Master of science thesis in physics

Non-Abelian Born-Infeld Theory from Strings

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Abstract

In this thesis we find the low energy non-Abelian Born-Infeld effective action in bosonic string theory to order α'^2 describing coincident D25-branes, which is equivalent to the effective action of open strings with non-dynamical Chan-Paton degrees of freedom at the ends of the strings. To this order, the effective action is determined from string tree-level physics including four-point scattering amplitudes. We develop and summarize all essential tools required to calculate this action including functional formulation of quantum field theory and string scattering physics. The analysis is mostly focused on massless gauge bosons, but also on the scalar tachyons which at a first stage are easier to consider.

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1

Introduction

At small length scales all classical fields have to be modified, as the macroscopical description of nature merely reveals an approximate truth. In the 30's, M. Born and L. Infeld tries to formulate an alternative formulation of classical electrodynamics. It is in this context, as a non-linear generalization of Maxwell's theory of electrodynamics [1], that the Born-Infeld action first appears. In spite of several nice features not present in the existing linear theories, such as a regular electric field at $r = 0$ and a finite total energy, the non-linear classical theory did not seem to manifest itself in nature and the theory was discarded. So classically, it is sufficient to consider the ordinary linear theory with the Maxwell Lagrangian. It has the structure

$$\mathcal{L}_{\mathrm{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \propto \mathbf{H}^2 - \mathbf{E}^2, \tag{1.1}$$

where $F_{\mu\nu}$ is the antisymmetric field strength tensor, and $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric fields respectively. To understand how the Born-Infeld Lagrangian appears from ordinary electrodynamics an analogy with the classical physics of a point particle is in order. As a consequence of Einstein's special theory of relativity, the non-relativistic Newtonian Lagrangian $\mathcal{L}_{\mathrm{Newton}} = m\mathbf{v}^2/2$ of a free particle respecting invariance under Galilei transformations, is generalized to a Lagrangian with an upper limit c on the particle velocity $\mathbf{v}$, now respecting Lorentz invariance:

$$\mathcal{L}_{\mathrm{Einstein}} \propto mc^2(1 - \sqrt{1 - \mathbf{v}^2/c^2}). \tag{1.2}$$

This is one example of a theory which is non-linearized, reducing to the linear approximate version of the theory in a certain limit. In complete analogy, an upper limit on the field strength can be introduced, and accordingly we end up with a generalized Lagrangian of classical electrodynamics - the Born-Infeld Lagrangian:

$$\begin{aligned}\mathcal{L}_{\mathrm{BI}} &\propto \sqrt{1+\vartheta(1-\mathbf{E}^2/\mathbf{H}^2)}-1 \\ &\sim \sqrt{1+\vartheta' F_{\mu\nu}F^{\mu\nu}}-1. \end{aligned} \tag{1.3}$$

Clearly, both the Born-Infeld Lagrangian and the Maxwell's Lagrangian are Lorentz invariant. ϑ and ϑ' are constants chosen such that the Maxwell action is reproduced at small field strengths. Assuming ϑ and ϑ' to be positive, the Born-Infeld Lagrangian restricts the electric field of the point particle from becoming infinite.

After almost falling into oblivion for a long time, the Born-Infeld action re-enters into a completely different field of physics. It appears in string theory as an exact tree-level effective action of an Abelian vector field in the particular case when the field strength is constant and small [3]. Now the derivative-independent part of the action involves a determinant $\sqrt{\det(\delta_{\mu\nu}+2\pi\alpha' F_{\mu\nu})}$ and is defined in 26 space-time dimensions (in the bosonic string theory), but has the same structure as Eq. (1.3) up order ${\alpha'}^2$.

It is only recently, with the discovery of Dirichlet-branes, that the complete significance of the Born-Infeld action has been fully appreciated. It is an essential part of the low energy effective action of a non-perturbative extended object: a Dirichlet-brane, D-brane for short. In the discussion below, we focus on the massless fields of the theory. The following $(p+1)$-dimensional world-volume integral, the *Dirac-Born-Infeld action*, summarizes the low energy dynamics of a single Dp-brane embedded in a space-time with metric $G_{\mu\nu}$:

$$S_p = -T_p \int d^{p+1}x\; e^{-\phi}\sqrt{-\det(G_{mn}+B_{mn}+2\pi\alpha' F_{mn})}, \tag{1.4}$$

where T_p is the tension of the Dp-brane - the mass per unit volume. ϕ, G_{mn} and B_{mn}, $m=0,\ldots,p$ are the pullbacks of the dilaton, metric and the antisymmetric tensor, respectively, to the D-brane world-volume. Neglecting all derivatives acting on field strengths, this low energy effective action is a good approximation when the fields are varying slowly. When $B_{mn}=0$, the action is of quadratic order in the field strengths equivalent to the U(1) Maxwell action reduced to $(p+1)$ dimensions. Obviously, when $B_{mn}=F_{mn}=0$, this action is nothing but a generalized version of the Nambu-Goto action for higher dimensional objects.

Eq. (1.4), describing a single Dp-brane, is an *Abelian* action, *i.e.* all fields commute. When instead considering multiple D-branes, matters become more complicated. For instance, the world-volume fields become *non-Abelian* and take their values in the Lie algebra U(N), if the open strings stretching between the branes are oriented. Several attempts to generalize the Born-Infeld action describing one D-brane to a non-Abelian action describing a stack of them have been made. The proper (perhaps closed) form of it is however not known to date.

In addition to the massless gauge fields $A_m,\ m = 0, \ldots, p$ living on the Dp-brane, $(25-p)$ transverse scalars $X_s,\ s = (p+1), \ldots, 25$, also appear. The D25-brane action contains no transverse scalars and is consequently easiest to consider. In fact, since the open strings of the theory can end on the D25-brane anywhere in space, D25-branes are equivalent to free strings moving in 26-dimensional space-time. One D25-brane with fields described by the gauge group U(1) can be generalized to a collection of N *coincident* D25-branes with gauge group U(N). Then there is an equivalent description in terms of open strings with Chan-Paton factors - extra non-dynamical degrees of freedom living at the ends of the strings.

The correspondence between bosonic D25-branes and open bosonic strings with "charges" attached to the endpoints suggests a perturbative course of action to obtain the non-Abelian effective action of the coincident D25-branes due to the implied equivalence between the S-matrices. One simply obtains the effective action describing both of these systems from calculating the open string scattering amplitudes of non-Abelian massless gauge fields up to a given order. The effective action describes a point field theory which contains the stringy behavior up to the order under consideration. This approach will be pursued in this thesis. Things become rather messy when going to high orders in the field strength, so this method is convenient only at low orders. Apparently not only the effective actions of the D25-branes are of interest; there also exist lower-dimensional Dp-branes in the theory with $p < 25$. The effective actions of these lower-dimensional Dp-branes are obtained by dimensional reduction from the D25-brane by standard procedure.

The thesis is organized as follows. Chapter 2 gives a rather thorough introduction to the quantum field theory of non-Abelian fields using the functional-integral formalism. In this chapter we also suggest the non-Abelian Lagrangian terms which catch the low energy dynamics of the lowest excitations of open bosonic strings with Chan-Paton degrees of freedom. In Chapter 3 we proceed to actually calculate the scattering amplitudes of the lowest-lying excitations of the open bosonic string, namely the tachyon and the massless gauge boson. Before doing this, a short introduction to the physics of the bosonic string is presented. In Chapter 4 we arrive at the interesting questions concerning whether the proposed Ansätze of effective actions were sufficient to reproduce the low energy string amplitudes. Finding that this is indeed the case, when α' goes to zero, we comment on the problems arising in non-Abelian gauge theories, mostly due to the ordering problems and that there is no covariant notion of slowly-varying fields. Also, a connection to D-branes is made and generic properties of the Born-Infeld action are summarized and commented on. In the Appendices conventions and notations used in the thesis are summarized as well as some useful relations which are inconvenient to include in the text.

2

QFT of Non-Abelian Scalar and Vector Fields

When working with non-Abelian gauge theories it is convenient to adopt the functional-integral formalism. In this chapter, a detailed derivation of the correlation functions, and consequently the Feynman rules, of a tachyonic scalar theory coupled to a non-Abelian gauge theory, with effective actions[1] describing low energy string physics, is carried out. Only tree-level amplitudes are treated in the analysis, so we do not have to worry about any ghost contributions arising from loop diagrams. In particular, it is not necessary to pursue the Faddeev-Popov procedure in this generalized Yang-Mills theory of non-Abelian gauge fields, since the lowest order ghost vertex involving one gauge boson and two ghosts is absent at tree-level.

The effective actions are chosen such that they represent the low energy behaviour of the open bosonic string amplitudes analysed in Chapter 3. In the low energy limit, the only relevant states are the massless and, if present, the tachyonic ones, so the analysis below of the massive scalar fields associated with the string ground state will essentially serve as illustration. The calculations are as generic as possible, specifying to explicit Lie algebras only when comparing to string amplitudes. The Feynman rules for the different theories coming out of these calculations, are summarized in Appendix A.4.

[1]By slight abuse of terminology we alternately use the words "Lagrangian" and "action". Evidently they differ by an integration over space-time.

2.1 Tachyon Effective Lagrangian

Open bosonic string theory inevitably contains a particle state with a negative mass-squared, viz. the tachyon

$$m^2 = -\frac{1}{\alpha'} < 0, \tag{2.1}$$

which will be verified later in Chapter 3. The tachyon field is denoted as $\varphi(x)$, and is carrying the adjoint representation of some Lie algebra $\mathfrak{g}$ with generators T^a:

$$\varphi \equiv \varphi^a T^a. \tag{2.2}$$

For a discussion of allowed groups for non-Abelian gauge fields obtained from open strings, see Section 3.3. We now begin with deducing the Lagrangian for a free non-Abelian scalar field. To have a stable ground state, the Hamiltonian of the theory

$$\mathcal{H} = T + V, \tag{2.3}$$

where T is the kinetic energy density and V is the potential energy density, has to be bounded from below, *i.e.* it has to be positive definite. Further, the second derivative of the potential V at $\varphi = 0$, or any other extremum, is a measure of the mass of the particle associated with the field. This implies that the tachyon theory has an unstable vacuum at $\varphi = 0$, since the potential has a maximum there. This is an unpleasant feature which will be overlooked in the sequel, since the bosonic theory merely serves as a toy-model. A few basic constraints must be put on the effective action for it to make sense. Amongst these are reality, Lorentz and gauge invariance. Analogously to ordinary scalar field theory, we put the free Lagrangian equal to (generally $\mathcal{L} = T - V$)

$$\mathcal{L}^{\text{free}} = -\frac{1}{2}\text{tr}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\text{tr}\varphi^2, \tag{2.4}$$

where the trace is in the fundamental representation of some gauge group $\mathfrak{G}$ corresponding to the Lie algebra $\mathfrak{g}$.

Our goal is to find an effective action that describes the string scattering amplitudes of the tachyon, the lowest lying particle in the open bosonic string spectrum, to order ${\alpha'}^2$. We will do this in a systematic approach. The Lagrangian terms which describe the interaction have to be gauge invariant, which greatly reduces the number of possible terms. For instance, terms of the type $(\varphi\varphi)^m(D\varphi D\varphi)^{2n},\ m, n = 0, 1, \ldots,$

are possible, since we know that under a local gauge transformation $\varphi(x) \to U\varphi(x)$, the covariant derivative acting on the field transforms as

$$D_\mu\varphi(x) \to UD_\mu\varphi(x), \tag{2.5}$$

where U is the operator

$$U = \exp(iT^a\theta^a(x)), \tag{2.6}$$

with $\theta^a(x)$ some Lie algebra-valued parameter, not necessarily infinitesimal. As a general Ansatz, consider the following interaction terms

$$\begin{aligned}\mathcal{L}^{\text{int}} &= \kappa_1 \mathrm{g}_\mathrm{o}'(\alpha')^{-1/2}\mathrm{tr}\varphi^3 + \kappa_2 {\mathrm{g}_\mathrm{o}'}^2\mathrm{tr}\varphi^4 \\ &+ \kappa_3\alpha'{\mathrm{g}_\mathrm{o}'}^2\mathrm{tr}\varphi^2 D_\mu\varphi D^\mu\varphi \\ &+ \kappa_4{\alpha'}^2{\mathrm{g}_\mathrm{o}'}^2\mathrm{tr}D_\mu D_\nu\varphi D^\mu\varphi D^\nu\varphi\varphi. \end{aligned} \tag{2.7}$$

The effective Lagrangian should only contain terms which are field redefinition independent, since the S-matrix is invariant under such redefinitions. It seems plausible that the Lagrangian terms above are sufficient to reproduce the low energy tachyon string amplitudes. It turns out that it gives rise to the required number of unknowns when comparing to the string amplitude. The Lagrangian in Eq. (2.7) is normalized to make the corresponding action dimensionless (in natural units). In Appendix A.1 a table of length dimensions of the fields and constants (in 26 space-time dimensions) frequently used in the thesis is given. For convenience the Lagrangian is normalized with respect to g_o' - the gauge boson coupling constant in open bosonic string theory. In string theory, it can be shown to be related to the tachyon coupling constant g_o by the simple relation[2]

$$\mathrm{g}_\mathrm{o} = \sqrt{2\alpha'}\mathrm{g}_\mathrm{o}', \tag{2.8}$$

see for example Ref. [7]. Note that an α' is needed for every two derivatives in the perturbative expansion. Of course, when considering interaction between tachyons, the only part of D_μ which is of interest is ∂_μ. As will be shown, the terms above, with properly chosen coefficients, are sufficient to reproduce the string amplitudes up to order ${\alpha'}^2$. Note that these terms are not unique; there are other possible combinations of the tachyon fields that give rise to the same string amplitudes in the limit when $\alpha' \to 0$.

To get a sensible low energy tachyon field theory, gauge bosons must be included in it. For example, the lowest order four-tachyon exchange diagrams contain a propagating gauge boson. These exchange diagrams are

[2]The string gauge boson coupling constant g_o' is later identified with the ordinary Yang-Mills coupling constant 'g'.

constructed by the vertex connecting one gauge boson and two tachyons. The term that describes this interaction is included in $D\varphi D\varphi$, as seen in Eq. (2.9) below. Furthermore, the vertex with one tachyon and two gauge bosons originate from the term $FF\varphi$. This term is not needed for our purposes, but is included below to illustrate how to construct vertices. Apparently, one field is needed for each scattered particle, and derivatives indicates higher order corrections. We obtain the Lagrangian of mixed fields to lowest order in α':

$$\begin{aligned} \mathcal{L}^{\text{mix}} &= \gamma_1 \mathrm{tr} D_\mu \varphi D^\mu \varphi\Big|_{int} + \gamma_2 \mathrm{g}_\mathrm{o}' \sqrt{\alpha'} \mathrm{tr} \varphi F_{\mu\nu} F^{\mu\nu} = \\ &= 2\gamma_1 \mathrm{g}_\mathrm{o}' \partial_\mu \varphi^a A^{b\mu} \varphi^c f^{abc} + 2\gamma_2 \mathrm{g}_\mathrm{o}' \sqrt{\alpha'} \mathrm{tr} T^{abc} \\ &\times \Big[\varphi^a \partial_\mu A^b_\nu \partial^\mu A^{c\nu} - \varphi^a \partial_\mu A^b_\nu \partial^\nu A^{c\mu}\Big] + \dots, \end{aligned} \tag{2.9}$$

where $D\varphi D\varphi\Big|_{int}$ only includes the interaction part. Next, we turn to evaluating correlation functions in the theory under consideration, using the functional-integral formalism.

2.2 Tachyon Scattering Amplitudes

In the functional-integral formalism all information about the theory is included in the generating functional $Z[J]$. The Green functions which characterize the quantum field theory are all generated by it. The generating functional for the non-Abelian scalar field is given by

$$Z[J] \equiv \int \mathcal{D}\varphi \, \exp\Big(i \int d^D x (\mathcal{L} + \varphi^a J^a)\Big), \tag{2.10}$$

where the integration measure is defined as

$$\mathcal{D}\varphi \equiv \prod_x \prod_a d\varphi^a(x), \tag{2.11}$$

and an 'i' has been included in the exponent of $Z[J]$ since we are working in Minkowski space[3]. An artificial source term $J^a(x)$ has been introduced in the functional; it is an auxiliary field that is put to zero at the end of the calculations. Correlation functions are easily evaluated by functional differentiation with respect to this source term:

$$\begin{aligned} &\langle \varphi^{a_1}(x_1) \varphi^{a_2}(x_2) \dots \varphi^{a_m}(x_m) \rangle = \\ &= \frac{1}{Z[0]} \frac{\delta}{i\delta J^{a_1}(x_1)} \frac{\delta}{i\delta J^{a_2}(x_2)} \cdots \frac{\delta}{i\delta J^{a_m}(x_m)} Z[J]\Big|_{J=0}, \end{aligned} \tag{2.12}$$

[3]Recall that in bosonic string theory consistency requires a 26-dimensional space-time, *i.e.* $D = 26$.

where time-ordering of the fields in the first line is implicitly understood. The propagator is merely the two-point function, and takes the anticipated form

$$\Delta^{ab}(x_1, x_2) \equiv \int \frac{d^D k}{(2\pi)^D} \Big(\frac{-i}{k^2 + m^2} \delta^{ab} \Big) e^{-ik(x_1 - x_2)}. \tag{2.13}$$

It is a Green function and satisfies

$$(\Box - m^2)\Delta^{ab}(x_1, x_2) = -i\delta^D(x_1 - x_2). \tag{2.14}$$

By making a change of variables

$$\varphi'^a(x) = \varphi^a(x) - i \int d^D y \Delta^{ab}(x, y) J^a(y), \tag{2.15}$$

and using Eq. (2.14), Eq. (2.10) can be rewritten in a more convenient form as

$$Z[J^a] \;\; = \;\; Z_0 \exp\Big(i\mathcal{S}_{\text{int}}[\varphi^a] \Big)\Big|_{\varphi^a = \frac{\delta}{i\delta J^a}} \mathcal{Z}[J^a], \tag{2.16}$$

where Z_0 is the generating functional for the free field without any interaction terms

$$Z_0 \;\; = \;\; \int \mathcal{D}\varphi' \exp\Big(\frac{i}{2} \int d^D x \varphi'^a (\Box - m^2) \varphi'^a \Big), \tag{2.17}$$

