilon_{q,-m,m}(\theta,\phi) = 0. \tag{C.46}$$

Inserting (C.44) into this equation yields

$$0 = \mathrm{e}^{-i\phi}\left(-\partial_\theta + \cot\theta(i\partial_\phi - A_\phi) + q\sin\theta + \frac{\sigma_3}{2\sin\theta}\right)\mathrm{e}^{i(m\mp q)\phi}\begin{pmatrix} \alpha(\theta) \\ \beta(\theta) \end{pmatrix}, \tag{C.47}$$

[2] Δ_{qj} and Υ_{qjm} with suppressed $\pm$ denotes either of the two eigenvalues or eigenspinors.

which implies

$$\begin{aligned} 0 &= \left(-\partial_\theta + q\sin\theta + \frac{\sigma_3}{2\sin\theta} + \cot\theta(-A_\phi - m \pm q)\right)\begin{pmatrix}\alpha(\theta)\\ \beta(\theta)\end{pmatrix} \\ &= \left(-\partial_\theta + q\sin\theta + \frac{\sigma_3}{2\sin\theta} - \cot\theta(m - q\cos\theta)\right)\begin{pmatrix}\alpha(\theta)\\ \beta(\theta)\end{pmatrix}. \end{aligned} \tag{C.48}$$

This equation can be put in a simpler form by making the variable substitution $x = \cos\theta$. Expressed in x the equation reads

$$\left(\sqrt{1-x^2}\partial_x - \frac{-q + mx - \frac{1}{2}\sigma_3}{\sqrt{1-x^2}}\right)\begin{pmatrix}a(x)\\ b(x)\end{pmatrix} = 0, \tag{C.49}$$

for two arbitrary functions $a(x)$ and $b(x)$. Due to the σ_3-matrix we get different equations for the upper and lower spinor components, namely:

$$a'(x) = \frac{mx - q - \frac{1}{2}}{1-x^2}a(x) \tag{C.50}$$

and

$$b'(x) = \frac{mx - q + \frac{1}{2}}{1-x^2}b(x). \tag{C.51}$$

Starting with (C.50), we note that the equation is separable and we can write

$$\frac{1}{a}da = \frac{mx - q - \frac{1}{2}}{1-x^2}dx. \tag{C.52}$$

Integrating both sides gives

$$\ln a = \frac{1}{4}(1 - 2m + 2q)\ln(1-x) + \frac{1}{4}(-1 - 2m - 2q)\ln(1+x) + C_a, \tag{C.53}$$

where C_a is an integration constant (possibly dependent on q and m) that will later be fixed by normalization. Finally, taking the exponent of (C.53) yields the solution

$$a(x) = C_a(1-x)^{\frac{1}{4}(1-2m+2q)}(1+x)^{\frac{1}{4}(-1-2m-2q)}. \tag{C.54}$$

In precicely the same way one also finds

$$b(x) = C_b(1-x)^{\frac{1}{4}(-1-2m+2q)}(1+x)^{\frac{1}{4}(1-2m-2q)}. \tag{C.55}$$

Since x is a more convenient variable to work with, we will keep using it throughout the calculations, rather than changing back to θ. Using expressions (C.54) and (C.55) we can write the lowest weight eigenspinor as

$$\Upsilon_{q,-m,m}(x,\phi) = \mathrm{e}^{i(m\mp q)\phi}\begin{pmatrix} C_a(1-x)^{\frac{1}{4}(1-2m+2q)}(1+x)^{\frac{1}{4}(-1-2m-2q)} \\ C_b(1-x)^{\frac{1}{4}(-1-2m+2q)}(1+x)^{\frac{1}{4}(1-2m-2q)}\end{pmatrix}. \tag{C.56}$$

Now, all that remains is to fix the constants C_a and C_b. First, changing variables to x in the eigenvalue equation of the Dirac operator (C.37) and inserting (C.56) gives us

$$\begin{aligned} & \sigma_1\left(-\sqrt{1-x^2}\partial_x+\frac{1}{2}\frac{x}{\sqrt{1-x^2}}\right)\Upsilon^{\pm}_{q,-m,m}(x,\phi)+ \\ & +\frac{\sigma_2}{\sqrt{1-x^2}}\left(\partial_\phi+iq(\pm 1-x)\right)\Upsilon^{\pm}_{q,-m,m}(x,\phi) \\ = \; & i\Delta^{\pm}_{q,-m}\Upsilon^{\pm}_{q,-m,m}(x,\phi). \end{aligned} \tag{C.57}$$

