Kaluza-Klein mass spectra, p-branes and AdS/CFT

A study of higher dimensional supergravities

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Abstract

In this thesis I study 11D supergravity and type IIB supergravity, which are low energy limits of M-theory and type IIB superstring theory, respectively. The p-brane metric and field strength are derived for the M2, M5 and D3-brane solutions. There are also a discussion about various properties of the p-branes. For example the near horizon geometry is seen to be $AdS_{p+2} \times S^{D-(p+2)}$ for a p-brane, where AdS_{p+2} means a $p+2$-dimensional anti de-Sitter space. I also obtain mass operators and mass spectra for various compactifications of the two supergravity theories above, for example $AdS_4 \times S^7$ and $AdS_5 \times S^5$. The AdS/CFT correspondence which is a duality between conformal field theories in Minkowski$_d$ space and superstring theories compactified on $AdS_{d+1} \times M^{D-d-1}$ is "derived" and various consequences of the correspondence are analyzed. Especially how deformations in a supergravity theory drives a renormalization group flow on the field theory side. Here I find some unexpected results which I try to explain. I also show how interconnected all the results in this thesis are.

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Contents

1

Introduction

In this introductory chapter I will give a brief introduction to superstring theory and its history, and how it is related to various supergravity theories. I will also have a short discussion about the organization of this thesis.

String theory is a theory where all the fundamental particles (fermions) and all the force mediating particles (bosons) of nature, are described as different vibrations of a fundamental string. This is analogous to a piano string, where different notes correspond to different modes of the piano string. Pursuing this analogy further, we could say that the basic particles in the universe corresponds to the musical notes of the superstring, and the laws of physics corresponds to the harmonies that these notes obey, and the universe itself corresponds to a symphony of superstrings (and perhaps higher dimensional objects).

The bosonic string theory was first developed in the late 1960s as a model for strong interactions, but was abandoned in the middle of the 1970s for the quark model. This first string theory had many good features, but also a few unwanted ones like: Tachyons with negative mass square and a mysterious massless spin 2 particle. Another striking fact about the theory was that it was only consistent in 26 dimensions, unlike an ordinary quantum field theory which can be formulated in any dimension.

In the mid 70s a new string theory was invented. This theory was not a bosonic string theory but a superstring theory. Here super in superstring means a supersymmetric string theory, i.e. a theory with both bosons and fermions in its spectrum. This theory was shown to be consistent in only 10 dimensions. I will explain below how a 10 dimensional theory can describe physics in 4 dimensions. Unlike the first string theories this theory did not try to explain only the strong force, but all forces and particles in the universe. The spin 2 particle in the bosonic string theory is also present in the superstring theory and is interpreted as the graviton, which is the particle that mediates the gravitational force. This massless spin 2 particle is included in every consistent superstring theory, which means that superstring the-

Name	IIA	IIB	I	HetA	HetB
string type	closed oriented	closed oriented	open and closed non-oriented	closed oriented	closed oriented
10D SUSY	$N = 2$ non-chiral	$N = 2$ chiral	$N = 1$	$N = 1$	$N = 1$
10D gauge group	none	none	$SO(32)$	$SO(32)$	$E_8 \times E_8$

Table 1.1: The 5 consistent string theories
Here SUSY means the number of supersymmetries. HetA is the heterotic string theory with gauge group $SO(32)$, while HetB is the one with $E_8 \times E_8$.

ory automatically includes a quantum theory of gravity. It was also shown that the theory was ultraviolet finite, which means that there are no unwanted infinities in the theory. This is very good since earlier these kinds of infinities had ruined a lot of possible quantum gravity theories.

In 1984 when the so called *first superstring revolution* started, M.B. Green and J. Schwarz showed that a special type of string theories, called the type I string theory, is free of anomalies if the 10 dimensional gauge group is uniquely $SO(32)$. Later, D. Gross, J. Harvey, E. Martinec and R. Rohm discovered a new consistent string theory called heterotic (hybrid) string theory. There are two versions of this string theory, one with gauge group $SO(32)$ and the other one with $E_8 \times E_8$. Finally, P. Candelas, G. Horowitz, A. Strominger and E. Witten showed that a Kaluza-Klein compactification of the $E_8 \times E_8$ theory from 10 dimensions down to 4 dimensions leads to a theory which seems to include the standard model and gravity. This was a great success which showed that string theory is much more than an exotic idea of some theoretical physicists. Here compactification from 10 to 4 dimensions means that 6 of the spatial dimensions are "curled" up and becomes very small, with a size of about 10^{-33}cm. The reason that these "extra" dimensions are not visible is because they are so small. An analogy is if one look at the opening of a water hose, which from a great distance looks like a point but is in fact a circle if one look more closely. So with the help of compactification a 10D string theory can describe physics in 4 dimensions.

In the middle of the 80s after the two heterotic string theories were found, it was clear that there existed five different superstring theories. These five theories and some of their properties are listed in table 1.1. For a thorough technical introduction to these five theories see [1, 2].

Now one might wonder, why is there five consistent string theories and not only one? This is a good question since we want to have one fundamental theory and not five. The answer to this question is that the five theories that are perturbatively different might be five different perturbative theories of a more fundamental non-perturbative 11 dimensional theory called M-theory, where M stands for membrane, mystery, magical or mother theory. This so far unknown theory was conjectured by

Witten in 1995. What we know about this theory is that 11 dimensional supergravity is its low energy limit. The five 10-dimensional superstring theories also have supergravity theories as low energy limits. In this thesis I am going to study type IIB and 11D supergravity, which are the low energy limits of type IIB superstring theory and M-theory.

The reason for this new idea of M-theory, was that it was found that the various string theories were connected via different types of dualities. This was the start of the *second superstring revolution* (1995$\rightarrow$). The different types of dualities are:

- T-duality
  This perturbative duality is between a theory A which is compactified on a space with large volume, and a theory B which is compactified on a space with small volume. If a theory D is self-dual then the duality implies that D compactified on a small volume is equivalent to D compactified on a large volume, i.e. the two different compactifications describe the same physics. On the other hand, if A and B are two unrelated theories when they are not compactified, then the duality implies that A and B compactified on a small and large space, respectively, can be viewed as different limits in the moduli space of the compactified theories. For example, the T-duality between the type IIA and IIB string theories compactified on a circle with radius R and α'/R,[1] respectively, means that IIA and IIB are the end points ($R = \infty$, $R = 0$) in the moduli space of compactified theories. The moduli space can be seen as a line from $R = 0$ (IIB) to $R = \infty$ (IIA) where every value of R gives a unique theory. It is very important to realize that type IIA and IIB compactified on a circle *do not* describe the same physics, which is the case for a self-dual theory. This is seen from the fact that the momentum and winding states constitute *inequivalent* $N = 2$, D=9 supermultiplets at any given compactification radius, (see [3]).

- S-duality
  This non-perturbative duality describes the quantum equivalence of two superstring theories A and B which are perturbatively distinct. Its central idea is that the strong coupling limit of A is equivalent to the weak coupling limit of B, and vice versa. This means that perturbative excitations of A are mapped to non-perturbative excitations of the dual theory B, and vice versa, making S-duality a non-perturbative duality. An example of this duality is type IIB string theory which is self-dual. Also D=11 M-theory is S-dual to the D=10 type IIA superstring as discovered by Witten.

- U-duality
  U-duality is a combination of S and T-duality.

The web of dualities that connects the five string theories and M-theory are shown in figure 1.1. We see in figure 1.1 how M-theory is connected with the $E_8 \times$

[1]Here $\alpha' \sim \frac{1}{T}$, where T is the string tension.

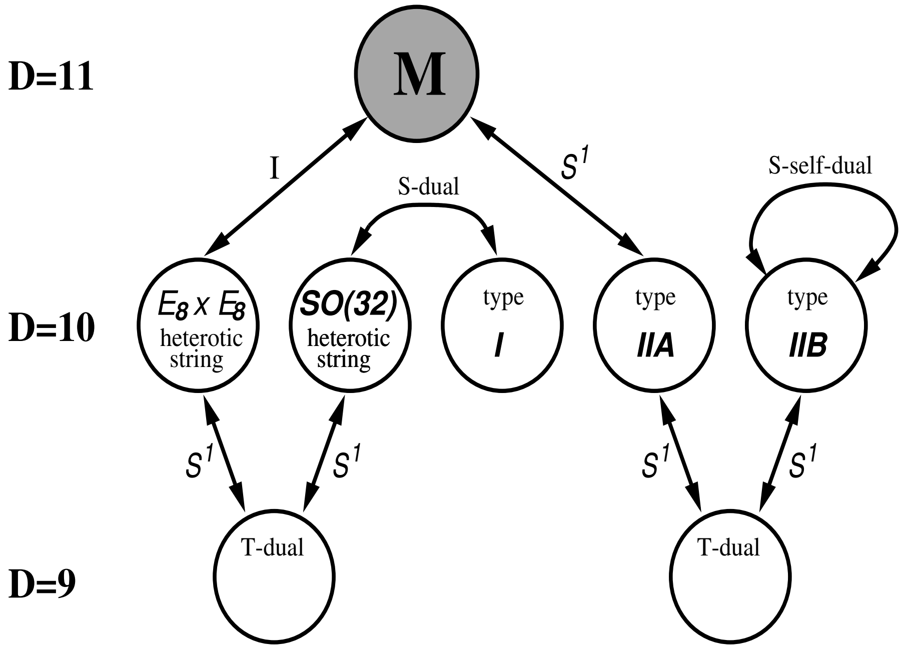

Figure 1.1: The net of string dualities in higher dimensions. As it is shown, the perturbatively different string theories in ten dimensions are connected by duality transformations using certain compactifications. Moreover, in eleven dimensions a new theory arises, which is called M–theory and whose compactification on the circle S^1 or the finite line interval I gives rise to type IIA and $E_8 \times E_8$ heterotic string theory, respectively. This figure is taken from [4].

E_8 heterotic theory and the type IIA theory by compactification on a finite line interval and on a circle, respectively. One evidence of these connections is that 11D supergravity compactified on a line interval and a circle gives the supergravity theories that corresponds to the $E_8 \times E_8$ heterotic theory and the type IIA theory, respectively.

In order to test all these dualities one had to use strings but also new objects. These new objects are the p-branes, which were discovered in the end of the 80s and the beginning of the 90s. Theses p-branes are extended objects with spatial dimension p, for example, p=0 gives a particle, p=1 a string, p=2 a membrane and so on. In 11 dimensions there are 2 different p-branes: the M2-brane and the M5-brane (M for M-theory), while in 10 dimensions there are branes for p=0-9. The M2-brane is a fundamental brane[2], while the M5-brane which is the dual brane of the M2-brane is a solitonic brane. In 10 dimensions there are also fundamental and solitonic branes, plus so called D-branes, (D for Dirichlet). The special thing with the D-branes is that open strings which obey Dirichlet boundary conditions can end on D-branes. D-branes are the most important branes when checking the various dualities.

The most recent important discovery in string theory is the AdS/CFT[3] correspondence (sometimes called the Maldacena conjecture)[4]. This correspondence implies that a conformal field theory in d-dimensional Minkowski spacetime is dual to a superstring theory or M-theory compactified on $AdS_{d+1} \times M^{D-d-1}$, see chapter 5 for a short introduction.

For a technical introduction to perturbative string theory see [1], string theory and M-theory, see [2]. For a non-technical introduction see [4, 5, 6].

In this thesis I am going to study 11D supergravity and type IIB supergravity, which are low energy limits of M-theory and type IIB superstring theory, respectively.

In chapter 2 I derive the explicit form of the metric and the field strength for the M2, M5 and D3-brane solutions. There is also a discussion about multiple p-brane solutions and various properties of the p-branes. One important result that is obtained is the near horizon geometry $AdS_{p+2} \times S^{D-p-2}$ of p-branes, which is later used in chapter 5 in the "derivation" of the AdS/CFT correspondence.

Chapter 3 is a very long chapter were I first show that 11D supergravity and type IIB supergravity can be spontaneously compactified down to $AdS_d \times M^{D-d}$ (D=11,10), by using the Freund-Rubin ansatz. I also obtain the bosonic mass operators for the following compactifications: $AdS_4 \times M^7$, $Minkowski_4 \times M^7$, $AdS_7 \times M^4$ and $AdS_5 \times M^5$.

In chapter 4, I am using the mass operators obtained in chapter 3 to obtain the following mass spectra: $AdS_4 \times S^7$, $AdS_7 \times S^4$, $AdS_5 \times S^5$, $Minkowski_4 \times T^7$ and

[2] Note that the M2-brane is not fundamental in the same sense as the fundamental string in 10-dimensions, since the M2-brane has a horizon and can not be quantized perturbatively.

[3] AdS is short for anti de-Sitter spacetime and CFT for conformal field theory.

[4] The importance of non-commutative geometry in string theory is an even more recent important discovery, but this topic will not be discussed in this thesis.

$Minkowski_4 \times (K3 \times T^3)$. I also show that the various mass spectra are equivalent to various supersymmetric supermultiplets.

In chapter 5 I analyze various consequences of the AdS/CFT correspondence. In this chapter most of the results in the earlier chapters are very important. I hope that the reader after reading this final chapter realize how interconnected all the results in this thesis are.

Chapter 6 is conclusions.

At the end of the thesis there are 4 appendices.

Here in the end of the introduction I wish the reader good luck in reading this thesis, and I hope that he/she will find it interesting and have enough patience to read all the way to the end.

2

p-branes

In this chapter, I will derive the metric and the field strength for three different p-branes. The M2-brane and the M5-brane will be derived from 11D supergravity [7], and the D3-brane from type IIB 10D supergravity [8, 9, 10]. To calculate the metric and the field strength one first has to make ansätze for the metric and the field strength, and insert them into the bosonic equations of motion. These are then solved for the unknown functions introduced by the ansätze. The three different p-brane solutions that I am going to obtain are preserving half of the supersymmetry of the corresponding supergravity theory (this will not be shown here, see for example [11]).

A p-brane is a p-dimensional extended object that is a solution to the equations of motion of a supergravity theory. A p-brane can carry "electric" charge (M2-brane), "magnetic" charge (M5-brane), or both (D3-brane). An "electric" p-brane couples to a p+1-form potential, while a "magnetic" p-brane couples to a D−(p+3)-form potential. This is a direct generalization of the case of point particles in D=4, like the electron and a magnetic monopole, which both couples to a 1-form potential A_μ and a 2-form field strength $F_{\mu\nu}$, (p+2=D−(p+2)=2, p=0, D=4), where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The reason that the D3-brane can have both type of charges is because its self-dual 5-form field strength is the field strength corresponding to both an "electric" and a "magnetic" 3-brane (p+2=D−(p+2)=5, when D=10, p=3). It's easy to see that the M2-brane and the M5-brane couples to the same field strength, (because p+2=4 when p=2 and D−(p+2)=4 when D=11, p=5).

In section 2.1 I am going to give the ansatz for the metric, and calculate the Ricci tensor R_{MN} and the Ricci scalar R for a general p-brane.

In section 2.2-2.4 I will present an ansatz for the field strength F_{MNPQ} of the M2- and M5-brane, as well as for the field strength G_{MNPQR} of the D3-brane. I will also obtain the metric and the field strength for the various p-branes. Then the solutions will be generalized to N p-branes located at different places in D-dimensional spacetime. Finally, I am going to discuss the behavior of the solutions at the horizon (r=0).

2.1 The metric ansatz and calculation of R_{MN} and R

The ansatz that I will use for the metric will be required to have $(Poincare)_d \times SO(D-d)$ symmetry. Here $d = p+1$ and D=11 for 11D sugra (supergravity) and D=10 for type IIB 10D sugra. This means that we have isotropic symmetry in the directions transverse to the p-brane. Now we split the spacetime coordinates into two ranges: $x^M = (x^\mu, y^m)$ where x^μ ($\mu = 0, 1, \cdots, p = d-1$) are the p-brane coordinates adapted to the $(Poincare)_d$ isometries on the world volume, and y^m ($m = p+1, \cdots, D-1$) are the coordinates transverse to the p-brane.

An ansatz for the metric that respects the required symmetries is

$$ds^2 = e^{A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{B(r)}\delta_{mn}dy^m dy^n \tag{2.1}$$

where $r = \sqrt{\delta_{mn}y^n y^m}$, $\eta_{\mu\nu} = diag(-1, 1, 1, \cdots)$ and $\delta_{mn} = diag(1, 1, \cdots)$. The metric (2.1) will be required to approach 11D (or 10D for the D3-brane) Minkowski spacetime when $r \to \infty$. This means that $e^{A(r)}$, $e^{B(r)} \to 1$ as $r \to \infty$.

(2.1) gives

$$\begin{aligned} g_{\mu\nu} &= e^{A(r)}\eta_{\mu\nu}, \quad g_{mn} = e^{B(r)}\delta_{mn}, \quad g_{\mu n} = g^{\mu n} = 0 \\ g^{\mu\nu} &= e^{-A(r)}\eta^{\mu\nu}, \quad g^{mn} = e^{-B(r)}\delta^{mn} \end{aligned} \tag{2.2}$$

Before presenting the ansatz for the field strength I am going to calculate the Ricci tensor R_{MN} and the Ricci scalar R for an arbitrary p-brane. R_{MN} is the LHS (left hand side) of (2.24) and (2.70) below, while R will be useful when checking if the horizon is singular or not. First, recall the definition

$$R_{MN} = R^Q{}_{MQN} = \partial_Q\Gamma^Q_{MN} - \partial_N\Gamma^Q_{MQ} + \Gamma^R_{MN}\Gamma^Q_{RQ} - \Gamma^R_{MQ}\Gamma^Q_{RN} \tag{2.3}$$

As one can see from this equation, before we can calculate R_{MN} we need to obtain Γ^P_{MN}, which is defined as follows:

$$\Gamma^P_{MN} = \frac{1}{2}g^{PQ}[\partial_M g_{QN} + \partial_N g_{QM} - \partial_Q g_{MN}] \tag{2.4}$$

I am going to use (2.2), (2.4) and that $\partial_\mu g_{MN} = 0$ in the calculation of Γ^P_{MN}. One finds

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho Q}[\partial_\mu g_{Q\nu} + \partial_\nu g_{Q\mu} - \partial_Q g_{\mu\nu}] = 0 \tag{2.5}$$

$$\begin{aligned} \Gamma^m_{\mu\nu} &= \frac{1}{2}g^{mQ}[\partial_\mu g_{Q\nu} + \partial_\nu g_{Q\mu} - \partial_Q g_{\mu\nu}] = -\frac{1}{2}g^{mq}\partial_q g_{\mu\nu} \\ &= -\frac{1}{2}\partial_q A e^{A-B}\delta^{mq}\eta_{\mu\nu} \end{aligned} \tag{2.6}$$

$$\Gamma^\nu_{m\mu} = \Gamma^\nu_{\mu m} = \frac{1}{2}g^{\nu Q}[\partial_\mu g_{Qm} + \partial_m g_{Q\mu} - \partial_Q g_{\mu m}] = \frac{1}{2}g^{\nu\rho}\partial_m A e^A \eta_{\mu\rho}$$

$$
\begin{aligned}
&= \frac{1}{2}\partial_m A \delta^\nu_\mu && (2.7)\\
\Gamma^n_{m\mu} &= \Gamma^n_{\mu m} = \frac{1}{2} g^{nQ}[\partial_\mu g_{Qm} + \partial_m g_{Q\mu} - \partial_Q g_{\mu m}] = 0 && (2.8)\\
\Gamma^\mu_{mn} &= \frac{1}{2} g^{\mu Q}[\partial_m g_{Qn} + \partial_n g_{Qm} - \partial_Q g_{mn}] = 0 && (2.9)\\
\Gamma^p_{mn} &= \frac{1}{2} g^{pQ}[\partial_m g_{Qn} + \partial_n g_{Qm} - \partial_Q g_{mn}]\\
&= \frac{1}{2} g^{pq}[\partial_m B \delta_{qn} + \partial_n B \delta_{qm} - \partial_q B \delta_{mn}] e^B\\
&= \frac{1}{2}[\partial_m B \delta^p_n + \partial_n B \delta^p_m - \partial_q B \delta_{mn} \delta^{pq}] && (2.10)
\end{aligned}
$$

I will now calculate R_{MN} using (2.2)-(2.10), $\eta^{\mu\nu}\eta_{\mu\nu} = d = p+1$, $\quad \delta^{mn}\delta_{mn} = d_1 = D - d$, $\Gamma^p_{mp} = \frac{d_1}{2}\partial_m B$ and $\Gamma^\mu_{m\mu} = \frac{d}{2}\partial_m A$. The result is

$$
\begin{aligned}
R_{\mu\nu} &= \partial_P \Gamma^P_{\mu\nu} - \partial_\nu \Gamma^P_{\mu P} + \Gamma^S_{\mu\nu}\Gamma^P_{SP} - \Gamma^S_{\mu P}\Gamma^P_{S\nu}\\
&= \partial_p \Gamma^p_{\mu\nu} + \Gamma^s_{\mu\nu}[\Gamma^p_{sp} + \Gamma^\rho_{s\rho}] - [\Gamma^\sigma_{\mu p}\Gamma^p_{\sigma\nu} + \Gamma^s_{\mu\rho}\Gamma^\rho_{s\nu}]\\
&= -\frac{1}{2}\delta^{pq}[\partial_p\partial_q A + \partial_p A \partial_q A - \partial_p A \partial_q B] e^{A-B}\eta_{\mu\nu}\\
&- \frac{1}{4} e^{A-B}\partial_q A \delta^{sq}\eta_{\mu\nu}[d_1 \partial_s B + d\partial_s A] + \frac{1}{2} e^{A-B}\partial_p A \partial_q A \delta^{pq}\eta_{\mu\nu}\\
&= -\frac{1}{4} e^{A-B}\eta_{\mu\nu}\delta^{pq}[2\partial_p\partial_q A + d\partial_p A \partial_q A + (d_1 - 2)\partial_p A \partial_q B] && (2.11)
\end{aligned}
$$

$$
\begin{aligned}
R_{\mu m} &= \partial_P \Gamma^P_{\mu m} - \partial_m \Gamma^P_{\mu P} + \Gamma^S_{\mu m}\Gamma^P_{SP} - \Gamma^S_{\mu P}\Gamma^P_{Sm}\\
&= \Gamma^\sigma_{\mu m}\Gamma^P_{\sigma P} - [\Gamma^\sigma_{\mu p}\Gamma^p_{\sigma m} + \Gamma^s_{\mu\sigma}\Gamma^\sigma_{sm}] = 0 && (2.12)
\end{aligned}
$$

$$
\begin{aligned}
R_{mn} &= \partial_P \Gamma^P_{mn} - \partial_n \Gamma^P_{mP} + \Gamma^S_{mn}\Gamma^P_{SP} - \Gamma^S_{mP}\Gamma^P_{Sn}\\
&= \partial_p \Gamma^p_{mn} - \partial_n[\Gamma^p_{mp} + \Gamma^\rho_{m\rho}] + \Gamma^s_{mn}[\Gamma^p_{sp} + \Gamma^\rho_{s\rho}] - [\Gamma^s_{mp}\Gamma^p_{sn} + \Gamma^\sigma_{m\rho}\Gamma^\rho_{\sigma n}]\\
&= \frac{1}{2}[2\partial_n\partial_m B - \delta_{mn}\delta^{pq}\partial_p\partial_q B] - \frac{1}{2}\partial_n\partial_m[d_1 B + dA]\\
&+ \frac{1}{4}\partial_s[d_1 B + dA][\partial_m B \delta^s_n + \partial_n B \delta^s_m - \delta_{mn}\delta^{st}\partial_t B]\\
&- \frac{1}{4}\Big[(\partial_m B \delta^s_p + \partial_p B \delta^s_m - \delta_{mp}\delta^{st}\partial_t B)(\partial_s B \delta^p_n + \partial_n B \delta^p_s - \delta_{sn}\delta^{pt}\partial_t B)\\
&+ d\partial_m A \partial_n A\Big]\\
&= \frac{1}{4}\Big\{2(2 - d_1)\partial_m\partial_n B - 2d\partial_m\partial_n A - 2\delta_{mn}\delta^{pq}\partial_p\partial_q B\\
&+ (d_1 - 2)[\partial_m B \partial_n B - \delta_{mn}\delta^{pq}\partial_p B \partial_q B]\\
&+ d[\partial_n B \partial_m A + \partial_m B \partial_n A - \delta_{mn}\delta^{pq}\partial_p B \partial_q A] - d\partial_m A \partial_n A\Big\} && (2.13)
\end{aligned}
$$

To simplify these expressions I am going to use the chain rule to write all derivatives with respect to $r = \sqrt{\delta_{mn} y^n y^m}$. That is, using

$$\begin{aligned}
\partial_m A &= \frac{\partial r}{\partial y^m}\frac{\partial A}{\partial r} = \frac{y_m}{r} A' \\
\partial_m \partial_n A &= \frac{y_m y_n}{r^2} A'' + \frac{A'}{r}\left(\delta_{mn} - \frac{y_m y_n}{r^2}\right) \\
\delta^{pq}\partial_p\partial_q A &= A'' + \frac{A'}{r}(d_1 - 1), \quad \delta^{pq}\partial_p A \partial_q A = A'^2 \qquad (2.14)
\end{aligned}$$

where $A' = \frac{\partial A}{\partial r}$ and $A'' = \frac{\partial^2 A}{\partial r^2}$, in (2.11) and (2.13) gives

$$R_{\mu\nu} = -\frac{1}{4} e^{A-B} \eta_{\mu\nu}\left[2A'' + \frac{2A'}{r}(d_1 - 1) + dA'^2 + (d_1 - 2)A'B'\right] \qquad (2.15)$$

$$\begin{aligned}
R_{mn} &= -\frac{1}{4}\delta_{mn}\left[2B'' + (4d_1 - 6)\frac{B'}{r} + (d_1 - 2)B'^2 + 2d\frac{A'}{r} + dA'B'\right] \\
&- \frac{1}{4}\frac{y_m y_n}{r^2}\Big[2(d_1 - 2)(B'' - \frac{B'}{r}) - (d_1 - 2)B'^2 \\
&+ 2d(A'' - \frac{A'}{r} - A'B') + dA'^2\Big] \qquad (2.16)
\end{aligned}$$

I am now assuming that the functions A(r) and B(r) can be written as one function $\Delta(r)$, but with different exponents, i.e, I set

$$e^A = \Delta^\alpha, \quad e^B = \Delta^\beta \qquad (2.17)$$

which means

$$e^A = \Delta^\alpha \Rightarrow A' = \alpha\frac{\Delta'}{\Delta}, \quad A'' = \alpha\left(\frac{\Delta''}{\Delta} - \frac{\Delta'^2}{\Delta^2}\right) \qquad (2.18)$$

where α and β are constants. That this assumption has to be made follows if the solution is required to preserve half of the supersymmetry [11].