$\mathcal{S}_{\text{int}}[\varphi^a]$ includes the interaction terms from $\mathcal{L}^{\text{int}}$

$$\mathcal{S}_{\text{int}}[\varphi^a] = \kappa_1 \frac{\mathrm{g}_o'}{\sqrt{\alpha'}} \int d^D x \; \text{tr}\varphi^3 + \dots, \tag{2.18}$$

and $\mathcal{Z}[J^a]$ contains the source terms and the propagator for the free field

$$\mathcal{Z}[J^a] \;\; = \;\; \exp\Big(-\frac{1}{2} \int d^D x_1 d^D x_2 J^a(x_1) \Delta^{ab}(x_1, x_2) J^b(x_2) \Big). \tag{2.19}$$

Notice that the argument in $\mathcal{S}_{\text{int}}[\varphi^a]$ in Eq. (2.16) is mapped as

$$\mathcal{S}_{\text{int}}[\varphi^a(x)] \to \mathcal{S}_{\text{int}}\Big[\frac{\delta}{i\delta J^a(x)} \Big]. \tag{2.20}$$

It turns out to be more convenient to work in momentum space than in configuration space. For a summary of the conventions used in momentum space, consult Appendix A.5. The propagator can be expressed in its Fourier components as

$$\Delta^{ab}(x_1, x_2) = \int \frac{d^D k_1 d^D k_2}{(2\pi)^{2D}} e^{-ik_1 x_1} e^{-ik_2 x_2} \tilde{\Delta}^{ab}(k_1, k_2). \tag{2.21}$$

Hence comparing to Eq. (2.13), the transformed propagator takes the form

$$\tilde{\Delta}^{ab}(k_1,k_2) \equiv \frac{-i}{{k_1}^2+m^2}\delta^{ab}\delta(k_1+k_2), \tag{2.22}$$

with the momentum pointing inwards, and $\mathcal{Z}[\tilde{J}^a]$

$$\mathcal{Z}[\tilde{J}^a] = \exp\Big(-\frac{1}{2}\int\frac{d^Dk_1d^Dk_2}{(2\pi)^{2D}}\tilde{J}^a(k_1)\tilde{\Delta}^{ab}(k_1,k_2)\tilde{J}^b(k_2)\Big). \tag{2.23}$$

Note that in the expressions above, the exponentials are defined by their Taylor expansions. When calculating correlation functions in momentum space, the functional differentiations are performed with respect to the Fourier transformed source $\tilde{J}^a(k)$, *e.g.* for the propagator we have

$$\langle\tilde{\varphi}^a(k_1)\tilde{\varphi}^b(k_2)\rangle = \frac{1}{Z_0}\frac{\delta}{i\delta\tilde{J}^a(k_1)}\frac{\delta}{i\delta\tilde{J}^b(k_2)}\mathcal{Z}\Big|_{\tilde{J}=0} = \tilde{\Delta}^{ab}(k_1,k_2). \tag{2.24}$$

The remaining correlation functions are now obtained by applying functional derivatives with respect to $\tilde{J}^a(k)$ to $Z[\tilde{J}^a]$, with $\mathcal{S}_{\text{int}}[\tilde{\varphi}^a]$ containing the interaction terms for the theory under consideration: for an n-point function, $(\frac{\delta}{i\delta\tilde{J}})^n$ is applied. To calculate the three-point function, the exponential containing the interaction terms have to be expanded to linear order in the fields $\tilde{\varphi}^a$:

$$\begin{aligned}\langle\tilde{\varphi}^a(k_1)\tilde{\varphi}^b(k_2)\tilde{\varphi}^c(k_3)\rangle\Big|_{g_o'} &= \Bigg\{\frac{1}{Z_0}\frac{\delta}{i\delta\tilde{J}^a(k_1)}\frac{\delta}{i\delta\tilde{J}^b(k_2)}\frac{\delta}{i\delta\tilde{J}^c(k_3)} \\ &\quad\times\ \exp\Big\{i\mathcal{S}_{\text{int}}\Big[\frac{\delta}{i\delta\tilde{J}^a(k)}\Big]\Big\}\Big|_{linear}\mathcal{Z}\Big|_{\tilde{J}=0}\Bigg\}\Bigg|_{g_o'} \\ &= -3\kappa_1\frac{g_o'}{\sqrt{\alpha'}}\prod_{i=1}^{3}\frac{1}{k_i^2+m^2}\text{tr}\Big(T^a\{T^b,T^c\}\Big).\end{aligned} \tag{2.25}$$

To save space here and in the following, the obvious delta functions coming from momentum conservation are not explicitly displayed. Note that the terms in the expansion of the exponential not including exactly three tachyon fields, either will be put to zero by the condition $\tilde{J}=0$, or be "differentiated away".

The four-point function corresponding to the three interaction terms involving four tachyon fields chosen, is calculated to be

$$\begin{aligned}
\langle\tilde{\varphi}^a(k_1)\tilde{\varphi}^b(k_2)\tilde{\varphi}^c(k_3)\tilde{\varphi}^d(k_4)\rangle\Big|_{\mathrm{g}_o'^2} &= i\mathrm{g}_o'^2\prod_{i=1}^{4}\frac{1}{k_i^2+m^2}\Bigg\{24\kappa_2\mathrm{str}T^{abcd}\\
&-2\kappa_3\alpha'\Big[k_1\cdot k_2\mathrm{tr}(T^{abcd}+T^{adcb}+T^{abdc}+T^{acdb})\\
&+k_1\cdot k_3\mathrm{tr}(T^{abdc}+T^{acdb}+T^{adbc}+T^{acbd})\\
&+k_1\cdot k_4\mathrm{tr}(T^{abcd}+T^{adcb}+T^{adbc}+T^{acbd})\Big]\\
&+2\kappa_4\alpha'^2\Big[k_1\cdot k_2k_1\cdot k_3\mathrm{tr}(T^{abdc}+T^{acdb}+T^{adbc}+T^{acbd})\\
&+k_1\cdot k_2k_1\cdot k_4\mathrm{tr}(T^{abdc}+T^{acdb}+T^{adbc}+T^{acbd})\\
&+k_1\cdot k_3k_1\cdot k_4\mathrm{tr}(T^{abcd}+T^{adcb}+T^{abdc}+T^{acdb})\Big]\Bigg\}, \qquad (2.26)
\end{aligned}$$

where the short-hand notation $T^{a_1\ldots a_n}\equiv T^{a_1}\ldots T^{a_n}$ is used. Finally, the correlation function coming from the remaining relevant interaction term become:

$$\langle\tilde{A}^{a\mu}(k_1)\tilde{\varphi}^b(k_2)\tilde{\varphi}^c(k_3)\rangle\Big|_{\mathrm{g}_o'} = -2i\mathrm{g}_o'\gamma_1 f^{abc}\frac{1}{k_1^2}\frac{1}{k_2^2+m^2}\frac{1}{k_3^2+m^2}(k_2^\mu-k_3^\mu) \qquad (2.27)$$

To obtain the Feynman rules, all we have to do is to amputate the Green functions calculated above. Amputation is defined as eliminating all the propagators from the external legs. The Feynman rules for the non-Abelian tachyon field theory are summarized in Appendix A.4.

2.3 Yang-Mills Lagrangian and Higher Order Contributions

In this section we want to find an Ansatz for the effective action describing the lowest order scattering amplitudes of the massless gauge bosons in open bosonic string theory, which are the first particle excitations in open bosonic string theory. When working with non-Abelian gauge theories, it is crucial to keep track of the different characteristics of the Lagrangian terms. It is especially noticeable when one is concerned with vector fields, which unlike the scalar fields, also carry Lorentz indices that must be contracted properly. Contrary to Abelian theories, where the gauge fields commute and the ordering of them is insignificant, here one has to distinguish between different orderings of the fields.

Another ambiguity arises for non-Abelian gauge theories. Crudely, the effective action contains a part dependent on covariant derivatives and a non-derivative dependent part only containing field strengths. Since we have that $[D_\mu, D_\nu]F_{\rho\sigma} \sim [F_{\mu\nu}F_{\rho\sigma}]$, the terms in the different parts can be traded for one another, which makes the ordering procedure even more difficult. To start with, we will resolve this difficulty by not taking terms including covariant derivatives into account in the effective Lagrangian.

The effective action has to satisfy a few basic constraints to make sense. First of all, the fields have to be properly contracted, to produce scalars. Furthermore, reality, Lorentz and gauge invariance are required as in the scalar field theory. The combination $\alpha'\mathrm{g}F_{\mu\nu}$ is dimensionless in any number of dimensions[4]. Hence, starting from the Yang-Mills Lagrangian $\sim \mathrm{tr}F_{\mu\nu}F^{\mu\nu}$, terms at any order, without derivatives, can be constructed by dimensional considerations. For instance, the terms at order α'^4 has the structure $\alpha'^4\mathrm{g}^4F^6$ ($[\alpha'^4\mathrm{g}^4F^6]$=(length)$^{-26}$ in 26 dimensions). The most general Ansatz, satisfying the requirements above, without any derivatives acting on the gauge field strengths[5], takes the form

$$\mathcal{L} = \sum_{i=2}^{\infty}\sum_{j}(\alpha'\mathrm{g})^{i-2}\mathcal{L}_i^{(j)}, \tag{2.28}$$

where $\mathcal{L}_i^{(j)}$ contains F to the ith power, and the inner sum takes care of the j inequivalent permutations of the Lorentz indices at a given order (fixed i). The first term in the expansion is the Yang-Mills Lagrangian. The number of inequivalent terms grows rapidly with order; already at order $\alpha'^4\mathrm{g}^4F^6$ there are 28 of them. When analysing higher order contributions, it is therefore necessary to classify the terms, in order to keep track of them, see for example Ref. [13], where *permutation group theory* is used. In Section 4.4 another basis is investigated, including symmetrized traces, in which the Lagrangian is expanded.

The Lagrangian term that gives rise to the correct kinetic energy for

[4]In $D = 26$ dimensions the coupling constant and the field strength have the length dimensions $+11$ and -13 respectively, see Appendix A.1.

[5]One might include gauge invariant terms like $\alpha' D^\mu F_{\mu\nu}D_\rho F^{\rho\nu}$, $\alpha'^2\mathrm{g}F_{\mu\nu}D_\rho F^{\mu\nu}D_\sigma F^{\sigma\rho}$, $\alpha'^2\mathrm{g}D_\mu F^{\mu\nu}D_\rho F^{\rho\sigma}F_{\nu\sigma}$, $\alpha'^2 D^\mu D^\nu F_{\nu\rho}D_\mu D_\sigma F^{\sigma\rho}$, etc. However the coefficients connected to these terms are field redefinition dependent, and can therefore be given arbitrary values, see Tseytlin Ref. [12]. By choosing the simplest effective action, the "minimal scheme", the coefficients can be put to zero.

the non-Abelian gauge field A^a_μ is the well-known Yang-Mills Lagrangian:

$$\begin{aligned}\mathcal{L}_2 &= -\frac{1}{4}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\partial_\mu A^a_\nu\partial^\nu A^{a\mu} + \\ &- \frac{1}{2}gf^{abc}A^b_\mu A^c_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\ &- \frac{1}{4}g^2 f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}. \end{aligned} \tag{2.29}$$

For a summary of the conventions used for non-Abelian gauge fields, see Appendix A.3. At order α', there is only one way to contract three F's; all permutations can be reduced to the same form using the cyclic property of the trace. To order A^4, the expansion of $\mathcal{L}_3$ looks like

$$\begin{aligned}\mathcal{L}_3 &= k_3\alpha'\mathrm{g}\mathrm{tr}F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu = \\ &= k_3\alpha'\mathrm{g}if^{abc}[\partial_\mu A^{a\nu}\partial_\nu A^{b\rho}\partial_\rho A^{c\mu} + 3\partial^\mu A_{a\rho}\partial_\mu A^{b\nu}\partial^\rho A_{c\nu}] \\ &+ k_3\alpha'\mathrm{g}^2 f^{cde}\left[\frac{3}{2}if^{abc} + \mathrm{tr}\Big(T^a\{T^b,T^c\}\Big)\right]A^d_\rho A^{e\mu} \\ &\times \left[\partial_\mu A^{a\nu}\partial_\nu A^{b\rho} + \partial^\nu A^a_\mu\partial^\rho A^b_\nu - \partial_\mu A^{a\nu}\partial^\rho A^b_\nu - \partial^\nu A^a_\mu\partial_\nu A^b_\rho\right] \\ &+ \mathcal{O}(A^5), \end{aligned} \tag{2.30}$$

where no Lie group in particular is under consideration. Next, at order α'^2, the first nontrivial combinations of field strengths appear; there are four inequivalent permutations:

$$\mathcal{L}_4^{(1)} = k_4^{(1)}(\alpha'\mathrm{g})^2\mathrm{tr}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \tag{2.31}$$

$$\mathcal{L}_4^{(2)} = k_4^{(2)}(\alpha'\mathrm{g})^2\mathrm{tr}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}, \tag{2.32}$$

$$\mathcal{L}_4^{(3)} = k_4^{(3)}(\alpha'\mathrm{g})^2\mathrm{tr}F_{\mu\nu}F_{\rho\sigma}F^{\nu\mu}F^{\sigma\rho}, \tag{2.33}$$

$$\mathcal{L}_4^{(4)} = k_4^{(4)}(\alpha'\mathrm{g})^2\mathrm{tr}F^{\mu\nu}F^{\rho\sigma}F_{\nu\rho}F_{\sigma\mu}. \tag{2.34}$$

The independence of these terms can be checked explicitly. If the traces are exchanged for symmetric ones, some of these terms become linearly dependent, see Section 4.4. It turns out that, to reproduce the low energy string amplitudes, the terms at order α'^2 in our Ansatz combine into two parts: one part that can be rewritten as a symmetric trace, and one term that combines into a commutator of the field strengths. The single term at order α' and the latter term at order α'^2 can be regarded as part of the derivative-dependent part of the effective action, see the discussion in Section 4.2.

2.4 Gauge Boson Scattering Amplitudes

Starting with the Yang-Mills Lagrangian in Eq. (2.29), and the higher order corrections in Eqs. (2.30) - (2.34), the corresponding Feynman rules can be calculated. This is most easily done using functional-integral methods as for the scalar fields. However, we have to be a little bit more careful here. Since the action

$$\int d^D x \mathcal{L}_2 \tag{2.35}$$

is invariant under the local gauge transformation

$$A_\mu \to U A_\mu U^{-1} - \frac{i}{\mathrm{g}}(\partial_\mu U) U^{-1}, \tag{2.36}$$

(for a proof, see Appendix A.3) where U is the operator

$$U = \exp(i\mathrm{g} T^a \varepsilon^a(x)), \tag{2.37}$$

an element of some gauge group $\mathfrak{G}$, A_μ must be restricted by means of some gauge condition, in order to eliminate the arbitrariness in it and to make the functional integral finite and well-defined. Here, a natural choice of this "gauge fixing condition" is the Lorentz condition $\partial^\mu A^a_\mu = 0$. After gauge fixing the Yang-Mills Lagrangian, the generating functional takes the form

$$Z[J^a_\mu] \equiv \int \mathcal{D}A \, \exp\Big(i \int d^D x (\mathcal{L}_2^{\mathrm{gf}} + A^a_\mu J^{a\mu})\Big), \tag{2.38}$$

where $\mathcal{D}A \equiv \prod_x \prod_{a,\mu} dA^a_\mu(x)$, and the source term $J^a_\mu(x)$ now has been assigned a Lorentz index as well. After some manipulations, Eq. (2.38) can analogously to the scalar field be written as

$$Z[J^a_\mu] \;=\; Z_0 \, \exp\Big(i\mathcal{S}_{\mathrm{int}}[A^a_\mu]\Big)\Big|_{A^a_\mu = \frac{\delta}{i\delta J^{a\mu}}} \mathcal{Z}[J^a_\mu], \tag{2.39}$$

where Z_0 is the generating functional of the free field, *i.e.*

$$Z_0 \;=\; \int \mathcal{D}A' \, \exp\Big(\frac{i}{2} \int d^D x A'^a_\mu \Box A'^{a\mu}\Big), \tag{2.40}$$

$\mathcal{S}_{\mathrm{int}}[A^a_\mu]$ includes the interaction terms at order α'^0 (if we consider the Yang-Mills term)

$$\begin{aligned} \mathcal{S}_{\mathrm{int}}[A^a_\mu(x)]\Big|_{\alpha'^0} \;&=\; -\frac{\mathrm{g}}{2} f^{abc} \int d^D x (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} \\ &\quad - \frac{\mathrm{g}^2}{4} f^{abe} f^{cde} \int d^D x A^a_\mu A^b_\mu A^{c\mu} A^{d\nu}, \end{aligned} \tag{2.41}$$

and $\mathcal{Z}[J^a_\mu]$ contains the source term $J^a_\mu(x)$ and the gauge boson propagator $\Delta^{ab}_{\mu\nu}(x_1, x_2)$:

$$\mathcal{Z}[J^a_\mu] = \exp\Big(- \frac{1}{2} \int d^D x_1 d^D x_2 J^{a\mu}(x_1) \Delta^{ab}_{\mu\nu}(x_1, x_2) J^{b\nu}(x_2) \Big), \quad (2.42)$$

cf. Eqs. (2.17)-(2.19). The propagator takes in configuration space the form

$$\Delta^{ab}_{\mu\nu}(x_1, x_2) \equiv \int \frac{d^D k}{(2\pi)^D} \Big(\frac{-i\eta_{\mu\nu}}{k^2} \delta^{ab} \Big) e^{-ik(x_1 - x_2)}, \quad (2.43)$$

and in momentum space

$$\tilde{\Delta}^{ab}_{\mu\nu}(k_1, k_2) \equiv \frac{-i\eta_{\mu\nu}}{k_1^2} \delta^{ab} \delta^D(k_1 + k_2), \quad (2.44)$$

since the propagator can be expressed in its Fourier components as

$$\Delta^{ab}_{\mu\nu}(x_1, x_2) = \int \frac{d^D k_1 d^D k_2}{(2\pi)^{2D}} e^{-ik_1 x_1} e^{-ik_2 x_2} \tilde{\Delta}^{ab}_{\mu\nu}(k_1, k_2). \quad (2.45)$$

Furthermore, $\mathcal{Z}[J^a_\mu]$ must be transformed to momentum space

$$\mathcal{Z}[\tilde{J}^a_\mu] = \exp\Big(- \frac{1}{2} \int \frac{d^D k_1 d^D k_2}{(2\pi)^{2D}} \tilde{J}^{a\mu}(k_1) \tilde{\Delta}^{ab}_{\mu\nu}(k_1, k_2) \tilde{J}^{b\nu}(k_2) \Big). \quad (2.46)$$