Let us look at the upper component of this equation:

$$\begin{aligned} & C_b\left(\sqrt{1-x^2}\frac{1}{4}(-1-2m+2q)(1-x)^{-1}-\right. \\ & -\sqrt{1-x^2}\frac{1}{4}(1-2m-2q)(1+x)^{-1}+ \\ & \left.+\frac{1}{2}\frac{x}{\sqrt{1-x^2}}-\frac{i}{\sqrt{1-x^2}}(i(m\mp q)+iq(\pm 1-x))\right)\times \\ & \times(1-x)^{\frac{1}{4}(-1-2m+2q)}(1+x)^{\frac{1}{4}(1-2m-2q)} \\ = \; & C_a i\Delta_{q,-m}(1-x)^{\frac{1}{4}(1-2m+2q)}(1+x)^{\frac{1}{4}(-1-2m-2q)}. \end{aligned} \tag{C.58}$$

By dividing with $\sqrt{1-x^2}(1-x)^{\frac{1}{4}(-1-2m+2q)}(1+x)^{\frac{1}{4}(1-2m-2q)}$ we arrive at

$$\begin{aligned} & C_a i\Delta_{q,-m}(1-x) \\ = & C_b\left(\frac{1}{4}(-1+2m+2q)(1+x)-\frac{1}{4}(1-2m-2q)(1-x)+\frac{1}{2}x+m-qx\right), \end{aligned} \tag{C.59}$$

which simplifies to

$$C_b\left(m-\frac{1}{2}+q\right)=C_a i\Delta_{q,-m}. \tag{C.60}$$

Inserting the expressions (C.41) for $\Delta_{q,-m}$ finally gives us

$$\frac{C_b}{C_a}=\mp i\sqrt{\frac{1-2m+2q}{1-2m-2q}} \tag{C.61}$$

for $\Upsilon^{\pm}_{q,-m,m}(x,\phi)$. Having determined the relationship between C_a and C_b, we can fix the absolute value by normalization. The overall phase will still be arbitrary. We normalize the spinors in the following way:

$$\int_0^{2\pi} d\phi\int_0^{\pi}\left(\Upsilon^{\pm}_{q,-m,m}(x,\phi)\right)^{\dagger}\Upsilon^{\pm}_{q,-m,m}(x,\phi)\sin\theta\, d\theta=1. \tag{C.62}$$

Inserting (C.56) into this equation, changing integration variables to x and using (C.61) yields

$$\begin{aligned} 1 \; = \; & 2\pi|C_a|^2\int_{-1}^{1}\left[(1-x)^{\frac{1}{2}(1-2m+2q)}(1+x)^{\frac{1}{2}(-1-2m-2q)}\right. \\ & \left.+\frac{1-2m+2q}{1-2m-2q}(1-x)^{\frac{1}{2}(-1-2m+2q)}(1+x)^{\frac{1}{2}(1-2m-2q)}\right] dx. \end{aligned} \tag{C.63}$$

Evaluating the integral, we arrive at

$$|C_a| = \frac{1}{\sqrt{2\pi}} \frac{(\frac{1}{2})^{1-m}\sqrt{\frac{1}{2}-m}\sqrt{1-2m-2q}\sqrt{(-2m)!}}{\sqrt{\Gamma(\frac{3}{2}-m+q)\Gamma(\frac{3}{2}-m-q)}}, \tag{C.64}$$

which means that the lowest weight eigenspinors are, up to a total phase, given by

$$\begin{aligned}\Upsilon^{\pm}_{q,-m,m}(x,\phi) =& \frac{(\frac{1}{2})^{1-m}\sqrt{\frac{1}{2}-m}\sqrt{(-2m)!}}{\sqrt{\Gamma(\frac{3}{2}-m+q)\Gamma(\frac{3}{2}-m-q)}} \frac{\mathrm{e}^{i(m\mp q)\phi}}{\sqrt{2\pi}} \times \\ &\times \begin{pmatrix} \sqrt{1-2m-2q}(1-x)^{\frac{1}{4}(1-2m+2q)}(1+x)^{\frac{1}{4}(-1-2m-2q)} \\ \mp i\sqrt{1-2m+2q}(1-x)^{\frac{1}{4}(-1-2m+2q)}(1+x)^{\frac{1}{4}(1-2m-2q)} \end{pmatrix},\end{aligned} \tag{C.65}$$