Using (2.17)-(2.18), (2.15) and (2.16) read

$$\begin{aligned}
R_{\mu\nu} &= -\frac{1}{4}\eta_{\mu\nu}\Delta^{\alpha-\beta}\left[\frac{2\alpha}{\Delta}\left(\Delta'' + \frac{\Delta'}{r}(d_1 - 1)\right)\right. \\
&+ \left.\alpha\frac{\Delta'^2}{\Delta^2}\Big(d\alpha + (d_1 - 2)\beta - 2\Big)\right] \qquad (2.19) \\
R_{mn} &= -\frac{1}{4}\delta_{mn}\left[\frac{1}{\Delta}\left(2\beta\Delta'' + \frac{\Delta'}{r}\Big((4d_1 - 6)\beta + 2d\alpha\Big)\right)\right. \\
&+ \left.\beta\frac{\Delta'^2}{\Delta^2}\Big((d_1 - 2)\beta + d\alpha - 2\Big)\right] \\
&- \frac{1}{4}\frac{y_m y_n}{r^2}\left[\frac{1}{\Delta}\Big(2(d_1 - 2)\beta + 2d\alpha\Big)\left(\Delta'' - \frac{\Delta'}{r}\right)\right. \\
&+ \left.\frac{\Delta'^2}{\Delta^2}\Big(d\alpha(\alpha - 2\beta - 2) - (d_1 - 2)\beta(\beta + 2)\Big)\right] \qquad (2.20)
\end{aligned}$$

From these equations we can now easily obtain the Ricci scalar. It is

$$\begin{aligned}
R &= g^{MN}R_{MN} = -\frac{1}{4}e^{-B}\Big\{4d\Big[A'' + \frac{A'}{r}(d_1-1)\Big] \\
&+ 4(d_1-1)\Big[B'' + \frac{B'}{r}(d_1-1)\Big] + d(d+1)A'^2 \\
&+ (d_1-2)(d_1-1)B'^2 + 2d(d_1-2)A'B'\Big\} \\
&= -\frac{1}{4}\Delta^{-\beta}\Big\{\frac{4}{\Delta}\Big(d\alpha + (d_1-1)\beta\Big)\Big(\Delta'' + \frac{\Delta'}{r}(d_1-1)\Big) \\
&+ \frac{\Delta'^2}{\Delta^2}\Big(d\alpha((d+1)\alpha + 2(d_1-2)\beta - 4) + (d_1-1)\beta((d_1-2)\beta-4)\Big)\Big\} \qquad (2.21)
\end{aligned}$$

2.2 The M2-brane

In this section I'm going to obtain the metric and the field strength for the M2-brane, and discuss some of its properties.

The M2-brane and the M5-brane can be obtained from the bosonic D=11 sugra equations of motion [7, 12]:

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{3}\Big[F_{MPQR}F_N{}^{PQR} - \frac{1}{8}g_{MN}F_{PQRS}F^{PQRS}\Big] \qquad (2.22)$$

$$\nabla_M F^{MNPQ} = -\frac{1}{576}\epsilon^{M_1\ldots M_8NPQ}F_{M_1M_2M_3M_4}F_{M_5M_6M_7M_8} \qquad (2.23)$$

(2.22) can be rewritten as

$$\begin{aligned}
R_{MN} &= \frac{1}{3}\Big[F_{MPQR}F_N{}^{PQR} - \frac{1}{12}g_{MN}F_{PQRS}F^{PQRS}\Big] \\
&= \frac{1}{3}\Big[g_{NS}F_{MPQR}F^{SPQR} - \frac{1}{12}g_{MN}F_{PQRS}F^{PQRS}\Big] \qquad (2.24)
\end{aligned}$$

where I have multiplied (2.22) with g^{MN} to get $R = \frac{1}{36}F_{PQRS}F^{PQRS}$, and then put the result back into (2.22).

I will now make the following ansatz for the four-form field strength [13][14]:

$$F_{m\mu_1\mu_2\mu_3} = t\varepsilon_{\mu_1\mu_2\mu_3}\partial_m e^{C(r)}, \quad \text{others zero}, \quad t = \text{constant} \qquad (2.25)$$

with $\varepsilon_{\mu_1\mu_2\mu_3}$ the Levi-Civita tensor density defined as $\varepsilon_{012} = -\varepsilon^{012} = 1$ (see also appendix B). Note that in (2.23) $\epsilon^{M_1\ldots M_8NPQ}$ is a tensor, not a tensor density.

This is the so called "electric" ansatz. The $SO(D-d)$ isotropicity and the $(Poincare)_d$ symmetry are guaranteed by this ansatz because the function $C(r)$ only depends on the transverse radial coordinate r.

Multiplying $F_{M'N'P'Q'}$ with $g^{MM'}g^{NN'}g^{PP'}g^{QQ'}$ gives

$$F^{m\mu_1\mu_2\mu_3} = t\varepsilon^{\mu_1\mu_2\mu_3}\partial^m e^C e^{-3A-B} = t\varepsilon^{\mu_1\mu_2\mu_3}\partial^m C e^{C-3A-B} \qquad (2.26)$$

I will now obtain $A(r)$, $B(r)$, $C(r)$ and t by using (2.24).

Calculating the RHS of (2.24) gives

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{3}\Big[g_{\nu S}F_{\mu PQR}F^{SPQR} - \frac{1}{12}g_{\mu\nu}F_{PQRS}F^{PQRS}\Big] \\ &= -\frac{4}{3}t^2C'^2e^{2C-2A-B}\eta_{\mu\nu} \end{aligned} \tag{2.27}$$

where I have used (2.2) and

$$\begin{aligned} F_{PQRS}F^{PQRS} &= 4F_{m\mu\nu\rho}F^{m\mu\nu\rho} = -24t^2\delta^{mn}\partial_mC\partial_nCe^{2C-3A-B} \\ &= -24t^2C'^2e^{2C-3A-B} \end{aligned} \tag{2.28}$$

$$\begin{aligned} F_{PQR\rho}F^{PQR\sigma} &= 3F_{m\mu\nu\rho}F^{m\mu\nu\sigma} = -6t^2\delta^{mn}\partial_mC\partial_nCe^{2C-3A-B}\delta_\rho^\sigma \\ &= -6t^2C'^2e^{2C-3A-B}\delta_\rho^\sigma \end{aligned} \tag{2.29}$$

To obtain these expressions (2.14), (B.5) and (B.6) were used. Finally,

$$\begin{aligned} R_{mn} &= \frac{1}{3}\Big[g_{nS}F_{mPQR}F^{SPQR} - \frac{1}{12}g_{mn}F_{PQRS}F^{PQRS}\Big] \\ &= \frac{2}{3}t^2C'^2e^{2C-3A}\left[\delta_{mn} - 3\frac{y_my_n}{r^2}\right] \end{aligned} \tag{2.30}$$

where (2.28) and

$$g_{nS}F_{mPQR}F^{SPQR} = -6\frac{y_my_n}{r^2}t^2C'^2e^{2C-3A} \tag{2.31}$$

have been used. Calculating both the RHS and the LHS of (2.24) gives three equations to solve. One from the $\mu\nu$ equation and two from the mn equation (the δ_{mn} part and the $\frac{y_my_n}{r^2}$ part). In these three equations, I will assume that $C(r)$ can be rewritten as $e^{C(r)} = \Delta^\gamma(r)$ where γ is a constant. This means that all the unknown functions $A(r)$, $B(r)$ and $C(r)$ are written in terms of the same function $\Delta(r)$, but with different exponents. As already mentioned this is required in order to preserve half of the supersymmetry in the solution.

Inserting $d = 3$ and $d_1 = 8$ into (2.24) gives

$$\mu\nu \quad : \quad 2\alpha\left(\Delta'' + 7\frac{\Delta'}{r}\right) + \alpha\frac{\Delta'^2}{\Delta}(3\alpha + 6\beta - 2) = \frac{16}{3}t^2\gamma^2\frac{\Delta'^2}{\Delta}\Delta^{2\gamma-3\alpha} \tag{2.32}$$

$$\begin{aligned} \delta_{mn} \quad : \quad & 2\beta\Delta'' + \frac{\Delta'}{r}\Big(26\beta + 6\alpha\Big) + \beta\frac{\Delta'^2}{\Delta}\Big(6\beta + 3\alpha - 2\Big) \\ & = -\frac{8}{3}t^2\gamma^2\frac{\Delta'^2}{\Delta}\Delta^{2\gamma-3\alpha} \end{aligned} \tag{2.33}$$

$$\begin{aligned} \frac{y_my_n}{r^2} \quad : \quad & \Big(4\beta + 2\alpha\Big)\left(\Delta'' - \frac{\Delta'}{r}\right) + \frac{\Delta'^2}{\Delta}\Big(\alpha(\alpha - 2\beta - 2) - 2\beta(\beta + 2)\Big) \\ & = \frac{8}{3}t^2\gamma^2\frac{\Delta'^2}{\Delta}\Delta^{2\gamma-3\alpha} \end{aligned} \tag{2.34}$$

By looking at (2.32)-(2.34) one can see that it is probably a smart choice to assume that terms of the type Δ'' and $\frac{\Delta'}{r}$ cancel each other, and that terms like $\frac{\Delta'^2}{\Delta}$ and the RHS of (2.32)-(2.34) cancel.

Assuming that terms like Δ'' and $\frac{\Delta'}{r}$ cancel in (2.32) gives

$$\Delta'' + 7\frac{\Delta'}{r} = 0 \tag{2.35}$$

which have the following solution

$$\Delta(r) = 1 + \frac{k}{r^6}, \quad k > 0, \quad r > 0 \tag{2.36}$$

By rewriting (2.35) with (2.14) one can easily see that $\Delta(r)$ is a harmonic function ($\delta^{mn}\partial_m\partial_n\Delta{=}0$). The constant k sets the mass scale of the solutions. The constant is taken to be positive to make sure that there is no naked singularity in the metric at finite r. This positivity restriction is similar to the usual restriction to a positive mass parameter M in the standard Schwarzschild black hole solution [14].

(2.33) gives

$$\Delta'' + \left(13 + 3\frac{\alpha}{\beta}\right)\frac{\Delta'}{r} = 0, \Rightarrow \alpha = -2\beta \tag{2.37}$$

were I have used (2.35).

(2.34) gives no new information, but is satisfied.

Cancellation of $\frac{\Delta'^2}{\Delta}$ and the RHS of (2.32), gives

$$\alpha(3\alpha + 6\beta - 2) = \frac{16}{3}t^2\gamma^2\Delta^{2\gamma-3\alpha} \tag{2.38}$$

The LHS of (2.38) is independent of r, which means that the RHS also have to be independent of r. This gives

$$\gamma = \frac{3}{2}\alpha, \quad \alpha = -\frac{1}{6t^2} \tag{2.39}$$

(2.33) gives no new information, but is satisfied.

(2.34) gives

$$t = \pm\frac{1}{2} \Rightarrow \alpha = -\frac{2}{3}, \quad \beta = \frac{1}{3} \quad \gamma = -1 \tag{2.40}$$

Inserting (2.36) and (2.40) into (2.1) and (2.25) gives the metric and the field strength for a M2-brane. Their explicit forms are

$$ds^2 = \left(1 + \frac{k}{r^6}\right)^{-\frac{2}{3}}\eta_{\mu\nu}dx^\mu dx^\nu + \left(1 + \frac{k}{r^6}\right)^{\frac{1}{3}}\delta_{mn}dy^m dy^n \tag{2.41}$$

$$F_{m\mu\nu\rho} = \pm\frac{1}{2}\varepsilon_{\mu\nu\rho}\partial_m\left(1 + \frac{k}{r^6}\right)^{-1}, \quad \text{others zero.} \tag{2.42}$$

where the plus sign in (2.42) gives a brane, and the minus sign gives an anti-brane.

I am now going to check that (2.41)-(2.42) satisfies (2.23). It is easy to see that the RHS of (2.23) is zero. It is also easy to see that all terms except one most be zero in the LHS; the calculation of the final term gives

$$\begin{aligned}\nabla_m F^{m\mu\nu\rho} &= \frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}F^{m\mu\nu\rho}\right)\\ &= \frac{1}{\sqrt{-g}}\partial_m\left(e^{\frac{3}{2}A+4B}F^{m\mu\nu\rho}\right)\\ &= e^{-\frac{3}{2}A-4B}t\varepsilon^{\mu\nu\rho}\delta^{mn}\partial_m\partial_n\Delta = 0 \end{aligned} \tag{2.43}$$

Here I have used that $g = -e^{3A+8B}$ is the determinant of the metric, (2.14) and that Δ is a harmonic function. This means that (2.41)-(2.42) satisfies (2.23).

The single brane case above is easy to generalize to a solution for N M2-branes located at different places in spacetime. The only thing that is different compared to one M2-brane is the $\Delta(r)$-function, which is changed to

$$\Delta(r) = 1 + \sum_{n=1}^{N}\left(\frac{k_n}{|\vec{r}-\vec{r}_n|^6}\right) \tag{2.44}$$

where $\vec{r}_n$ corresponds to the position of the n:th brane. I will now show that this new $\Delta(r)$ satisfy the equations of motions. (I will only do that for two M2-branes).

The only thing that has to be checked is that $\Delta(r)$ satisfy $\delta^{mn}\partial_m\partial_n\Delta(r) = 0$, i.e, $\Delta(r)$ is still a harmonic function:

$$\begin{aligned}\delta^{mn}\partial_m\partial_n\Delta &= \delta^{mn}\partial_m\partial_n\left(1+\frac{k}{r^6}\right) + \delta^{mn}\partial_m\partial_n\left(\frac{k_1}{|\vec{r}-\vec{r}_1|^6}\right)\\ &= \delta^{mn}\partial_m\partial_n\left(\frac{k_1}{r'^6}\right) = \delta^{mn}\partial'_m\partial'_n\left(\frac{k_1}{r'^6}\right) = 0 \end{aligned} \tag{2.45}$$

where I have used (2.14), (2.35) and $r' = |\vec{r}-\vec{r}_1|$.

One can see by looking at (2.44) that the solution for N M2-branes is a superposition of solutions of N individual M2-branes. This means that the branes do not attract or repel each other (they behave like solitons). This is possible because the gravitational attraction between branes, is exactly cancelled by an equal and opposite force generated by the field strength F_{MNPQ} [13].

Now it is time to see what happens when $r = 0$. One can see that $r = 0$ in (2.41) gives a singularity in the metric. Then the question is; is this a real singularity in spacetime, or is it only a coordinate singularity. To check this, I have to obtain the Ricci scalar R at $r = 0$. If R is finite, then it's only a coordinate singularity. But if R is infinite, then it's a spacetime singularity.

Inserting $d = 3$ and $d_1 = 8$ into (2.21) using (2.35) gives

$$R = -\frac{1}{6}\Delta'^2\Delta^{-\frac{7}{3}} \tag{2.46}$$

$$R(r\to 0) \propto -k^{-\frac{1}{3}} \tag{2.47}$$

The implication of this equation is that the singularity is a coordinate singularity, and not a spacetime singularity. This situation is similar to the horizon of a black hole, were there also is a coordinate singularity.

If we now go inside the horizon ($R < 0$ in (2.48) below) it is possible to show [14] that there is a spacetime singularity at $R = -\infty$. This singularity is timelike and not spacelike as for an ordinary black hole. This implies that one could consider coupling a δ-function *source* to the solution at $R = -\infty$, which can be seen to originate from the D=11 supermembrane action, which is a generalization of the Nambu-Goto action for the string. This is the unique matter system that can consistently couple to 11D supergravity.

I will now perform a coordinate change, and show that when r is close to zero, it's possible to write the metric as a product of two spaces $AdS_4 \times S^7$, which is a four dimensional anti de-Sitter spacetime, times a seven sphere.

Setting $r = k^{\frac{1}{6}} R^{\frac{1}{2}}/(1 - R^3)^{\frac{1}{6}}$ in (2.41), for $0 < R < 1$ (corresponding to $0 < r < \infty$) gives [14]

$$ds^2 = R^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{k^{\frac{1}{3}}}{(1 - R^3)^{\frac{7}{3}}} \left[\frac{dR^2}{4R^2} + d\Omega_7^2 \right] \tag{2.48}$$

where $d\Omega_7$ is the line element of S^7. If we now let $R \to 0$ in the metric (same as $r \to 0$), and also re-scale R as $R \to \frac{k^{\frac{1}{6}}}{2} R$, then the metric will have the following form

$$ds^2 = \frac{k^{\frac{1}{3}}}{4} \left[R^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{R^2} dR^2 \right] + k^{\frac{1}{3}} d\Omega_7^2 \tag{2.49}$$

where the first part is AdS_4 in Poincare coordinates (see appendix D), and the second part is S^7. This means that the metric is asymptotic to $AdS_4 \times S^7$ when r is close to zero. We also know that the metric is asymptotic to 11D Minkowski spacetime when r is large. The interpretation of (2.49) is therefore that the M2-brane solution to the bosonic equations of motion is a soliton in the sense that it is an interpolating solution between different vacua, namely $AdS_4 \times S^7$ and 11D Minkowski spacetime.

2.3 The M5-brane

Next, I will derive the metric and the field strength for the M5-brane, and discuss some of its properties.

The calculations in this section are very similar to the calculations in section 2.2. The big difference will be the ansatz for the field strength F_{MNPQ}.

I will make the following ansatz for the four-form field strength [14]:

$$F_{m_1 m_2 m_3 m_4} = \lambda \varepsilon_{m_1 m_2 m_3 m_4 p} \frac{y^p}{r^5}, \quad \text{others zero}, \quad \lambda = \text{constant} \tag{2.50}$$

where $\varepsilon_{m_1 m_2 m_3 m_4 p}$ is the Levi-Civita tensor density defined as $\varepsilon_{012345} = \varepsilon^{012345} = 1$ (see also appendix B). This is the so called "magnetic" ansatz. The five in the exponent of r is required by the Bianchi identity $\partial_{[M} F_{NPQR]} = 0$.

Multiplying $F_{M'N'P'Q'}$ with $g^{MM'}g^{NN'}g^{PP'}g^{QQ'}$ gives

$$F^{m_1m_2m_3m_4p} = \lambda\varepsilon^{m_1m_2m_3m_4p}\frac{y_p}{r^5}e^{-4B} \tag{2.51}$$

I will now obtain A(r), B(r) and λ by using (2.24).

Calculating the RHS of (2.24) gives

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{3}\Big[g_{\nu S}F_{\mu PQR}F^{SPQR} - \frac{1}{12}g_{\mu\nu}F_{PQRS}F^{PQRS}\Big] \\ &= -\frac{2}{3}\frac{e^{A-4B}}{r^8}\lambda^2\eta_{\mu\nu} \end{aligned} \tag{2.52}$$

where I have used (2.2) and

$$\begin{aligned} F_{M_1M_2M_3M_4}F^{M_1M_2M_3M_4} &= \lambda^2\frac{y^py_s}{r^{10}}\varepsilon_{m_1m_2m_3m_4p}\varepsilon^{m_1m_2m_3m_4s}e^{-4B} \\ &= 24\lambda^2\frac{e^{-4B}}{r^8} \end{aligned} \tag{2.53}$$

where (B.7) were used in the second line. Finally,

$$\begin{aligned} R_{mn} &= \frac{1}{3}\Big[g_{nS}F_{mPQR}F^{SPQR} - \frac{1}{12}g_{mn}F_{PQRS}F^{PQRS}\Big] \\ &= \frac{2}{3}\frac{e^{-3B}}{r^8}\lambda^2\bigg(2\delta_{mn} - 3\frac{y_my_n}{r^2}\bigg) \end{aligned} \tag{2.54}$$

Here I have used (2.2), (2.53) and

$$\begin{aligned} F_{M_1M_2M_3M_4}F^{M_1M_2M_3M_4'} &= \lambda^2\frac{y^py_s}{r^{10}}\varepsilon_{m_1m_2m_3m_4p}\varepsilon^{m_1m_2m_3m_4's}e^{-4B} \\ &= 6\lambda^2\frac{e^{-4B}}{r^8}\bigg(\delta^{m_4'}_{m_4} - \frac{y^{m_4'}y_{m_4}}{r^2}\bigg) \end{aligned} \tag{2.55}$$

where (B.7) were used in the second line.

Calculating both the RHS and the LHS of (2.24) gives three equations to solve. One from the $\mu\nu$ equation and two from the mn equation (the δ_{mn} part and the $\frac{y_my_n}{r^2}$ part).

Inserting $d = 6$ and $d_1 = 5$ into (2.24) gives

$$\mu\nu \quad : \quad 2\alpha\bigg(\Delta'' + 4\frac{\Delta'}{r}\bigg) + \alpha\frac{\Delta'^2}{\Delta}(6\alpha + 3\beta - 2) = \frac{8}{3}\frac{\Delta^{-3\beta+1}}{r^8}\lambda^2 \tag{2.56}$$

$$\begin{aligned} \delta_{mn} \quad : \quad & 2\beta\Delta'' + \frac{\Delta'}{r}\Big(14\beta + 12\alpha\Big) + \beta\frac{\Delta'^2}{\Delta}\Big(3\beta + 6\alpha - 2\Big) \\ & = -\frac{16}{3}\frac{\Delta^{-3\beta+1}}{r^8}\lambda^2 \end{aligned} \tag{2.57}$$

$$\begin{aligned} \frac{y_my_n}{r^2} \quad : \quad & \Big(2\beta + 4\alpha\Big)\bigg(\Delta'' - \frac{\Delta'}{r}\bigg) + \frac{\Delta'^2}{\Delta}\Big(2\alpha(\alpha - 2\beta - 2) - \beta(\beta + 2)\Big) \\ & = \frac{8}{3}\frac{\Delta^{-3\beta+1}}{r^8}\lambda^2 \end{aligned} \tag{2.58}$$

I am now doing the same assumption that I did in section 2.2, which is that terms like Δ'' and $\frac{\Delta'}{r}$ cancel each other, and that terms like $\frac{\Delta'^2}{\Delta}$ and the RHS of (2.56)-(2.58) cancel.