The two-point function in Eq. (2.43) is calculated by keeping only the "one" (no interaction) in the expansion of the exponential in Eq. (2.39), and thereafter applying two functional derivatives to $\mathcal{Z}[\tilde{J}^a_\mu]$:

$$\begin{aligned} \langle \tilde{A}^a_\mu(k_1) \tilde{A}^b_\nu(k_2) \rangle &= \frac{1}{Z_0} \frac{\delta}{i\delta \tilde{J}^{a\mu}(k_1)} \frac{\delta}{i\delta \tilde{J}^{b\mu}(k_2)} \mathcal{Z}[\tilde{J}^a_\mu] \Big|_{\tilde{J}=0} \\ &= \tilde{\Delta}^{ab}_{\mu\nu}(k_1, k_2). \end{aligned} \quad (2.47)$$

The contribution of the linear term in $\exp(i\mathcal{S}_{\text{int}}\Big|_{\alpha'^0})$ gives the coupling between three massless vector particles:

$$\begin{aligned} \langle \tilde{A}^a_\mu(k_1) \tilde{A}^b_\nu(k_2) \tilde{A}^c_\rho(k_3) \rangle \Big|_{\alpha'^0} &= \frac{1}{Z_0} \frac{\delta}{i\delta \tilde{J}^{a\mu}(k_1)} \frac{\delta}{i\delta \tilde{J}^{b\mu}(k_2)} \frac{\delta}{i\delta \tilde{J}^{c\rho}(k_3)} \\ &\times \exp\Big\{ i\mathcal{S}_{\text{int}} \Big[\frac{\delta}{i\delta \tilde{J}^{a\mu}(k)} \Big] \Big\} \Big|_{linear} \mathcal{Z}[\tilde{J}^a_\mu] \Big|_{\tilde{J}=0} \\ &= ig f^{abc} \frac{1}{k_1^2 k_2^2 k_3^2} [\eta_{\mu\nu}(k_1 - k_2)_\rho \\ &+ \eta_{\rho\mu}(k_3 - k_1)_\nu \\ &+ \eta_{\nu\rho}(k_2 - k_3)_\mu]. \end{aligned} \quad (2.48)$$

The four-point function is calculated in the same manner, but with the term involving four A's as the interaction term:

$$
\begin{aligned}
&\langle \tilde{A}^a_\mu(k_1)\tilde{A}^b_\nu(k_2)\tilde{A}^c_\rho(k_3)\tilde{A}^d_\sigma(k_4)\rangle\Big|_{\alpha'^0} = \\
&= \frac{1}{Z_0}\frac{\delta}{i\delta\tilde{J}^{a\mu}(k_1)}\frac{\delta}{i\delta\tilde{J}^{b\mu}(k_2)}\frac{\delta}{i\delta\tilde{J}^{c\rho}(k_3)}\frac{\delta}{i\delta\tilde{J}^{d\sigma}(k_4)} \\
&\times\exp\Big\{i\mathcal{S}_{\text{int}}\Big[\frac{\delta}{i\delta\tilde{J}^{a\mu}(k)}\Big]\Big\}\Big|_{linear}\mathcal{Z}[\tilde{J}^a_\mu]\Big|_{\tilde{J}=0} \\
&= -i\mathrm{g}^2\frac{1}{k_1^2k_2^2k_3^2k_4^2}[f^{abe}f^{cde}(\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\sigma}\eta_{\nu\rho}) \\
&\qquad\qquad + f^{ace}f^{bde}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\sigma}\eta_{\nu\rho} \\
&\qquad\qquad + f^{ade}f^{bce}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\rho}\eta_{\nu\sigma})].
\end{aligned}
\tag{2.49}
$$

This amplitude contains a product of two antisymmetric tensors f^{abc}. Due to the bosonic nature of the correlation function it should be symmetric under any interchange of the external particles. If the underlying gauge symmetry is $\mathrm{U}(n)$ the product $f^{abe}f^{cde}$ can be rewritten in terms of symmetric tensors d^{abc}, see Appendix C. Next, the correlation functions and Feynman rules arising from the interaction terms from $\mathcal{L}_3$ are to be calculated. The generating functional in Eq. (2.39) can still be used, but the interaction part $\mathcal{S}_{\text{int}}$ have to be replaced by the new interaction terms. The three-point function becomes

$$
\begin{aligned}
&\langle \tilde{A}^a_\mu(k_1)\tilde{A}^b_\nu(k_2)\tilde{A}^c_\rho(k_3)\rangle\Big|_{\alpha'} = 3k_3\alpha'\mathrm{g}f^{abc}\frac{1}{k_1^2k_2^2k_3^2} \\
&\times\Big\{2k_{2\mu}k_{3\nu}k_{1\rho} \\
&\quad + [\eta_{\mu\nu}(k_2\cdot k_3k_{1\rho}-k_1\cdot k_3k_{2\rho}) \\
&\quad + \eta_{\nu\rho}(k_3\cdot k_1k_{2\mu}-k_2\cdot k_1k_{3\mu}) \\
&\quad + \eta_{\rho\mu}(k_1\cdot k_2k_{3\nu}-k_3\cdot k_2k_{1\nu})]\Big\},
\end{aligned}
\tag{2.50}
$$

where constant k_3 in the first line must not be confused the the momentum k_3. When calculating four-point functions, one must bear in mind that it is not necessary to check all terms when one is to compare the string amplitudes with correlation functions in the quantum field theory. This is due to the fact that the structure of the string vertices is fixed by gauge invariance, and by permutation symmetry from the bosonic nature of the particles. Therefore, to make life easier, from now on we will focus only on the terms of the type $\xi_1\cdot\xi_2\xi_3\cdot\xi_4\equiv\eta_{\mu\nu}\eta_{\rho\sigma}\xi_1^\mu\xi_2^\nu\xi_3^\rho\xi_4^\sigma$.

At order α', the only Lagrangian term that produces terms of the type $\xi \cdot \xi\xi \cdot \xi$ is

$$-\frac{3i}{2}k_3\alpha' \mathrm{g}^2 f^{abc}f^{cde}\partial^\nu A^a_\mu \partial_\nu A^b_\rho A^d_\rho A^{e\mu}, \tag{2.51}$$

cf. Eq. (2.30). Note that the term including the anti-commutator, vanishes by antisymmetry. Evaluating the four-point function, we obtain

$$\begin{aligned}&\langle \tilde{A}^a_\mu(k_1)\tilde{A}^b_\nu(k_2)\tilde{A}^c_\rho(k_3)\tilde{A}^d_\sigma(k_4)\rangle\Big|_{\alpha'} = -6k_3\alpha' \mathrm{g}^2\eta_{\mu\nu}\eta_{\rho\sigma}\frac{1}{k_1^2k_2^2k_3^2k_4^2}\\ &\times[f^{ace}f^{dbe}k_1\cdot k_3 + f^{ade}f^{cbe}k_1\cdot k_4] + \ldots\end{aligned} \tag{2.52}$$

Written explicitly, the terms at α'^2-level that give rise to $\xi \cdot \xi\xi \cdot \xi$-terms are

$$\mathcal{L}^{(1)}_4|_{(\xi\cdot\xi)^2} = 4k^{(1)}_4(\alpha'\mathrm{g})^2\partial_\mu A^a_\nu \partial^\mu A^{b\nu}\partial_\rho A^c_\sigma \partial^\rho A^{d\sigma}\mathrm{tr}T^{(ab)(cd)} \tag{2.53}$$

$$\mathcal{L}^{(2)}_4|_{(\xi\cdot\xi)^2} = 2k^{(2)}_4(\alpha'\mathrm{g})^2\partial_\nu A^a_\mu \partial^\nu A^{b\rho}\partial_\sigma A^c_\rho \partial^\sigma A^{d\mu}\mathrm{tr}T^{abcd} \tag{2.54}$$

$$\mathcal{L}^{(3)}_4|_{(\xi\cdot\xi)^2} = 4k^{(3)}_4(\alpha'\mathrm{g})^2\partial_\mu A^a_\nu \partial_\rho A^b_\sigma \partial^\mu A^{c\nu}\partial^\rho A^{d\sigma}\mathrm{tr}T^{abcd} \tag{2.55}$$

$$\begin{aligned}\mathcal{L}^{(4)}_4|_{(\xi\cdot\xi)^2} &= k^{(4)}_4(\alpha'\mathrm{g})^2[\partial^\mu A^{a\nu}\partial^\rho A^{b\sigma}\partial_\rho A^c_\nu \partial_\mu A^d_\sigma\\ &+ \partial^\nu A^{a\mu}\partial^\sigma A^{b\rho}\partial_\nu A^c_\rho \partial_\sigma A^d_\mu]\mathrm{tr}T^{abcd}\end{aligned} \tag{2.56}$$

The part of the four-point correlation function corresponding to the terms in Eqs. (2.53) - (2.56), containing $\xi_1 \cdot \xi_2\xi_3 \cdot \xi_4$-terms, becomes

$$\begin{aligned}&\langle A^a_\mu(k_1)A^b_\nu(k_2)A^c_\rho(k_3)A^d_\sigma(k_4)\rangle\Big|_{\alpha'^2} = i(\alpha'\mathrm{g})^2\frac{1}{k_1^2k_2^2k_3^2k_4^2}\eta_{\mu\nu}\eta_{\rho\sigma}\\ &\times\mathrm{tr}\Big\{32k^{(1)}_4 k_1\cdot k_2 k_3\cdot k_4 T^{(ab)(cd)}\\ &+4k^{(2)}_4\Big[k_1\cdot k_3 k_2\cdot k_4(T^{abdc}+T^{acdb})\\ &+k_1\cdot k_4 k_2\cdot k_3(T^{abcd}+T^{adcb})\Big]\\ &+16k^{(3)}_4 k_1\cdot k_2 k_3\cdot k_4(T^{acbd}+T^{adbc})\\ &+2k^{(4)}_4\Big[k_1\cdot k_4 k_2\cdot k_3(T^{acbd}+T^{adbc}+T^{abdc}+T^{acdb})\\ &+k_1\cdot k_3 k_2\cdot k_4(T^{abcd}+T^{adcb}+T^{acbd}+T^{adbc})\Big]\Big\} + \ldots\end{aligned} \tag{2.57}$$

With that, all the essential correlation functions needed for the comparison with the low energy string amplitudes are evaluated. In Appendix A.4, the Feynman rules, with the coefficients $k^{(j)}_i$ specified to the specific theory, are summarized. Next, we turn to the string calculations.

3

Open Bosonic String Theory

This chapter is meant to give an introduction to the physics of the open bosonic string. Starting on more elementary grounds by analysing the action principle of the bosonic string, we end up with evaluating open bosonic four-gauge-boson tree-level scattering amplitudes including Chan-Paton factors. Throughout the chapter we mainly focus on the lowest excitations of the open string - the tachyons and the massless gauge bosons.

3.1 Introducing the Concepts

It is an established fact that the Polyakov action

$$S[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_{-\infty}^{+\infty}\int_0^L d\tau d\sigma \sqrt{-\det\gamma_{ij}}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} \tag{3.1}$$

describes the dynamics of the bosonic string, where $X^\mu(\tau,\sigma)$ is the space-time coordinates of the string world-sheet and γ_{ij} is the world-sheet metric, the intrinsic metric on the surface. This action is Poincaré invariant, *i.e.* it is invariant under space-time rotations and space-time translations. Further, the Polyakov action is invariant under diffeomorphisms of the world-sheet, due to its geometrical nature, and conformal rescalings of the world-sheet metric γ_{ij}, Weyl invariance[1]. Varying this action with respect to X^μ, leads to the equation of motion

$$\Box X^\mu = 0 \tag{3.2}$$

[1]Recall that the critical dimension of a string theory with no supersymmetries, *i.e.* $\mathcal{N}=0$, is $D=26$.

in the interior, and a boundary term, which vanishes if we choose, *e.g.*, the Neumann boundary conditions

$$\partial_n X^\mu(\tau, \sigma = 0) = \partial_n X^\mu(\tau, \sigma = L) = 0 \tag{3.3}$$

at the ends of the string. Here $\partial_n X^\mu$ is the normal derivative with respect to the boundary. This corresponds to an open string that moves freely in space-time; the boundary conditions prevent momentum to flow out of the ends of the string. Note that an open string is topologically equivalent to the unit interval.

The stress tensor is defined analogously to general relativity as

$$T_{ij} \equiv -\frac{4\pi}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{ij}}, \tag{3.4}$$

and turns out to play an important role in string theory. Varying the Polyakov action with respect to the intrinsic metric γ_{ij}, gives the classical equation of motion, $T^{ij} = 0$. Upon quantization, as we will see, this turns into $(\hat{T}^{ij})^+ |phys\rangle = 0$, where the plus-sign denotes the destruction part of the stress tensor and $|phys\rangle$ is a physical state. This condition imply that the physical states are ghost-free in the critical dimension. The most general solution to Laplace's equation in Eq. (3.2) (in Minkowski space) with the boundary condition in Eq. (3.3) is the mode expansion

$$X^\mu(\tau, \sigma) = x^\mu - \frac{2\pi i c \alpha'}{L} p^\mu \tau + \sqrt{2\alpha'} i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \exp\left(-\frac{n\pi c \tau}{L}\right) \cos \frac{n\pi\sigma}{L}, \tag{3.5}$$

which consists of x^μ and p^μ, the centers of position and momentum of the string, and a superposition of infinitely many plane wave solutions. L is the length of the string. Equal time canonical commutation relations imply for the center-of-mass variables and the harmonic oscillators α_n^μ the commutation relations

$$\begin{aligned} [\hat{x}^\mu, \hat{p}^\nu] &= i\eta^{\mu\nu}, \\ [\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] &= m\eta^{\mu\nu}\delta_{m+n,0}, \end{aligned} \tag{3.6}$$

where the hat emphasizes that the Fourier components now are operators. In addition, they have to satisfy the hermiticity conditions

$$\hat{x}^\dagger = \hat{x}, \tag{3.7}$$

$$\hat{p}^\dagger = \hat{p} \tag{3.8}$$

$$(\hat{\alpha}_n^\mu)^\dagger = \hat{\alpha}_{-n}^\mu, \quad n > 0, \tag{3.9}$$

in order for X^μ to be real. The Fock space is defined by

$$\begin{aligned} \langle 0|0\rangle &= 1, && (3.10)\\ \hat{p}^\mu|0\rangle = \hat{\alpha}^\mu_n|0\rangle &= 0, \quad \forall n>0 && (3.11)\\ \langle 0|\hat{x}^\mu = \langle 0|\hat{\alpha}^\mu_{-n} &= 0, \quad \forall n>0. && (3.12)\end{aligned}$$

Further, we define the lightest string state $|0;p\rangle$, such that

$$|0;p\rangle \equiv \exp(i\hat{p}\cdot\hat{x})|0\rangle, \quad \hat{p}^\mu|0;p\rangle = p^\mu|0;p\rangle, \tag{3.13}$$

where p^μ is the eigenvalue of $\hat{p}^\mu$. The lightest state is defined to be annihilated by the lowering operators

$$\hat{\alpha}^\mu_n|0;p\rangle = 0, \quad \forall n>0. \tag{3.14}$$

By applying creation operators $(\hat{\alpha}^\mu_n)^\dagger, n>0$, to $|0;p\rangle$, excited string states are created. The first excited state becomes

$$|0;p,\xi\rangle = \xi_\mu(p)\hat{\alpha}^\mu_{-1}|0;p\rangle. \tag{3.15}$$

Here $\xi_\mu(p)$ is the dimensionless polarization tensor associated with the state. In conformal field theory it is convenient to introduce the complex coordinates

$$z = \exp\frac{\pi}{L}(c\tau + i\sigma), \qquad \bar{z} = \exp\frac{\pi}{L}(c\tau - i\sigma). \tag{3.16}$$

From Eq. (3.5) we then obtain

$$\hat{X}^\mu(z,\bar{z}) = \hat{x}^\mu - i\alpha'\hat{p}^\mu \ln z\bar{z} + \sqrt{\frac{\alpha'}{2}}\, i\sum_{n\neq 0}^{\infty}\frac{1}{n}\hat{\alpha}^\mu_n(z^{-n}+\bar{z}^{-n}). \tag{3.17}$$

We can form the invariant mass

$$m^2 = -p^2 = \frac{1}{\alpha'}(N-1), \tag{3.18}$$

where N is the level:

$$N = \sum_{n=1}^{\infty}\hat{\alpha}_{n\mu}\hat{\alpha}^\mu_{-n}. \tag{3.19}$$

The eigenvalues of N is the occupation number N_m, which counts the number of excitations in a state, weighted by the oscillator number m:

$$N = \sum_{m=1}^{\infty} mN_m. \tag{3.20}$$

The states in Eqs. (3.13) and (3.15) have the levels 0 and 1 respectively; they are the tachyon with negative mass-squared $-(\alpha')^{-1}$, and the massless gauge boson. The -1 in Eq. (3.18) can be interpreted as the zero-point energy of the bosons.

The stress tensor $T(z)$ is a holomorphic field, and can accordingly be expanded in a Laurent series. Since the stress tensor has the conformal dimension 2, the Laurent modes, better known as the Virasoro generators, become

$$L_n = \oint \frac{dz}{2\pi i} T(z) z^{n+1}, \tag{3.21}$$

in which the stress tensor can be expanded:

$$T(z) = \sum_{n \in \mathbf{Z}} L_n z^{-n-2}. \tag{3.22}$$

The quantum Virasoro generators generate the Virasoro algebra with a central extension:

$$[L_m, L_n] = \frac{c}{12}(m^3 - m)\delta_{m+n,0} + (m-n)L_{m+n}. \tag{3.23}$$

The coefficient c in the central term turns out to be the space-time dimension D. It is by means of these Virasoro generators we can put constraints on states defining physical states, see below.