or expressed in j

$$\begin{aligned}\Upsilon^{\pm}_{q,j,-j}(x,\phi) =& \frac{(\frac{1}{2})^{1+j}\sqrt{\frac{1}{2}+j}\sqrt{(2j)!}}{\sqrt{\Gamma(\frac{3}{2}+j+q)\Gamma(\frac{3}{2}+j-q)}} \frac{\mathrm{e}^{i(-j\mp q)\phi}}{\sqrt{2\pi}} \times \\ &\times \begin{pmatrix} \sqrt{1+2j-2q}(1-x)^{\frac{1}{4}(1+2j+2q)}(1+x)^{\frac{1}{4}(-1+2j-2q)} \\ \mp i\sqrt{1+2j+2q}(1-x)^{\frac{1}{4}(-1+2j+2q)}(1+x)^{\frac{1}{4}(1+2j-2q)} \end{pmatrix}.\end{aligned} \tag{C.66}$$

C.4.2 Eigenspinors for General m

Having found the normalized lowest weight SU(2) eigenspinors, we now want to generalize to other values of m by using the J_+-operator. In general, the action of a step operator on an angular momentum eigenstate $|j,m\rangle$ is given by

$$|j,m+1\rangle = \frac{1}{\sqrt{(j-m)(j+m+1)}} J_+ |j,m\rangle\,. \tag{C.67}$$

We obtain the general state $|j,m\rangle$ if we let J_+ act on the lowest weight state $(j+m)$ times. Thus, we pick up an extra normalization factor of

$$\frac{1}{\sqrt{(2j)(2j-1)...(j-m+1)\times(j+m)!}} = \sqrt{\frac{(j-m)!}{(2j)!(j+m)!}}. \tag{C.68}$$

Next, we turn to the action of J_+ on the eigenspinors. Changing variables to x in (C.26) we have

$$J_+ = \mathrm{e}^{i\phi}\left(-\sqrt{1-x^2}\partial_x + \frac{x}{\sqrt{1-x^2}}(i\partial_\phi \mp q + qx) + q\sqrt{1-x^2} + \frac{\sigma_3}{2\sqrt{1-x^2}}\right). \tag{C.69}$$

To avoid writing out the constant factors of (C.66) all the time, we define $P_{q,j,-j}$ by

$$e^{i(-j\mp q)\phi} P_{q,j,-j} = e^{i(-j\mp q)\phi} \begin{pmatrix} (1-x)^{\frac{1}{4}(1+2j+2q)}(1+x)^{\frac{1}{4}(-1+2j-2q)} \\ (1-x)^{\frac{1}{4}(-1+2j+2q)}(1+x)^{\frac{1}{4}(1+2j-2q)} \end{pmatrix}, \quad \text{(C.70)}$$

which we, by defining $j_{ab} = j + a\frac{1}{2} + bq$, further can streamline to

$$e^{i(-j\mp q)\phi} P_{q,j,-j} = e^{i(-j\mp q)\phi} \begin{pmatrix} (1-x)^{\frac{j_{++}}{2}}(1+x)^{\frac{j_{--}}{2}} \\ (1-x)^{\frac{j_{-+}}{2}}(1+x)^{\frac{j_{+-}}{2}} \end{pmatrix}. \quad \text{(C.71)}$$

Our task is now to generalize (C.71) to any value of m. The ϕ-dependent part is easily handled. Because of the $e^{i\phi}$-factor in (C.69) we see that applying J_+ $(j+m)$ times to (C.71) changes the exponential part into $e^{i(m\mp q)\phi}$.

Next, we turn to the more complicated x-dependent part. Our strategy will be to apply J_+ to (C.71) a couple of times, until we can see a pattern and make a qualified guess about the general form of P_{qjm}. By induction, we will then prove that our assumption is true.