Assuming that terms like Δ'' and $\frac{\Delta'}{r}$ cancel in (2.56)-(2.58) gives

$$\mu\nu \quad : \quad \Delta'' + 4\frac{\Delta'}{r} = 0, \Rightarrow \Delta(r) = 1 + \frac{k}{r^3}, \quad k > 0, \quad r > 0 \tag{2.59}$$

$$\delta_{mn} \quad : \quad \beta = -2\alpha \tag{2.60}$$

Here (2.59) means that Δ is a harmonic function. Cancellation of $\frac{\Delta'^2}{\Delta}$ and the RHS of (2.56)-(2.58) gives

$$\mu\nu : \quad \beta = \frac{2}{3}, \quad \alpha = -\frac{1}{3}, \quad \lambda = \pm\frac{3}{2}k \tag{2.61}$$

(2.57)-(2.58) are satisfied, but gives no new information. Inserting (2.59) and (2.61) into (2.1) and (2.50) gives the metric and the field strength for a M5-brane. Their explicit forms are

$$ds^2 = \left(1 + \frac{k}{r^3}\right)^{-\frac{1}{3}} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{k}{r^3}\right)^{\frac{2}{3}} \delta_{mn} dy^m dy^n \tag{2.62}$$

$$F_{m_1 m_2 m_3 m_4} = \pm\frac{3}{2}k\varepsilon_{m_1 m_2 m_3 m_4 p}\frac{y^p}{r^5}, \quad \text{others zero.} \tag{2.63}$$

where the plus sign in (2.63) gives a brane, and the minus sign gives a anti-brane.

I will now check that (2.62)-(2.63) satisfies (2.23). It is easy to see that the RHS of (2.23) is zero. It is also easy to see that all terms except one most be zero in the LHS; the calculation of the final term gives

$$\begin{aligned} \nabla_m F^{mnpq} &= \frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}F^{mnpq}\right) \\ &= e^{-3A-\frac{5}{2}B}\lambda\varepsilon^{mnpqr}\partial_m\left(\frac{y_r}{r^5\Delta^2}\right) = 0 \end{aligned} \tag{2.64}$$

where I have used that $g = -e^{6A+5B}$ is the determinant of the metric, (2.14), $\varepsilon^{mnpqr}\partial_m y_r = 0$ and $\varepsilon^{mnpqr} y_m y_r = 0$. This means that (2.62)-(2.63) satisfies (2.23).

In exactly the same way as for the M2-brane solution, it is also possible to extend the M5-brane solution to N M5-branes located at different places in spacetime. The only thing that is different compared to one M5-brane is the $\Delta(r)$-function, which is changed to

$$\Delta(r) = 1 + \sum_{n=1}^{N}\left(\frac{k_n}{|\vec{r} - \vec{r}_n|^3}\right) \tag{2.65}$$

where $\vec{r}_n$ corresponds to the position of the n:th brane. This new $\Delta(r)$ function satisfy the equations of motion, which is shown in the same way as for the M2-branes.

I am now going to see if the singularity at $r = 0$ in the metric, is a coordinate singularity or a spacetime singularity.

Calculation of the Ricci scalar R, with $d = 6$ and $d_1 = 5$ gives

$$R = \frac{1}{6}\Delta'^2\Delta^{-\frac{8}{3}} \tag{2.66}$$

$$R(r \to 0) \propto k^{-\frac{2}{3}} \tag{2.67}$$

The implication of this equation is that the singularity is a coordinate singularity.

If we now go inside the horizon ($R < 0$ in (2.68) below) it is possible to show [14] that there is no spacetime singularity inside the horizon. This is seen from the fact that (2.68) is fully symmetric under a discrete isometry $R \to -R$, which implies that the inside of the horizon can not have a singularity as long as there is no one at the outside. Since we know that there is no singularity outside the horizon we also know that the M5-brane solution is completely non-singular.

I will now perform a coordinate change, and show that when r is close to zero, it's possible to write the metric as a product of two spaces $AdS_7 \times S^4$, which is a seven dimensional anti de-Sitter spacetime, times a four sphere.

Setting $r = k^{\frac{1}{3}}R^2/(1-R^6)^{\frac{1}{3}}$ in (2.62), for $0 < R < 1$ (corresponding to $0 < r < \infty$) gives [14]

$$\begin{aligned} ds^2 &= R^2\eta_{\mu\nu}dx^\mu dx^\nu + k^{\frac{2}{3}}\left[\frac{4}{R^2(1-R^6)^{\frac{8}{3}}}dR^2\right. \\ &+ \left.\frac{1}{(1-R^6)^{\frac{2}{3}}}d\Omega_4^2\right] \end{aligned} \tag{2.68}$$

where $d\Omega_4$ is the line element of S^4. If we now let $R \to 0$ in the metric (same as $r \to 0$), and also re-scale R as $R \to 2k^{\frac{1}{3}}R$, then the metric will have the following form

$$ds^2 = 4k^{\frac{2}{3}}\left[R^2\eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{R^2}dR^2\right] + k^{\frac{2}{3}}d\Omega_4^2 \tag{2.69}$$

where the first part is AdS_7 in Poincare coordinates (see appendix D), and the second part is S^4. This means that the metric is asymptotic to $AdS_7 \times S^4$ when r is close to zero. We also know that the metric is asymptotic to 11D Minkowski spacetime when r is large. The interpretation of (2.69) is therefore that the M5-brane solution to the bosonic equations of motion is a soliton in the sense that it is an interpolating solution between different vacua, namely $AdS_7 \times S^4$ and 11D Minkowski spacetime.

2.4 The D3-brane

Concluding this chapter the metric and the field strength for the D3-brane will be derived. There will also be a discussion about some of its properties.

The D3-brane can be obtained from two of the bosonic type IIB D=10 sugra equations of motion [8, 9, 10].

$$R_{MN} = \frac{1}{96}G_M{}^{PQRS}G_{NPQRS} = \frac{1}{96}g_{MT}G^{TPQRS}G_{NPQRS} \tag{2.70}$$

$$G_{MNPQR} = *G_{MNPQR} = \frac{1}{5!}\sqrt{-g}\varepsilon_{MNPQRS_1S_2S_3S_4S_5}G^{S_1S_2S_3S_4S_5} \tag{2.71}$$

where (2.71) means that G_{MNPQR} is self-dual. In (2.70) all the bosonic fields have been set to zero except G_{MNPQR}. The other fields are set to zero because I am only interested in the D3-brane solution (this is a consistent truncation).

I will now make the following ansatz for the five-form field strength:

$$\begin{aligned} G_{m\mu\nu\rho\sigma} &= z\varepsilon_{\mu\nu\rho\sigma}\partial_m e^{C(r)} \\ G_{mnpqr} &= z_1\varepsilon_{mnpqrs}\partial^s e^{D(r)}, \quad \text{others zero.} \end{aligned} \tag{2.72}$$

where z and z_1 are constants, and $\varepsilon_{\mu\nu\rho\sigma}$ is defined as in section 2.2, and ε_{mnpqrs} as in 2.3. Multiplying $G_{M'N'P'Q'R}$ with $g^{MM'}g^{NN'}g^{PP'}g^{QQ'}g^{RR'}$ gives

$$\begin{aligned} G^{m\mu\nu\rho\sigma} &= z\varepsilon^{\mu\nu\rho\sigma}\partial^m e^{C-4A-B} \\ G^{mnpqr} &= z_1\varepsilon^{mnpqrs}\partial_s e^{D-5B}, \quad \text{others zero.} \end{aligned} \tag{2.73}$$

When I have obtained $C(r)$ and $D(r)$, I am going to show that the first line in (2.72) correspond to an "electric" ansatz, and that the second line correspond to a "magnetic" ansatz. I will now obtain $A(r)$, $B(r)$, $C(r)$, $D(r)$, z and z_1 by using (2.70).

Calculating the RHS of (2.70) gives

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{96}g_{\mu\tau}G^{\tau PQRS}G_{\nu PQRS} = \frac{1}{24}z^2 g_{\mu\tau}\varepsilon^{\tau\rho\sigma\chi}\varepsilon_{\nu\rho\sigma\chi}\delta^{mn}\partial_m C\partial_n C e^{2C-4A-B} \\ &= -\frac{1}{4}z^2C'^2e^{2C-3A-B}\eta_{\mu\nu} \end{aligned} \tag{2.74}$$

and

$$\begin{aligned} R_{mn} &= \frac{1}{96}g_{mt}G^{tPQRS}G_{nPQRS} = \frac{1}{96}g_{mt}\Big[G^{tpqrs}G_{npqrs} + G^{t\rho\sigma\tau\chi}G_{n\rho\sigma\tau\chi}\Big] \\ &= \frac{1}{4}\Big[z_1^2D'^2e^{2D-4B}\Big(\delta_{mn} - \frac{y_my_n}{r^2}\Big) - z^2C'^2e^{2C-4A}\frac{y_my_n}{r^2}\Big] \end{aligned} \tag{2.75}$$

where I have used (2.2), (2.14), (B.6) and (B.7).

Calculating both the RHS and the LHS of (2.70) gives three equations to solve. One from the $\mu\nu$ equation and two from the mn equation (the δ_{mn} part and the $\frac{y_my_n}{r^2}$ part). In these three equations, I will assume that $C(r)$ and $D(r)$ can be rewritten as $e^{C(r)} = \Delta^\gamma(r)$ and $e^{D(r)} = \Delta^\delta(r)$, where γ and δ are constants. This means that all the unknown functions $A(r)$, $B(r)$, $C(r)$ and $D(r)$ are written in terms of the same function $\Delta(r)$, but with different exponents. As already mentioned this is required in order to preserve half of the supersummetry in the solution.

Inserting $d = 4$ and $d_1 = 6$ into (2.70) gives

$$\mu\nu \quad : \quad 2\alpha\Big(\Delta'' + 5\frac{\Delta'}{r}\Big) + 2\alpha\frac{\Delta'^2}{\Delta}(2\alpha + 2\beta - 1) = z^2\gamma^2\frac{\Delta'^2}{\Delta}\Delta^{2\gamma-4\alpha} \tag{2.76}$$

$$\delta_{mn} \quad : \quad 2\beta\Delta'' + \frac{\Delta'}{r}\Big(18\beta + 8\alpha\Big) + 2\beta\frac{\Delta'^2}{\Delta}\Big(2\beta + 2\alpha - 1\Big)$$

$$= -z_1^2\delta^2\frac{\Delta'^2}{\Delta}\Delta^{2\delta-4\beta} \tag{2.77}$$

$$\frac{y_m y_n}{r^2} \quad : \quad \Big(8\beta+8\alpha\Big)\left(\Delta''-\frac{\Delta'}{r}\right)+4\frac{\Delta'^2}{\Delta}\Big(\alpha(\alpha-2\beta-2)-\beta(\beta+2)\Big)$$

$$= \frac{\Delta'^2}{\Delta}\Big(z^2\gamma^2\Delta^{2\gamma-4\alpha}+z_1^2\delta^2\Delta^{2\delta-4\beta}\Big) \tag{2.78}$$

I am now doing the same assumption that I did in section 2.2 and 2.3, which is that terms like Δ'' and $\frac{\Delta'}{r}$ cancel each other, and that terms like $\frac{\Delta'^2}{\Delta}$ and the RHS of (2.76)-(2.78) cancel.

Assuming that terms like Δ'' and $\frac{\Delta'}{r}$ cancel in (2.76)-(2.78) gives

$$\mu\nu \quad : \quad \Delta''+5\frac{\Delta'}{r}=0, \Rightarrow \Delta(r)=1+\frac{k}{r^4}, \quad k>0, \quad r>0 \tag{2.79}$$

$$\delta_{mn} \quad : \quad \beta=-\alpha \tag{2.80}$$

Here (2.79) means that Δ is a harmonic function. Cancellation of $\frac{\Delta'^2}{\Delta}$ and the RHS of (2.76)-(2.78) gives

$$\mu\nu \quad : \quad \gamma=2\alpha, \quad \alpha=-\frac{1}{2z^2} \tag{2.81}$$

$$\delta_{mn} \quad : \quad \delta=2\beta, \quad \alpha=-\frac{1}{2z_1^2}\rightarrow z^2=z_1^2 \tag{2.82}$$

$$\frac{y_m y_n}{r^2} \quad : \quad z=\pm 1 \Rightarrow z_1=\pm 1 \tag{2.83}$$

Inserting (2.80) and (2.83) into (2.81)-(2.82) gives

$$\alpha=-\frac{1}{2}, \quad \beta=\frac{1}{2}, \quad \gamma=-1, \quad \delta=1 \tag{2.84}$$

Inserting (2.83) and (2.84) into (2.1) and (2.72) gives the metric and the field strength for a D3-brane. Their explicit forms are

$$ds^2 = \left(1+\frac{k}{r^4}\right)^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu+\left(1+\frac{k}{r^4}\right)^{\frac{1}{2}}\delta_{mn}dy^m dy^n \tag{2.85}$$

$$\begin{aligned} G_{m\mu\nu\rho\sigma} &= \pm\varepsilon_{\mu\nu\rho\sigma}\partial_m\left(1+\frac{k}{r^4}\right)^{-1} \\ G_{mnpqr} &= \pm\varepsilon_{mnpqrs}\partial^s\left(1+\frac{k}{r^4}\right)=\pm 4k\varepsilon_{mnpqrs}\frac{y^s}{r^6} \end{aligned} \tag{2.86}$$

If we compare (2.86) with (2.42) and (2.63), we easily see that the first line in (2.86) correspond to the "electric" ansatz, and that the second line correspond to the "magnetic" ansatz. This is one way to understand why the D3-brane have both "electric" and "magnetic" charge.

I am now going to check that (2.85)-(2.86) satisfies (2.71). (2.71) implies that $G^2=G_{MNPQR}G^{MNPQR}=0$. This is seen from [15]

$$\frac{1}{n!}G^2=-\frac{1}{(D-n)!}(*G)^2 \tag{2.87}$$

which with D=10 and n=5 give $G^2 = -(*G)^2$, and implies that $G^2 = 0$, since G_{MNPQR} is self-dual. Calculating G^2 gives

$$\begin{aligned} G_{MNPQR}G^{MNPQR} &= 5!e^{-B}\Big(\delta^{mn}\partial_m D\partial_n De^{2D-4B} - \delta^{mn}\partial_m C\partial_n Ce^{2C-4A}\Big) \\ &= 5!\Delta^{-\frac{1}{2}}(D'^2 - C'^2) = 5!\frac{\Delta'^2}{\Delta^{\frac{5}{2}}}\Big(1^2 - (-1)^2\Big) = 0 \end{aligned} \tag{2.88}$$

where I have used (2.14), (B.5) and (B.6). This means that (2.85)-(2.86) satisfies (2.71).

In exactly the same way as for the M2- and M5-branes, it is also possible to extend the D3-brane solution to N D3-branes located at different places in spacetime. The only thing that is different compared to one D3-brane is the $\Delta(r)$-function, which is changed to

$$\Delta(r) = 1 + \sum_{n=1}^{N}\left(\frac{k_n}{|\vec{r}-\vec{r}_n|^4}\right) \tag{2.89}$$

where $\vec{r}_n$ corresponds to the position of the n:th brane. This new $\Delta(r)$ function satisfy the equations of motion, which is shown in the same way as for the M2-branes.

I am now going to see if the singularity at $r = 0$ in the metric, is a coordinate singularity or a spacetime singularity.

Calculation of the Ricci scalar R, with $d = 4$ and $d_1 = 6$ gives

$$R = 0 \tag{2.90}$$

The implication of this equation is that the singularity is a coordinate singularity. This also means that the D3-brane is non-singular inside the horizon, since $R = 0$ for all r.

I will now perform a coordinate change, and show that when r is close to zero, it's possible to write the metric as a product of two spaces $AdS_5 \times S^5$, which is a five dimensional anti de-Sitter spacetime, times a five sphere.

Setting $r = k^{\frac{1}{4}}R/(1-R^4)^{\frac{1}{4}}$ in (2.85), for $0 < R < 1$ (corresponding to $0 < r < \infty$) gives

$$\begin{aligned} ds^2 &= R^2\eta_{\mu\nu}dx^\mu dx^\nu + k^{\frac{1}{2}}\Bigg[\frac{1}{R^2(1-R^4)^{\frac{5}{2}}}dR^2 \\ &+ \frac{1}{(1-R^4)^{\frac{1}{2}}}d\Omega_5^2\Bigg] \end{aligned} \tag{2.91}$$

where $d\Omega_5$ is the line element of S^5. If we now let $R \to 0$ in the metric (same as $r \to 0$), and also re-scale R as $R \to k^{\frac{1}{4}}R$, then the metric will have the following form

$$ds^2 = k^{\frac{1}{2}}\Big[R^2\eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{R^2}dR^2\Big] + k^{\frac{1}{2}}d\Omega_5^2 \tag{2.92}$$

where the first part is AdS_5 in Poincare coordinates (see appendix D), and the second part is S^5. This means that the metric is asymptotic to $AdS_5 \times S^5$ when r is close to zero. We also know that the metric is asymptotic to 10D Minkowski spacetime when r is large. The interpretation of (2.92) is therefore that the D3-brane solution to the bosonic equations of motion is a soliton in the sense that it is an interpolating solution between different vacua, namely $AdS_5 \times S^5$ and 10D Minkowski spacetime.

3

Bosonic mass operators for $AdS_{4,5,7} \times M^{7,5,4}$

In this chapter, I am going to calculate the bosonic mass operators for various anti de-Sitter spacetime. $AdS_{4,7}$ (means AdS_4 or AdS_7) are obtained by compactification of 11D sugra on an Einstein space $M^{7,4}$, where the Einstein space can be, for example $S^{4,7}$ or $T^{4,7}$. AdS_5 is obtained by compactification of type IIB sugra on an Einstein space M^5. An Einstein space is defined as $R_{MN} \sim g_{MN}$, which means that it is compact and have positive curvature. In this chapter the Einstein space $M^{7,5,4}$ will be left arbitrary, since it is not necessary to know which space it is in order to obtain the mass operators. The only difference between Einstein spaces that is important, is if they have zero or nonzero second betti number b_2, (this will be explained in 3.3). Later in chapter 4, I will show that the isometry group of the internal space (M) is equivalent to the gauge group in the compactified theory.

Derivation of the bosonic mass operators resulting from the Freund-Rubin compactification on an arbitrary $M^{7,5,4}$, is performed by first expanding the bosonic equations of motion to linear order in the fluctuations, and then inserting the Freund-Rubin ansatz, followed by substituting the various harmonic expansions of the D=11,10 fields into the bosonic equations of motion. The bosonic mass operators for $AdS_{4,5,7}$ are obtained by diagonalizing the mass matrix for each type of field, mode by mode. There will also be a few modes that are not included in the general analysis. These modes will be dealt with after the general analysis.

In section 3.1, I am going to solve the bosonic equations of motion for vacua, based on the Freund-Rubin ansatz [16]. Section 3.2 will deal with linearization of the bosonic equations of motion. In section 3.3-3.6, I am presenting the calculations of the bosonic mass operators for $AdS_{4,5,7} \times M^{7,5,4}$ $(m \neq 0)$, and $Minkowski_4 \times M^7$ $(m = 0)$ (in section 3.4), where M^7 is T^7 or $K3 \times T^3$, which are the only Ricci flat spaces in 7 dimensions.

3.1 Spontaneous compactification to $d = 4, 5, 7$

Spontaneous compactification can be achieved by assuming that the field strength F_{MNPQ} or G_{MNPQR} obey the Freund-Rubin ansatz. The Freund-Rubin ansatz breaks the D=11,10 spacetime into a ground state that is a product of a spacetime and an Einstein space. We want to find a solution that have maximal spacetime symmetry. This means that the spacetime vacuum should be invariant under $SO(1, d)$, $Poincare_d$ or $SO(2, d-1)$, $(d = 4, 5, 7)$. This corresponds to de-Sitter ($\Lambda > 0$), Minkowski (Λ =0) or Anti de-Sitter ($\Lambda < 0$) spacetime, where Λ is the cosmological constant.

The 11 dimensional bosonic equations of motion are [7, 12]:

$$\begin{aligned} R_{MN} - \frac{1}{2} g_{MN} R &= \frac{1}{3}\Big[F_{MPQR}F_N{}^{PQR} - \frac{1}{8} g_{MN} F_{PQRS}F^{PQRS}\Big] && (3.1) \\ \nabla_M F^{MNPQ} &= -\frac{1}{576}\epsilon^{M_1 \ldots M_8 NPQ} F_{M_1M_2M_3M_4}F_{M_5M_6M_7M_8} && (3.2) \end{aligned}$$

where the field strength $F_{MNPQ} = 4\partial_{[M}A_{NPQ]}$ and A_{NPQ} is the potential. I will now look for solutions of the direct product form $M_4 \times M^7$ and $M_7 \times M^4$ that have maximal spacetime symmetry. The metric VEV (Vacuum Expectation Value) is

$$< g_{\mu\nu} >= \overset{\circ}{g}_{\mu\nu}(x), \quad < g_{mn} >= \overset{\circ}{g}_{mn}(y), \quad < g_{\mu n} >= 0 \tag{3.3}$$

where x^μ are the spacetime coordinates and y^m are the Einstein space coordinates. The x-independence of $\overset{\circ}{g}_{mn}$ and y-independence of $\overset{\circ}{g}_{\mu\nu}$ in (3.3) are necessary if g_{MN} is to be a metric on a product space. The four-form VEV is

$$\begin{aligned} < F_{\mu\nu\rho\sigma} > &= \overset{\circ}{F}_{\mu\nu\rho\sigma}(x), \quad < F_{mnpq} > = \overset{\circ}{F}_{mnpq}(y) \\ < F_{\mu\nu\rho q} > &= < F_{\mu\nu pq} > = < F_{\mu npq} > = 0 \end{aligned} \tag{3.4}$$

where the x-independence of $\overset{\circ}{F}_{mnpq}$ and y-independence of $\overset{\circ}{F}_{\mu\nu\rho\sigma}$ are required by the Bianchi identity $\partial_{[M}\overset{\circ}{F}_{NPQR]}= 0$, i.e.,

$$\begin{aligned} \partial_{[m}\overset{\circ}{F}_{\mu\nu\rho\sigma]} &= 0 \Rightarrow \partial_m \overset{\circ}{F}_{\mu\nu\rho\sigma} = 0 \Rightarrow < F_{\mu\nu\rho\sigma} > = \overset{\circ}{F}_{\mu\nu\rho\sigma}(x) \\ \partial_{[\mu}\overset{\circ}{F}_{mnpq]} &= 0 \Rightarrow \partial_\mu \overset{\circ}{F}_{mnpq} = 0 \Rightarrow < F_{mnpq} > = \overset{\circ}{F}_{mnpq}(y) \end{aligned} \tag{3.5}$$

I am now going to solve (3.1)-(3.2) for the $M_4 \times M^7$ case. The Freund-Rubin ansatz is [16]

$$\overset{\circ}{F}_{\mu\nu\rho\sigma}= 3m\overset{\circ}{\epsilon}_{\mu\nu\rho\sigma}, \quad \overset{\circ}{F}_{mnpq}= 0, \quad \text{others zero} \tag{3.6}$$

where $\mu = 0, 1, 2, 3$, $m = 4, 5, \ldots, 10$, m is a real constant, $\overset{\circ}{\epsilon}_{\mu\nu\rho\sigma}$ is the Levi-Cevita tensor defined as, $\overset{\circ}{\epsilon}_{0123}= \sqrt{-\overset{\circ}{g}}$, (see also appendix B) and the factor of 3 is chosen for future convenience. Inserting (3.6) into (3.1), multiplying with $\overset{\circ}{g}^{MN}$ and solving for $\overset{\circ}{R}$ gives

$$\overset{\circ}{R}= \frac{1}{36}\overset{\circ}{F}_{MNPQ}\overset{\circ}{F}^{MNPQ} = -6m^2 \tag{3.7}$$

Here I have used (B.5). Inserting (3.6)-(3.7) into (3.1) gives

$$\begin{aligned}\overset{\circ}{R}_{\mu\nu} &= \frac{1}{3}\Big[\overset{\circ}{g}_{\nu S}\overset{\circ}{F}_{\mu PQR}\overset{\circ}{F}{}^{SPQR} - \frac{1}{8}\overset{\circ}{g}_{\mu\nu}\overset{\circ}{F}_{PQRS}\overset{\circ}{F}{}^{PQRS}\Big] - 3m^2\overset{\circ}{g}_{\mu\nu} \\ &= -12m^2\overset{\circ}{g}_{\mu\nu} \end{aligned} \tag{3.8}$$

$$\begin{aligned}\overset{\circ}{R}_{mn} &= \frac{1}{3}\Big[\overset{\circ}{g}_{nS}\overset{\circ}{F}_{mPQR}\overset{\circ}{F}{}^{SPQR} - \frac{1}{8}\overset{\circ}{g}_{mn}\overset{\circ}{F}_{PQRS}\overset{\circ}{F}{}^{PQRS}\Big] - 3m^2\overset{\circ}{g}_{mn} \\ &= 6m^2\overset{\circ}{g}_{mn} \end{aligned} \tag{3.9}$$

$$\overset{\circ}{R}_{\mu n} = 0 \tag{3.10}$$

where I have used (B.5)-(B.6) and $\overset{\circ}{g}_{\mu n}= 0$. The constant in front of $\overset{\circ}{g}_{\mu\nu}$ in (3.8) is the cosmological constant Λ. We see here that Λ is negative, which means that the 4 dimensional spacetime is Anti de-Sitter ($m \neq 0$), or Minkowski ($m = 0$). In the $m = 0$ case the compactification of the extra dimensions is no longer mandatory, but solutions exist, (see section 3.2 and 4.4). (3.9) is the Ricci tensor for an 7-dimensional Einstein space. The implication of (3.8)-(3.10) is that we have achieved spontaneous compactification from D=11 spacetime down to d=4 spacetime.