Next, we introduce the notion of primary fields $\phi(z)$; primary fields, such as vertex operators which will be introduced shortly, define primary states as in- and out states at $\tau \to \pm\infty$:

$$|\phi\rangle \equiv \lim_{z\to 0} \phi(z)|0\rangle, \qquad \langle\phi| \equiv \lim_{z\to\infty} \langle 0|\phi(z). \tag{3.24}$$

Primary states are annihilated by L_n modes with $n > 0$. For L_n modes with $n < 0$, a tower of descendant states above the primary state $|\phi\rangle$ can be generated. This follows since application of $-nL_n = [L_0, L_n]$, following from the Virasoro algebra in Eq. (3.23), means that the eigenvalue of L_0 is raised by $|n|$ units for $n < 0$; and the eigenvalue of the zeroth mode L_0, is nothing but the energy of the state. Vertex operators $V_\Lambda(k,z)$ of physical fields are defined as primary fields with conformal dimension 1. In addition we have to demand that the vertex operators map physical states $|\omega\rangle$ into new physical states $|\omega'\rangle$:

$$\begin{cases} V_\Lambda(k,z)|\omega\rangle = |\omega'\rangle \\ \lim_{z\to 0} V_\Lambda(k,z)|0\rangle = |\Lambda; k\rangle. \end{cases} \tag{3.25}$$

The vertex operators for the tachyon and the massless gauge boson are

$$V(k,z) \equiv i\mathrm{g}_\mathrm{o} : \exp[ik_\mu X^\mu(z)] :, \tag{3.26}$$

$$V(\xi,k,z) \equiv i\mathrm{g}'_\mathrm{o}\xi_\mu : \partial_z X^\mu(z) \exp[ik_\lambda X^\lambda(z)] :, \tag{3.27}$$

where the colons denote normal ordering. There is only one independent coupling constant in the open bosonic string theory; in other words, g_o and g'_o have to be proportional. From the operator product expansion (OPE) of the stress tensor and the tachyon vertex operator

$$T(z)V(k,w) \sim \alpha' k^2 \frac{1}{(z-w)^2} V(k,w) + \frac{1}{z-w}\partial_w V(k,w) \tag{3.28}$$

we have to demand for the tachyon that $k^2 = (\alpha')^{-1}$ to get the conformal dimension 1. Similarly, from the OPE $T(z)V(\xi,k,w)$, we obtain a term $\sim k_\mu \xi^\mu(z) V(\xi,k,w)/(z-w)^3$ and a term $\sim (1+\alpha' k^2)V(\xi,k,w)/(z-w)^2$. So for the massless boson we have to demand that $k^2 = 0$ and $\xi \cdot k = 0$ to get conformal dimension 1 and in order for $V(\xi,k,w)$ to be a primary field. By the limiting procedure we find for the tachyon

$$\lim_{z\to 0} V(k,z)|0\rangle = i\mathrm{g}_\mathrm{o}|0;k\rangle, \tag{3.29}$$

which is just the lightest string state introduced earlier, and for the massless vector

$$\begin{aligned}\lim_{z\to 0} V(\xi,k,z)|0\rangle &= i\mathrm{g}'_\mathrm{o}\xi_\mu \alpha^\mu_{-1}|0;k\rangle \\ &\equiv i\mathrm{g}'_\mathrm{o}|\xi;0;k\rangle.\end{aligned} \tag{3.30}$$

These states which have the conformal dimension $h = 1$ satisfy

$$L_n|\Lambda;k\rangle = \begin{cases} 0 & \text{if } n>0, \\ h|\Lambda;k\rangle & \text{if } n=0, \end{cases} \tag{3.31}$$

as required. We have now accounted for all the basic ingredients for evaluating string scattering amplitudes as we will turn to next.

Incidentally, the graviton is missing in the free open string spectrum, a rather unattractive feature in a theory supposed to explain quantum gravity. However, when interactions are added, the open string can join its ends, by self-interaction, and thereby produce one.

3.2 Tree-Level Open Bosonic String Scattering Amplitudes

In string theory, scattering amplitudes are obtained perturbatively as a sum over surfaces of all possible topologies with fixed initial and final external on-shell states at $\tau \to \pm\infty$. Here we will focus on tree amplitudes, where the open string world-sheets contain no holes.

Scattering amplitudes for particles in string theory are obtained from correlation functions of vertex operators, modulo some divergent factor arising from invariance under Möbius transformations; transformations under the non-compact Möbius group, $SL(2,R)$, generated by the Virasoro generators L_{-1}, L_0 and L_1. All external states in the scattering amplitudes are physical particles on their mass-shell; to each external particle on the world-sheet, one vertex operator is associated. Since the particles can be emitted from any point on the world-sheet, one must integrate over all possible points on it. World-sheets for open string tree-level scattering processes are conformally equivalent to the unit disk, with the vertex operators inserted on the boundary. The string tree-amplitude S-matrix for n particles takes the generic form

$$S_n = i\mathrm{g}_{\mathrm{os}}^{n-2}\langle\Lambda_1;k|V_2\Delta V_3\ldots V_{n-2}\Delta V_{n-1}|\Lambda_n;k\rangle, \tag{3.32}$$

where g_{os} is the open string coupling constant associated with the scattered particle, and the propagator Δ is defined to be

$$\Delta \equiv \int_0^1 x^{L_0-2}dx = \frac{1}{L_0-1}. \tag{3.33}$$

The $SL(2,R)$ Möbius group volume factor has to be divided out. In the end, this means that three of the integration variables can be set to arbitrary values. After taking the Möbius invariance into account, Eq. (3.32) can be written as [4]:

$$\begin{aligned} S_n &= i\mathrm{g}_{\mathrm{os}}^{n-2}\mathcal{K}\int\prod_k dz_k\delta(z_1-z_1^0)\delta(z_2-z_2^0)\delta(z_3-z_3^0) \\ &\times \left(\prod_l \theta(z_{l-1}-z_l)\right)(z_1-z_2)(z_1-z_3)(z_2-z_3) \\ &\times \langle 0;0|\frac{V(k_1,z_1)}{z_1}\ldots\frac{V(k_n,z_n)}{z_n}|0;0\rangle, \end{aligned} \tag{3.34}$$

where $\mathcal{K}$ is a normalization constant and Heaviside's step function $\theta(z)$ takes care of the ordering of the coordinates on the boundary. The term cubic in z is simply the determinant arising from the infinitesimal $SL(2,R)$ transformation

$$z \to z + \lambda_{-1} + \lambda_0 z + \lambda_1 z^2. \tag{3.35}$$

As already pointed out, three of the integration variables can be given arbitrary values, conventionally taken to be

$$z_1^0 \to \infty, \quad z_2^0 \to 1, \quad z_3^0 \to 0. \tag{3.36}$$

In particular, for n on-shell tachyons we find the general expression

$$\begin{aligned} S_{n\varphi} &= i\mathrm{g}_\mathrm{o}^{n-2}\mathcal{K}_\varphi \int \prod_k dz_k \delta(z_1 - z_1^0)\delta(z_2 - z_2^0)\delta(z_3 - z_3^0) \\ &\times \prod_l \theta(z_{l-1} - z_l)(z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \\ &\times \prod_{1 \le i \le j \le n} (z_i - z_j)^{2\alpha' k_i \cdot k_j}, \end{aligned} \tag{3.37}$$

known as the Koba-Nielsen formula. 'g_o' is the tachyon coupling constant, cf. Eq. (3.26). The three-tachyon amplitude is easily evaluated to be

$$S_{3\varphi} = \frac{2i\mathrm{g}_\mathrm{o}}{\alpha'}, \tag{3.38}$$

where $2\alpha' k_i \cdot k_j = -1, \ \forall\ i,j$ has been used. From here on $\mathcal{K}_\varphi$ is put to $1/\alpha'$. The additional factor of 2 arises as a consequence of the two possible cyclic inequivalent orderings of the vertex operators on the boundary of the disk. The delta-function $\delta(\sum_i k_i)$, coming from momentum conservation, is here and in the following implicitly understood. The four-tachyon amplitude is calculated to be

$$\begin{aligned} S_{4\varphi} &= \frac{2i\mathrm{g}_\mathrm{o}^2}{\alpha'} \\ &\times \Big(B(-\alpha_\mathrm{o}(s), -\alpha_\mathrm{o}(t)) + B(-\alpha_\mathrm{o}(s), -\alpha_\mathrm{o}(u)) + B(-\alpha_\mathrm{o}(t), -\alpha_\mathrm{o}(u))\Big), \end{aligned} \tag{3.39}$$

which is known as the Veneziano amplitude. There are six inequivalent permutations of four tachyon vertex operators on the boundary of the disk giving rise to six terms in the S-matrix. However just three of them are independent, explaining the factor of 2 in the amplitude. Above we have introduced the Euler Beta function

$$B(x,y) \equiv \int_0^1 dv v^{x-1}(1-v)^{y-1} \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3.40}$$

with the argument $\alpha_o(w) = 1 + \alpha' w$, called the Regge trajectory. For a definition of the Mandelstam variables s, t and u, see Appendix A. The Euler Beta function has simple poles when the Γ-functions in the numerator have the arguments $0, -1, -2, -3, \ldots$. This means that the poles will lie on the Regge trajectories with equal spacings. The first pole in the s-channel in Eq. (3.39) occurs when $\alpha_o(s) = 0$. This is a particle with invariant mass $(\alpha')^{-1}$, namely a tachyon. The next propagating resonance at this Regge trajectory is a massless gauge boson. Thereafter, there are only massive states with masses at the Planck scale.

Scattering amplitudes of the gauge bosons can be calculated most easily by rewriting the vertex operator in Eq. (3.27) as

$$V(\xi, k, z) = i g'_o : \exp[i k_\lambda X^\lambda(z) + \xi_\mu \partial_z X^\mu(z)]\Big|_{linear\ in\ \xi_i} :, \qquad (3.41)$$

where only the linear term in the expansion of the exponential function is to be kept. Evaluating the matrix-element $\langle 0; 0 | V/z \ldots V/z | 0; 0\rangle$ generically, for the massless gauge bosons, we get

$$\begin{aligned} S_{nA} &= i {g'_o}^{n-2} \mathcal{K}_A \int \prod_k dz_k \prod_l \theta(z_{l-1} - z_l) \delta(z_1 - z_1^0) \delta(z_2 - z_2^0) \delta(z_3 - z_3^0) \\ &\times (z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \prod_{i<j} (z_i - z_j)^{2\alpha' k_i \cdot k_j} \\ &\times \exp\left\{ \sum_{i \neq j} \left(\frac{1}{2} \frac{\xi_i \cdot \xi_j}{(z_i - z_j)^2} - \frac{k_i \cdot \xi_j}{z_i - z_j} \right) \right\} \Bigg|_{multi-linear\ in\ \xi_i}, \end{aligned} \qquad (3.42)$$

where $\mathcal{K}_A$ is a normalization constant. Continuing with a concrete example, the three-gauge-boson amplitude becomes

$$\begin{aligned} S_{3A} &= 2 i g'_o \xi^a_{1\mu} \xi^b_{2\nu} \xi^c_{3\rho} \Big(k_2^\mu \eta^{\nu\rho} + k_3^\nu \eta^{\mu\rho} + k_1^\rho \eta^{\mu\nu} + \frac{\alpha'}{2} k_2^\mu k_3^\nu k_1^\rho \Big) \\ &+ [2 \leftrightarrow 3] = 0 \end{aligned} \qquad (3.43)$$

where $\mathcal{K}_A$ has been put to 2. Note that this amplitude vanishes when summed over cyclic permutations; it is only present in theories with a non-Abelian symmetry groups, see below. Also note that the last term is a contribution at α'-level only present in the bosonic string theory. Proceeding to compute the four-gauge-boson amplitude is straightforward. Suppressing the step functions, the $\xi_1 \cdot \xi_2 \xi_3 \cdot \xi_4$-part of the amplitude is explicitly calculated as

$$
\begin{aligned}
S_{4A}\Big|_{\xi_1\cdot\xi_2\xi_3\cdot\xi_4} &= 2ig_{\mathrm{o}}'^{\,2}\int_0^1 dz_1dz_2dz_3dz_4\delta(z_1-\infty)\delta(z_2-1)\delta(z_4-0) \\
&\times\; z_{12}^{2\alpha' k_1\cdot k_2}z_{13}^{2\alpha' k_1\cdot k_3}z_{14}^{2\alpha' k_1\cdot k_4}z_{23}^{2\alpha' k_2\cdot k_3}z_{24}^{2\alpha' k_2\cdot k_4}z_{34}^{2\alpha' k_3\cdot k_4}z_{12}z_{14}z_{24} \\
&\times\; \xi_1\cdot\xi_2\xi_3\cdot\xi_4 z_{12}^{-2}z_{34}^{-2} \\
&= 2ig_{\mathrm{o}}'^{\,2}\xi_1\cdot\xi_2\xi_3\cdot\xi_4\int_0^1 dz z^{-\alpha' s-2}(1-z)^{-\alpha' u} \\
&= 2ig_{\mathrm{o}}'^{\,2}\xi_1\cdot\xi_2\xi_3\cdot\xi_4 B(-\alpha_{\mathrm{o}}(s),-\alpha_{\mathrm{o}}(u)+2),
\end{aligned}
\tag{3.44}
$$

where we have defined $z_{ij}\equiv z_i-z_j$. It should be clear where the different factors originate from. In particular, the factors in the third line come from the $\xi_1\cdot\xi_2\xi_3\cdot\xi_4$-part in the expansion of the exponential. The full amplitude becomes

$$
\begin{aligned}
\tilde{S}_{4A} &= 2ig_{\mathrm{o}}'^{\,2}\alpha'^2\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' u)}{\Gamma(-\alpha' t+1)} \\
&\times\; \Big\{4\alpha' s\xi_1\cdot k_3\xi_2\cdot k_3(\xi_3\cdot k_1\xi_4\cdot k_1+\xi_3\cdot k_2\xi_4\cdot k_2) \\
&+\; \frac{4\alpha' s}{3}(\xi_1\cdot k_2\xi_2\cdot k_3\xi_3\cdot k_1-\xi_1\cdot k_3\xi_2\cdot k_1\xi_3\cdot k_2)(\xi_4\cdot k_1-\xi_4\cdot k_2) \\
&-\; 2s(\xi_1\cdot k_4\xi_3\cdot k_2\xi_2\cdot\xi_4+\xi_2\cdot k_3\xi_4\cdot k_1\xi_1\cdot\xi_3 \\
&+\; \xi_1\cdot k_3\xi_4\cdot k_2\xi_2\cdot\xi_3+\xi_2\cdot k_4\xi_3\cdot k_1\xi_1\cdot\xi_4) \\
&+\; \frac{tu}{\alpha' s+1}(\xi_1\cdot\xi_2-2\alpha'\xi_1\cdot k_2\xi_2\cdot k_1)(\xi_3\cdot\xi_4-2\alpha'\xi_3\cdot k_4\xi_4\cdot k_3) \\
&+\; [3\leftrightarrow 1]+[3\leftrightarrow 2]\Big\},
\end{aligned}
\tag{3.45}
$$

where the tilde on S indicates that we have not yet summed over the six permutations of the vertex operators on the boundary. For convenience the full amplitude is not written out - the square brackets in Eq. (3.45) have the same structure as the expression displayed, but with all subscripts exchanged according to the rule exhibited. Due to the bosonic nature of the amplitude, it is invariant under the interchange of the pairs $(\xi_m,k_m)\leftrightarrow(\xi_n,k_n)$, for any m,n. In addition it is gauge invariant, *i.e.* it vanishes if we make the replacement $\xi_m\to k_m$ for any m. Note that since every factor in the amplitude is proportional to the momenta, it vanishes when they all are put to zero, *i.e.* it is Abelian.

To add non-Abelian gauge interactions to the theory, it has to be modified in some way. By attaching extra non-dynamical degrees of freedom, Chan-Paton factors, to the ends of the open strings, we generate a self-interacting gauge theory, see Fig. 3.1. These factors can be interpreted as "group" charges. The particle wave functions are decomposed

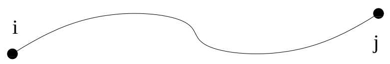


Figure 3.1: To get a non-Abelian gauge theory, Chan-Paton factors transforming under some group symmetry have to be attached to the ends of the string.

into constituents as

$$|\Lambda; k; a\rangle = \sum_{i,j} T^a_{ij} |\Lambda; k; ij\rangle, \tag{3.46}$$

where Λ is a particular state with momentum k, and T^a_{ij} are the generators of the group under which the the charges transform. i and j denote the states in which the two endpoints are. In the calculations, this means that group theoretic factors involving traces of the generators multiply the open string scattering amplitudes according to how the vertex operators on the boundary are ordered. For example, when computing an n-point amplitude, with the n vertex operators ordered on the boundary as $(1, 2, \ldots, n)$, this part of the amplitude is to be multiplied by $\mathrm{tr} T^{a_1 \cdots a_n}$, where as before $T^{a_1 \cdots a_n} \equiv T^{a_1} \ldots T^{a_n}$.