A simple calculation shows that acting with J_+ on (C.71) yields

$$P_{q,j,-j+1} = (1-x^2)^{-\frac{1}{2}} \begin{pmatrix} (2jx+1+2q)(1-x)^{\frac{j_{++}}{2}}(1+x)^{\frac{j_{--}}{2}} \\ (2jx-1+2q)(1-x)^{\frac{j_{-+}}{2}}(1+x)^{\frac{j_{+-}}{2}} \end{pmatrix}, \quad \text{(C.72)}$$

which can be rewritten as

$$P_{q,j,-j+1} = -(1-x^2)^{\frac{1}{2}} \begin{pmatrix} \left((1-x)^{j_{++}}(1+x)^{j_{--}}\right)^{-\frac{1}{2}} \partial_x (1-x)^{j_{++}}(1+x)^{j_{--}} \\ \left((1-x)^{j_{-+}}(1+x)^{j_{+-}}\right)^{-\frac{1}{2}} \partial_x (1-x)^{j_{-+}}(1+x)^{j_{+-}} \end{pmatrix}. \quad \text{(C.73)}$$

Likewise, we can show that acting with J_+ an additional time gives

$$P_{q,j,-j+2} = (1-x^2) \begin{pmatrix} \left((1-x)^{j_{++}}(1+x)^{j_{--}}\right)^{-\frac{1}{2}} \partial_x^2 (1-x)^{j_{++}}(1+x)^{j_{--}} \\ \left((1-x)^{j_{-+}}(1+x)^{j_{+-}}\right)^{-\frac{1}{2}} \partial_x^2 (1-x)^{j_{-+}}(1+x)^{j_{+-}} \end{pmatrix}. \quad \text{(C.74)}$$

As a generalization of (C.73) and (C.74) to any value of m, we make the ansatz

$$\begin{aligned} P_{qjm} =& (-1)^{j+m}(1-x^2)^{\frac{m+j}{2}} \times \\ & \times \begin{pmatrix} \left((1-x)^{j_{++}}(1+x)^{j_{--}}\right)^{-\frac{1}{2}} \partial_x^{j+m} (1-x)^{j_{++}}(1+x)^{j_{--}} \\ \left((1-x)^{j_{-+}}(1+x)^{j_{+-}}\right)^{-\frac{1}{2}} \partial_x^{j+m} (1-x)^{j_{-+}}(1+x)^{j_{+-}} \end{pmatrix} \\ =& (-1)^{j+m} \begin{pmatrix} (1-x)^{\frac{m_{--}}{2}}(1+x)^{\frac{m_{++}}{2}} \partial_x^{j+m} (1-x)^{j_{++}}(1+x)^{j_{--}} \\ (1-x)^{\frac{m_{+-}}{2}}(1+x)^{\frac{m_{-+}}{2}} \partial_x^{j+m} (1-x)^{j_{-+}}(1+x)^{j_{+-}} \end{pmatrix}, \end{aligned} \quad \text{(C.75)}$$

where we, analogously to j_{ab}, have defined $m_{ab} = m + a\frac{1}{2} + bq$. Clearly, (C.75) reduces to $P_{q,j,-j}$ in (C.71) for $m = -j$. Let us now act with J_+

on $e^{i(m\mp q)\phi}P_{qjm}$. The partial ϕ-derivative brings down a factor of $i(m\mp q)$ from the exponential. Not writing out the exponential, we have for the upper component

$$\begin{aligned}
&\left[-\sqrt{1-x^2}\partial_x+\frac{1}{\sqrt{1-x^2}}(-mx+q+\frac{1}{2})\right]\times\\
&\times(-1)^{j+m}(1-x)^{\frac{m_{--}}{2}}(1+x)^{\frac{m_{++}}{2}}\partial_x^{j+m}(1-x)^{j_{++}}(1+x)^{j_{--}}\\
=&(-1)^{j+m+1}(1-x^2)^{\frac{1}{2}}\left(\frac{m_{++}}{2}(1+x)^{-1}-\frac{m_{--}}{2}(1-x)^{-1}\right)\times\\
&\times(1-x)^{\frac{m_{--}}{2}}(1+x)^{\frac{m_{++}}{2}}\partial_x^{j+m}(1-x)^{j_{++}}(1+x)^{j_{--}}\\
&+(-1)^{j+m+1}(1-x^2)^{\frac{1}{2}}(1-x)^{\frac{m_{--}}{2}}(1+x)^{\frac{m_{++}}{2}}\partial_x^{j+m+1}(1-x)^{j_{++}}(1+x)^{j_{--}}\\
&+(1-x^2)^{-\frac{1}{2}}(-mx+q+\frac{1}{2})(1-x)^{\frac{m_{--}}{2}}(1+x)^{\frac{m_{++}}{2}}\partial_x^{j+m}(1-x)^{j_{++}}(1+x)^{j_{--}}\\
=&(-1)^{j+m+1}(1-x)^{\frac{(m+1)_{--}}{2}}(1+x)^{\frac{(m+1)_{++}}{2}}\partial_x^{j+m+1}(1-x)^{j_{++}}(1+x)^{j_{--}}+\\
&+(-1)^{j+m}(1-x^2)^{-\frac{1}{2}}\left(-\frac{m_{++}}{2}(1-x)+\frac{m_{--}}{2}(1+x)-mx+q+\frac{1}{2}\right)\\
=&(-1)^{j+m+1}(1-x)^{\frac{(m+1)_{--}}{2}}(1+x)^{\frac{(m+1)_{++}}{2}}\partial_x^{j+m+1}(1-x)^{j_{++}}(1+x)^{j_{--}}.
\end{aligned}\tag{C.76}$$