I will now check that (3.2) is satisfied by the Freund-Rubin ansatz. It is easy to see that most terms are zero. The harder terms are

$$\begin{aligned} RHS \quad &: \quad -\frac{1}{576}\,\overset{\circ}{\epsilon}{}^{\mu\nu\rho\sigma\mu'\nu'\rho'\sigma' NPQ}\,\overset{\circ}{F}_{\mu\nu\rho\sigma}\overset{\circ}{F}_{\mu'\nu'\rho'\sigma'}= 0 \\ LHS \quad &: \quad \nabla_\mu \overset{\circ}{F}{}^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-\overset{\circ}{g}}}\partial_\mu\Big(\sqrt{-\overset{\circ}{g}}\,\overset{\circ}{F}{}^{\mu\nu\rho\sigma}\Big) \\ &\qquad = \frac{3m}{\sqrt{-\overset{\circ}{g}}}\partial_\mu\Big(\sqrt{-\overset{\circ}{g}}\,\overset{\circ}{\epsilon}{}^{\mu\nu\rho\sigma}\Big) = 0 \end{aligned} \tag{3.11}$$

Here I have used that $\overset{\circ}{g}$ is the determinant of the vacuum metric, and (B.2).

Now it is time solve (3.1)-(3.2) for the $M_7 \times M^4$ case. The Freund-Rubin ansatz is [16]

$$\overset{\circ}{F}_{\mu\nu\rho\sigma}= 0, \quad \overset{\circ}{F}_{mnpq}= \frac{3m}{2}\overset{\circ}{\epsilon}_{mnpq}, \quad \text{others zero} \tag{3.12}$$

where $\mu = 0, 1, 2, \ldots, 6$, $m = 7, 8, 9, 10$, m is a real constant, $\overset{\circ}{\epsilon}_{mnpq}$ is the Levi-Cevita tensor defined as $\overset{\circ}{\epsilon}_{0123}= \sqrt{-\overset{\circ}{g}}$ (see appendix B) and the factor of $\frac{3}{2}$ is chosen for future convenience. Inserting (3.12) into (3.1), multiplying with $\overset{\circ}{g}{}^{MN}$ and solving for $\overset{\circ}{R}$ gives

$$\overset{\circ}{R}= \frac{1}{36}\overset{\circ}{F}_{MNPQ}\overset{\circ}{F}{}^{MNPQ} = \frac{3}{2}m^2 \tag{3.13}$$

Here I have used (B.5). Inserting (3.12)-(3.13) into (3.1) gives

$$\begin{aligned}
\overset{\circ}{R}_{\mu\nu} &= \frac{1}{3}\Big[\overset{\circ}{g}_{\nu S}\overset{\circ}{F}_{\mu PQR}\overset{\circ}{F}{}^{SPQR} - \frac{1}{8}\overset{\circ}{g}_{\mu\nu}\overset{\circ}{F}_{PQRS}\overset{\circ}{F}{}^{PQRS}\Big] - \frac{3}{4}m^2\overset{\circ}{g}_{\mu\nu} \\
&= -\frac{3}{2}m^2\overset{\circ}{g}_{\mu\nu} \qquad (3.14)
\end{aligned}$$

$$\begin{aligned}
\overset{\circ}{R}_{mn} &= \frac{1}{3}\Big[\overset{\circ}{g}_{nS}\overset{\circ}{F}_{mPQR}\overset{\circ}{F}{}^{SPQR} - \frac{1}{8}\overset{\circ}{g}_{mn}\overset{\circ}{F}_{PQRS}\overset{\circ}{F}{}^{PQRS}\Big] - \frac{3}{4}m^2\overset{\circ}{g}_{mn} \\
&= 3m^2\overset{\circ}{g}_{mn} \qquad (3.15)
\end{aligned}$$

$$\overset{\circ}{R}_{\mu n} = 0 \qquad (3.16)$$

where I have used (B.5)-(B.6) and $\overset{\circ}{g}_{\mu n}=0$. We see here from (3.14)-(3.16) that we have achieved spontaneous compactification from D=11 spacetime down to d=7 Anti de-Sitter spacetime ($AdS_7 \times M^4$).

(3.2) is satisfied by the Freund-Rubin ansatz, which is easily shown in the same way as for the $M_4 \times M^7$ case.

I am now going to solve the bosonic equations of motion for the $M_5 \times M^5$ case. The bosonic equations of motion are given by (3.47)-(3.50) below. In these equations I set all fields to zero except G_{MNPQR}, in order to get maximal spacetime symmetry. The result of this is that two equations of motion remains. The two equations are

$$\begin{aligned}
R_{MN} &= \frac{1}{96}G_M{}^{PQRS}G_{NPQRS} = \frac{1}{96}g_{MT}G^{TPQRS}G_{NPQRS} \qquad (3.17) \\
G_{MNPQR} &= *G_{MNPQR} = \frac{1}{5!}\epsilon_{MNPQRS_1S_2S_3S_4S_5}G^{S_1S_2S_3S_4S_5} \qquad (3.18)
\end{aligned}$$

where $G_{MNPQR} = 5\partial_{[M}I_{NPQR]}$ is the self-dual field strength and I_{NPQR} is the potential. I will now look for solutions of the direct product form $M_5 \times M^5$ with maximal spacetime symmetry. The metric VEV is the same as (3.3). The five-form VEV is similar to the four-form VEV in (3.4), i.e,

$$\begin{aligned}
< G_{\mu\nu\rho\sigma\tau} > &= \overset{\circ}{G}_{\mu\nu\rho\sigma\tau}(x), \quad < G_{mnpqr} > = \overset{\circ}{G}_{mnpqr}(y) \\
< G_{\mu\nu\rho\sigma r} > &= < G_{\mu\nu\rho qr} > = < G_{\mu\nu pqr} > = < G_{\mu npqr} > = 0 \qquad (3.19)
\end{aligned}$$

where the x-independence of $\overset{\circ}{G}_{mnpqr}$ and y-independence of $\overset{\circ}{G}_{\mu\nu\rho\sigma\tau}$ are required by the Bianchi identity $\partial_{[M}\overset{\circ}{G}_{NPQRS]}=0$.

In this case the Freund-Rubin ansatz is

$$\overset{\circ}{G}_{\mu\nu\rho\sigma\tau}= 4m\overset{\circ}{\epsilon}_{\mu\nu\rho\sigma\tau}, \quad \overset{\circ}{G}_{mnpqr}= 4m\overset{\circ}{\epsilon}_{mnpqr}, \quad \text{others zero} \qquad (3.20)$$

where $\mu = 0, 1, 2, 3, 4,\ m = 5, 6, 7, 8, 9,\ m$ is a real constant, $\overset{\circ}{\epsilon}_{MNPQR}$ is the Levi-Cevita tensor defined as, $\overset{\circ}{\epsilon}_{01234}= \sqrt{-\overset{\circ}{g}}$, (see also appendix B) and the factor of 4 is chosen for future convenience. In this case, it's not possible to only have one term nonzero since G_{MNPQR} is self-dual. I will now check that (3.20) satisfy (3.18).

$$
\begin{aligned}
*\overset{\circ}{G}_{\mu\nu\rho\sigma\tau} &= \frac{1}{5!}\overset{\circ}{\epsilon}_{\mu\nu\rho\sigma\tau S_1S_2S_3S_4S_5}\ G^{S_1S_2S_3S_4S_5} \\
&= \frac{1}{5!}\Big(\overset{\circ}{\epsilon}_{\mu\nu\rho\sigma\tau s_1s_2s_3s_4s_5}\,4m\overset{\circ}{\epsilon}{}^{s_1s_2s_3s_4s_5} \\
&+ \overset{\circ}{\epsilon}_{\mu\nu\rho\sigma\tau\chi_1\chi_2\chi_3\chi_4\chi_5}\,4m\overset{\circ}{\epsilon}{}^{\chi_1\chi_2\chi_3\chi_4\chi_5}\Big) = 4m\overset{\circ}{\epsilon}_{\mu\nu\rho\sigma\tau}
\end{aligned}
\tag{3.21}
$$

$\overset{\circ}{G}_{mnpqr}$ is shown in the same way. This means that (3.18) is satisfied by (3.20).

Inserting (3.20) into (3.17) gives

$$\overset{\circ}{R}_{\mu\nu} = \frac{1}{6}m^2\,\overset{\circ}{g}_{\mu\xi}\overset{\circ}{\epsilon}_{\nu\rho\sigma\tau\chi}\overset{\circ}{\epsilon}{}^{\xi\rho\sigma\tau\chi} = -4m^2\,\overset{\circ}{g}_{\mu\nu} \tag{3.22}$$

$$\overset{\circ}{R}_{mn} = \frac{1}{6}m^2\,\overset{\circ}{g}_{mt}\overset{\circ}{\epsilon}_{tpqrs}\overset{\circ}{\epsilon}{}^{npqrs} = 4m^2\,\overset{\circ}{g}_{mn} \tag{3.23}$$

$$\overset{\circ}{R}_{\mu n} = 0 \tag{3.24}$$

Here I have used (B.6) and $\overset{\circ}{g}_{\mu n}= 0$. We see from (3.22)-(3.24) that we have achieved spontaneous compactification from D=10 spacetime down to $d = 5$ Anti de-Sitter spacetime ($AdS_5 \times M^5$).

3.2 Linearization of the bosonic equations of motion

In this section, I am going to linearize the bosonic equations of motion for 11D sugra and type IIB 10D sugra. I will do this because I want to solve the equations of motion for a background with small fluctuations, in order to obtain the bosonic mass operators for $AdS_{4,5,7} \times M^{7,5,4}$.

I will start with the 11D sugra bosonic equations of motion. I define the fluctuations in the following way

$$g_{MN}(x,y) = \overset{\circ}{g}_{MN}(x,y) + h_{MN}(x,y) \tag{3.25}$$

$$A_{MNP}(x,y) = \overset{\circ}{A}_{MNP}(x,y) + a_{MNP}(x,y) \tag{3.26}$$

$$F_{MNPQ}(x,y) = \overset{\circ}{F}_{MNPQ}(x,y) + f_{MNPQ}(x,y) \tag{3.27}$$

where $\overset{\circ}{F}_{MNPQ}= 4\partial_{[M}\overset{\circ}{A}_{NPQ]}$ and $f_{MNPQ} = 4\partial_{[M}a_{NPQ]}$. Here $\overset{\circ}{A}_{MNP}$ is the Freund-Rubin background potential, and a_{MNP}, f_{MNPQ} and h_{MN} are the fluctuations, where a_{MNP} and f_{MNPQ} are completely antisymmetric, and h_{MN} is symmetric. Next, I will calculate some relations that will be useful below. These are

$$\delta g_{MN} = h_{MN} \tag{3.28}$$

and

$$\delta g^{MN} = -h^{MN} \tag{3.29}$$

where I have used $g^{MN} = \overset{\circ}{g}{}^{MN} - h^{MN}$. Here the minus sign makes sure that $g^{MN} g_{MN} = \delta^M_M = D$ when only keeping terms linear in h_{MN}.

$$
\begin{aligned}
\delta F_{MNPQ} &= f_{MNPQ} &(3.30)\\
\delta F^{MNPQ} &= \delta\Big(g^{MM'} g^{NN'} g^{PP'} g^{QQ'} F_{M'N'P'Q'}\Big) \\
&= f^{MNPQ} - h^{MM'} \overset{\circ}{F}_{M'}{}^{NPQ} - h^{NN'} \overset{\circ}{F}{}^{M}{}_{N'}{}^{PQ} \\
&- h^{PP'} \overset{\circ}{F}{}^{MN}{}_{P'}{}^{Q} - h^{QQ'} \overset{\circ}{F}{}^{MNP}{}_{Q'} \\
&= f^{MNPQ} - 4h^{M'[M} \overset{\circ}{F}_{M'}{}^{NPQ]} &(3.31)
\end{aligned}
$$

In (3.31) I have used (3.29)-(3.30). Next,

$$\delta F_N{}^{PQR} = f_N{}^{PQR} - 3h^{P'[P} \overset{\circ}{F}_{NP'}{}^{QR]} \tag{3.32}$$

which was calculated in the same way as (3.31).

I am now going to start linearizing (3.1). Equation (3.1) can be rewritten as

$$R_{MN} = \frac{1}{3}\Big[g_{NS} F_{MPQR} F^{SPQR} - \frac{1}{12} g_{MN} F_{PQRS} F^{PQRS}\Big] \tag{3.33}$$

Linearization is achieved by doing a variation of the field and only retaining terms up to linear order in the fluctuations. The variation of R_{MN} was calculated in appendix B. Inserting $\delta g_{MN} = h_{MN}$ into (B.28) and only keeping terms linear in the fluctuations gives

$$\delta R_{MN} = \frac{1}{2} \overset{\circ}{\Delta}_L h_{MN} + \overset{\circ}{\nabla}_{(M} \overset{\circ}{\nabla}{}^P h_{N)P} - \frac{1}{2} \overset{\circ}{\nabla}_M \overset{\circ}{\nabla}_N h_P{}^P \tag{3.34}$$

where I have used that

$$
\begin{aligned}
\nabla_M h_{NP} &= \partial_M h_{NP} - \Gamma^Q_{MN} h_{QP} - \Gamma^Q_{MP} h_{QN} \\
&= \partial_M h_{NP} - \overset{\circ}{\Gamma}{}^Q_{MN} h_{QP} - \overset{\circ}{\Gamma}{}^Q_{MP} h_{QN} + \Big(\text{higher terms in } h_{MN}\Big) \\
&= \overset{\circ}{\nabla}_M h_{NP} + \Big(\text{higher terms in } h_{MN}\Big) &(3.35)
\end{aligned}
$$

and that (3.35) gives $\Delta_L h_{MN} = \overset{\circ}{\Delta}_L h_{MN} +$ (higher terms in h_{MN}), (see (B.25)).

The RHS of (3.1) (which I will name T_{MN}) is

$$
\begin{aligned}
\delta T_{MN} &= \frac{1}{3}\Big[\delta F_{MPQR} F_N{}^{PQR} + F_{MPQR} \delta F_N{}^{PQR} - \frac{1}{12}\Big(\delta g_{MN} F_{PQRS} F^{PQRS} \\
&+ g_{MN} \delta F_{PQRS} F^{PQRS} + g_{MN} F_{PQRS} \delta F^{PQRS}\Big)\Big] &(3.36)
\end{aligned}
$$

Inserting (3.25), (3.27)-(3.28) and (3.30)-(3.32) into (3.36) and keeping terms linear in the fluctuations gives

$$\delta T_{MN} = \frac{1}{3}\Big[f_{MPQR} \overset{\circ}{F}_N{}^{PQR} + \Big(\overset{\circ}{F}_{MPQR} + f_{MPQR}\Big)\Big(f_N{}^{PQR} - 3h^{P'[P} \overset{\circ}{F}_{NP'}{}^{QR]}\Big)$$

$$
\begin{aligned}
&- \frac{1}{12}\Big[h_{MN}\overset{\circ}{F}_{PQRS}\overset{\circ}{F}{}^{PQRS}+\overset{\circ}{g}_{MN}\ f_{PQRS}\overset{\circ}{F}{}^{PQRS}\\
&+ \overset{\circ}{g}_{MN}\Big(\overset{\circ}{F}_{PQRS}+f_{PQRS}\Big)\Big(f^{PQRS}-4h^{P'[P}\overset{\circ}{F}_{P'}{}^{QRS]}\Big)\Big]\Big]\\
&= \frac{2}{3}\overset{\circ}{F}_{(M}{}^{PQR}f_{N)PQR}-\frac{1}{36}h_{MN}\overset{\circ}{F}_{PQRS}\overset{\circ}{F}{}^{PQRS}-\frac{1}{18}\overset{\circ}{g}_{MN}\overset{\circ}{F}_{PQRS}f^{PQRS}\\
&- h^{P'[P}\overset{\circ}{F}_{NP'}{}^{QR]}\overset{\circ}{F}_{MPQR}+\frac{1}{9}\overset{\circ}{g}_{MN}h^{P'[P}\overset{\circ}{F}_{P'}{}^{QRS]}\overset{\circ}{F}_{PQRS} \qquad (3.37)
\end{aligned}
$$

Term 4 and 5 in (3.37) can be rewritten in the following way

$$
\begin{aligned}
h^{S[P}\overset{\circ}{F}_{NS}{}^{QR]}\overset{\circ}{F}_{MPQR} &= h^{SP}\overset{\circ}{F}_{NS}{}^{QR}\overset{\circ}{F}_{MPQR}\\
&= \overset{\circ}{F}_{(M}{}^{PQR}\overset{\circ}{F}_{N)}{}^{S}{}_{QR}h_{PS} \qquad (3.38)\\
\overset{\circ}{g}_{MN}h^{T[P}\overset{\circ}{F}_{T}{}^{QRS]}\overset{\circ}{F}_{PQRS} &= \overset{\circ}{g}_{MN}h^{PT}\overset{\circ}{F}_{T}{}^{QRS}\overset{\circ}{F}_{PQRS} \qquad (3.39)
\end{aligned}
$$

where I have used that $\overset{\circ}{F}_{MNPQ}$ is completely antisymmetric. Inserting (3.38)-(3.39) into (3.37) and combining with (3.34) gives

$$
\begin{aligned}
&\frac{1}{2}\overset{\circ}{\Delta}_L h_{MN}+\overset{\circ}{\nabla}_{(M}\overset{\circ}{\nabla}{}^{P}h_{N)P}-\frac{1}{2}\overset{\circ}{\nabla}_M\overset{\circ}{\nabla}_N h_P{}^P\\
&= \frac{2}{3}\overset{\circ}{F}_{(M}{}^{PQR}f_{N)PQR}-\frac{1}{36}h_{MN}\overset{\circ}{F}_{PQRS}\overset{\circ}{F}{}^{PQRS}-\frac{1}{18}\overset{\circ}{g}_{MN}\overset{\circ}{F}_{PQRS}f^{PQRS}\\
&- \overset{\circ}{F}_{(M}{}^{PQR}\overset{\circ}{F}_{N)}{}^{S}{}_{QR}h_{PS}+\frac{1}{9}\overset{\circ}{g}_{MN}h^{PT}\overset{\circ}{F}_{T}{}^{QRS}\overset{\circ}{F}_{PQRS} \qquad (3.40)
\end{aligned}
$$

I will now linearize (3.2). The RHS (which I will call S^{MNP}) is

$$
\begin{aligned}
\delta S^{MNP} &= \delta\Big(-\frac{1}{576}\epsilon^{MNPQ_1\ldots Q_8}F_{Q_1Q_2Q_3Q_4}F_{Q_5Q_6Q_7Q_8}\Big)\\
&= -\frac{1}{576}\epsilon^{MNPQ_1\ldots Q_8}\Big(\overset{\circ}{F}_{Q_1Q_2Q_3Q_4}f_{Q_5Q_6Q_7Q_8}+f_{Q_1Q_2Q_3Q_4}\overset{\circ}{F}_{Q_5Q_6Q_7Q_8}\Big)\\
&- \frac{1}{576}\delta\epsilon^{MNPQ_1\ldots Q_8}\overset{\circ}{F}_{Q_1Q_2Q_3Q_4}\overset{\circ}{F}_{Q_5Q_6Q_7Q_8}\\
&= -\frac{1}{288}\overset{\circ}{\epsilon}{}^{MNPQ_1\ldots Q_8}\overset{\circ}{F}_{Q_1Q_2Q_3Q_4}f_{Q_5Q_6Q_7Q_8}\\
&+ \frac{1}{1152}\overset{\circ}{\epsilon}{}^{MNPQ_1\ldots Q_8}\overset{\circ}{F}_{Q_1Q_2Q_3Q_4}\overset{\circ}{F}_{Q_5Q_6Q_7Q_8}h_R{}^R \qquad (3.41)
\end{aligned}
$$

where I have used (3.30), (B.17) and $\delta\sqrt{-g}=\frac{1}{2}\sqrt{-g}g^{PQ}\delta g_{PQ}$ (see [37]). The LHS is

$$
\delta\Big(\nabla_Q F^{QMNP}\Big)=\delta\Big(\partial_Q F^{QMNP}+\Gamma^{Q}_{QR}F^{RMNP}\Big) \qquad (3.42)
$$

where I have used that

$$
\Gamma^{M}_{QR}F^{QRNP}=0 \qquad (3.43)
$$

because Γ^{M}_{QR} is symmetric in QR and F^{QRNP} is completely antisymmetric. Continuing with the LHS gives

$$
\delta\Big(\nabla_Q F^{QMNP}\Big) = \partial_Q\Big(f^{QMNP}-4h^{T[Q}\overset{\circ}{F}_{T}{}^{MNP]}\Big)
$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\,\overset{\circ}{g}{}^{QT}\Big(\,\overset{\circ}{\nabla}_Q h_{TR}+\overset{\circ}{\nabla}_R h_{TQ}-\overset{\circ}{\nabla}_T h_{QR}\Big)\,\overset{\circ}{F}{}^{RMNP}\\
&+ \quad \overset{\circ}{\Gamma}{}^Q_{QR}\Big(f^{RMNP}-4h^{T[R}\,\overset{\circ}{F}_R{}^{MNP]}\Big)\\
&= \quad \overset{\circ}{\nabla}_Q f^{QMNP}+\frac{1}{2}\,\overset{\circ}{F}{}^{QMNP}\,\overset{\circ}{\nabla}_Q h_R{}^R\\
&- \quad 4\Big(\partial_Q h^{T[Q}\,\overset{\circ}{F}_T{}^{MNP]}+\overset{\circ}{\Gamma}{}^Q_{QR}h^{T[R}\,\overset{\circ}{F}_T{}^{MNP]}\Big) \qquad (3.44)
\end{aligned}
$$

Here I have used (3.31) and (B.19). I am now going to rewrite term 3 and 4 in the following way

$$
\begin{aligned}
&4\partial_Q h^{T[Q}\,\overset{\circ}{F}_T{}^{MNP]}+4\overset{\circ}{\Gamma}{}^Q_{QR}h^{T[R}\,\overset{\circ}{F}_T{}^{MNP]}\\
=\ &\overset{\circ}{\nabla}_Q\left(h^{QR}\,\overset{\circ}{F}_R{}^{MNP}\right)+3\,\overset{\circ}{\nabla}{}^Q\Big(h^{S[M}\,\overset{\circ}{F}_{QS}{}^{NP]}\Big) \qquad (3.45)
\end{aligned}
$$

where I have used that Γ^P_{MN} and h_{MN} are symmetric in MN and that F^{MNPQ} is completely antisymmetric. The result of the linearization of (3.2) is

$$
\begin{aligned}
&\overset{\circ}{\nabla}_Q f^{QMNP}+\frac{1}{2}\,\overset{\circ}{F}{}^{QMNP}\,\overset{\circ}{\nabla}_Q h_R{}^R-\overset{\circ}{\nabla}_Q\left(h^{QR}\,\overset{\circ}{F}_R{}^{MNP}\right)-3\,\overset{\circ}{\nabla}{}^Q\Big(h^{S[M}\,\overset{\circ}{F}_{QS}{}^{NP]}\Big)\\
= \quad &-\frac{1}{1154}\,\overset{\circ}{\epsilon}{}^{MNPQ_1\ldots Q_8}\Big(4\,\overset{\circ}{F}_{Q_1Q_2Q_3Q_4}\,f_{Q_5Q_6Q_7Q_8}\\
- \quad &\overset{\circ}{F}_{Q_1Q_2Q_3Q_4}\overset{\circ}{F}_{Q_5Q_6Q_7Q_8}\;h_R{}^R\Big) \qquad (3.46)
\end{aligned}
$$

Next I will linearize the type IIB 10D sugra bosonic equations of motion. The four equations of motion are [8, 9, 10]

$$
\begin{aligned}
R_{MN} &= 2\bar{P}_{(M}P_{N)}+\bar{F}_{(M}{}^{PQ}F_{N)PQ}-\frac{1}{12}g_{MN}\bar{F}_{PQR}F^{PQR}\\
&+ \frac{1}{96}G_M{}^{PQRS}G_{NPQRS} \qquad (3.47)
\end{aligned}
$$

$$\nabla^M\bar{P}_M = \frac{1}{6}\bar{F}_{MNP}\bar{F}^{MNP} \qquad (3.48)$$

$$\nabla^P\bar{F}_{MNP} = -F_{MNP}\bar{P}^P+\frac{i}{6}G_{MNPQR}\bar{F}^{PQR} \qquad (3.49)$$

$$G_{MNPQR} = *G_{MNPQR}=\frac{1}{5!}\epsilon_{MNPQRS_1S_2S_3S_4S_5}G^{S_1S_2S_3S_4S_5} \qquad (3.50)$$

where all the fields are field strengths and complex, except G_{MNPQR} which is real.