The simplest example involving these new extra degrees of freedom is the generalization of the three-tachyon amplitude calculated before. With Chan-Paton factors the amplitude becomes (the amplitudes involving Chan-Paton factors are primed)

$$S'_{3\varphi} = \frac{i g_o}{\alpha'} \mathrm{tr}\Big(T^a \{T^b, T^c\} \Big) \tag{3.47}$$

since there are two inequivalent orderings of the vertex operators on the boundary as noted before. Note that this amplitude reduces to the amplitude calculated in Eq. (3.38) if we consider the Abelian case, where the generators (crudely) approach the limiting value $T^a \to 1$, and further ignore the trace. Continuing, the four-tachyon coupling with Chan-Paton factors generalizes to

$$\begin{aligned} S'_{4\varphi} \quad = \quad & \frac{i g_o^2}{\alpha'} \Big(\mathrm{tr}(T^{abdc} + T^{acdb}) B(-\alpha_o(s), -\alpha_o(t)) \\ & + \mathrm{tr}(T^{acbd} + T^{adbc}) B(-\alpha_o(t), -\alpha_o(u)) \\ & + \mathrm{tr}(T^{abcd} + T^{adcb}) B(-\alpha_o(s), -\alpha_o(u)) \Big). \end{aligned} \tag{3.48}$$

By taking the "Abelian limit", $T^a \to 1$, we see that the three-gauge-boson amplitude is absent in the Abelian case:

$$
\begin{aligned}
&S'_{3A} = 2i\mathrm{g}'_{\mathrm{o}}\xi^a_{1\mu}\xi^b_{2\nu}\xi^c_{3\rho}\mathrm{tr}\Big(T^a[T^b,T^c]\Big) \\
&\times\Big((k_2^\mu - k_3^\mu)\eta^{\nu\rho} + (k_3^\nu - k_1^\nu)\eta^{\mu\rho} + (k_1^\rho - k_2^\rho)\eta^{\mu\nu} + 4\alpha' k_2^\mu k_3^\nu k_1^\rho\Big),
\end{aligned}
\tag{3.49}
$$

as pointed out before. Note that the polarization tensors now have been assigned group indices as well in analogy with the non-Abelian gauge fields $A^a_\mu(x)$ in the quantum field theory. The antisymmetry in the group indices in the vertex involving one gauge boson and two tachyons

$$
S'_{A,2\varphi} = -i\mathrm{g}'_{\mathrm{o}}\xi^a_{1\mu}(k_2^\mu - k_3^\mu)\mathrm{tr}\Big(T^a[T^b,T^c]\Big), \tag{3.50}
$$

is also manifest[2]. We do not verify this amplitude explicitly. For the four-gauge-boson amplitude, there are six inequivalent orderings of the vertex operators on the disk. With Chan-Paton factors, the $\xi_1\cdot\xi_2\xi_3\cdot\xi_4$-part of the amplitude becomes

$$
\begin{aligned}
S'_{4A}\Big|_{\xi_1\cdot\xi_2\xi_3\cdot\xi_4} = \; & 2i{\mathrm{g}'_{\mathrm{o}}}^2\xi^a_{1\mu}\xi^b_{2\nu}\xi^c_{3\rho}\xi^d_{4\sigma}\eta^{\mu\nu}\eta^{\rho\sigma} \\
&\times\Big(\mathrm{tr}(T^{abdc}+T^{acdb})B(-\alpha_{\mathrm{o}}(s),-\alpha_{\mathrm{o}}(t)+2) \\
&+\mathrm{tr}(T^{acbd}+T^{adbc})B(-\alpha_{\mathrm{o}}(t)+2,-\alpha_{\mathrm{o}}(u)+2) \\
&+\mathrm{tr}(T^{abcd}+T^{adcb})B(-\alpha_{\mathrm{o}}(s),-\alpha_{\mathrm{o}}(u)+2)\Big).
\end{aligned}
\tag{3.51}
$$

This is the only part of the string amplitude needed for the comparison with the field theory amplitudes as seen later on.

3.3 Allowed Gauge Groups

So far the internal non-dynamical degrees of freedom at the ends of the string do not respect any symmetry in particular. To get a feeling for which symmetries that are allowed we note that due to the cyclic property of the trace the scattering amplitudes obtained in the last section are left invariant under the transformation

$$
T^a \to VT^aV^\dagger, \tag{3.52}
$$

[2] Recall that $\mathrm{tr}\Big(T^a[T^b,T^c]\Big) = if^{abc}$.

where V is an element of the unitary Lie group $\mathrm{U}(n)$. For details about this group see Appendix C. Open strings come in two different shapes, *oriented* and *unoriented*, which in turn determine the allowed gauge group structure. The only allowed group for the oriented open string, with which we are mostly concerned in this thesis is $\mathrm{U}(n)$, with the property above. For the unoriented open string the two gauge groups $\mathrm{SO}(n)$ and $\mathrm{USp}(n)$ are possible. As we will see in the next chapter, in a system with N *coincident* parallel D-branes with oriented strings stretching between them, the gauge fields and scalars, living parallel and transverse to the brane respectively, are respecting symmetry under the gauge group $\mathrm{U}(N)$. The Chan-Paton indices i and j indicate on which branes the open strings start and end, cf. Eq. (3.46). The fields contained in this system become massless, since the masses of the particles are proportional to the distance between the branes on which the strings have their endpoints on. If the branes become separated, but still parallel, the $\mathrm{U}(N)$ group breaks spontaneously to $\mathrm{U}(1)^N$ with Abelian $\mathrm{U}(1)$ gauge fields residing on each of the N branes.

Both the gauge field living on the Dp-brane and the $(25-p)$ transverse scalars naturally take their values in the adjoint representation of $\mathrm{U}(N)$. The Chan-Paton factors necessarily take their values in the fundamental representations of the symmetry groups. For oriented strings the two endpoints transform in N and $\bar{N}$ of $\mathrm{U}(N)$, which naturally gives the adjoint representation $N \otimes \bar{N}$.

4

Non-Abelian Effective Actions From String Amplitudes

In this chapter we determine the low energy tachyon and gauge boson effective actions by matching the coefficients of the proposed Ansätze made in Chapter 2 to the string scattering amplitudes. To be able to make this comparison we must take the low energy limit of the string amplitudes, in which the Regge slope α' approaches zero[1], and Taylor expand them. It is worth emphasizing that the obtained actions are not uniquely determined as several actions are related by various field re-definitions and identities and consequently produce the same low energy scattering amplitudes.

At the end of the chapter a comparison with the open superstring is made. We conclude that due to the few constraints put on it, bosonic string theory generally has a richer structure than superstring theory. Concretely, this means that the scattering amplitudes in the bosonic string theory contain more terms, cf. the α'-contribution in the three-gauge-boson amplitude which is not present in superstring theory due to the $D = 10$, $\mathcal{N} = 1$ supersymmetry [7]. The structure of the non-Abelian Born-Infeld action is similar in the two theories except for a covariant derivative part only present in the bosonic theory. Further we describe the problems with the non-Abelian Born-Infeld action and finally how it can be interpreted as an action describing overlapping D25-branes.

[1]More precisely, we take the limit of the dimensionless quantity $\alpha' k^2$.

4.1 Non-Abelian Tachyon Action

We start by identifying the Yang-Mills coupling constant 'g' with the gauge boson coupling constant g'_o in the string theory. Further, the relation between the gauge boson coupling constant and the tachyon coupling constant can be shown to be

$$\mathrm{g}_\mathrm{o} = \sqrt{2\alpha'}\mathrm{g}'_\mathrm{o}, \tag{4.1}$$

giving g_o the dimension $(\mathrm{length})^{12}$ in 26 space-time dimensions. Beginning the analysis with the 3-point vertex coming from the lowest order interaction term $D\varphi D\varphi|_{int}$, involving two tachyons and one gauge boson, we get the coefficient

$$\gamma_1 = -1/2 \tag{4.2}$$

by comparing Eq. (2.27) with the string amplitude in Eq. (3.50). Of course, this was expected from Eq. (2.4), since the non-interacting part of $D\varphi D\varphi$ is nothing but the kinetic term. Furthermore, κ_1 is trivially determined to be

$$\kappa_1 = \frac{\sqrt{2}}{3} \tag{4.3}$$

from Eq. (2.25). Next we turn to the 4-point amplitude. To compare with the quantum field theory, the Euler Beta functions need to be expanded in α'. For $B(-\alpha_\mathrm{o}(s), -\alpha_\mathrm{o}(t))$ we get

$$\begin{aligned} B(-\alpha' s - 1, -\alpha' t - 1) &= -\frac{2}{\alpha'}\frac{s+t}{st} - \frac{(s+t)^2}{st} + 2\alpha'(s+t) \\ &- \alpha'^2(s+t)^2 + 2\alpha'^2\zeta(3)(s+t)^2 \\ &+ \frac{\pi^2}{3}\Big(\alpha'(s+t) + \frac{\alpha'^2}{2}(s+t)^2\Big) \\ &+ \alpha'^3(s+t)\Big((s^2+t^2) - \frac{\pi^2}{3}st + \zeta(3)(s+t)^2\Big) \\ &+ \mathcal{O}(\alpha'^4), \end{aligned} \tag{4.4}$$

where the Riemann zeta function $\zeta(p)$ is defined to be the infinite series[2]

$$\zeta(p) \equiv \sum_{n=1}^{\infty} n^{-p}. \tag{4.5}$$

[2]Note that that the function $\zeta(2) = \pi^2/6$ also appears in the the expansion of the Euler Beta function in Eq. (4.4).

The momenta of four scattered particles are generically related via the linear dependence

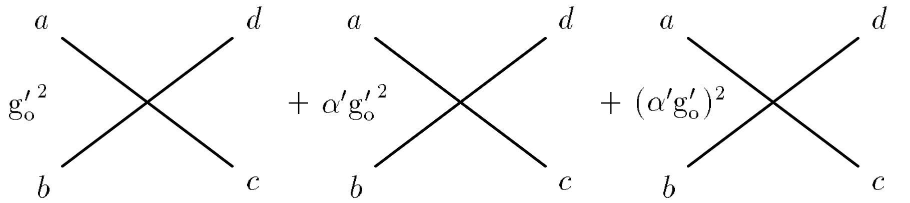
$$s + t + u = \sum_{i=1}^{4} m_i^2. \tag{4.6}$$

In particular, the relation reads $s+t+u = -4(\alpha')^{-1}$ for tachyons. Looking at Eq. (4.4) we conclude that terms at order ${\alpha'}^3$ can be reduced to terms at lower orders in α'. We ignore this fact and determine a consistent effective action to order ${\alpha'}^2$ without "reducing" from order ${\alpha'}^3$. It is now convenient (though not necessary) to restrict the generators T^a to the group $\mathrm{U}(N)$, see Appendix C for some useful relations. By symmetry in the external legs it is sufficient to analyse the $d^{abe}d^{cde}$-terms of the scattering amplitude. Thus the expansion of the string amplitude is deduced from Eqs. (3.48) and (4.4) to be

$$\begin{aligned} S'_{4\varphi}\Big|_{d^{abe}d^{cde}} &= \frac{i g_o^2 d^{abe} d^{cde}}{2\alpha'} \Big[B(-\alpha_o(s), -\alpha_o(t)) - B(-\alpha_o(t), -\alpha_o(u)) \\ &+ B(-\alpha_o(s), -\alpha_o(u)) \Big] \\ &= \frac{i g_o^2 d^{abe} d^{cde}}{2\alpha'} \Big[\frac{s^2 + t^2 + u^2}{tu} + \frac{4s}{\alpha' tu} - 1 + 12\alpha' s + 2{\alpha'}^2 tu \\ &- \frac{\pi^2}{3}(2s\alpha' + {\alpha'}^2 tu) - 4\zeta(3)(4s\alpha' + {\alpha'}^2 tu) \Big] + \mathcal{O}({\alpha'}^3). \end{aligned} \tag{4.7}$$

From the point particle theory, five Feynman diagrams contribute to this part of the low energy 4-point string amplitude. Two kinds of diagrams occur - the *exchange diagrams* where the in-going particles interact by exchanging virtual particles, and the *contact terms* where the particles interact in a single point. To lowest order the tachyons exchange massless gauge bosons.

Two diagrams correspond to the t- and u-channel of the exchange diagram with two vertices coming from the $D\varphi D\varphi|_{int}$ term. The remaining three diagrams are contact terms coming from the three Lagrangian terms involving four tachyon fields. After fixing γ_1, the two exchange diagrams exactly reproduces the first two terms in the square bracket in Eq. (4.7). The contribution from the contact terms becomes explicitly



$$= \quad i{g'_o}^2 d^{abe} d^{cde} \Big[2(\kappa_2+\kappa_3+\kappa_4) + \alpha'\Big(s\kappa_3 + \kappa_4(t+u)\Big) + \frac{{\alpha'}^2\kappa_4}{2} tu \Big], \tag{4.8}$$

and they reproduce the remaining terms in Eq. (4.7) if the coefficients are chosen to be

$$\begin{aligned}
\kappa_2 &= -\frac{25}{2} + \frac{2\pi^2}{3} + 16\zeta(3) \\
\kappa_3 &= 16 - \frac{4\pi^2}{3} - 24\zeta(3) \\
\kappa_4 &= 4 - \frac{2\pi^2}{3} - 8\zeta(3).
\end{aligned} \tag{4.9}$$

Thus, the Lagrangian that reproduces the low energy non-Abelian tachyon string amplitudes up to order ${\alpha'}^2$ has been obtained and can be summarized as

$$\begin{aligned}
\hat{\mathcal{L}}^{\rm NT} &= -\frac{1}{2}{\rm tr}(\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi^2) \\
&+ \sqrt{\frac{2}{\alpha'}}\frac{g'_o}{3}{\rm tr}\varphi^3 - g'_o\partial_\mu\varphi^a A^{b\mu}\varphi^c f^{abc} \\
&+ {g'_o}^2\Big(-\frac{25}{2} + \frac{2\pi^2}{3} + 16\zeta(3)\Big){\rm tr}\varphi^4 \\
&+ {g'_o}^2\Big(16 - \frac{4\pi^2}{3} - 24\zeta(3)\Big){\rm tr}\varphi^2\partial_\mu\varphi\partial^\mu\varphi \\
&- {g'_o}^2\Big(\frac{2\pi^2}{3} + 8\zeta(3)\Big){\rm tr}\partial_\mu\partial_\nu\varphi\partial^\mu\varphi\partial^\nu\varphi\varphi \\
&+ \mathcal{O}({\alpha'}^3).
\end{aligned} \tag{4.10}$$

Note that only the terms that we have checked explicitly have been written out. The Lagrangian in Eq. (4.10) is not gauge invariant as it stands and we indicate this by the hat. By making it gauge invariant, *i.e.* making the replacement $\partial_\mu \to D_\mu$, we know that the extra interaction terms appearing have to be present in the theory. For instance, ${\rm tr}\varphi^2 D_\mu\varphi D^\mu\varphi$ contains a vertex involving four tachyons and two gauge bosons. Let us now turn to the more interesting case involving the gauge bosons.

4.2 Non-Abelian Born-Infeld Action

From the Yang-Mills Lagrangian $\mathcal{L}_2$ the three-gauge-boson amplitude follows to lowest order. However, to get the higher order contribution to the amplitude, we have to add the $\mathcal{L}_3$-term. The non-Abelian term in the three-gauge-boson amplitude at order α' is reproduced by the effective action if

$$k_3 = -\frac{2i}{3}, \tag{4.11}$$

as seen from the Eqs. (2.50) and (3.49). Note that the i is there to make the action at order α' real, as required. This is a consequence of the generators of the fundamental representation being Hermitian. The four-gauge-boson string amplitude is again written in a closed form, now including Euler Beta functions of two types. Expanding $B(-\alpha_{\rm o}(s), -\alpha_{\rm o}(t)+2)$ and $B(-\alpha_{\rm o}(s)+2, -\alpha_{\rm o}(t)+2)$ yields

$$\begin{aligned} B(-\alpha' s-1, -\alpha' t+1) &= -\Big(1+\frac{t}{s}\Big)+\alpha'(s+t) \\ &+ \alpha'^2\Big(s-\frac{1}{6}\pi^2 t\Big)u+\mathcal{O}(\alpha'^3) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} B(-\alpha' s+1, -\alpha' t+1) &= 1+\alpha'(s+t)+\alpha'^2\Big(u^2-\frac{1}{6}\pi^2 st\Big) \\ &+ \mathcal{O}(\alpha'^3), \end{aligned} \tag{4.13}$$

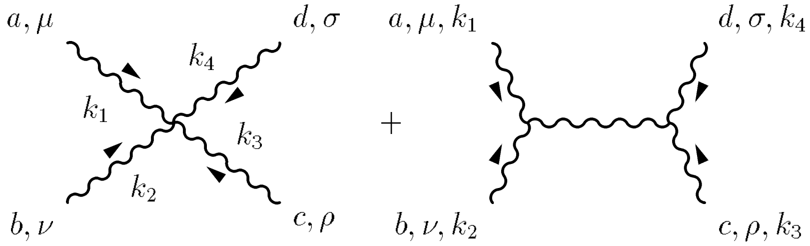
where the relation $s+t+u=0$, which is valid for massless fields, has been used. Expanding the string amplitude to order α'^2, using Eqs. (3.51), (4.12) and (4.13) we obtain

$$\begin{aligned} S'_{4A} &= 2i{\rm g}_{\rm o}'^2 \xi_1^a\cdot\xi_2^b\xi_3^c\cdot\xi_4^d \Bigg\{ \Big[\frac{u}{s}{\rm tr}(T^{abdc}+T^{acdb})+\frac{t}{s}{\rm tr}(T^{abcd}+T^{adcb}) \\ &+ {\rm tr}(T^{acbd}+T^{adbc})\Big]+\alpha'\Big[(s+t){\rm tr}(T^{abdc}+T^{acdb}) \\ &+ (t+u){\rm tr}(T^{acbd}+T^{adbc})+(s+u){\rm tr}(T^{abcd}+T^{adcb})\Big] \\ &+ \alpha'^2\Big[-\pi^2 tu\ {\rm str}T^{abcd}+{\rm tr}(T^{abcd}+T^{adcb})st \\ &+ {\rm tr}(T^{abdc}+T^{acdb})su+{\rm tr}(T^{acbd}+T^{adbc})s^2\Big]\Bigg\}+\mathcal{O}(\alpha'^3). \end{aligned} \tag{4.14}$$

It is now convenient to specify to the algebra U(N). Using the relations in Appendix C, the amplitude can be completely expressed in terms of the symmetric tensor d^{abc} as

$$
\begin{aligned}
S'_{4A} &= i g_o'^2 \xi_1^a \cdot \xi_2^b \xi_3^c \cdot \xi_4^d \Big\{ -2\Big[d^{abe}d^{cde} + \frac{t}{s} d^{ace}d^{bde} + \frac{u}{s} d^{ade}d^{bce} \Big] \\
&+ 2\alpha' \Big[s d^{abe}d^{cde} + t d^{ace}d^{bde} + u d^{ade}d^{bce} \Big] \\
&+ \alpha'^2 \Big[-\frac{\pi^2}{6} tu (d^{abe}d^{cde} + d^{ace}d^{bde} + d^{ade}d^{bce}) \\
&- 2 (s^2 d^{abe}d^{cde} + st d^{ace}d^{bde} + su d^{ade}d^{bce}) \Big] \Big\} + \mathcal{O}(\alpha'^3).
\end{aligned}
\tag{4.15}
$$

To lowest order, α'^0, the coefficients are already fixed by the Yang-Mills term, so the analysis below of the terms in the first two square brackets will merely serve as an independent check. For an explicit evaluation of an exchange diagram, see Eq. (4.18) below. The terms in the first square bracket of Eq. (4.15) are reproduced by the sum of the contact term coming from $\mathcal{L}_2$ and the exchange diagram in the s-channel[3] with two vertices from $\mathcal{L}_2$, see Eq. (4.16) below. To lowest order, we consider the processes where the particles interact by exchanging a massless gauge boson.