In the same way, we can show that the lower component turns into

$$(-1)^{j+m+1}(1-x)^{\frac{(m+1)_{+-}}{2}}(1+x)^{\frac{(m+1)_{-+}}{2}}\partial_x^{j+m+1}(1-x)^{j_{-+}}(1+x)^{j_{+-}}.\tag{C.77}$$

Thus, (C.76) and (C.77) are indeed of the same form as P_{qjm} in our ansatz (C.75), with the m-value shifted by 1. This completes the proof that our ansatz was correct.
Putting everything together, we are now ready to give the complete explicit expressions for the monopole spinor harmonics:

$$\begin{aligned}
\Upsilon^{\pm}_{qjm}(x,\phi)=&\frac{(-1)^{j+m}(\frac{1}{2})^{j+1}\sqrt{j+\frac{1}{2}}}{\sqrt{\Gamma(\frac{3}{2}+j+q)\Gamma(\frac{3}{2}+j-q)}}\sqrt{\frac{(j-m)!}{(j+m)!}}\frac{\mathrm{e}^{i(m\mp q)\phi}}{\sqrt{2\pi}}\times\\
&\times\begin{pmatrix}\sqrt{1+2j-2q}(1-x)^{\frac{m_{--}}{2}}(1+x)^{\frac{m_{++}}{2}}\partial_x^{j+m}(1-x)^{j_{++}}(1+x)^{j_{--}}\\ \mp i\sqrt{1+2j+2q}(1-x)^{\frac{m_{+-}}{2}}(1+x)^{\frac{m_{-+}}{2}}\partial_x^{j+m}(1-x)^{j_{-+}}(1+x)^{j_{+-}}\end{pmatrix}.
\end{aligned}\tag{C.78}$$

C.5 Some Useful Properties

In this section we record some useful properties of the monopole spinor harmonics. It is immediately clear that the spinors satisfy

$$\gamma^{\tau}\Upsilon^{\pm}_{qjm}=\Upsilon^{\mp}_{qjm}.\tag{C.79}$$

As said before, there is only one independent zero mode eigenspinor for each value of q and m. We define this spinor by

$$\Upsilon^0_{qm} \equiv \frac{1}{\sqrt{2}} \left(\Upsilon^+_{qjm} + \mathrm{sign}(q) \Upsilon^-_{qjm} \right)_{j=|q|-\frac{1}{2}}, \tag{C.80}$$

from which we directly can infer

$$\gamma^\tau \Upsilon^0_{qjm} = \mathrm{sign}(q) \Upsilon^0_{qjm}. \tag{C.81}$$

The spinors can also be shown to be orthogonal:

$$\int d\Omega\, \Upsilon^{0\dagger}_{qm} \Upsilon^0_{qm'} \quad = \quad \delta_{mm'} \tag{C.82}$$

$$\int d\Omega\, \Upsilon^{\varepsilon\dagger}_{qjm} \Upsilon^{\varepsilon'}_{qj'm'} \quad = \quad \delta^{\varepsilon\varepsilon'} \delta_{jj'} \delta_{mm'}. \tag{C.83}$$

Finally, we have the completeness relations

$$\sum_m \Upsilon^0_{qm}(\Omega) \Upsilon^{0\dagger}_{qm}(\Omega') + \sum_{jm\varepsilon} \Upsilon^\varepsilon_{qjm}(\Omega) \Upsilon^{\varepsilon\dagger}_{qjm}(\Omega') = \delta^2(\Omega - \Omega'), \tag{C.84}$$

which allows us to expand a general spinor on S^2 using the monopole spinor harmonics.

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