I define the fluctuations in the following way

$$g_{MN}(x,y) = \overset{\circ}{g}_{MN}(x,y)+h_{MN}(x,y) \qquad (3.51)$$

$$\Phi(x,y) = \overset{\circ}{\Phi}(x,y)+\phi(x,y) \qquad (3.52)$$

$$P_M(x,y) = \overset{\circ}{P}_M(x,y) + p_M(x,y) \tag{3.53}$$

$$A_{MN}(x,y) = \overset{\circ}{A}_{MN}(x,y) + a_{MN}(x,y) \tag{3.54}$$
$$F_{MNP}(x,y) = \overset{\circ}{F}_{MNP}(x,y) + f_{MNP}(x,y) \tag{3.55}$$

$$\hat{I}_{MNPQ}(x,y) = \overset{\circ}{\hat{I}}_{MNPQ}(x,y) + I_{MNPQ}(x,y) \tag{3.56}$$
$$G_{MNPQR}(x,y) = \overset{\circ}{G}_{MNPQR}(x,y) + l_{MNPQR}(x,y) \tag{3.57}$$

where $p_M = \partial_M \phi$, $f_{MNP} = 3\partial_{[M} a_{NP]}$ and $l_{MNPQR} = 5\partial_{[M} I_{NPQR]}$.

I am starting with equation (3.48). The RHS is

$$\begin{aligned} \delta\Big(\frac{1}{6}\bar{F}_{MNP}\bar{F}^{MNP}\Big) &= \frac{1}{6}\Big(\delta\bar{F}_{MNP}\bar{F}^{MNP} + \bar{F}_{MNP}\delta\bar{F}^{MNP}\Big) \\ &= \frac{1}{6}\Big(2\,\overset{\circ}{\bar{F}}_{MNP}\bar{f}^{MNP} - 3h^{TM}\,\overset{\circ}{\bar{F}}_{T}{}^{NP}\,\overset{\circ}{\bar{F}}_{MNP}\Big) \end{aligned} \tag{3.58}$$

where I have used

$$\delta\bar{F}_{MNP} = \bar{f}_{MNP}, \tag{3.59}$$

and

$$\delta\bar{F}^{MNP} = \bar{f}^{MNP} - 3h^{T[M}\,\overset{\circ}{\bar{F}}_{T}{}^{NP]} \tag{3.60}$$

The LHS is

$$\begin{aligned} \delta(\nabla^M\bar{P}_M) &= \delta(g^{PM}\nabla_P\bar{P}_M) = -h^{PM}\nabla_P\bar{P}_M + g^{PM}\delta(\nabla_P\bar{P}_M) \\ &= -h^{PM}\overset{\circ}{\nabla}_P\overset{\circ}{\bar{P}}_M + g^{PM}\Big(\partial_P\bar{p}_M - \overset{\circ}{\Gamma}{}^{Q}_{PM}\bar{p}_Q \\ &- \frac{1}{2}g^{QR}[\overset{\circ}{\nabla}_P h_{RM} + \overset{\circ}{\nabla}_M h_{RP} - \overset{\circ}{\nabla}_R h_{PM}]\overset{\circ}{\bar{P}}_Q\Big) \\ &= -h^{PM}\overset{\circ}{\nabla}_P\overset{\circ}{\bar{P}}_M + \overset{\circ}{\nabla}_M\bar{p}^M - \frac{1}{2}[2\,\overset{\circ}{\nabla}_M h^{RM} + \overset{\circ}{\nabla}{}^R h_P{}^P]\overset{\circ}{\bar{P}}_R \end{aligned} \tag{3.61}$$

where I have used (B.19) and

$$\delta\bar{P}_M = \bar{p}_M \tag{3.62}$$

The result of the linearization of (3.48) is

$$\begin{aligned} &-h^{PM}\overset{\circ}{\nabla}_P\overset{\circ}{\bar{P}}_M + \overset{\circ}{\nabla}_M\bar{p}^M - \frac{1}{2}[2\,\overset{\circ}{\nabla}_M h^{RM} + \overset{\circ}{\nabla}{}^R h_P{}^P]\overset{\circ}{\bar{P}}_R \\ &= \frac{1}{6}\Big(2\,\overset{\circ}{\bar{F}}_{MNP}\bar{f}^{MNP} - 3h^{TM}\,\overset{\circ}{\bar{F}}_{T}{}^{NP}\,\overset{\circ}{\bar{F}}_{MNP}\Big) \end{aligned} \tag{3.63}$$

Next over to equation (3.47). The LHS was obtained in (3.34). The RHS (which I will call T_{MN}) is

$$\delta T_{MN} = \delta\bar{P}_M P_N + \bar{P}_M\delta P_N + \delta\bar{P}_N P_M + \bar{P}_N\delta P_M$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\Big(\delta\bar{F}_M{}^{PQ}F_{NPQ} + \bar{F}_M{}^{PQ}\delta F_{NPQ} + \delta\bar{F}_N{}^{PQ}F_{MPQ} + \bar{F}_N{}^{PQ}\delta F_{MPQ}\Big) \\
&- \quad \frac{1}{12}\Big(h_{MN}\bar{F}_{PQR}F^{PQR} + g_{MN}\delta\bar{F}_{PQR}F^{PQR} + g_{MN}\bar{F}_{PQR}\delta F^{PQR}\Big) \\
&+ \quad \frac{1}{96}\Big(\delta G_M{}^{PQRS}G_{NPQRS} + G_M{}^{PQRS}\delta G_{NPQRS}\Big) \\
&= \quad 2\bar{p}_{(M}\overset{\circ}{P}_{N)} + 2\overset{\circ}{\bar{P}}_{(M}p_{N)} + \bar{f}_{(M}{}^{PQ}\overset{\circ}{F}_{N)PQ} + \overset{\circ}{\bar{F}}_{(M}{}^{PQ}f_{N)PQ} - 2h_{TP}\overset{\circ}{\bar{F}}_{(M}{}^{TQ}\overset{\circ}{F}_{N)}{}^P{}_Q \\
&- \quad \frac{1}{12}\Big(h_{MN}\overset{\circ}{\bar{F}}_{PQR}\overset{\circ}{F}{}^{PQR} + \overset{\circ}{g}_{MN}\bar{f}_{PQR}\overset{\circ}{F}{}^{PQR} + \overset{\circ}{g}_{MN}\overset{\circ}{\bar{F}}_{PQR}f^{PQR} \\
&- \quad 3\overset{\circ}{g}_{MN}h^{TP}\overset{\circ}{F}_T{}^{QR}\overset{\circ}{\bar{F}}_{PQR}\Big) + \frac{1}{48}\Big(\overset{\circ}{G}_{(M}{}^{PQRS}l_{N)PQRS} \\
&- \quad 2h^{T[P}\overset{\circ}{G}_{MT}{}^{QRS]}\overset{\circ}{G}_{NPQRS}\Big) \qquad (3.64)
\end{aligned}
$$

where (3.62), (3.59) and

$$
\begin{aligned}
\delta\bar{F}_M{}^{PQ} &= \bar{f}_M{}^{PQ} - 2h^{T[P}\overset{\circ}{\bar{F}}_{MT}{}^{Q]} && (3.65) \\
\delta G_M{}^{PQRS} &= l_M{}^{PQRS} - 4h^{T[P}\overset{\circ}{G}_{MT}{}^{QRS]} && (3.66) \\
\delta\bar{P}^M &= \bar{p}^M - h^{TP}\overset{\circ}{\bar{P}}_T && (3.67) \\
\delta G_{MNPQR} &= l_{MNPQR} && (3.68)
\end{aligned}
$$

were used. Combining (3.34) and (3.64) gives

$$
\begin{aligned}
&\quad \frac{1}{2}\overset{\circ}{\Delta}_L h_{MN} + \overset{\circ}{\nabla}_{(M}\overset{\circ}{\nabla}{}^P h_{N)P} - \frac{1}{2}\overset{\circ}{\nabla}_M\overset{\circ}{\nabla}_N h_P{}^P \\
&= \quad 2\bar{p}_{(M}\overset{\circ}{P}_{N)} + 2\overset{\circ}{\bar{P}}_{(M}p_{N)} + \bar{f}_{(M}{}^{PQ}\overset{\circ}{F}_{N)PQ} + \overset{\circ}{\bar{F}}_{(M}{}^{PQ}f_{N)PQ} - 2h_{TP}\overset{\circ}{\bar{F}}_{(M}{}^{TQ}\overset{\circ}{F}_{N)}{}^P{}_Q \\
&- \quad \frac{1}{12}h_{MN}\overset{\circ}{\bar{F}}_{PQR}\overset{\circ}{F}{}^{PQR} - \frac{1}{12}\overset{\circ}{g}_{MN}\bar{f}_{PQR}\overset{\circ}{F}{}^{PQR} - \frac{1}{12}\overset{\circ}{g}_{MN}\overset{\circ}{\bar{F}}_{PQR}f^{PQR} \\
&+ \quad \frac{1}{4}\overset{\circ}{g}_{MN}h^{TP}\overset{\circ}{F}_T{}^{QR}\overset{\circ}{\bar{F}}_{PQR} + \frac{1}{48}\overset{\circ}{G}_{(M}{}^{PQRS}l_{N)PQRS} \\
&- \quad \frac{1}{24}h^{T[P}\overset{\circ}{G}_{MT}{}^{QRS]}\overset{\circ}{G}_{NPQRS} \qquad (3.69)
\end{aligned}
$$

Linearization of the RHS of (3.49) (which I will call S_{MN}) gives

$$
\begin{aligned}
\delta S_{MN} &= \delta\Big(-F_{MNP}\bar{P}^P + \frac{i}{6}G_{MNPQR}\bar{F}^{PQR}\Big) \\
&= -f_{MNP}\overset{\circ}{\bar{P}}{}^P - \overset{\circ}{F}_{MNP}\bar{p}^P + \overset{\circ}{F}_{MNP}h^{TP}\overset{\circ}{\bar{P}}_T + \frac{i}{6}\Big(l_{MNPQR}\overset{\circ}{\bar{F}}{}^{PQR} \\
&+ \overset{\circ}{G}_{MNPQR}\bar{f}^{PQR} - 3\overset{\circ}{G}_{MNPQR}h^{SP}\overset{\circ}{\bar{F}}_S{}^{QR}\Big) \qquad (3.70)
\end{aligned}
$$

where I have used (3.59)-(3.60) and (3.67)-(3.68). The LHS is

$$
\delta\Big(\nabla^P\bar{F}_{MNP}\Big) = \delta\Big(g^{RP}\nabla_R\bar{F}_{MNP}\Big) = -h^{PR}\overset{\circ}{\nabla}_R\overset{\circ}{\bar{F}}_{MNP} + g^{PR}\delta\Big(\partial_R\bar{F}_{MNP}
$$

$$
\begin{aligned}
&- \quad \Gamma^Q_{RM}\bar{F}_{QNP} - \Gamma^Q_{RN}\bar{F}_{MQP} - \Gamma^Q_{RP}\bar{F}_{MNQ}\Big) \\
&= \quad -h^{PR}\overset{\circ}{\nabla}_R \overset{\circ}{\bar{F}}_{MNP} + \overset{\circ}{\nabla}_P \bar{f}_{MN}{}^P - \frac{1}{2}(2\overset{\circ}{\nabla}_{[P}\, h_{Q]M} + \overset{\circ}{\nabla}_M\, h_{QP})\, \overset{\circ}{\bar{F}}_N{}^{PQ} \\
&- \quad \frac{1}{2}(2\overset{\circ}{\nabla}_{[P}\, h_{Q]N} + \overset{\circ}{\nabla}_N\, h_{QP})\, \overset{\circ}{\bar{F}}_M{}^{QP} \\
&- \quad \frac{1}{2}(2\overset{\circ}{\nabla}_P\, h_Q{}^P - \overset{\circ}{\nabla}_Q\, h_P{}^P)\, \overset{\circ}{\bar{F}}_{MN}{}^Q
\end{aligned} \tag{3.71}
$$

where I have used (B.19) and (3.59). The result of the linearization of (3.49) is

$$
\begin{aligned}
&- \quad \overset{\circ}{\nabla}_R\, (h^{PR}\, \overset{\circ}{\bar{F}}_{MNP}) + \overset{\circ}{\nabla}_P \bar{f}_{MN}{}^P - \frac{1}{2}(2\overset{\circ}{\nabla}_{[P}\, h_{Q]M} + \overset{\circ}{\nabla}_M\, h_{QP})\, \overset{\circ}{\bar{F}}_N{}^{PQ} \\
&- \quad \frac{1}{2}(2\overset{\circ}{\nabla}_{[P}\, h_{Q]N} + \overset{\circ}{\nabla}_N\, h_{QP})\, \overset{\circ}{\bar{F}}_M{}^{QP} + \frac{1}{2}\, \overset{\circ}{\nabla}_Q\, h_P{}^P\, \overset{\circ}{\bar{F}}_{MN}{}^Q \\
&= \quad -f_{MNP}\, \overset{\circ}{\bar{P}}{}^P - \overset{\circ}{F}_{MNP}\bar{p}^P + \overset{\circ}{F}_{MNP}\, h^{TP}\, \overset{\circ}{\bar{P}}_T + \frac{i}{6}\Big(l_{MNPQR}\, \overset{\circ}{\bar{F}}{}^{PQR} \\
&+ \quad \overset{\circ}{G}_{MNPQR}\, \bar{f}^{PQR} - 3\, \overset{\circ}{G}_{MNPQR}\, h^{SP}\, \overset{\circ}{\bar{F}}_S{}^{QR}\Big)
\end{aligned} \tag{3.72}
$$

Finally, the linearization of (3.50) is

$$
\begin{aligned}
l_{MNPQR} \quad &= \quad \delta\Big(\frac{1}{5!}\sqrt{-g}\varepsilon_{MNPQRS_1S_2S_3S_4S_5} G^{S_1S_2S_3S_4S_5}\Big) \\
&= \quad \frac{1}{5!}\epsilon_{MNPQRS_1S_2S_3S_4S_5}\Big(\frac{1}{2}h_P{}^P\, \overset{\circ}{G}{}^{S_1S_2S_3S_4S_5} + l^{S_1S_2S_3S_4S_5} \\
&- \quad 5h^{T[S_1}\, \overset{\circ}{G}_T{}^{S_2S_3S_4S_5]}\Big)
\end{aligned} \tag{3.73}
$$

Here (3.68), $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{PQ}\delta g_{PQ}$ and

$$
\delta G^{MNPQR} = l^{MNPQR} - 5h^{T[M}\, \overset{\circ}{G}_T{}^{NPQR]} \tag{3.74}
$$

were used.

3.3 The $AdS_4 \times M^7$ bosonic mass operators

In this section, I am going to obtain the $AdS_4 \times M^7$ bosonic mass operators. I will first split the D=11 indices into spacetime $(\mu, \nu, \ldots)$ and internal indices $(m, n, \ldots)$, and insert the Freund-Rubin ansatz (3.6) into the bosonic equations of motion (3.40) and (3.46), followed by substituting the various harmonic expansions of the D=11 fields into the bosonic equations of motion. Then the bosonic mass operators are obtained by diagonalizing the mass matrix for each type of field, mode by mode. Finally, some modes[1] that were excluded in the general analysis will be analyzed.

[1]The massless graviton will be seen to be one of these modes.

I will now drop the superscript "$\circ$", since it is superfluous at the linearized level. Inserting (3.6) into (3.40) and splitting the indices into, $\mu = 0, 1, 2, 3$, and $m = 4, 5, \ldots, 10$ ((3.40) is multiplied with 2 below), the RHS with $\mu\nu$ indices is

$$\begin{aligned} 2T_{\mu\nu} &= 2m\Big(g_{\mu\chi}\epsilon^{\chi\rho\sigma\tau}f_{\nu\rho\sigma\tau} + g_{\nu\chi}\epsilon^{\chi\rho\sigma\tau}f_{\mu\rho\sigma\tau}\Big) - \frac{m}{3}g_{\mu\nu}\epsilon^{\rho\sigma\tau\chi}f_{\rho\sigma\tau\chi} \\ &- 9m^2\Big(g_{\mu\xi}\epsilon^{\xi\rho\sigma\tau}\epsilon_{\nu\chi\sigma\tau}h_\rho{}^\chi + g_{\nu\xi}\epsilon^{\xi\rho\sigma\tau}\epsilon_{\mu\chi\sigma\tau}h_\rho{}^\chi\Big) - \frac{m^2}{2}h_{\mu\nu}\epsilon_{\rho\sigma\tau\chi}\epsilon^{\rho\sigma\tau\chi} \\ &= 24m^2(g_{\mu\nu}h^\rho{}_\rho - h_{\mu\nu}) + \frac{2m}{3}g_{\mu\nu}\epsilon^{\rho\sigma\tau\chi}f_{\rho\sigma\tau\chi} \end{aligned} \tag{3.75}$$

where I have used (B.5), (B.7) and

$$g_{\mu\chi}\epsilon^{\chi\rho\sigma\tau}f_{\nu\rho\sigma\tau} = \frac{1}{4}g_{\mu\nu}\epsilon^{\chi\rho\sigma\tau}f_{\chi\rho\sigma\tau} \tag{3.76}$$

which were obtained from

$$0 = 5g_{\mu[\chi}f_{\nu\rho\sigma\tau]} = 4g_{\mu[\chi}f_{|\nu|\rho\sigma\tau]} - g_{\mu\nu}f_{\chi\rho\sigma\tau} \tag{3.77}$$

Here $|\nu|$ means that ν is not included in the antisymmetrization. This means that (3.40) with $\mu\nu$ indices reads

$$\begin{aligned} &\Delta_L h_{\mu\nu} + 2\nabla_{(\mu}\nabla^\rho h_{\nu)\rho} + 2\nabla_{(\mu}\nabla^p h_{\nu)p} - \nabla_\mu\nabla_\nu h^\rho{}_\rho - \nabla_\mu\nabla_\nu h^p{}_p \\ &= 24m^2(g_{\mu\nu}h^\rho{}_\rho - h_{\mu\nu}) + \frac{2m}{3}g_{\mu\nu}\epsilon^{\rho\sigma\tau\chi}f_{\rho\sigma\tau\chi}, \end{aligned} \tag{3.78}$$

and (3.40) with μn indices is

$$\begin{aligned} &\Delta_L h_{\mu n} + 2\nabla_{(\mu}\nabla^\rho h_{n)\rho} + 2\nabla_{(\mu}\nabla^p h_{n)p} - \nabla_\mu\nabla_n h^\rho{}_\rho - \nabla_\mu\nabla_n h^p{}_p \\ &= 12m^2 h_{\mu n} + 2m\epsilon_\mu{}^{\rho\sigma\tau}f_{n\rho\sigma\tau} \end{aligned} \tag{3.79}$$

where I have used (B.5). (3.40) with mn indices is

$$\begin{aligned} &\Delta_L h_{mn} + 2\nabla_{(m}\nabla^\rho h_{n)\rho} + 2\nabla_{(m}\nabla^p h_{n)p} - \nabla_m\nabla_n h^\rho{}_\rho - \nabla_m\nabla_n h^p{}_p \\ &= 12m^2(h_{mn} - g_{mn}h^\rho{}_\rho) - \frac{m}{3}g_{mn}\epsilon^{\rho\sigma\tau\chi}f_{\rho\sigma\tau\chi} \end{aligned} \tag{3.80}$$

where (B.5) were used.