(4.16)

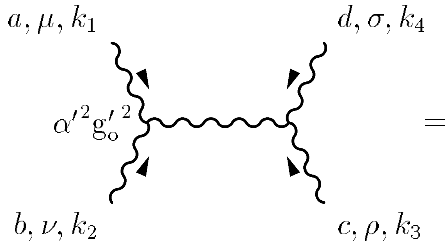
At order α', the coefficient is already fixed by Eq. (4.11), and the string amplitude corresponds to the sum of the contact term coming from $\mathcal{L}_3$ and the exchange diagram with one vertex from $\mathcal{L}_2$ and one from $\mathcal{L}_3$. These terms are due to the tachyon poles and are not present in the superstring theory where the tachyons are absent.

[3]Obviously we have to sum over all possible exchange diagrams to get the full amplitude, *i.e.* include the t- and u-channel as well. However, since only the $\xi_1 \cdot \xi_2 \xi_3 \cdot \xi_4$-term is considered in our present analysis, the 's-diagram' is sufficient to reproduce this part of the amplitude. The reason is of course that the diagram is proportional to $\eta_{\mu\nu}$ from the left-hand vertex, and to $\eta_{\rho\sigma}$ from the right-hand vertex. This holds for *all* orders in α'.

The term at α'^2-level proportional to strT^{abcd} is reproduced by the four contact terms coming from $\mathcal{L}_4^{(i)}$, $i = 1, \ldots, 4$, if the coefficients are chosen to be

$$
\begin{aligned}
k_4^{(1)} &\propto -\frac{\pi^2}{12}, \qquad & k_4^{(2)} &\propto \frac{\pi^2}{6} \\
k_4^{(3)} &\propto -\frac{\pi^2}{24}, \qquad & k_4^{(4)} &\propto \frac{\pi^2}{3}.
\end{aligned} \tag{4.17}
$$

Of course, another possible Lagrangian term that reproduces this string amplitude can be written as a symmetrised trace, see next section. Finally, the three remaining terms at α'^2-level are to be calculated. To illustrate how an exchange diagram is evaluated with gauge boson fields, let us compute the diagram including two k^3-vertices, again only focusing on the $\xi_1 \cdot \xi_2 \xi_3 \cdot \xi_4$-term:



$=$

$$
\begin{aligned}
= \;& (2\alpha' \mathrm{g}_\mathrm{o}')^2 \xi^a_\mu \xi^b_\nu \xi^c_\rho \xi^d_\sigma f^{abe} f^{cdf} \eta^{\mu\nu} \Bigg[k_2 \cdot (k_3 + k_4) k_{1\lambda} - k_1 \cdot (k_3 + k_4) k_{2\lambda} \Bigg] \\
\times \;& \Big(- i \frac{\eta^{\lambda\kappa} d^{ef}}{(k_3 + k_4)^2} \Big) \\
\times \;& \eta^{\rho\sigma} \Bigg[- k_4 \cdot (k_3 + k_4) k_{3\kappa} + k_3 \cdot (k_3 + k_4) k_{4\kappa} \Bigg] + \ldots \\
= \;& i(\alpha' \mathrm{g}_\mathrm{o}')^2 \xi^a_\mu \xi^b_\nu \xi^c_\rho \xi^d_\sigma \eta^{\mu\nu} \eta^{\rho\sigma} s(u - t)[d^{ace} d^{bde} - d^{ade} d^{bce}] + \ldots
\end{aligned} \tag{4.18}
$$

Using this s-channel diagram and the contact terms coming from $\mathcal{L}_4^{(1)}$ and $\mathcal{L}_4^{(3)}$, the remaining terms at α'^2-level are reproduced if the corresponding coefficients are chosen to be

$$
k_4^{(1)} \propto -\frac{1}{2}, \qquad k_4^{(3)} \propto \frac{1}{2}. \tag{4.19}
$$

Accordingly, one possible effective action that reproduces all gauge boson amplitudes to order α'^2 is

$$\begin{aligned}
\mathcal{L}^{\rm NBI} &= -\frac{1}{4}{\rm tr}F_{\mu\nu}F^{\mu\nu} - \frac{2i}{3}(\alpha' {\rm g}_{\rm o}'){\rm tr}F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu \\
&- \Big(\frac{\pi^2}{12}+\frac{1}{2}\Big)(\alpha' {\rm g}_{\rm o}')^2{\rm tr}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \\
&+ \frac{\pi^2}{6}(\alpha' {\rm g}_{\rm o}')^2{\rm tr}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} \\
&+ \Big(-\frac{\pi^2}{24}+\frac{1}{2}\Big)(\alpha' {\rm g}_{\rm o}')^2{\rm tr}F_{\mu\nu}F_{\rho\sigma}F^{\nu\mu}F^{\sigma\rho} \\
&+ \frac{\pi^2}{3}(\alpha' {\rm g}_{\rm o}')^2{\rm tr}F^{\mu\nu}F^{\rho\sigma}F_{\nu\rho}F_{\sigma\mu} + \mathcal{O}(\alpha'^3).
\end{aligned} \tag{4.20}$$

Unlike the tachyon action obtained earlier this action is manifestly gauge invariant. The F^4-terms not including π^2 combine into a commutator, and the F^3-term can be re-expressed as a commutator:

$$\begin{aligned}
\mathcal{L}^{\rm NBI} &= {\rm tr}\bigg[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\tilde\alpha'^2}{96}\Big(-F_{\mu\nu}F_{\rho\sigma}F^{\nu\mu}F^{\sigma\rho} \\
&- 2F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + 4F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} + 8F^{\mu\nu}F^{\rho\sigma}F_{\nu\rho}F_{\sigma\mu}\Big)\bigg] \\
&- \frac{i}{3}(\alpha' {\rm g}_{\rm o}'){\rm tr}F_\mu{}^\nu[F_\nu{}^\rho, F_\rho{}^\mu] + \frac{1}{2}(\alpha' {\rm g}_{\rm o}')^2{\rm tr}F^{\mu\nu}F^{\rho\sigma}[F_{\mu\nu}, F_{\rho\sigma}] \\
&+ \mathcal{O}(\alpha'^3),
\end{aligned} \tag{4.21}$$

where $\tilde\alpha' \equiv 2\pi\alpha' {\rm g}_{\rm o}'$ is a constant of dimension (length)^{13} in 26 space-time dimensions. Hence, since we have the relation

$$[F_{\mu\nu}, F_{\rho\sigma}] = \frac{i}{{\rm g}_{\rm o}'}[D_\mu, D_\nu]F_{\rho\sigma}, \tag{4.22}$$

these terms can be traded for covariant derivative terms

$$F_\mu{}^\nu[F_\nu{}^\rho, F_\rho{}^\mu] = \frac{i}{{\rm g}_{\rm o}'}F_\mu{}^\nu[D_\nu, D^\rho]F_\rho^\mu \tag{4.23}$$

$$F^{\mu\nu}F^{\rho\sigma}[F_{\mu\nu}, F_{\rho\sigma}] = \frac{i}{{\rm g}_{\rm o}'}F^{\mu\nu}F^{\rho\sigma}[D_\mu, D_\nu]F_{\rho\sigma}. \tag{4.24}$$

It can be shown that *all* Lagrangian terms of odd powers of F belong to the covariant derivative-dependent part of the effective action, see *e.g.* Ref. [18]. So how are we to interpret this covariant derivative-dependent part of the effective action? One alternative is to neglect them in a first approximation, assuming that the covariant derivatives are much smaller than the powers of F, *i.e.* the variations of the fields are small. But this leads to a contradiction as pointed out in Ref. [14]. Small variations of large fields imply that the region over which they are non-zero is of considerable extent. To avoid the configuration of turning into a black hole, the field strength must fall to zero in a small region of space-time, implying appreciable derivatives. Hence they are really not negligible. Incidentally, this argument is valid in the Abelian case.

Finally, we notice that the non-Abelian Born-Infeld action (NBI) can be rewritten in a variety of ways using various field redefinitions.

4.3 Comparison With the Open Superstring

In superstring theory, which is consistent in ten space-time dimensions, we have in addition to bosonic fields X also fermionic fields χ. Ultimately, this implies that the tachyon is absent in this theory. The Euler Beta functions have a different structure [12], and as a consequence the expansion of the four-boson string amplitude contains fewer terms at every order:

$$\begin{aligned} S^{ss}_{4A} &= 2i{\mathrm{g_s}}^2\xi_1\cdot\xi_2\xi_3\cdot\xi_4\left[\frac{u}{s}\mathrm{tr}(T^{abdc}+T^{acdb})+\frac{t}{s}\mathrm{tr}(T^{abcd}+T^{adcb})\right.\\ &+ \left.\mathrm{tr}(T^{abdc}+T^{acdb})-{\alpha'}^2\pi^2 tu\ \mathrm{str}T^{abcd}\right]+\mathcal{O}({\alpha'}^4). \end{aligned} \tag{4.25}$$

Notice that the amplitude only involves even powers of α'. The bosonic part of the effective Lagrangian takes to order ${\alpha'}^2$ the form

$$\begin{aligned} \mathcal{L}^{ss}_{\mathrm{bos}} &= -\frac{1}{4}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}\\ &+ \frac{(2\pi\alpha'\mathrm{g_s})^2}{8}\mathrm{str}\Big(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big)+\mathcal{O}({\alpha'}^4), \end{aligned} \tag{4.26}$$

where the string coupling constant $\mathrm{g_s}$ and the field strength F have the dimensions $(\mathrm{length})^3$ and $(\mathrm{length})^{-5}$ respectively in ten space-time dimensions. See next section for a derivation of the term quartic in the field strengths for the bosonic string. The Lagrangian in Eq. (4.26) can to order ${\alpha'}^2$ be written as

$$(2\pi\alpha'\mathrm{g_s})^{-2}\mathrm{str}\Big[1-\sqrt{-\det(\eta_{\mu\nu}+2\pi\alpha'\mathrm{g_s}F_{\mu\nu})}\Big], \tag{4.27}$$

a simple and "natural" generalization of the Abelian Born-Infeld action [11]. At order α'^4, however, discrepancies from the true action appear, see *e.g.* Ref. [23]. Obviously, one has to define what a "true" action means in this context. In this particular case, the symmetrized non-Abelian effective action does not reproduce the correct mass spectrum of certain D-brane configurations[4]. Hence, the non-Abelian effective action in Eq. (4.27) is a good approximation, in the superstring case, when F is "small" in some sense, *e.g.* when the field strengths are "approximately Abelian", *i.e.* when they nearly commute. Even though covariant derivatives are absent up to this order, this does not mean that the ordering problem is solved. The covariant derivatives cannot be neglected as noted in the previous section.

The covariant derivative-dependent part of the effective action in the bosonic theory is a consequence of the tachyons being present. The bosonic theory has quite generally a richer structure than the superstring theory, due to the few (physical) constraints put on it.

Higher derivative corrections to the *Abelian* Born-Infeld action (BI) for the superstring have been calculated, see *e.g.* recent papers by N. Wyllard [19, 20], where an action correct up to four derivatives is deduced using the Seiberg-Witten map. Obviously, in the Abelian case there is no ambiguity in which terms belong to the derivative-dependent and which belong to the derivative-independent part of the effective action. For the covariant derivative terms in Eq. (4.21), it is not clear whether they belong to the derivative-dependent or the derivative-independent part of the bosonic action. The classification of the terms is ambigious.

4.4 Symmetrized Trace

Instead of making the Ansatz in Eqs. (2.29)-(2.34) for the effective action, we can replace the ordinary trace with a symmetric one. The Feynman rules for the F^4-terms will simply be modified by symmetric traces instead of ordinary ones, see Appendix A.4. However, caution must be taken, since terms with an odd number of field strengths vanish under "symmetrization". As we will see, these terms have a rather different structure; for now, the ordinary trace is kept for the F^3-term. Also, a simple calculation shows the linear dependence

$$\mathrm{str} F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} = \mathrm{str} F_{\mu\nu} F_{\rho\sigma} F^{\nu\mu} F^{\sigma\rho}. \tag{4.28}$$

[4]It may be possible to express the *full* NBI action, in a particular basis, as symmetrized traces over fields strengths and covariant derivatives.

This implies that it is impossible to express the part of the amplitude not including the π^2-terms as a symmetric trace only involving field strengths. After these observations one can easily show that the action that reproduces the string amplitudes to order α'^2 is given by ($\tilde{\alpha}' \equiv 2\pi\alpha' g'_o$)

$$
\begin{aligned}
\mathcal{L}^{\rm NBI} &= -\frac{1}{4}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} - \frac{2i}{3}(\alpha' g'_o)\mathrm{tr}F_{\mu}{}^{\nu}F_{\nu}{}^{\rho}F_{\rho}{}^{\mu} \\
&+ \frac{1}{2}(\alpha' g'_o)^2\mathrm{tr}\Big(F_{\mu\nu}F_{\rho\sigma}F^{\nu\mu}F^{\sigma\rho} - F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&+ \frac{1}{8}\tilde{\alpha}'^2\mathrm{str}\Big(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&+ \mathcal{O}(\alpha'^3),
\end{aligned}
\tag{4.29}
$$

using the relations

$$
\begin{aligned}
\mathrm{str}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} &= \frac{1}{3}\mathrm{tr}\Big(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} + 2F^{\mu\nu}F^{\rho\sigma}F_{\nu\rho}F_{\sigma\mu}\Big) \\
\mathrm{str}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} &= \frac{1}{3}\mathrm{tr}\Big(2F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + F_{\mu\nu}F_{\rho\sigma}F^{\nu\mu}F^{\sigma\rho}\Big).
\end{aligned}
\tag{4.30}
$$

Note that the symmetric trace part of the action is identical to the effective action of the open superstring, cf. Eq. (4.26). Of course, some of the terms can be re-expressed in terms of field strength commutators, and then further as covariant derivatives:

$$
\begin{aligned}
\mathcal{L}^{\rm NBI} &= \mathrm{str}\Bigg[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&+ \tilde{\alpha}'^2\Big(\frac{1}{8}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{32}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big)\Bigg] \\
&+ \frac{1}{3}\alpha'\mathrm{tr}F_{\mu}{}^{\nu}[D_\nu, D^\rho]F_{\rho}{}^{\mu} + \frac{i}{2}\alpha'^2 g'_o\mathrm{tr}F^{\mu\nu}F^{\rho\sigma}[D_\mu, D_\nu]F_{\rho\sigma} \\
&+ \mathcal{O}(\alpha'^3).
\end{aligned}
\tag{4.31}
$$

Going one step further, the non-derivative part of this form of the action can be expressed as a symmetrized trace of a non-Abelian generalization of the Born-Infeld action as for the superstring since

$$
\begin{aligned}
&(\tilde{\alpha}')^{-2}\mathrm{str}\Big[1-\sqrt{-\det(\eta_{\mu\nu}+\tilde{\alpha}'F_{\mu\nu})}\Big]\bigg|_{\alpha'^2} \\
=\;& \mathrm{str}\bigg[\frac{1}{4}F_{\mu\nu}F^{\nu\mu}+\tilde{\alpha}'^2\Big(\frac{1}{8}F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}-\frac{1}{32}F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big)\bigg] \\
=\;& \mathrm{tr}\bigg[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\tilde{\alpha}'^2}{96}\Big(-F_{\mu\nu}F_{\rho\sigma}F^{\nu\mu}F^{\sigma\rho} \\
-\;& 2F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}+4F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}+8F^{\mu\nu}F^{\rho\sigma}F_{\nu\rho}F_{\sigma\mu}\Big)\bigg],
\end{aligned}
\tag{4.32}
$$

using the "Cayley-Hamilton expansion" of the determinant inside a square root examined in detail in Appendix B. To verify the correspondance between the expansion in Eq. (4.32) to order α'^6 and scattering amplitudes, we need to calculate 6-point functions which is very tedious.

4.5 Connections to Dirichlet-branes

As noted in the introduction, D-branes naturally appear in string theory by replacing some of the open string Neumann boundary conditions in the Polyakov action by Dirichlet ones. Physically, this means that the endpoints of the open strings are restricted to move on lower-dimensional submanifolds in the directions $m = 0, \ldots, p$. These extended objects on which open strings are allowed to end are called Dp-branes. The Dp-branes are not generically rigid, but are rather dynamical objects with finite tensions which can fluctuate and move around in space-time. In particular, the Dirichlet boundary conditions of the strings ending on the branes imply that the momenta of the strings are not conserved, but can be exchanged between the strings and the branes. In the following we consider only oriented strings.

Two kinds of fields appear on a Dp-brane: the embedding coordinate $X^\mu(x)$, $\mu = 0, \ldots, 25$ and the gauge field $A_m(x)$, $m = 0, \ldots, p$, where x^m are coordinates on the brane. In the case of a single D-brane, the massless string excitations A_m appear as an U(1) gauge field on the brane and $(25-p)$ world-brane scalars X_s describing the oscillations of the brane in the transverse directions. The action describing this configuration in a trivial background is

$$
-T_p\int d^{p+1}x\sqrt{-\det(\eta_{mn}+\partial_m X^s\partial_n X^s+2\pi\alpha' F_{mn})}, \tag{4.33}
$$

where T_p is the tension of the Dp-brane[5]. Note that Eq. (4.33) is reproduced by the more general expression in Eq. (1.4) if we replace the curved background $G_{\mu\nu}$ with a flat one, put the antisymmetric tensor $B_{\mu\nu}$ to zero and use the diffeomorphism invariance on the brane world-sheet to put $X^m = x^m$, *i.e.* use the static gauge:

$$G_{mn} = \frac{\partial X^\mu}{\partial x^m}\frac{\partial X^\nu}{\partial x^n}G_{\mu\nu} = \eta_{mn} + \partial_m X^s \partial_n X^s. \tag{4.34}$$

Generalizing to a cluster of parallel m D-branes, we obtain an $\mathrm{U}(1)^m$ gauge symmetry describing the massless fields, since each brane has an U(1) gauge field living on it. The masses of the fields associated to the strings stretching between the branes are proportional to the "interbrane" distance. Taking the limit when the branes *coincide*, these fields become massless; specifically for N Dp-branes, with oriented strings stretching between them, the gauge fields become U(N)-valued and we end up with a non-Abelian gauge theory. The massless gauge fields, arising from the N coincident branes, are now carrying additional group indices (ij), $i, j = 1, \ldots, N$ indicating on which branes the stretched strings start and end. There are N^2 possibilities for the strings to start and end on the parallel branes, and thus the massless fields live in the adjoint representation of U(N). In the bosonic string theory there are no physical constraints which fix the number of coinciding branes, due to its intrinsic unphysical nature. In a more physical theory with 10 spacetime dimensions, like type I superstring theory, the number of coinciding D-branes is limited to 32. This stems from the fact that U(N) contains SO(N) as a subgroup, which appear after some careful restrictions of the theory, *e.g.* an orientfold projection. Further, SO(32) has the advantage of giving rise to an *anomaly-free* theory - there are no divergences at the one-loop level (*e.g.* the tadpoles sum up to zero).