I will now insert (3.6) into (3.46) and split the indices. The RHS with $\mu\nu\rho$ indices is zero. The LHS is

$$\begin{aligned} S^{\mu\nu\rho} &= \nabla_\tau f^{\tau\mu\nu\rho} + \nabla_t f^{t\mu\nu\rho} + \frac{3m}{2}\epsilon^{\tau\mu\nu\rho}\nabla_\tau(h^\nu{}_\nu + h^n{}_n) - 3m\nabla_\tau(h^\tau{}_\xi\epsilon^{\xi\mu\nu\rho}) \\ &- 3m\nabla_t(h^t{}_\xi\epsilon^{\xi\mu\nu\rho}) - 9m\nabla^\tau(h^{\epsilon[\mu}\epsilon_{\tau\epsilon}{}^{\nu\rho]}) \end{aligned} \tag{3.81}$$

Multiplication with $\epsilon_{\sigma\mu\nu\rho}$ gives

$$\begin{aligned} \epsilon_{\sigma\mu\nu\rho}S^{\mu\nu\rho} &= \epsilon_{\sigma\mu\nu\rho}(\nabla_\tau f^{\tau\mu\nu\rho} + \nabla_t f^{t\mu\nu\rho}) - 9m\nabla_\sigma(h^\nu{}_\nu + h^n{}_n) + 18m\nabla_\tau h^\tau{}_\sigma \\ &+ 18m\nabla_t h^t{}_\sigma + 18m\nabla_\tau h^\mu{}_\epsilon(\delta^\tau_\sigma\delta^\epsilon_\mu - \delta^\tau_\mu\delta^\epsilon_\sigma) \\ &= \epsilon_{\sigma\mu\nu\rho}(\nabla_\tau f^{\tau\mu\nu\rho} + \nabla_t f^{t\mu\nu\rho}) + 9m\nabla_\sigma(h^\nu{}_\nu - h^n{}_n) \\ &+ 18m\nabla_t h^t{}_\sigma \end{aligned} \tag{3.82}$$

where (B.6)-(B.7) were used. This means that (3.46) with "$\mu\nu\rho$" indices is

$$\epsilon_{\sigma\mu\nu\rho}(\nabla_\tau f^{\tau\mu\nu\rho} + \nabla_t f^{t\mu\nu\rho}) + 9m\nabla_\sigma(h^\nu{}_\nu - h^n{}_n) + 18m\nabla_t h^t{}_\sigma = 0 \tag{3.83}$$

(3.46) with $m\nu\rho$ indices is

$$\nabla_\sigma f^{\sigma m\nu\rho} + \nabla_q f^{qm\nu\rho} - 3m\epsilon^{\sigma\tau\nu\rho}\nabla_\sigma h^m{}_\tau = 0, \tag{3.84}$$

and (3.46) with $mn\rho$ indices is

$$\nabla_\sigma f^{\sigma mn\rho} + \nabla_q f^{qmn\rho} = 0. \tag{3.85}$$

Finally, (3.46) with mnp indices is

$$\nabla_\sigma f^{\sigma mnp} + \nabla_q f^{qmnp} = \frac{1}{4}m\epsilon^{mnpqrst}f_{qrst} \tag{3.86}$$

where I have used that

$$\epsilon^{mnpqrst\sigma_1\sigma_2\sigma_3\sigma_4}\epsilon_{\sigma_1\sigma_2\sigma_3\sigma_4} = -4!\epsilon^{mnpqrst} \tag{3.87}$$

I am now going to make a harmonic expansion of the fields $h_{MN}(x,y)$ and $a_{MNP}(x,y)$. The expansions will take a simple form if the following gauge conditions are imposed

$$\nabla^m h_{m\nu} = 0 \tag{3.88}$$

$$\nabla^m(h_{mn} - \frac{1}{7}g_{mn}h^p{}_p) = 0 \tag{3.89}$$

$$\nabla^m a_{mNP} = 0 \tag{3.90}$$

These gauge conditions give the following harmonic expansions of $h_{MN}(x,y)$ and $a_{MNP}(x,y)$ [12]

$$h_{\mu\nu}(x,y) = h_{\mu\nu}(x)Y(y) \tag{3.91}$$

$$h_{\mu n}(x,y) = B_\mu(x)Y_n(y) \tag{3.92}$$

$$h_{mn}(x,y) = S(x)X_{mn}(y) + \frac{1}{7}g_{mn}\hat{S}(x)Y(y) \tag{3.93}$$

$$a_{\mu\nu\rho}(x,y) = a_{\mu\nu\rho}(x)Y(y) \tag{3.94}$$

$$a_{\mu\nu p}(x,y) = a_{\mu\nu}(x)Y_p(y) \tag{3.95}$$

$$a_{\mu np}(x,y) = a_\mu(x)Y_{np}(y) \tag{3.96}$$

$$a_{mnp}(x,y) = a(x)Y_{mnp}(y) \tag{3.97}$$

Here the p-forms $Y_{m_1\ldots m_p}(y)$ are transverse modes of Δ_p, which is the Hodge-de Rham operator on p-forms, see (B.37)-(B.40). Y is a spherical harmonic, Y_m is a vector spherical harmonic, Y_{mn} and Y_{mnp} are tensor spherical harmonics. X_{mn} are symmetric, transverse traceless modes of the Lichnerowicz operator Δ_L, see (B.23). I have suppressed all summation symbols and the corresponding isometry indices in the RHS of (3.91)-(3.97). From now on, the Lichnerowicz operator and the Hodge-de Rham operator are both denoted Δ. It will be clear from the context if Δ is the Hodge-de Rham operator acting on p-forms, or the Lichnerowicz operator acting on a symmetric, transverse traceless tensor. I will also drop the explicit coordinate dependence, because in the background metric (3.3) all operators are splitting up in a sum of one operator on AdS_4, and one on M^7. It can be seen from the gauge condition (3.90) that the p-forms $Y_{m_1 \dots m_p}$ must be transverse, because

$$\nabla^m a_{mNP} = 0 \Rightarrow \nabla^{m_1} Y_{m_1 \dots m_p} = 0, \quad p = 1, 2, 3 \tag{3.98}$$

I am now going to start analyzing the field equations for a_{MNP}. Inserting (3.95)-(3.97) into (3.85) and lowering the indices gives

$$\begin{aligned} 0 &= 4\nabla^\sigma \partial_{[\sigma} a_{mn\rho]} + 4\nabla^q \partial_{[q} a_{mn\rho]} = 2\nabla^\sigma \nabla_{[\sigma} a_{\rho]mn} + 2\nabla^\sigma \nabla_{[n} a_{m]\rho\sigma} \\ &+ 3\nabla^q \nabla_{[q} a_{mn]\rho} - \nabla^q \nabla_\rho a_{qmn} = 2\nabla^\sigma \nabla_{[\sigma} a_{\rho]} Y_{mn} + 2\nabla_{[n} Y_{m]} \nabla^\sigma a_{\rho\sigma} \\ &+ 3\nabla^q \nabla_{[q} Y_{mn]} a_\rho - \nabla^q Y_{qmn} \nabla_\rho a = -(\Delta a_\rho + \nabla_\rho \nabla^\sigma a_\sigma) Y_{mn} - (\Delta Y_{mn}) a_\rho \\ &+ 2\nabla_{[n} Y_{m]} \nabla^\sigma a_{\rho\sigma} \end{aligned} \tag{3.99}$$

where I have used (3.98), (B.41), (B.45) and

$$\partial_{[M} a_{NPQ]} = \nabla_{[M} a_{NPQ]} \tag{3.100}$$

which is valid because Γ^P_{MN} is symmetric in the lower indices. I will now split (3.99) in two equations because the two first terms are orthogonal to the third term, see (B.50). This gives

$$(\Delta a_\rho) Y_{mn} + (\Delta Y_{mn}) a_\rho = 0 \tag{3.101}$$

$$\nabla^\sigma a_{\rho\sigma} = 0 \tag{3.102}$$

where I have used (3.108) below in (3.101).

Inserting (3.92) and (3.94)-(3.96) into (3.84) and lowering the indices gives

$$\begin{aligned} 0 &= 4\nabla^\sigma \nabla_{[\sigma} a_{m\nu\rho]} + 4\nabla^q \nabla_{[q} a_{m\nu\rho]} - 3m\epsilon_{\sigma\tau\nu\rho} \nabla^\sigma B^\tau Y_m \\ &= 3\nabla^\sigma \nabla_{[\sigma} a_{\nu\rho]} Y_m - \nabla_m Y \nabla^\sigma a_{\nu\rho\sigma} + 2\nabla^q \nabla_{[q} Y_{m]} a_{\nu\rho} - 3m\epsilon_{\sigma\tau\nu\rho} \nabla^\sigma B^\tau Y_m \\ &= -(\Delta a_{\nu\rho}) Y_m - (\Delta Y_m) a_{\nu\rho} - 3m\epsilon_{\sigma\tau\nu\rho} \nabla^\sigma B^\tau Y_m - \nabla_m Y \nabla^\sigma a_{\nu\rho\sigma} \end{aligned} \tag{3.103}$$

where I have used (3.98), (3.100), (3.102), (B.42) and (B.45). This equation can be split into two equations due to orthogonality of the various mode functions, see (B.50). Splitting the equation gives

$$(\Delta a_{\nu\rho}) Y_m + (\Delta Y_m) a_{\nu\rho} + 3m\epsilon_{\sigma\tau\nu\rho} \nabla^\sigma B^\tau Y_m = 0 \tag{3.104}$$

$$\nabla^\sigma a_{\nu\rho\sigma} = 0 \tag{3.105}$$

Inserting (3.96)-(3.97) into (3.86) and lowering the indices gives

$$\begin{aligned} 0 &= 4\nabla^\sigma \nabla_{[\sigma} a_{mnp]} + 4\nabla^q \nabla_{[q} a_{mnp]} - m\epsilon_{mnpqrst} \nabla^{[q} a^{rst]} \\ &= \nabla^\sigma \nabla_\sigma a Y_{mnp} - 3\nabla^\sigma a_\sigma \nabla_{[m} Y_{np]} + 4\nabla^q \nabla_{[q} Y_{mnp]} a - m\epsilon_{mnpqrst} \nabla^q a^{rst} \\ &= -(\Delta a) Y_{mnp} - (\Delta Y_{mnp}) a - m\epsilon_{mnpqrst} \nabla^q a^{rst} - 3\nabla^\sigma a_\sigma \nabla_{[m} Y_{np]} \end{aligned} \tag{3.106}$$

Here (3.100), (B.37) and (B.48) have been used. Splitting the equation gives

$$\begin{aligned}(\Delta a)Y_{mnp} + (\Delta Y_{mnp})a + m\epsilon_{mnpqrst}\nabla^q a^{rst} &= 0 & (3.107)\\ \nabla^\sigma a_\sigma &= 0 & (3.108)\end{aligned}$$

The equations (3.101)-(3.102), (3.104)-(3.105) and (3.107)-(3.108) are valid for all modes, except the zero eigenvalue of zero-, one- and two-forms, i.e, modes that satisfy $\Delta Y = 0$, $\Delta Y_m = 0$ or $\Delta Y_{mn} = 0$. The first case $\Delta Y = 0$ always have a solution on any compact internal space, and the solution is just $Y = constant$. The $\Delta Y_m = 0$ case never have non-trivial solutions on a positive curvature Einstein space, which can be seen from

$$\Delta Y_m = -\Box Y_m + R_m{}^n Y_n = (-\Box + 6m^2)Y_m > 0, \quad Y_m \neq 0 \tag{3.109}$$

where $-\Box$ is positive semidefinite. This also means that no such internal space have non-zero first betti number b_1, (see appendix B.4). The third case $\Delta Y_{mn} = 0$ only have non-trivial solutions when the internal space have non-zero second betti number b_2. Modes corresponding to the first or the third case will be treated after the general analysis.

I will now insert (3.91)-(3.95) into (3.83) and lower the indices. This gives

$$\begin{aligned}0 &= \epsilon^{\mu\nu\rho\sigma}(4\nabla^\tau\nabla_{[\tau}a_{\nu\rho\sigma]} + 4\nabla^t\nabla_{[t}a_{\nu\rho\sigma]}) + 9m\nabla^\mu(h^\nu{}_\nu - h^n{}_n) + 18m\nabla_n h^{n\mu}\\ &= \epsilon^{\mu\nu\rho\sigma}(4\nabla^\tau\nabla_{[\tau}a_{\nu\rho\sigma]}Y + \nabla^t\nabla_t Y a_{\nu\rho\sigma}) + 9m\nabla^\mu(h^\nu{}_\nu - \hat{S})Y\\ &= \epsilon^{\mu\nu\rho\sigma}(-(\Delta a_{\nu\rho\sigma})Y - (\Delta Y)a_{\nu\rho\sigma}) + 9m\nabla^\mu(H - \hat{S})Y & (3.110)\end{aligned}$$

where I have used (3.88), (3.98), (3.100), (B.37), (B.48) and $H = h^\nu{}_\nu$, i.e, H is the $\eta_{\mu\nu}$-trace of $h_{\mu\nu}$. I am now defining a new field θ, which is defined as

$$\partial_\mu\theta = m\epsilon_{\mu\nu\rho\sigma}a^{\nu\rho\sigma} \tag{3.111}$$

Inserting (3.111) into (3.110) gives

$$(\Delta\theta)Y + (\Delta Y)\theta - 9m^2(H - \hat{S})Y = 0 \tag{3.112}$$

where I have used that Δ commutes with ∇_μ. This last statement follows easily if we recall that $\Delta = d\delta + \delta d$.

I am now going to analyze the field equations for g_{MN}. Inserting (3.91)-(3.94) into (3.80) gives the following result for the LHS (which I will call G_{mn})

$$G_{mn} = \Delta(SX_{mn} + \frac{1}{7}g_{mn}\hat{S}Y) - \frac{1}{7}(5\hat{S} + 7H)\nabla_m\nabla_n Y + 2\nabla_{(m}Y_{n)}\nabla^\rho B_\rho \tag{3.113}$$

where I have used

$$\nabla^n h_{np} = \frac{1}{7}\hat{S}\nabla_n Y \tag{3.114}$$

The RHS is

$$T_{mn} = 12m^2(SX_{mn} + \frac{1}{7}g_{mn}\hat{S}Y - g_{mn}HY) + \frac{4}{3}g_{mn}(\Delta\theta)Y \tag{3.115}$$

where (3.111) and (B.37) were used. I will now take the trace of the equation that results from (3.113) and (3.115). The trace part will give one equation and the traceless part will give another one. The trace part gives

$$(\Delta\hat{S})Y + (\Delta Y)H + \frac{12}{7}\hat{S}(\Delta Y) = 12m^2(\hat{S} - 7H)Y + \frac{28}{3}(\Delta\theta)Y \tag{3.116}$$

where I have used that X_{mn} is traceless, (3.98) and (B.37). The traceless part gives

$$(\Delta S)X_{mn} + (\Delta X_{mn})S + 2\nabla_{(m}Y_{n)}\nabla^\rho B_\rho = 12m^2 S X_{mn} \tag{3.117}$$

This equation can be split into two equations because X_{mn} and $\nabla_m Y_n$ are orthogonal[2]. The two equations are

$$(\Delta S)X_{mn} + (\Delta X_{mn})S = 12m^2 S X_{mn} \tag{3.118}$$
$$\nabla^\rho B_\rho = 0 \tag{3.119}$$

I will now put (3.116) and (3.118)-(3.119) back into (3.113) and (3.115) and take the divergence of the resulting equation. This will produce one new equation and a few new modes that will be excluded for now, and analyzed later. Inserting (3.116) and (3.118)-(3.119) into (3.113) and (3.115) gives

$$\frac{1}{7}g_{mn}(\Delta Y)\hat{S} - \nabla_m\nabla_n Y\Big(\frac{5}{7}\hat{S} + H\Big) = \frac{1}{7}g_{mn}(\Delta Y)\Big(H + \frac{12}{7}\hat{S}\Big) \tag{3.120}$$

The divergence of this equation gives

$$(7H + 5\hat{S})\Big(\Box(\nabla_n Y) + \frac{1}{7}\nabla_n(\Delta Y)\Big) = 0 \tag{3.121}$$

This equation can be rewritten with the help of

$$\Box(\nabla_n Y) = [\nabla^m, \nabla_n]\nabla_m Y + \nabla_n(\Box Y) = 6m^2\nabla_n Y - \nabla_n(\Delta Y) \tag{3.122}$$

where (B.37) and (B.43) were used. Inserting this equation into (3.121) gives

$$(7H + 5\hat{S})\nabla_n\Big(\Delta Y - 7m^2 Y\Big) = 0 \tag{3.123}$$

The implication of this equation is either that $\Delta Y = 7m^2 Y$ or

$$7H + 5\hat{S} = 0, \quad \Delta Y \neq 7m^2 Y \tag{3.124}$$

This means that we have more modes that has to be excluded, i.e, modes that satisfy $\Delta Y = 7m^2 Y$ or $\Delta Y = 0$, the latter because $Y = \textit{constant} \Rightarrow \nabla_n Y = 0$. Finally, modes that satisfy $\Delta Y_n = 12m^2 Y_n$ must also be excluded. This happens because when we take the divergence of (3.113) and (3.115), $\nabla^\rho B_\rho = 0$ is not possible to derive, if the following equation is satisfied

$$0 = 2\nabla^m\nabla_{(m}Y_{n)} = \Box Y_n + \nabla^m\nabla_n Y_m = (-\Delta + 12m^2)Y_n \tag{3.125}$$

[2] This can be seen from (B.50) using that X_{mn} is transverse.

where (B.37) and (B.43) have been used. This means that we now have to exclude all modes that satisfy any of the following equations

$$\Delta Y = 0, \quad \Delta Y = 7m^2 Y, \quad \Delta Y_m = 12m^2 Y_m, \quad \Delta Y_{mn} = 0 \tag{3.126}$$

I am now going to analyze (3.79). Inserting (3.91)-(3.95) gives the following LHS

$$G_{\mu n} = (\Delta B_\mu) Y_n + (\Delta Y_n) B_\mu - \nabla_\mu \Big(\frac{6}{7}\hat{S} + H\Big) \nabla_n Y + \nabla^\rho h_{\mu\rho} \nabla_n Y \tag{3.127}$$

where I have used (3.98), (3.114) and (3.119). The RHS is

$$\begin{aligned} T_{\mu n} &= 12m^2 B_\mu Y_n + 2m \epsilon_\mu{}^{\rho\sigma\tau} (4\nabla_{[n} a_{\rho\sigma\tau]}) = 12m^2 B_\mu Y_n + 2m\epsilon_{\mu\rho\sigma\tau} (\nabla_n Y a^{\rho\sigma\tau} \\ &- 3\nabla^{[\rho} a^{\sigma\tau]} Y_n) = 12m^2 B_\mu Y_n + 2\partial_\mu \theta \nabla_n Y - 6m\epsilon_{\mu\rho\sigma\tau} \nabla^\rho a^{\sigma\tau} Y_n \end{aligned} \tag{3.128}$$

where I have used (3.100) and (3.111). The equation that results from (3.127) and (3.128) can be split into two equations, because Y_n and $\nabla_n Y$ are orthogonal, see (B.50). The two equations are

$$(\Delta B_\mu) Y_n + (\Delta Y_n) B_\mu = 12m^2 B_\mu Y_n - 6m\epsilon_{\mu\rho\sigma\tau} \nabla^\rho a^{\sigma\tau} Y_n \tag{3.129}$$

$$\nabla^\rho h_{\mu\rho} = \nabla_\mu \Big(\frac{6}{7}\hat{S} + H + 2\theta\Big) \tag{3.130}$$

Before I analyze the last equation of motion I will decompose $h_{\mu\nu}$. The decomposition of $h_{\mu\nu}$ is [12]

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \nabla_{(\mu} V^T_{\nu)} + (\nabla_\mu \nabla_\nu + \frac{1}{4} g_{\mu\nu}\Delta)\tilde{\phi} + \frac{1}{4} g_{\mu\nu} H \tag{3.131}$$

Here V^T_μ is transverse and $h^{TT}_{\mu\nu}$ is transverse and traceless. Using this equation and (3.130) makes it possible to show that $V^T_\mu = 0$, which is done in the following way

$$-\frac{1}{2}(\Delta + 12m^2) V^T_\mu + \nabla^\nu (\nabla_\mu \nabla_\nu + \frac{1}{4} g_{\mu\nu} \Delta)\tilde{\phi} + \frac{1}{4}\nabla_\mu H = \nabla_\mu \Big(\frac{6}{7}\hat{S} + H + 2\theta\Big) \tag{3.132}$$

where I have used that $h^{TT}_{\mu\nu}$ and V^T_μ are transverse and (B.43). The first term in this equation is orthogonal with respect to the other terms, which gives a new equation[3] with only V^T_μ in it. The equation is

$$(\Delta + 12m^2) V^T_\mu = 0 \rightarrow V^T_\mu = 0 \tag{3.133}$$

where I have integrated the LHS on AdS_4 and used that the surface term is zero.

Finally, it is time to analyze the last equation of motion (3.78). Inserting (3.91)-(3.94) gives

$$\begin{aligned} &(\Delta h_{\mu\nu}) Y + (\Delta Y) h_{\mu\nu} + 2\nabla_{(\mu} \nabla^\rho h_{\nu)\rho} Y - \nabla_\mu \nabla_\nu (H + \hat{S}) \\ &= -\frac{8}{3} g_{\mu\nu} (\Delta\theta) Y + 24m^2 (g_{\mu\nu} H - h_{\mu\nu}) Y \end{aligned} \tag{3.134}$$

[3]The orthogonality gives of course two equations, but only the one with V^T_μ gives anything interesting.

where I have used (3.98), (3.119) and (B.37). This equation will give a total of three equations; one from the trace, one from the divergence and one from the traceless part. Taking the trace gives

$$2(\Delta H)Y + (\Delta Y)H + (\Delta\hat{S})Y + 2m^2GY = 72m^2HY - \frac{32}{3}(\Delta\theta)Y \tag{3.135}$$

where $m^2G = \nabla^\mu\nabla^\nu h_{\mu\nu}$. Taking the divergence of (3.134) gives

$$\begin{aligned} & \nabla^\mu(\Delta h_{\mu\nu})Y + (\Delta Y)\nabla^\mu h_{\mu\nu} + \Box\nabla^\rho h_{\nu\rho}Y + \nabla^\mu\nabla_\nu\nabla^\rho h_{\mu\rho}Y - \Box\nabla_\nu(H+\hat{S})Y \\ = \; & -\frac{8}{3}\nabla_\nu(\Delta\theta)Y + 24m^2(\nabla_\nu H - \nabla^\mu h_{\mu\nu})Y \end{aligned} \tag{3.136}$$

This equation can be rewritten by using

$$\Box\nabla_\nu H = [\nabla^\mu, \nabla_\nu]\nabla_\mu H + \nabla_\nu\Box H = -\nabla_\nu(\Delta + 12m^2)H \tag{3.137}$$

and

$$\nabla^\mu\nabla_\nu\nabla^\rho h_{\mu\rho} = m^2\nabla_\nu G - 12m^2\nabla^\mu h_{\mu\nu} \tag{3.138}$$

where (3.8) and (B.43) were used. These two equations inserted in (3.136) gives

$$\begin{aligned} & \nabla^\mu h_{\mu\nu}(\Delta Y) + \nabla_\nu[(\Delta - 12m^2)H + (\Delta + 12m^2)\hat{S} + m^2G + \frac{8}{3}\Delta\theta]Y \\ = \; & -\nabla^\mu(\Delta h_{\mu\nu} + \nabla_\mu\nabla^\rho h_{\nu\rho} + 12m^2h_{\mu\nu})Y \end{aligned} \tag{3.139}$$

I am now going to show that the RHS of this equation is zero. The RHS (excluding Y) multiplied with -1 is

$$\begin{aligned} & \nabla^\mu(-\Box h_{\mu\nu} - 2R_{\mu\rho\nu\sigma}h^{\rho\sigma} + R_{(\mu}{}^\rho h_{\nu)\rho} + \nabla_\mu\nabla^\rho h_{\nu\rho}) \\ = \; & -\nabla^\mu\Box h_{\mu\nu} + \nabla^\mu\Box h_{\mu\nu} + R_{\mu\rho\nu\sigma}\nabla^\mu h^{\rho\sigma} - \nabla^\mu(R_{\mu\rho\nu\sigma}h^{\rho\sigma}) \\ = \; & R_{\mu\rho\nu\sigma}\nabla^\mu h^{\rho\sigma} - R_{\mu\rho\nu\sigma}\nabla^\mu h^{\rho\sigma} = 0 \end{aligned} \tag{3.140}$$

where I have used

$$\Box\nabla^\rho h_{\nu\rho} = \nabla^\mu(R_{\mu\rho\nu\sigma}h^{\rho\sigma} - R_{(\mu}{}^\rho h_{\nu)\rho} + \nabla^\rho\nabla_\mu h_{\nu\rho}) \tag{3.141}$$

and

$$\begin{aligned} \nabla^\mu\nabla^\rho\nabla_\mu h_{\nu\rho} &= \nabla^\mu\Box h_{\mu\nu} + R_{\mu\sigma}(\nabla^\sigma h_\nu{}^\mu - \nabla^\mu h_\nu{}^\sigma) + R_{\mu\rho\nu\sigma}\nabla^\mu h^{\rho\sigma} \\ &= \nabla^\mu\Box h_{\mu\nu} + R_{\mu\rho\nu\sigma}\nabla^\mu h^{\rho\sigma} \end{aligned} \tag{3.142}$$

In these two equations (B.46) was used in (3.141), and (B.49) in (3.142). I have also used $R_{\mu\rho\nu\sigma} \sim (g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\rho\nu})$ (see appendix D) and the metric postulate $\nabla_\mu g_{\nu\rho} = 0$ in (3.140). The result of taking the divergence of (3.134) is therefore

$$\nabla^\mu h_{\mu\nu}(\Delta Y) + \nabla_\nu[(\Delta - 12m^2)H + (\Delta + 12m^2)\hat{S} + m^2G + \frac{8}{3}\Delta\theta]Y = 0 \tag{3.143}$$

Now to the traceless part, using (3.131) and (3.133) in the traceless part of (3.134) gives

$$\begin{aligned}
& \Delta[h^{TT}_{\mu\nu} + (\nabla_\mu\nabla_\nu + \frac{1}{4}g_{\mu\nu}\Delta)\tilde{\phi}]Y + (\Delta Y)[h^{TT}_{\mu\nu} + (\nabla_\mu\nabla_\nu + \frac{1}{4}g_{\mu\nu}\Delta)\tilde{\phi}]Y \\
+ & \nabla_\mu\nabla^\rho(\nabla_\nu\nabla_\rho + \frac{1}{4}g_{\nu\rho}\Delta)\tilde{\phi}Y + \nabla_\nu\nabla^\rho(\nabla_\mu\nabla_\rho + \frac{1}{4}g_{\mu\rho}\Delta)\tilde{\phi}Y \\
= & -24m^2[h^{TT}_{\mu\nu} + (\nabla_\mu\nabla_\nu + \frac{1}{4}g_{\mu\nu}\Delta)\tilde{\phi}]Y \qquad (3.144)
\end{aligned}$$

This equation splits into two equations, because the $h^{TT}_{\mu\nu}$ terms are orthogonal with respect to the $\tilde{\phi}$ terms. The interesting equation is

$$(\Delta h^{TT}_{\mu\nu})Y + (\Delta Y)h^{TT}_{\mu\nu} + 24m^2 h^{TT}_{\mu\nu}Y = 0 \qquad (3.145)$$

The reason that we have obtained this equation and not a similar equation for $h_{\mu\nu}$, is that it is only the transverse part of the field that is physical[4]. The spin 2 field $h_{\mu\nu}$ must also be traceless because the trace part H is a spin 0 field, which is treated separately.

I am now going to obtain the bosonic mass operators for the different fields, by using the definitions[5] of the mass matrix M below. All modes that satisfy any of the equations in (3.126) are excluded for now, but will be analyzed later after the general analysis. The definition for spin 2 fields $h_{\mu\nu}$ are

$$\Delta_L h_{\mu\nu} + 2\nabla_{(\mu}\nabla^\rho h_{\nu)\rho} - \nabla_{(\mu}\nabla_{\nu)}h^\rho{}_\rho + 24m^2 h_{\mu\nu} + M^2 h_{\mu\nu} = 0 \qquad (3.146)$$

and the definition for spin 1 fields B_μ are

$$\Delta B_\mu + \nabla_\mu\nabla^\rho B_\rho + M^2 B_\mu = 0 \qquad (3.147)$$

and finally, the definition for spin 0 fields S are

$$\Delta S - 8m^2 S + M^2 S = 0 \qquad (3.148)$$

The factor $-8m^2$ is included because I want $M^2 = 0$ to correspond to massless scalar fields[6].

I will begin with (3.145). Comparing this equation with the mass matrix definition (3.146), where terms two and three are zero because $h^{TT}_{\mu\nu}$ is transverse and traceless, gives that the mass operator for the spin 2 field $h^{TT}_{\mu\nu}$ is Δ_0.

I will now analyze (3.101). Comparing this equation with the mass matrix definition (3.147), where term 2 is zero because of (3.108), implies that the mass operator for the spin 1 field a_ρ is Δ_2.

Equation (3.118) is analyzed as follows. (3.118) can be rewritten as

$$(\Delta S)X_{mn} - 8m^2 SX_{mn} + S(\Delta - 4m^2)X_{mn} = 0 \qquad (3.149)$$

[4]This is also true for vector fields, i.e, spin 1 fields.
[5]The definitions are taken from [12].
[6]See the discussion about mass, i.e., at the end of this section.

Comparing this equation with the mass matrix definition (3.148), gives that $\Delta_L - 4m^2$ is the mass operator corresponding to the spin 0 field S.

The towers of modes corresponding to any of these three fields are denoted 2^+, 1^+ or $0^{+(2)}$ in table 3.1.

I am now going to analyze (3.104) and (3.129), which are two coupled equations, and therefore has to be solved together by diagonalization. But before that, I will first rename B_μ to B_μ^T, which is done because (3.119) gives B_μ transverse. I will also define a new field $\hat{B}_\mu^T$ as

$$\hat{B}_\mu^T = \frac{1}{m}\epsilon_{\mu\nu\rho\sigma}\nabla^\nu a^{\rho\sigma} \tag{3.150}$$

Raising the indices and multiplying (3.104) with $\frac{1}{m}\epsilon_{\mu\nu\rho\sigma}\nabla^\nu$ and inserting (3.150) into (3.129) gives the following coupled equations

$$\begin{aligned} (\Delta\hat{B}_\mu^T)Y_m + (\Delta Y_m)\hat{B}_\mu^T - 6(\Delta B_\mu^T)Y_m &= 0 && (3.151)\\ (\Delta B_\mu^T)Y_m + (\Delta Y_m)B_\mu^T - 12m^2 B_\mu^T Y_m + 6m^2\hat{B}_\mu^T Y_m &= 0 && (3.152)\end{aligned}$$

where I have used (B.7) and that Δ commutes with ∇^ν. Using (3.152) in (3.151) gives the following system of coupled equations

$$Y_m\Delta\begin{pmatrix}\hat{B}_\mu^T\\ B_\mu^T\end{pmatrix} + \begin{pmatrix}\Delta + 36m^2 & 6\Delta - 72m^2\\ 6m^2 & \Delta - 12m^2\end{pmatrix} Y_m \begin{pmatrix}\hat{B}_\mu^T\\ B_\mu^T\end{pmatrix} = 0 \tag{3.153}$$

If we compare this matrix equation with (3.147), where the second term is zero because the fields are transverse, we see that the eigenvalues to the two by two matrix gives the mass operators of the spin-1 fields. Calculating the eigenvalues gives the following mass operators

$$\Delta_1 + 12m^2 \pm 6m\sqrt{\Delta_1 + 4m^2} \tag{3.154}$$

In table 3.1, I have called the towers of modes that correspond to either of these mass operators $1^{-(1)}$ or $1^{-(2)}$, where (1) refers to the minus sign and (2) to the plus sign.

I will now start analyzing (3.107), by first rewriting the equation in the following way

$$(\Delta a)Y_{mnp} - 8m^2 aY_{mnp} + a(\Delta + 8m^2)Y_{mnp} + ma\epsilon_{mnpqrst}\nabla^q Y^{rst} = 0 \tag{3.155}$$

This equation looks very similar to (3.148) except for the last term in (3.155), which makes it harder to find the mass operator. To solve this tricky situation, I have to use the following equation [12]

$$QY_3 = \mu Y_3 \tag{3.156}$$

where Q is an operator, μ is the eigenvalue to Q and the index on Y_3 means that Y_3 is a 3-form. Writing this equation in component form gives

$$\frac{1}{6}\epsilon_{mnpqrst}\nabla^q Y^{rst} = \mu Y_{mnp} \tag{3.157}$$

Equation (3.156) implies that $\Delta_3 Y_{mnp} = \mu^2 Y_{mnp}$, which is shown in the following way

$$\begin{aligned} Q^2 Y_{mnp} &= \frac{1}{36}\epsilon_{mnpqrst}\nabla^q \epsilon^{rstuvxy}\nabla_u Y_{vxy} = 4\delta^{uvxy}_{[mnpq]}\nabla^q\nabla_u Y_{vxy} \\ &= 4\nabla^q\nabla_{[m}Y_{npq]} = \Delta_3 Y_{mnp} \end{aligned} \tag{3.158}$$

where I have used (B.7) and (B.48). Identifying (3.157)-(3.158) in (3.155) makes it easy to obtain the mass operator, which is

$$Q^2 + 6mQ + 8m^2 \tag{3.159}$$

In table 3.1 the tower of modes that corresponds to this mass operator are called $0^{-(1)}$ and $0^{-(2)}$, where (1) refers to the negative part of the spectrum of Q and (2) to the positive part.

The last two bosonic mass operators will be obtained from (3.112), (3.116) and (3.124)[7]. Inserting (3.124) into (3.112) and (3.116) gives two coupled equations, which are

$$(\Delta\theta)Y + (\Delta Y)\theta + \frac{108}{7}m^2\hat{S}Y = 0 \tag{3.160}$$

$$(\Delta\hat{S})Y + (\Delta Y)\hat{S} - 72m^2\hat{S}Y - \frac{28}{3}(\Delta\theta)Y = 0 \tag{3.161}$$

Rewriting (3.161) by using (3.160) gives

$$Y\Delta\begin{pmatrix}\theta \\ \hat{S}\end{pmatrix} - 8m^2 Y\begin{pmatrix}\theta \\ \hat{S}\end{pmatrix} + \begin{pmatrix}\Delta + 8m^2 & \frac{108}{7}m^2 \\ \frac{28}{3}\Delta & \Delta + 80m^2\end{pmatrix} Y\begin{pmatrix}\theta \\ \hat{S}\end{pmatrix} = 0 \tag{3.162}$$

If we compare this matrix equation with (3.148), we see that the mass operators correspond to the eigenvalues of the two by two matrix. Calculating the eigenvalues gives the following mass operators

$$\Delta_0 + 44m^2 \pm 12m\sqrt{\Delta_0 + 9m^2} \tag{3.163}$$

In table 3.1, I have called the towers of modes that correspond to either of these mass operators $0^{+(1)}$ or $0^{+(3)}$, where (1) refers to the minus sign and (3) to the plus sign.

Finally, I am going to show that the equations (3.130), (3.135) and (3.143) gives no new mass operators, but are satisfied. Inserting (3.124) and (3.130) into (3.143) and (3.124) into (3.135) gives

$$\hat{S}(\Delta Y) + 14\theta(\Delta Y) + 2(\Delta\hat{S})Y + 144m^2\hat{S}Y + 7m^2GY + \frac{56}{3}(\Delta\theta)Y = 0 \tag{3.164}$$

$$-3(\Delta\hat{S})Y - 5\hat{S}(\Delta Y) + 14m^2GY + 360m^2\hat{S}Y + \frac{224}{3}(\Delta\theta)Y = 0 \tag{3.165}$$

[7] These three equations give only two mass operators because (3.124) is an algebraic equation.

Multiplying (3.164) with 2, subtracting (3.165) and using (3.161) (or (3.160)), gives an equation that is exactly the same as equation (3.160) (or (3.161)). This implies that the equations (3.130), (3.135) and (3.143) gives no new mass operators, but are satisfied.

Now it is time to analyze the modes that were excluded in the analysis above. I will begin with M^7 spaces that have $b_2 \neq 0$, i.e, non-trivial solutions to $\Delta Y_{mn} = 0$. The implications of this are that the condition $\nabla^\mu a_\mu = 0$ is no longer derivable and that we get a gauge invariant field equation for a_μ, which is

$$\Delta Y_{mn} = 0 \to \Delta a_\mu + \nabla_\mu \nabla^\rho a_\rho + M^2 a_\mu = 0 \tag{3.166}$$

where $M^2 = 0$. But 1^+ also gives $M^2 = 0$, if $\Delta Y_{mn} = 0$, which implies that the modes corresponding to $\Delta Y_{mn} = 0$ can be included in 1^+.

Next, it is time to analyze modes that satisfy $\Delta Y = 0$. The graviton[8] case is analogous to the case treated above. $Y = constant$ gives

$$\Delta h^{TT}_{\mu\nu} + 24m^2 h^{TT}_{\mu\nu} + M^2 h^{TT}_{\mu\nu} = 0 \tag{3.167}$$

where $M^2 = 0$, which is also given by 2^+, if $\Delta Y = 0$. The implication of this, is that the mode satisfying $\Delta Y = 0$ can be included in 2^+.

I will now consider $0^{+(1),(3)}$ when $\Delta Y = 0$. In this case it is not possible to derive (3.124). Inserting $\Delta Y = 0$ in (3.112), (3.116), (3.135) and (3.143) and combining (3.135) and (3.143) gives

$$\begin{aligned} \Delta\theta - 9m^2(H - \hat{S}) &= 0 && (3.168)\\ \Delta\hat{S} - 12m^2(\hat{S} - 7H) - \frac{28}{3}\Delta\theta &= 0 && (3.169)\\ \Delta\hat{S} + 24m^2(\hat{S} + 2H) - \frac{16}{3}\Delta\theta &= 0 && (3.170) \end{aligned}$$

I am now going to show that we have to impose an extra gauge condition to get complete gauge fixing. Variation of $h_{\mu\nu}$ with respect to the gauge parameter gives (with Y constant)

$$\delta h_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} \Rightarrow \delta h_\mu{}^\mu = \delta H = 2\nabla^\mu \xi_\mu \tag{3.171}$$

The question now is if $\nabla^\mu \xi_\mu$ is completely undetermined by all gauge conditions. Variation of $h_{\mu n}$ gives

$$\delta h_{\mu n} = \nabla_\mu \xi_n + \nabla_n \xi_\mu \tag{3.172}$$

Using (3.88) and (3.177) below gives that both the RHS and LHS of (3.172) after taking the divergence are zero. If we now split the gauge parameter ξ_μ in one transverse component ξ^T_μ and its orthogonal complement $\nabla_\mu \xi$ and insert this into (3.171) we get

$$\delta H(x) = -2\Delta\xi \tag{3.173}$$

[8]The reason that I call this the graviton case, is because the mode that satisfy $\Delta Y = 0$ is the massless graviton, which is the particle that mediates gravity.

This equation gives two cases: $\Delta \neq 0$ and $\Delta = 0$. The first case implies that we can gauge H to zero. Inserting $H = 0$ in (3.168)-(3.170) and solving (3.168) for θ and inserting this result in (3.169)-(3.170) gives the same equation for both equations. The equation is

$$\Delta\hat{S} - 8m^2\hat{S} + M^2\hat{S} = 0 \tag{3.174}$$

where $M^2 = 80m^2$. Finally, the second case gives

$$H = \hat{S}, \quad \hat{S} = 7H, \quad 2H + \hat{S} = 0 \Rightarrow H = \hat{S} = 0 \tag{3.175}$$

We see here that $H = 0$ is all right in both cases. If we compare (3.174) with $0^{+(3)}$ we see that they give the same M^2, i.e, $0^{+(3)}$ give $M^2 = 80m^2$, if $\Delta Y = 0$. The implication of this, is that modes corresponding to $\Delta Y = 0$ can be included in $0^{+(3)}$, but not in $0^{+(1)}$.

Next, I turn to the case $1^{-(1),(2)}$ when $\Delta Y_m = 12m^2 Y_m$. In this case it is not possible to derive the condition $\nabla^\mu B_\mu = 0$. This means that we have to add an extra gauge condition to get complete gauge fixing. This can be seen from varying the fields h_{mn} with respect to the gauge parameter, which gives

$$\delta h_{mn}(x,y) = \nabla_m \xi_n(x,y) + \nabla_n \xi_m(x,y) = 2\epsilon\nabla_{(m}Y_{n)} + 2\tilde{\epsilon}\nabla_{(m}\nabla_{n)}Y \tag{3.176}$$

where I have used

$$\xi_m(x,y) = \epsilon(x)Y_m(y) + \tilde{\epsilon}(x)\nabla_m Y \tag{3.177}$$

Using the gauge condition (3.89) gives

$$\begin{aligned} 0 &= \nabla^m(\delta h_{mn} - \frac{1}{7}g_{mn}\delta h^m{}_m) = \epsilon(\Box Y_n + \nabla^m\nabla_n Y_m) + 2\tilde{\epsilon}(\nabla^m\nabla_n\nabla_m Y - \frac{1}{7}\nabla_n\Box Y) \\ &= -\epsilon(\Delta Y_n - 12m^2 Y_n) - \frac{12}{7}\tilde{\epsilon}\nabla_n(\Delta Y - 7m^2 Y) \end{aligned} \tag{3.178}$$

where (B.37)-(B.38) and (B.43) were used. This equation can be split in two because $\nabla_n Y$ and Y_n are orthogonal. The two equations are

$$\epsilon(\Delta Y_n - 12m^2 Y_n) = 0 \tag{3.179}$$

$$\tilde{\epsilon}\nabla_n(\Delta Y - 7m^2 Y) = 0 \tag{3.180}$$

We see here that the parameter ϵ is not gauge fixed when $\Delta Y_n = 12m^2 Y_n$. The implication of this is that we have to impose a new gauge condition on B_μ, which can be chosen to be $\nabla^\mu B_\mu = 0$. This means that modes corresponding to $\Delta Y_n = 12m^2 Y_n$, can be included in $1^{-(1),(2)}$.

Finally, I will analyze $0^{+(1),(3)}$ when $\Delta Y = 7m^2 Y$. In this case it is not possible to derive (3.124). Inserting $\Delta Y = 7m^2 Y$ in (3.112), (3.116), (3.135) and (3.143), and eliminating $m^2 G$ from (3.135) and (3.143), gives

$$\Delta\theta + 7m^2\theta - 9m^2(H - \hat{S}) = 0 \tag{3.181}$$

$$\Delta\hat{S} + 91m^2 H - \frac{28}{3}\Delta\theta = 0 \tag{3.182}$$

$$\Delta\hat{S} + 36m^2\hat{S} + 55m^2 H - \frac{16}{3}\Delta\theta + 28m^2\theta = 0 \tag{3.183}$$

Solving (3.182) for H and inserting that result into (3.181) and (3.183) gives

$$\frac{36}{91}\Delta\hat{S} + 36m^2\hat{S} + 4\Big(\frac{1}{13}\Delta\theta + 7m^2\theta\Big) = 0 \tag{3.184}$$

$$\Delta\theta + 91m^2\theta + \frac{13}{91}\Delta\hat{S} = 0 \tag{3.185}$$

By looking at (3.180) we see that $\tilde{\epsilon}$ is not gauge fixed when $\Delta Y = 7m^2 Y$, which can be taken care of by adding a new gauge condition for $\hat{S}$. I choose

$$h^m{}_m(x,y) = \hat{S}(x)Y(y) = 0 \rightarrow \hat{S} = 0, \quad \Delta Y = 7m^2 Y \tag{3.186}$$

as the new gauge condition. Inserting this into (3.180)-(3.181) gives

$$\Delta\theta - 8m^2\theta + M^2\theta = 0 \tag{3.187}$$

where $M^2 = 99m^2$. $0^{+(3)}$ also gives $M^2 = 99m^2$, if $\Delta Y = 7m^2 Y$. The implication of this, is that modes corresponding to $\Delta Y = 7m^2 Y$ can be included in $0^{+(3)}$, but not in $0^{+(1)}$.

The last thing I will do in this section is to discuss stability of the compactification[9]. The condition for classical stability, (i.e, no exponentially growing modes) for spin 2 and 1 is that the square of the mass matrix is positive definite, i.e, $M^2 \geq 0$, and for scalar fields $M^2 \geq -m^2$, which is called the Beitenlohner-Freedman bound. At a first look, the stability condition for scalar fields might look a bit strange. As we all know, if $M^2 < 0$ in Minkowski spacetime we have a tachyon, which makes the vacuum unstable. In AdS spacetime on the other hand we do not get a tachyon as long $M^2 \geq -m^2$. This happens because the kinetic energy is always larger then zero in AdS spacetime.

In table 3.1 all modes corresponding to spin 1 or spin 2 fields satisfy $M^2 \geq 0$. For the scalar fields all fields except $0^{+(2)}$ satisfy the Beitenlohner-Freedman bound,[10] for all internal spaces. Modes corresponding to $0^{+(2)}$ satisfy the Beitenlohner-Freedman bound as long as $\Delta_L \geq 3m^2$. This is always the case for a supersymmetric vacua.[11] Also their skew-whiffed[12] non-supersymmetric counterparts are stable. Examples of these spaces are: The left squashed S^7 with $N = 1$ supersymmetry and the right squashed S^7 with $N = 0$ supersymmetry. Examples of non stable vacua are compactifications where M^7 is a product metric $M^7 = M^d_{(1)} \times M^{7-d}_{(2)}$.

3.4 The $Minkowski_4 \times M^7$ bosonic mass operators

In this section I will give a short derivation of the bosonic mass operators for $Minkowski_4 \times M^7$, i.e, when $m = 0$ in the Freund-Rubin ansatz. Naively one

[9] The results are taken from [12, 18].

[10] There is equality only for modes corresponding to $\Delta Y = 27m^2 Y$ in $0^{+(1)}$, (this happens for example when $M^7 = S^7$).

[11] For example when $M^7 = S^7$ is $\Delta_L \geq 28m^2$.

[12] That is reversing the orientation of M^7.

Spin	Mass operator
2^+	Δ_0
$1^{-(1),(2)}$	$\Delta_1 + 12m^2 \pm 6m\sqrt{\Delta_1 + 4m^2}$
1^+	Δ_2
$0^{+(1),(3)}$	$\Delta_0 + 44m^2 \pm 12m\sqrt{\Delta_0 + 9m^2}$
$0^{+(2)}$	$\Delta_L - 4m^2$
$0^{-(1),(2)}$	$Q^2 + 6mQ + 8m^2$

Table 3.1: Bosonic mass operators from the Freund-Rubin ansatz
In this table modes corresponding to $\Delta Y = 0$ and $\Delta Y = 7m^2 Y$ are not included in $0^{+(1)}$.

might think that the only thing one has to do to obtain the mass operators is to put $m = 0$ in table 3.1. This is true for some of the mass operators, but not for all.

I will begin with the mass operator for $h^{TT}_{\mu\nu}$. By going through the derivation of the mass operator for $h^{TT}_{\mu\nu}$ in the last section, we see that nothing new happens if we let $m = 0$, which means that $h^{TT}_{\mu\nu}$ has the same mass operator Δ_0 as in the last section. It is also straightforward to include the special mode corresponding to $\Delta Y = 0$.

The mass operators corresponding to a_μ and S are easily seen to be Δ_2 and Δ_L as in the last section (with $m = 0$). Modes that correspond to any of the first three mass operators are called 2^+, 1^+ or $0^{+(2)}$ in table 3.2.

In the $m = 0$ case, the two equation that include B^T_μ and $a_{\rho\sigma}$ are no longer coupled. The equations are

$$(\Delta a_{\rho\sigma})Y_m + (\Delta Y_m)a_{\rho\sigma} = 0 \tag{3.188}$$
$$(\Delta B_\mu)Y_m + (\Delta Y_m)B_\mu = 0 \tag{3.189}$$

The second equation is easily seen to give Δ_1 as the mass operator. The first equation is a little trickier, because it has to be analyzed for two cases separately, which are $M^2 = 0$ (i.e. the massless case), and $M^2 > 0$ (i.e. the massive case). In the massless case, the antisymmetric tensor field $a_{\rho\sigma}$ correspond to a scalar field Θ. This can be seen by performing a dualization of $a_{\rho\sigma}$, which in four dimensions gives a scalar, (it is also easy to see that they have the same d.o.f. (degrees of freedom), i.e. one). The implication of this is that the mass operator obtained from (3.188), in the massless case, is Δ_1 for the scalar field Θ, which I will call $0^{+(3)}$ in table 3.2. In the massive case[13] we can define a new spin 1 field $\hat{B}^T_\mu$ defined as

$$\hat{B}^T_\mu = \epsilon_{\mu\nu\rho\sigma}\nabla^\nu a^{\rho\sigma} \tag{3.190}$$

Raising the indices and multiplying (3.188) with $\epsilon_{\mu\nu\rho\sigma}\nabla^\nu$ gives

$$(\Delta \hat{B}_\mu)Y_m + (\Delta Y_m)\hat{B}_\mu = 0 \tag{3.191}$$

[13] In this case the antisymmetric tensor field has the same amount of d.o.f. as a massive spin 1 field, i.e, 3 d.o.f.

Spin	Mass operator $M^2=0$	$M^2>0$
2^+	Δ_0	Δ_0
$1^{-(1)}$	Δ_1	Δ_1
$1^{-(2)}$	$-$	Δ_1
1^+	Δ_2	Δ_2
$0^{+(1)}$	Δ_0	Δ_0
$0^{+(2)}$	Δ_L	Δ_L
$0^{+(3)}$	Δ_1	$-$
$0^{-(1)}$	Δ_3	Δ_3
$0^{+(4)}$	$-$	Δ_0
Ant	Δ_0	$-$

Table 3.2: Bosonic mass operators from the Freund-Rubin ansatz ($m=0$)

which implies that the mass operator is Δ_1 for the spin 1 field $\hat{B}^T_\mu$. Modes that corresponds to this mass operator in table 3.2 are called $1^{-(2)}$.