The effective action of N coinciding D25-branes is equivalent to the open string action with Chan-Paton factors. To understand this we simply note that the D25-branes fill all of space, and correspondingly the open strings in the theory are free to end on the branes anywhere in spacetime. To link up with previous sections, the action obtained in Eq. (4.20) is accordingly the low energy action for a stack of D25-branes up to order α'^2 (with a proper normalization). To obtain the effective actions of the lower-dimensional Dp-branes, we simply perform a *dimensional reduction* of the D25-brane action by putting $A^\mu = (A^m, A^s)$, where the transverse gauge fields are identified with the fluctuations of the brane about some classical value, as noted above. These transverse coordinates X^s are in a generic configuration of branes non-Abelian matrices, which give rise to

[5] The coupling constant is here absorbed into the field strength tensor.

the notion of *noncommutative geometry*. This means that when the matrices are not simultaneously diagonalizable, the branes no longer have well-defined positions in terms of our classical notion of geometry [22].

In the Abelian case the dimensional reduction is performed by the following steps. Starting with the D25-brane action

$$-T_{25}\int d^{26}x\sqrt{-\det(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu})}, \tag{4.35}$$

we note that the the product of the tension and the measure, $T_p d^{p+1}x$, has the same length dimension for any p. The matrix $\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}$ can be expressed in block matrix form as

$$\begin{pmatrix} \eta_{mn}+T^{-1}F_{mn} & \partial_m X_s \\ -\partial_n X_r & \eta_{rs} \end{pmatrix}, \tag{4.36}$$

where the gauge potential has been decomposed into its longitudinal and transversal parts $A_\mu = (A_m, TX_s)$, and is assumed to depend only on x^m. By using the determinant formula

$$\det\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \det B \det(A - CB^{-1}D) \tag{4.37}$$

where A, B, C and D are block matrices (A and B are invertible), we obtain the dimensionally reduced action for a Dp-brane:

$$-T_p\int d^{p+1}x\sqrt{-\det(\eta_{mn}+\partial_m X^s\partial_n X^s+2\pi\alpha' F_{mn})}. \tag{4.38}$$

identical to the action in Eq. (4.33). The transverse scalars X^s are obviously absent for a D25-brane which makes the analysis much simpler. In fact, most of the current research about the non-Abelian Born-Infeld action is focusing on the D9-branes in superstring theory, which are the branes that fill space in ten dimensions.

It should be stressed that D-branes in the bosonic theory are not solitons in the same sense as in the superstring theory. In particular, the Dp-branes in superstring theory are charged and the effective actions contain Wess-Zumino terms which couple to Ramond-Ramond fields. There are no corresponding charges in the bosonic theory under which the branes are charged. Once again, the theory merely serve as a toy-model which contains several unattractive features.

To summarize this chapter, we present the effective action for a stack of bosonic D25-branes in the Born-Infeld form (needs appropiate normalization):

$$\begin{aligned}
\mathcal{L}^{\mathrm{NBI}} &= (\tilde{\alpha}')^{-2}\mathrm{str}\Big[1-\sqrt{-\det(\eta_{\mu\nu}+\tilde{\alpha}'F_{\mu\nu})}\Big]\Bigg|_{\alpha'^2} \\
&+ \frac{1}{3}\alpha'\mathrm{tr}F_{\mu}{}^{\nu}[D_\nu,D^\rho]F_{\rho}{}^{\mu}+\frac{i}{2}\alpha'^2\mathrm{g}_{\mathrm{o}}'\mathrm{tr}F^{\mu\nu}F^{\rho\sigma}[D_\mu,D_\nu]F_{\rho\sigma} \\
&+ \mathcal{O}(\alpha'^3). \qquad (4.39)
\end{aligned}$$

5

Conclusions and Comments

The notion of a non-Abelian Born-Infeld action is rather misleading. We do not know *a priori* whether the non-Abelian analog of the ordinary Abelian Born-Infeld action can be written in a closed form. Of course this depends on what system it is intended to describe. In this thesis we have found this non-Abelian Born-Infeld action in the bosonic theory up to order α'^2, see *e.g.* Eq. (4.31). Though there exist several methods to obtain this action, the approach pursued in this thesis is certainly the most straightforward and the easiest to grasp conceptually. However, the method relies on ordinary S-matrix calculations which tend to grow extremely complicated at high orders. Going one step further, to order α'^3, we have to calculate 5-point scattering amplitudes. These can certainly be calculated, but calculations of generic n-point amplitudes, including loop corrections, would require more efficient methods, computer calculations for example. Alternative indirect approaches have been pursued, *e.g.* β-function calculations, which like the former method also become unwieldy in the non-Abelian case at high orders. Other attempts include methods using supersymmetry, BPS-type solutions and Seiberg-Witten mapping.

The results obtained in this thesis are valid in the bosonic string theory, which contains several unattractive features. Among these one is that the lightest string state is a tachyon which in turn implies an unstable vacuum. To get a more acceptable theory, from the physical point of view, fermions has to be included in it. Furthermore, the tachyons have to be eliminated from the theory; this is accomplished by the GSO projection. The calculations of the string scattering amplitudes for the superstring are similar to the calculations performed in this thesis. As noted earlier, due to the physical restrictions put on it, the superstring theory generically has a more restricted structure than the pure bosonic theory. In this particular context this implies that the Born-Infeld action[1], obtained from the supersymmetric theory, contains fewer terms. In particular, at order α'^2, the bosonic part of the superstring Born-Infeld action only contains the part of the *bosonic* Born-Infeld action that can be expressed as symmetrized traces. The part involving commutators of covariant derivatives is absent in the superstring theory since they are due to the tachyons being present.

Hence it is more convenient to analyse the Born-Infeld action in the supersymmetric string theories, since there its structure is more constrained. In addition all the symmetries and dualities present in these theories can be made used of to obtain the Born-Infeld action by various indirect approaches.

Other interesting related topics concern the Abelian situation. We know that the tree-level scattering amplitudes are summarized to all orders in α' in the Abelian Born-Infeld action, if the field strength is regarded to be constant. This action can be exactly derived from path integral methods if one only considers the disk diagram, which is the lowest order contribution to the corresponding partition function. The question arises whether derivative corrections to the BI action also can be derived exactly from path-integral methods. This would then imply a natural non-Abelian generalization of the Born-Infeld action involving symmetrized traces. The reason why this would work is the following.

An exact closed form of the non-Abelian Born-Infeld action at *tree-level* can be derived from path integral methods [9], if we treat the field strengths as commuting, *i.e.* include a symmetrized trace in the generating functional

$$Z[A_\mu^a] = \langle \mathrm{str} \exp i \int d\vartheta \frac{\partial x^\mu}{\partial \vartheta} A_\mu(x) \rangle, \tag{5.1}$$

where x^μ is the string embedding field and ϑ parametrizes the boundary of the disk. Notice that when using symmetrized traces no path ordering is required. The NBI action is then calculated analogously to the Abelian case using *e.g.* the radial gauge, in which $\xi \cdot A(x) = 0$ and $A(x_0) = 0$, where $x = x_0 + \xi$ has been divided into a constant zero-mode part and a part describing the fluctuations about that value. Since $F_{\mu\nu}$ is constant, A_μ can be expressed simply as $\xi^\nu F_{\nu\mu}/2$. Under these assumptions we obtain the NBI Lagrangian in a closed form

$$\mathcal{L} \sim \mathrm{str} \sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}. \tag{5.2}$$

[1] Here we refer to the full effective action, not only the part which can be written on "Born-Infeld form".

From the symmetrized trace description, higher order derivative corrections to this action involve only symmetrized covariant derivatives by construction.

We know that discrepancies from the symmetrized trace description appear at the order α'^3 if the action is to describe a particular configuration of D-branes, so the discussion above seems to fail at some point.

A

Conventions and Notations

A.1 Units and Dimensions

As is customary in high energy physics, natural units are used, *i.e.*

$$c = \hbar = 1,$$

and all dimensions are given in terms of length (inverse mass dimension). The Regge slope α' has in this system the dimension of $(\text{length})^2$, and is of the order M_{Planck}^{-2}, where M_{Planck} is the Planck mass. In open bosonic string theory it is conventional to put the Regge slope to be equal to $1/2$. To avoid confusion all the α''s are explicitly shown in the thesis. The string tension

$$T = \frac{1}{2\pi\alpha'} \tag{A.1}$$

is introduced for dimensional reasons. To facilitate dimensional analysis, we here collect the dimensions of frequently used fields and constants in the thesis. For comparison, the dimensions in four-dimensional space-time are also presented in the table below.

	$D = 26$	$D = 4$
$F_{\mu\nu}(x)$	-13	-2
$A_\mu(x),\ \varphi(x)$	-12	-1
$\tilde{A}_\mu(p),\ \tilde{\varphi}(p)$	$+14$	$+3$
$\mathrm{g_o}$	$+12$	$+1$
$\mathrm{g_o}'$	$+11$	0
$\delta^D(x)$	-26	-4
$\delta^D(p)$	$+26$	$+4$

A.2 Tensors

We generally work in the critical dimension, 26-dimensional Minkowski space-time, with the metric $\eta_{\mu\nu} = (-, +, \ldots, +)$. This implies that the invariant mass of a particle is related to its momentum by $p^2 = -m^2$. Furthermore, the world-sheet metric, γ_{ij}, has the Lorentzian signature, $(-, +)$. Lorentz indices are denoted by the Greek letters $\mu, \nu, \rho, \sigma = 0, \ldots, 25$.

For Dp-branes the longitudinal directions are indicated by the indices $m, n = 0, \ldots, p$ and the transverse directions by $s, t = (p+1), \ldots, 25$.

The Mandelstam variables

$$s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_3)^2, \quad u = -(k_1 + k_4)^2 \tag{A.2}$$

are used to simplify four-particle amplitudes. Since $k^2 = 1/\alpha'$ for the tachyon and $k^2 = 0$ for the gauge boson, we obtain the often used relations

$$\begin{aligned} 2k_1 \cdot k_2 &= -s - \frac{2}{\alpha'} \\ 2k_1 \cdot k_3 &= -t - \frac{2}{\alpha'} \\ 2k_1 \cdot k_4 &= -u - \frac{2}{\alpha'} \end{aligned} \tag{A.3}$$

for the tachyons and

$$\begin{aligned} 2k_1 \cdot k_2 &= -s \\ 2k_1 \cdot k_3 &= -t \\ 2k_1 \cdot k_4 &= -u \end{aligned} \tag{A.4}$$

for the gauge bosons.

A.3 Non-Abelian Fields

Throughout the thesis the following conventions for the non-Abelian fields are used:

$$\varphi \equiv \varphi^a T^a \tag{A.5}$$
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i\mathrm{g}[A_\mu, A_\nu] \tag{A.6}$$
$$F_{\mu\nu} \equiv F^a_{\mu\nu} T^a \tag{A.7}$$
$$D_\mu \equiv \partial_\mu - i\mathrm{g}[A_\mu, .], \tag{A.8}$$

where 'g' is the Yang-Mills coupling constant, identified with the gauge boson string coupling constant g_o'. The fields are matrices and are expanded in a basis spanned by the generators T^a. Furthermore, these generators are defined to be Hermitian, *i.e.*

$$[T^a, T^b] = i f^{abc} T^c, \tag{A.9}$$

with the structure constants f^{abc} real. This yields the field strength components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{A.10}$$

which are "ordinary commuting numbers". The invariance of the ordinary Yang-Mills action under local gauge transformations is a very important tool in constructing Lagrangians of non-Abelian gauge fields. Therefore a proof of this using forms is presented here as promised in Section 2.4. The gauge transformation in Eq. (2.36) can be rewritten in the language of forms using the definitions

$$A \equiv A_\mu dx^\mu, \qquad F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA - i\mathrm{g} A \wedge A \tag{A.11}$$

to be

$$A \to U(A + \frac{i}{\mathrm{g}} d)U^{-1}. \tag{A.12}$$

Recall that the wedge product is defined as the antisymmetric product

$$dx^\mu \wedge dx^\nu \equiv dx^\mu dx^\nu - dx^\nu dx^\mu. \tag{A.13}$$

We also need the elementary relations:

$$d^2 = 0, \qquad 0 = (dU)U^{-1} + U dU^{-1}. \tag{A.14}$$

Hence, under a local gauge transformation the field strength changes as

$$\begin{aligned} F &\rightarrow d(UAU^{-1} + \frac{i}{g}UdU^{-1}) \\ &- ig(UAU^{-1} + \frac{i}{g}UdU^{-1})(UAU^{-1} + \frac{i}{g}UdU^{-1}) \\ &= U(dA - igA \wedge A)U^{-1} = UFU^{-1}, \end{aligned} \tag{A.15}$$

where we have used Eq. (A.14) and the fact that a minus sign appears when the differential operator d passes a one-form. From this it follows that

$$F_{\mu\nu} \rightarrow UF_{\mu\nu}U^{-1} \tag{A.16}$$

which verifies the assertion. In fact, due to the cyclic property of the trace, every Lagrangian term of the form

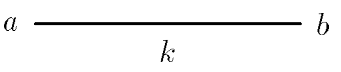
$$\mathrm{tr}F^q, \qquad q = 2, 3, 4, \ldots\,, \tag{A.17}$$

including inequivalent permutations, is gauge invariant. Obviously, terms with any number of covariant derivatives are also gauge invariant, by construction.

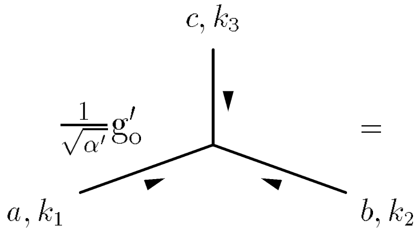
A.4 Feynman Rules

Tachyons are represented by plain lines and gauge bosons by wavy ones. The particles are assigned group indices $a, b, \ldots$, momenta $k_1, k_2, \ldots$, and Lorentz indices $\mu, \nu, \ldots$ in the case of gauge bosons. The momenta are all taken to be incoming, and momentum conservation $\sum_i k_i = 0$ is understood at each vertex.

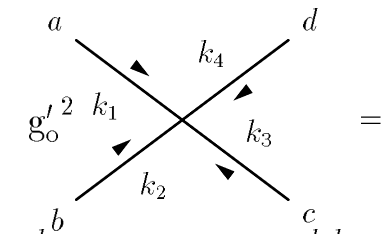
From the correlation functions calculated in Chapter 2, the Feynman rules with the coefficients shown below are deduced. Two different low energy effective actions for the gauge bosons are treated in the thesis: one including ordinary traces, and one including symmetric traces. The Feynman rules are given for the former one. It is trivial to obtain the Feynman rules for the latter one. Note also that no particular Lie algebra is considered. For the tachyon field theory we deduce the Feynman rules:



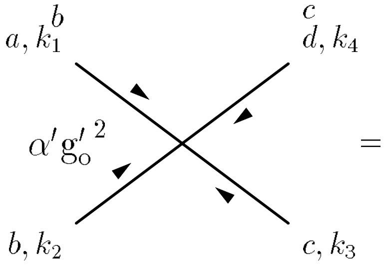
$$= \quad -i\delta^{ab}(k^2+m^2)^{-1} \tag{A.18}$$



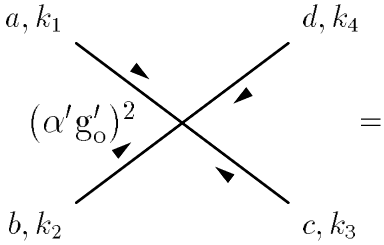
$$= \quad \sqrt{\tfrac{2}{\alpha'}}\, i\mathrm{g}'_{\mathrm{o}}\mathrm{tr}\Big(T^a\{T^b,T^c\}\Big) \tag{A.19}$$



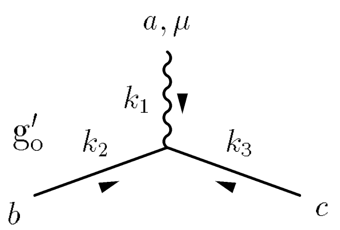
$$= \quad 4i\mathrm{g}'^{\,2}_{\mathrm{o}}[-75+4\pi^2+96\zeta(3)]\mathrm{str}T^{abcd} \tag{A.20}$$



$$= \quad \begin{aligned} &-8i\mathrm{g}'^{\,2}_{\mathrm{o}}\alpha'[4-\pi^2/3-6\zeta(3)] \\ &\times[k_1\cdot k_2\mathrm{tr}(T^{abcd}+T^{adcb}+T^{abdc}+T^{acdb}) \\ &+k_1\cdot k_3\mathrm{tr}(T^{adbc}+T^{acbd}+T^{abdc}+T^{acdb}) \\ &+k_1\cdot k_4\mathrm{tr}(T^{abcd}+T^{adcb}+T^{adbc}+T^{acbd})] \end{aligned} \tag{A.21}$$



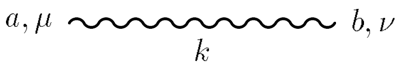
$$= \quad \begin{aligned} &4i\mathrm{g}'^{\,2}_{\mathrm{o}}\alpha'^2[2-\pi^2/3-4\zeta(3)][k_1\cdot k_3k_1\cdot k_4 \\ &\times\mathrm{tr}(T^{abcd}+T^{adcb}+T^{abdc}+T^{acdb}) \\ &+k_1\cdot k_2k_1\cdot k_4 \\ &\times\mathrm{tr}(T^{adbc}+T^{acbd}+T^{abdc}+T^{acdb}) \\ &+k_1\cdot k_2k_1\cdot k_3 \\ &\times\mathrm{tr}(T^{abcd}+T^{adcb}+T^{acbd}+T^{adbc})] \end{aligned} \tag{A.22}$$



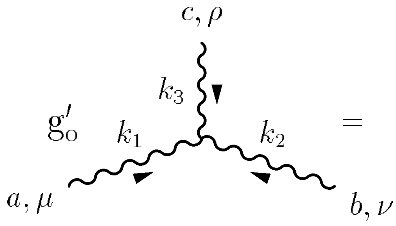
$$= \quad \mathrm{g}'_{\mathrm{o}}f^{abc}(k_2^\mu-k_3^\mu) \tag{A.23}$$

To get the full gauge boson scattering amplitude we have to attach polarization tensors to each external leg. The Yang-Mills action at order

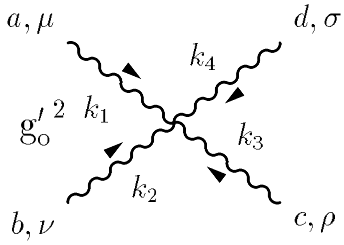
α'^0 gives the following Feynman rules:



$$= \quad -i\eta_{\mu\nu}\delta^{ab}/k^2 \tag{A.24}$$

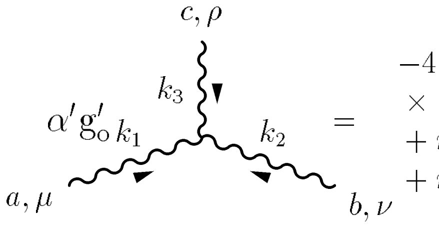


$$\begin{aligned} = \quad & g'_o f^{abc}[\eta_{\mu\nu}(k_1 - k_2)_\rho \\ & + \eta_{\rho\mu}(k_3 - k_1)_\nu \\ & + \eta_{\nu\rho}(k_2 - k_3)_\mu] \end{aligned} \tag{A.25}$$

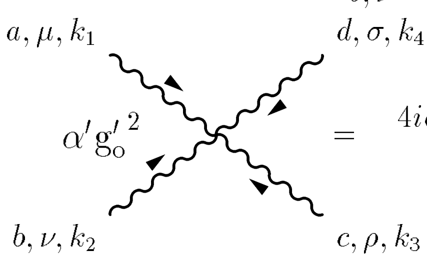


$$\begin{aligned} = \quad & -i{g'_o}^2[f^{abe}f^{cde}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ & + f^{ace}f^{bde}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ & + f^{ade}f^{bce}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma})] \end{aligned} \tag{A.26}$$

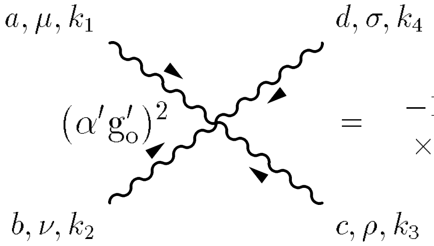
Proceeding with the higher order corrections at order α', with *one* four-point vertex, and α'^2 with *four* four-point vertices, we display for convenience only the $\eta_{\mu\nu}\eta_{\rho\sigma}$-terms. In fact, these are the only terms needed for the calculations in this thesis.



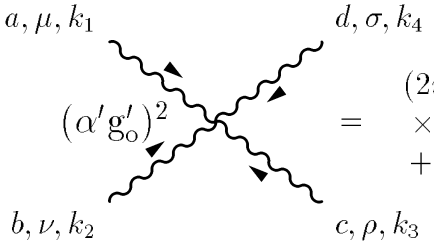
$$\begin{aligned} = \quad & -4\alpha' g'_o f^{abc} k_{2\mu} k_{3\nu} k_{1\rho} + 2\alpha' g'_o f^{abc} \\ & \times [\eta_{\mu\nu}(k_2 \cdot k_3 k_{1\rho} - k_1 \cdot k_3 k_{2\rho}) \\ & + \eta_{\nu\rho}(k_3 \cdot k_1 k_{2\mu} - k_2 \cdot k_1 k_{3\mu}) \\ & + \eta_{\rho\mu}(k_1 \cdot k_2 k_{3\nu} - k_3 \cdot k_2 k_{1\nu})] \end{aligned} \tag{A.27}$$



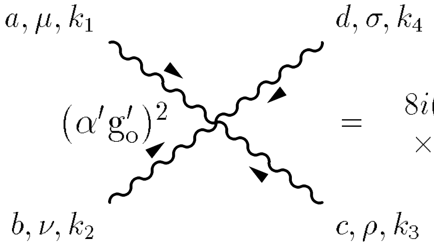
$$\begin{aligned} = \quad & 4i\alpha' {g'_o}^2 \eta_{\mu\nu}\eta_{\rho\sigma}[f^{ace}f^{dbe} k_1 \cdot k_3 \\ & + f^{ade}f^{cbe} k_1 \cdot k_4] + \ldots \end{aligned} \tag{A.28}$$



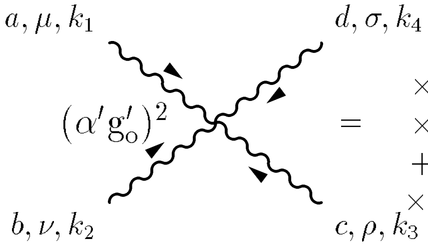
$$= \begin{array}{l} -16i(\pi^2/6+1)\alpha'^2 {\mathrm{g}'_{\mathrm{o}}}^2 \eta_{\mu\nu}\eta_{\rho\sigma} \\ \times\, k_1\cdot k_2 k_3\cdot k_4 \mathrm{tr} T^{(ab)(cd)} + \ldots \end{array} \tag{A.29}$$



$$= \begin{array}{l} (2i\pi^2\alpha'^2 {\mathrm{g}'_{\mathrm{o}}}^2/3)\eta_{\mu\nu}\eta_{\rho\sigma} \\ \times\, [k_1\cdot k_3 k_2\cdot k_4 \mathrm{tr}(T^{abdc}+T^{acdb}) \\ +\, k_1\cdot k_4 k_2\cdot k_3 \mathrm{tr}(T^{abcd}+T^{adcb})] + \ldots \end{array} \tag{A.30}$$



$$= \begin{array}{l} 8i(-\pi^2/12+1)\alpha'^2 {\mathrm{g}'_{\mathrm{o}}}^2 \eta_{\mu\nu}\eta_{\rho\sigma} \\ \times\, k_1\cdot k_2 k_3\cdot k_4 \mathrm{tr}(T^{acbd}+T^{adbc}) + \ldots \end{array} \tag{A.31}$$



$$= \begin{array}{l} (2i\pi^2\alpha'^2 {\mathrm{g}'_{\mathrm{o}}}^2/3)\eta_{\mu\nu}\eta_{\rho\sigma} \\ \times\, [k_1\cdot k_4 k_2\cdot k_3 \\ \times\, \mathrm{tr}(T^{acbd}+T^{adbc}+T^{abdc}+T^{acdb}) \\ +\, k_1\cdot k_3 k_2\cdot k_4 \\ \times \mathrm{tr}(T^{abcd}+T^{adcb}+T^{acbd}+T^{adbc})] + \ldots \end{array} \tag{A.32}$$

A.5 Fourier Transforms

The Fourier transforms are in this thesis defined as

$$\phi(x) = \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot x}\tilde{\phi}(p) \tag{A.33}$$

$$\tilde{\phi}(p) = \int d^D x e^{ip\cdot x}\phi(x), \tag{A.34}$$

and the integral representations of Dirac's delta function as:

$$\delta^D(x) = \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot x} \tag{A.35}$$

$$\delta^D(p) = \int d^D x e^{ip\cdot x}, \tag{A.36}$$

where $\delta^D(x) \equiv \delta(x_1)\ldots\delta(x_D)$ and $\delta^D(p) \equiv \delta(p_1)\ldots\delta(p_D)$. When Fourier-transforming, the derivatives acting in coordinate space are substituted for momentum by the rule $\partial_\mu \to -ip_\mu$.

B

Expansion of the Abelian Born-Infeld Action

We are here interested in evaluating $\det(\lambda\mathbf{1}-\mathbf{M})$, where $\mathbf{M}_{ij}$ is a general $n\times n$ matrix. Writing the determinant as a series expansion

$$\begin{aligned}\det(\lambda\mathbf{1}-\mathbf{M}) &= a_0\lambda^n+(-1)^1a_1\lambda^{n-1}+\ldots+(-1)^{n-1}a_{n-1}\lambda+(-1)^na_n\\ &= \sum_{l=0}^{n}(-1)^{n-l}a_{n-l}\lambda^l,\end{aligned}\tag{B.1}$$

where λ is a constant, the coefficients a_m can be expressed in terms of traces of $\mathbf{M}$ as

$$a_0=1,\qquad a_m=\frac{1}{m}\sum_{k=1}^{m}(-1)^{k+1}(\mathrm{tr}\mathbf{M}^{\mathrm{k}})a_{m-k},\tag{B.2}$$

using Cayley-Hamilton theory. For the *Abelian* Born-Infeld action $\mathcal{L}^{\mathrm{E}}_{\mathrm{BI}}=\sqrt{\det(\delta_{\mu\nu}+2\pi\alpha' F_{\mu\nu})}$ with Euclidean signature we get by letting $\lambda\to 1$ and $\mathbf{M}\to -2\pi\alpha' F$

$$\begin{aligned}\det(\delta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}) &= 1-\frac{1}{2}(2\pi\alpha')^2\mathrm{tr}F^2+\frac{1}{8}(2\pi\alpha')^4\Big((\mathrm{tr}F^2)^2-2\mathrm{tr}F^4\Big)\\ &+ \frac{1}{6}(2\pi\alpha')^6\Big(\frac{3}{4}\mathrm{tr}F^2\mathrm{tr}F^4-\frac{1}{8}(\mathrm{tr}F^2)^3-\mathrm{tr}F^6\Big)\\ &+ \mathcal{O}(F^8),\end{aligned}\tag{B.3}$$

where odd powers vanish due to the tracelessness of the field strength. Note that in the Abelian case the trace is over the Lorentz indices, contracting them as $\mathrm{tr}F^2=F^{\mu\nu}F_{\nu\mu}$ and $\mathrm{tr}F^4=F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}$, etc. After

Taylor expanding the square root $(T^{-1} \equiv 2\pi\alpha')$ we get

$$\begin{aligned} \mathcal{L}_{\rm BI}^{\rm E} &= 1 - \frac{1}{4}T^{-2}{\rm tr}F^2 - \frac{1}{8}T^{-4}{\rm tr}F^4 + \frac{1}{32}T^{-4}({\rm tr}F^2)^2 - \frac{1}{12}T^{-6}{\rm tr}F^6 \\ &+ \frac{1}{32}T^{-6}{\rm tr}F^2{\rm tr}F^4 - \frac{1}{384}T^{-6}({\rm tr}F^2)^3 + \mathcal{O}(F^8). \end{aligned} \tag{B.4}$$

This is true in any number of dimensions. However, the sum in Eq. (B.1) only contains $(n+1)$ terms, which means that in 4 dimensions for instance, Eq. (B.3) terminates at order α'^4 (recall that odd powers of F vanish). Of course when expanding the square root we still get an infinite sum.

Consider now the same expansion when we instead have Minkowski signature $\eta_{\mu\nu} = (-,+,\ldots,+)$. The determinant $\det(\varrho\eta_\mu{}^\nu - \mathbf{M}_\mu{}^\nu)$, with ϱ a constant, is modified by a minus sign since

$$\det(\varrho\eta_\mu{}^\nu - \mathbf{M}_\mu{}^\nu) = \det(\eta^{\nu\rho})\det(\varrho\delta_{\mu\rho} - \mathbf{M}_{\mu\rho}), \tag{B.5}$$

and $\det(\eta^{\nu\rho}) = -1$. So the expansion in terms of traces

$$\det(\varrho\eta_\mu{}^\nu - \mathbf{M}_\mu{}^\nu) = \sum_{l=0}^{n}(-1)^{n-l}b_{n-l}\varrho^l,$$

is accordingly modified with a minus sign:

$$b_0 = -1, \qquad\qquad b_m = \frac{1}{m}\sum_{k=1}^{m}(-1)^{k+1}({\rm tr}\mathbf{M}^{\rm k})b_{m-k}. \tag{B.6}$$

Finally, since the Abelian Born-Infeld action with Minkowski signature has to be modified with a minus sign in the square root to avoid being imaginary,

$$\mathcal{L}_{\rm BI}^{\rm M} = \sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}, \tag{B.7}$$

the Taylor expansion is identical to the one in Eq. (B.4).

C

The Unitary Group $\mathrm{U}(n)$

The gauge group $\mathrm{U}(n)$ is defined as

$$\mathrm{U}(n) = \{V \in \mathrm{GL}(n, \mathbb{C}) | VV^\dagger = V^\dagger V = 1\}, \tag{C.1}$$

where $V^\dagger = V^{-1}$. The associated Lie algebra $\mathfrak{u}(n)$ is the set of Hermitian $n \times n$ matrices. We use the Hermitian generators T^a with the properties

$$[T^a, T^b] = if^{abc}T^c, \qquad \{T^a, T^b\} = d^{abc}T^c, \tag{C.2}$$

where f^{abc} is totally antisymmetric and d^{abc} is totally symmetric. This implies that any product of two $\mathrm{U}(n)$ generators can be expanded into symmetric and antisymmetric pieces as

$$T^a T^b = \frac{1}{2}(d^{abc} + if^{abc})T^c. \tag{C.3}$$

The generators are normalized by the inner product

$$\mathrm{tr}(T^a T^b) = \delta^{ab}. \tag{C.4}$$

$\mathfrak{u}(n)$ is not a simple Lie algebra since it can be decomposed into subalgebras as

$$\mathfrak{u}(n) = \mathfrak{su}(n) \otimes \mathfrak{u}(1). \tag{C.5}$$

From the relations in Eqs. (C.2) and (C.4) we can derive the useful relations:

$$f^{bce}f^{aed} + f^{cae}f^{bed} + f^{abe}f^{ced} = 0 \tag{C.6}$$
$$f^{abe}f^{cde} = d^{ace}d^{bde} - d^{ade}d^{bce} \tag{C.7}$$
$$\mathrm{tr}\Big(T^a[T^b, T^c]\Big) = f^{abc} \tag{C.8}$$
$$\mathrm{tr}\Big(T^a\{T^b, T^c\}\Big) = d^{abc} \tag{C.9}$$
$$\mathrm{tr}(T^{abcd} + T^{adcb}) = \frac{1}{2}(d^{abe}d^{cde} + d^{ade}d^{bce} - d^{ace}d^{bde}) \tag{C.10}$$
$$\mathrm{tr}(T^{acbd} + T^{adbc}) = \frac{1}{2}(d^{ace}d^{bde} + d^{ade}d^{bce} - d^{abe}d^{cde}) \tag{C.11}$$
$$\mathrm{tr}(T^{abdc} + T^{acdb}) = \frac{1}{2}(d^{ace}d^{bde} + d^{abe}d^{cde} - d^{ade}d^{bce}) \tag{C.12}$$
$$\mathrm{tr}T^{(ab)(cd)} = \frac{1}{4}d^{abe}d^{cde} \tag{C.13}$$
$$\mathrm{str}T^{abc} = \frac{1}{2}d^{abc} \tag{C.14}$$
$$\mathrm{str}T^{abcd} = \frac{1}{12}(d^{abe}d^{cde} + d^{ace}d^{bde} + d^{ade}d^{bce}), \tag{C.15}$$

where $\mathrm{str}T^{a_1\ldots a_n} \equiv \mathrm{tr}T^{a_{(1}a_2\ldots a_{n-1}a_{n)}}$. The short-hand notation $T^{a_1\ldots a_n} \equiv T^{a_1}\ldots T^{a_n}$ is used throughout the thesis. Eq. (C.6) is nothing but the Jacobi identity of the Lie algebra.

Bibliography

[1] M. Born and L. Infeld, *Foundations of the new field theory*, Proc. Royal Soc. A144 (1934) 425.

[2] J.D. Jackson, *Classical Electrodynamics, 3rd edition.*

[3] E.S. Fradkin and A.A. Tseytlin, *Non-linear electrodynamics from quantized strings*, Phys. Letters B163 (1985) 123.

[4] M.B. Green, J.H. Schwarz, E. Witten, *Superstring theory Vol. I,II* (Cambridge U.P., Cambridge 1987).

[5] T. Muta, *Foundations of quantum chromodynamics* (World scientific lecture notes in physics, Vol. 57).

[6] M.E. Peskin, D.V. Schroeder, *An introduction to quantum field theory.*

[7] J. Polchinski, *String theory, Vol. I,II.*

[8] B. Hatfield, *Quantum field theory of point particles and strings* (Frontiers in physics, Vol. 75).

[9] A.A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, In the Yuri Golfand memorial volume, ed. M. Shifman, World Scientific (2000) `hep-th/9908105` (1999).

[10] E.A. Bergshoeff, M. de Roo, A. Sevrin, *On the supersymmetric non-Abelian Born-Infeld action*, `hep-th/0011264` (2000).

[11] A.A. Tseytlin, *On the non-Abelian generalization of Born-Infeld action in string theory*, Nucl. Phys. B501 (1997) 41 `hep-th/9701125`.

[12] A.A. Tseytlin, *Vector field effective action in the open superstring theory*, Nucl. Phys. B276 (1986) 391.

[13] A. Sevrin, J. Troost, W. Troost, *The non-Abelian Born-Infeld action at order F^6*, `hep-th/0101192` (2001).

[14] A. Bilal, *Higher-derivative corrections to the non-Abelian Born-Infeld action*, `hep-th/0106062` (2001).

[15] H. Kawai, D.C. Lewellen, S.-H.H. Tye, *A relation between tree amplitudes of closed and open strings*, Nucl. Phys. B269 (1986) 1.

[16] J. Polchinski, *Notes on D-branes*, `hep-th/9602052` (1996).

[17] L.J. Dixon, *Introduction to conformal field theory and string theory*, `slac-pub-5149` (1989).

[18] Y. Kitazawa, *Effective Lagrangian for the open superstring from a 5-point function*, Nucl. Phys. B289 (1987) 599.

[19] N. Wyllard, *Derivative corrections to the D-brane actions with constant background field*, `hep-th/0008125` (2000).

[20] N. Wyllard, *Derivative corrections to the D-brane Born-Infeld action: non-geodesic embeddings and the Seiberg-Witten map*, `hep-th/0107185` (2001).

[21] E.A. Bergshoeff, M. de Roo, *Non-Abelian Born-Infeld and kappa-symmetry*, Math. Phys. 42 (2001) `hep-th/0011018`.

[22] W. Taylor, *Lectures on D-branes, gauge theory and M(atrices)*, `hep-th/9801182` (1998).

[23] F. Denef, A. Sevrin, W. Troost, *Non-Abelian Born-Infeld versus string theory*, `hep-th/0002180` (2000).