To obtain the mass operator for the scalar field a we set $m=0$ in (3.107) and read of the mass operator, which is Δ_3. The modes that correspond to this mass operator in table 3.2 are called $0^{-(1)}$.

The equations that gave $0^{+(1),(3)}$ in the previous section are no longer coupled. Going through equations (3.80) and (3.83) with $m=0$, gives

$$(\Delta\hat{S})Y + (\Delta Y)\hat{S} = 0 \tag{3.192}$$

$$(\Delta a_{\mu\nu\rho})Y + (\Delta Y)a_{\mu\nu\rho} = 0 \tag{3.193}$$

where I have used $7H+5\hat{S}=0$ to obtain (3.192). The mass operators are easily seen to be Δ_0 for both fields. In table 3.2 modes corresponding to Δ_0 are called $0^{+(1)}$, for the scalar field $\hat{S}$. Modes corresponding to Δ_0 for the antisymmetric tensor field $a_{\mu\nu\rho}$ are called Ant[14] in the massless case, and $0^{+(4)}$ in the massive case. The latter happens because a new scalar field[15] θ can be defined in the massive case, which is defined as

$$\partial_\mu\theta = \epsilon_{\mu\nu\rho\sigma}a^{\nu\rho\sigma} \tag{3.194}$$

Finally, it is easy to see that the equations that correspond to (3.130), (3.135) and (3.143) give no new mass operators, but are satisfied. To see this, combine the three equations and use (3.124), which gives (3.192).

3.5 The $AdS_7 \times M^4$ bosonic mass operators

In this section, I will derive the $AdS_7 \times M^4$ bosonic mass operators. The derivation will be very similar to the derivation in section 3.3, except for a few differences. As

[14]Note that in d=4 a massless 3-form gauge potential has no d.o.f.

[15]That $a_{\mu\nu\rho}$ in the massive case can correspond to a scalar follows from the fact that the little group is $so(3)$.

in section 3.3, I will first split the D=11 indices into spacetime $(\mu, \nu, \ldots)$ and internal indices $(m, n, \ldots)$, and insert the Freund-Rubin ansatz (3.12) into the bosonic equations of motion (3.40) and (3.46), followed by substituting the various harmonic expansions of the D=11 fields into the bosonic equations of motion. Then the bosonic mass operators are obtained by diagonalizing the mass matrix for each type of field, mode by mode. Finally, some modes[16] that were excluded in the general analysis will be analyzed.

I will now drop the superscript "$\circ$", since it is superfluous at the linearized level. Inserting (3.12) into (3.40) and splitting the indices into, $\mu = 0, 1, 2, \ldots, 6$, and $m = 7, 8, 9, 10$, ((3.40) is multiplied with 2 below), gives that (3.40) with $\mu\nu$ indices is

$$\begin{aligned} & \Delta_L h_{\mu\nu} + 2\nabla_{(\mu}\nabla^\rho h_{\nu)\rho} + 2\nabla_{(\mu}\nabla^p h_{\nu)p} - \nabla_\mu\nabla_\nu h^\rho{}_\rho - \nabla_\mu\nabla_\nu h^p{}_p \\ = \;& 3m^2(g_{\mu\nu}h^p{}_p - h_{\mu\nu}) - \frac{m}{6}g_{\mu\nu}\epsilon^{pqrs}f_{pqrs} \end{aligned} \tag{3.195}$$

where I have used (B.5)-(B.6). Equation (3.40) with μn indices is

$$\begin{aligned} & \Delta_L h_{\mu n} + 2\nabla_{(\mu}\nabla^\rho h_{n)\rho} + 2\nabla_{(\mu}\nabla^p h_{n)p} - \nabla_\mu\nabla_n h^\rho{}_\rho - \nabla_\mu\nabla_n h^p{}_p \\ = \;& m\epsilon_n{}^{pqr}f_{\mu pqr} - 3m^2 h_{\mu n} \end{aligned} \tag{3.196}$$

where I have used (B.5). The RHS of (3.40) with mn indices is

$$\begin{aligned} 2T_{\mu\nu} &= m\Big(g_{ms}\epsilon^{spqr}f_{npqr} + g_{ns}\epsilon^{spqr}f_{mpqr}\Big) - \frac{m}{6}g_{mn}\epsilon_{pqrs}f^{pqrs} \\ &- \frac{9}{4}m^2\Big(g_{mt}\epsilon^{tpqr}\epsilon_{nsqr}h_p{}^s + g_{nt}\epsilon^{tpqr}\epsilon_{msqr}h_p{}^s\Big) - 3m^2h_{mn} + \frac{m^2}{2}g_{mn}h^p{}_t\epsilon_{pqrs}\epsilon^{tqrs} \\ &= 6m^2(h_{mn} - g_{mn}h^p{}_p) + \frac{m}{3}g_{mn}\epsilon^{pqrs}f_{pqrs} \end{aligned} \tag{3.197}$$

where I have used (B.5), (B.7) and

$$g_{ms}\epsilon^{spqr}f_{npqr} = \frac{1}{4}g_{mn}\epsilon^{spqr}f_{spqr}, \tag{3.198}$$

which were obtained from

$$0 = 5g_{m[s}f_{npqr]} = 4g_{m[s}f_{|n|pqr]} - g_{mn}f_{spqr} \tag{3.199}$$

Here $|n|$ means that n is not included in the antisymmetrization. This gives that (3.40) with mn indices is

$$\begin{aligned} & \Delta_L h_{mn} + 2\nabla_{(m}\nabla^\rho h_{n)\rho} + 2\nabla_{(m}\nabla^p h_{n)p} - \nabla_m\nabla_n h^\rho{}_\rho - \nabla_m\nabla_n h^p{}_p \\ = \;& 6m^2(h_{mn} - g_{mn}h^p{}_p) + \frac{m}{3}g_{mn}\epsilon^{pqrs}f_{pqrs} \end{aligned} \tag{3.200}$$

[16] The massless graviton will be seen to be one of these modes.

I will now insert (3.12) in (3.46) and split the indices. Equation (3.46) with $\mu\nu\rho$, $m\nu\rho$ and $mn\rho$ indices is

$$\begin{aligned} \nabla_\sigma f^{\sigma\mu\nu\rho} + \nabla_q f^{q\mu\nu\rho} + \frac{1}{8} m \epsilon^{\mu\nu\rho\sigma_1\sigma_2\sigma_3\sigma_4} f_{\sigma_1\sigma_2\sigma_3\sigma_4} &= 0, \qquad (3.201)\\ \nabla_\sigma f^{\sigma m\nu\rho} + \nabla_q f^{qm\nu\rho} &= 0, \qquad (3.202) \end{aligned}$$

and

$$\nabla_\sigma f^{\sigma mn\rho} + \nabla_q f^{qmn\rho} - \frac{3}{2} m \epsilon^{qsnm} \nabla_q h_s{}^\rho = 0 \qquad (3.203)$$

where I have used

$$\epsilon^{\mu\nu\rho\sigma_1\sigma_2\sigma_3\sigma_4 mnpq} \epsilon_{mnpq} = 4! \epsilon^{\mu\nu\rho\sigma_1\sigma_2\sigma_3\sigma_4} \qquad (3.204)$$

in (3.201). The RHS in (3.46) with mnp indices is zero, while the LHS is

$$\begin{aligned} S^{mnp} &= \nabla_\sigma f^{\sigma mnp} + \nabla_q f^{qmnp} + \frac{3m}{4} \epsilon^{qmnp} \nabla_q (h^\nu{}_\nu + h^n{}_n) - \frac{3}{2} m \epsilon^{rmnp} \nabla_q h^q{}_r \\ &- \frac{3}{2} m \epsilon^{rmnp} \nabla_\sigma h^\sigma{}_r - \frac{9}{2} m \nabla^q (h^{s[m} \epsilon_{qs}{}^{np]}) \qquad (3.205) \end{aligned}$$

Multiplication with ϵ_{tmnp} gives that (3.46) with "mnp" indices is

$$\epsilon_{tmnp} (\nabla_\sigma f^{\sigma mnp} + \nabla_q f^{qmnp}) + \frac{9}{2} m \nabla_t (h^\nu{}_\nu - h^n{}_n) - 9m \nabla_\sigma h^\sigma{}_t = 0 \qquad (3.206)$$

where (B.6)-(B.7) were used.

I am now going to make a harmonic expansion of the fields $h_{MN}(x,y)$ and $a_{MNP}(x,y)$. The expansions will take a simple form if the following gauge conditions are imposed[17]

$$\nabla^m h_{m\nu} = 0 \qquad (3.207)$$

$$\nabla^m (h_{mn} - \frac{1}{4} g_{mn} h^p{}_p) = 0 \qquad (3.208)$$

$$\nabla^m a_{mNP} = 0 \qquad (3.209)$$

These gauge conditions give the following harmonic expansions of $h_{MN}(x,y)$ and $a_{MNP}(x,y)$ [12]

$$\begin{aligned} h_{\mu\nu}(x,y) &= h_{\mu\nu}(x) Y(y) \qquad (3.210)\\ h_{\mu n}(x,y) &= B_\mu(x) Y_n(y) \qquad (3.211)\\ h_{mn}(x,y) &= S(x) X_{mn}(y) + \frac{1}{4} g_{mn} \hat{S}(x) Y(y) \qquad (3.212) \end{aligned}$$

$$\begin{aligned} a_{\mu\nu\rho}(x,y) &= a_{\mu\nu\rho}(x) Y(y) \qquad (3.213)\\ a_{\mu\nu p}(x,y) &= a_{\mu\nu}(x) Y_p(y) \qquad (3.214)\\ a_{\mu np}(x,y) &= a_\mu(x) Y_{np}(y) \qquad (3.215)\\ a_{mnp}(x,y) &= a(x) Y_{mnp}(y) \qquad (3.216) \end{aligned}$$

[17] As one can see below, I am imposing exactly the same gauge conditions and harmonic expansions as in the last section, except $\frac{1}{7} \rightarrow \frac{1}{4}$ in (3.208) and (3.212).

Here the p-forms $Y_{m_1 \dots m_p}(y)$ are transverse modes of Δ_p, which is the Hodge-de Rham operator on p-forms, see (B.37)-(B.40). Y is a spherical harmonic, Y_m is a vector spherical harmonic, Y_{mn} and Y_{mnp} are tensor spherical harmonics. X_{mn} are symmetric, transverse traceless modes of the Lichnerowicz operator Δ_L, see (B.23). I have suppressed all summation symbols and the corresponding isometry indices in the RHS of (3.210)-(3.216). From now on, the Lichnerowicz operator and the Hodge-de Rham operator are both denoted Δ. It will be clear from the context if Δ is the Hodge-de Rham operator acting on p-forms, or the Lichnerowicz operator acting on a symmetric, transverse traceless tensor. I will also drop the explicit coordinate dependence, because in the background metric (3.3) all operators are splitting up in a sum of one operator on AdS_7, and one on M^4. It can be seen from the gauge condition (3.209) that the p-forms $Y_{m_1 \dots m_p}$ must be transverse, because

$$\nabla^m a_{mNP} = 0 \Rightarrow \nabla^{m_1} Y_{m_1 \dots m_p} = 0, \quad p = 1, 2, 3 \tag{3.217}$$

I am now going to start analyzing the field equations for a_{MNP}. Some of the calculations will be very similar with calculations in section 3.3. In these cases I will show a very short derivation and refer back to the similar derivation in section 3.3.

Inserting (3.213)-(3.215) in (3.202) and splitting the resulting equation in two, because $\nabla_m Y$ and Y_m are orthogonal gives[18]

$$\begin{aligned} (\Delta a_{\nu\rho}) Y_m + (\Delta Y_m) a_{\nu\rho} &= 0 && (3.218) \\ \nabla^\sigma a_{\nu\rho\sigma} &= 0 && (3.219) \end{aligned}$$

where I have used (3.221) below in (3.218).

Inserting (3.211) and (3.214)-(3.116) in (3.203) and splitting the resulting equation in two, gives[19]

$$\begin{aligned} (\Delta a_\rho) Y_{mn} + (\Delta Y_{mn}) a_\rho + \frac{3}{2} m \epsilon_{qsnm} \nabla^q Y^s B_\rho &= 0 && (3.220) \\ \nabla^\sigma a_{\rho\sigma} &= 0 && (3.221) \end{aligned}$$

where I have used (3.225) below in (3.220).

Next, Inserting (3.213)-(3.214) in (3.201) gives[20]

$$(\Delta a_{\mu\nu\rho}) Y + (\Delta Y) a_{\mu\nu\rho} - \frac{1}{2} m \epsilon_{\mu\nu\rho\sigma_1\sigma_2\sigma_3\sigma_4} \nabla^{\sigma_1} a^{\sigma_2\sigma_3\sigma_4} Y = 0, \tag{3.222}$$

while (3.210)-(3.212) and (3.215)-(3.216) in (3.206) gives[21]

$$\epsilon^{tmnp}((\Delta a) Y_{mnp} + (\Delta Y_{mnp}) a + 3 \nabla^\rho a_\rho \nabla_{[m} Y_{np]}) - \frac{9}{2} m \nabla^t Y (H - \hat{S}) = 0 \tag{3.223}$$

[18] This derivation is exactly the same as the derivation of the two first terms in (3.103).

[19] This derivation is similar to the derivation of the two first terms in (3.99).

[20] The two first terms are derived analogous to the two first terms in (3.110).

[21] The derivation of the two first terms are the same as the derivation of the two first terms in (3.106).

where I have used $H = h^{\mu}{}_{\mu}$ and (3.232) below. This equation can be split in two equations, because term three is orthogonal with respect to the others, see (B.50). The two equations are

$$\begin{aligned} (\Delta\theta)a + (\Delta a)\theta - \frac{9}{4}m^2(H - \hat{S})Y &= 0 && (3.224) \\ \nabla^{\rho} a_{\rho} &= 0 && (3.225) \end{aligned}$$

where I have used that Δ commutes with ∇_t and defined a new field θ as

$$\partial_t \theta = \frac{m}{2}\epsilon_{tmnp}Y^{mnp} \tag{3.226}$$

The equations (3.218)-(3.221) and (3.224)-(3.225) are valid for all modes , except for modes that are solutions to $\Delta Y = 0$, $\Delta Y_m = 0$ or $\Delta Y_{mn} = 0$. The first case always have a solution on any compact internal space, and the solution is just $Y = constant$. The $\Delta Y_m = 0$ case never have non-trivial solutions on a positive curvature Einstein space, which can be seen from

$$\Delta Y_m = -\Box Y_m + R_m{}^n Y_n = (-\Box + 3m^2)Y_m > 0, \quad Y_m \neq 0 \tag{3.227}$$

where $-\Box$ is positive semidefinite. The third case $\Delta Y_{mn} = 0$ only have non-trivial solutions when the internal space have non-zero second betti number b_2. Modes corresponding to the first or the third case will be treated after the general analysis.

Now it is time to analyze the field equations for g_{MN}. Inserting (3.210)-(3.212) and (3.216) into (3.200), gives

$$\begin{aligned} &\Delta(SX_{mn} + \frac{1}{4}g_{mn}\hat{S}Y) - \frac{1}{2}(\hat{S} + 2H)\nabla_m\nabla_n Y + 2\nabla_{(m}Y_{n)}\nabla^{\rho}B_{\rho} \\ = \; &6m^2(SX_{mn} - \frac{3}{4}g_{mn}\hat{S}Y) - \frac{8}{3}g_{mn}a(\Delta\theta) && (3.228) \end{aligned}$$

where (3.226) and (B.37) were used. I will now take the trace of (3.228). The trace part will give one equation and the traceless part will give another one. The trace part gives

$$(\Delta\hat{S})Y + (\Delta Y)H + \frac{3}{2}\hat{S}(\Delta Y) = -18m^2\hat{S}Y - \frac{32}{3}a(\Delta\theta) \tag{3.229}$$

where I have used that X_{mn} is traceless, (3.217) and (B.37). The traceless part gives

$$(\Delta S)X_{mn} + (\Delta X_{mn})S + 2\nabla_{(m}Y_{n)}\nabla^{\rho}B_{\rho} = 6m^2 SX_{mn} \tag{3.230}$$

This equation can be split into two equations because X_{mn} and $\nabla_m Y_n$ are orthogonal[22]. The two equations are

$$\begin{aligned} (\Delta S)X_{mn} + (\Delta X_{mn})S &= 6m^2 SX_{mn} && (3.231) \\ \nabla^{\rho}B_{\rho} &= 0 && (3.232) \end{aligned}$$

[22]This can be seen from (B.50) using that X_{mn} is transverse.

I will now put (3.229) and (3.231)-(3.232) back into (3.228) and take the divergence of the resulting equation. This will produce one new equation and a few new modes that will be excluded for now, and analyzed later. Inserting (3.229) and (3.231)-(3.232) into (3.228) gives

$$\frac{1}{4}g_{mn}(\Delta Y)\hat{S} - \nabla_m\nabla_n Y\Big(\frac{1}{2}\hat{S} + H\Big) = \frac{1}{4}g_{mn}(\Delta Y)\Big(H + \frac{3}{2}\hat{S}\Big) \tag{3.233}$$

The divergence of this equation gives

$$(H + \frac{1}{2}\hat{S})\Big(\Box(\nabla_n Y) + \frac{1}{4}\nabla_n(\Delta Y)\Big) = 0 \tag{3.234}$$

This equation can be rewritten with the help of

$$\Box(\nabla_n Y) = [\nabla^m, \nabla_n]\nabla_m Y + \nabla_n(\Box Y) = 3m^2\nabla_n Y - \nabla_n(\Delta Y) \tag{3.235}$$

where (B.37) and (B.43) were used. Inserting this equation into (3.234) gives

$$(H + \frac{1}{2}\hat{S})\nabla_n\Big(\Delta Y - 4m^2 Y\Big) = 0 \tag{3.236}$$

The implication of this equation is either that $\Delta Y = 4m^2 Y$ or

$$H + \frac{1}{2}\hat{S} = 0, \quad \Delta Y \neq 4m^2 Y \tag{3.237}$$

This means that we have more modes that have to be excluded, i.e. modes that satisfy $\Delta Y = 4m^2 Y$ or $\Delta Y = 0$, the latter because $Y = constant \rightarrow \nabla_n Y = 0$. Finally, modes that satisfy $\Delta Y_n = 6m^2 Y_n$ must also be excluded. This happens because when we take the divergence of (3.228), $\nabla^\rho B_\rho = 0$ is not possible to derive, if the following equation is satisfied

$$0 = 2\nabla^m\nabla_{(m}Y_{n)} = \Box Y_n + \nabla^m\nabla_n Y_m = (-\Delta + 6m^2)Y_n \tag{3.238}$$

where (B.37) and (B.43) have been used. This means that we now have to exclude all modes that satisfy any of the following equations

$$\Delta Y = 0, \quad \Delta Y = 4m^2 Y, \quad \Delta Y_m = 6m^2 Y_m, \quad \Delta Y_{mn} = 0 \tag{3.239}$$

I am now going to analyze (3.196). Inserting (3.210)-(3.212) and (3.215)-(3.216) gives the following LHS (which I will call $G_{\mu n}$)

$$G_{\mu n} = (\Delta B_\mu)Y_n + (\Delta Y_n)B_\mu - \nabla_\mu\Big(\frac{3}{4}\hat{S} + H\Big)\nabla_n Y + \nabla^\rho h_{\mu\rho}\nabla_n Y \tag{3.240}$$

where I have used (3.217) and (3.232). The RHS is

$$\begin{aligned} T_{\mu n} &= -3m^2 B_\mu Y_n + m\epsilon_n{}^{pqr}(4\nabla_{[\mu}a_{pqr]}) = -3m^2 B_\mu Y_n + 2\nabla_n\theta\nabla_\mu a \\ &- 3m\epsilon_{npqr}\nabla^p Y^{qr} a_\mu \end{aligned} \tag{3.241}$$

where I have used (3.226). Combining the LHS and the RHS of (3.196) gives

$$\begin{aligned}
&(\Delta B_\mu)Y_n + (\Delta Y_n)B_\mu - \nabla_\mu\Big(\frac{3}{4}\hat{S} + H\Big)\nabla_n Y + \nabla^\rho h_{\mu\rho}\nabla_n Y \\
= \quad & -3m^2 B_\mu Y_n + 2\nabla_n\theta\nabla_\mu a - 3m\epsilon_{npqr}\nabla^p Y^{qr} a_\mu \qquad (3.242)
\end{aligned}$$

Before I try to split this equation in two, I have to show that $\theta \sim Y$. $\nabla_n\theta$ can be written as a linear combination of Y_n and $\nabla_n Y$, which are orthogonal. This gives

$$\nabla_n\theta = \frac{m}{2}\epsilon_{npqr}Y^{pqr} = b_1 Y_n + b_2\nabla_n Y \qquad (3.243)$$

where b_1 and b_2 are constants. I am now rewriting this equation in the following way

$$Y_{mnp} = \frac{1}{3m}\epsilon_{mnpq}\nabla^q\theta = \frac{1}{3m}\epsilon_{mnpq}(b_1 Y^q + b_2\nabla^q Y) \qquad (3.244)$$

If we now take the divergence of this equation and use (3.217) we see that $b_1 = 0$, which implies that

$$\nabla_n\theta = b_2\nabla_n Y \qquad (3.245)$$

I can now split (3.232) into two equations because Y_n and $\nabla_n Y$ are orthogonal. The two equations are

$$\begin{aligned}
(\Delta B_\mu)Y_n + (\Delta Y_n)B_\mu &= -3m^2 B_\mu Y_n - 3m\epsilon_{npqr}\nabla^p Y^{qr} a_\mu \qquad (3.246)\\
\nabla^\rho h_{\mu\rho} &= \nabla_\mu\Big(H + \frac{3}{4}\hat{S} + 2\tilde{a}\Big) \qquad (3.247)
\end{aligned}$$

where I have used $\tilde{a} = b_2 a$ and (3.245).

Before I analyze the last equation of motion I will decompose $h_{\mu\nu}$. The decomposition of $h_{\mu\nu}$ is [12]

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \nabla_{(\mu}V^T_{\nu)} + (\nabla_\mu\nabla_\nu + \frac{1}{7}g_{\mu\nu}\Delta)\tilde{\phi} + \frac{1}{7}g_{\mu\nu}H \qquad (3.248)$$

Here V^T_μ is transverse and $h^{TT}_{\mu\nu}$ is transverse and traceless. Using this equation and (3.247) makes it possible to show that $V^T_\mu = 0$, which is done in exactly the same way as in section 3.3.

Finally, it is time to analyze the last equation of motion (3.195). Inserting (3.210)-(3.212) and (3.216) gives

$$\begin{aligned}
&(\Delta h_{\mu\nu})Y + (\Delta Y)h_{\mu\nu} + 2\nabla_{(\mu}\nabla^\rho h_{\nu)\rho}Y - \nabla_\mu\nabla_\nu(H + \hat{S}) \\
= \quad & \frac{4}{3}g_{\mu\nu}\tilde{a}(\Delta Y) + 3m^2(g_{\mu\nu}\hat{S} - h_{\mu\nu})Y \qquad (3.249)
\end{aligned}$$

where I have used (3.217), (3.232), (B.37) and $(\Delta\theta)a = (\Delta Y)\tilde{a}$. This equation will give a total of three equations; one from the trace, one from the divergence and one from the traceless part. Taking the trace gives

$$2(\Delta H)Y + (\Delta Y)H + (\Delta\hat{S})Y + 2m^2 GY = 3m^2(7\hat{S} - H)Y + \frac{28}{3}\tilde{a}(\Delta Y) \qquad (3.250)$$

where $m^2G = \nabla^\mu\nabla^\nu h_{\mu\nu}$. Taking the divergence of (3.249) gives

$$\begin{aligned} & \nabla^\mu(\Delta h_{\mu\nu})Y + (\Delta Y)\nabla^\mu h_{\mu\nu} + \Box\nabla^\rho h_{\nu\rho}Y + \nabla^\mu\nabla_\nu\nabla^\rho h_{\mu\rho}Y - \Box\nabla_\nu(H+\hat{S})Y \\ = \; & \frac{4}{3}\nabla_\nu\tilde{a}(\Delta Y) + 3m^2(\nabla_\nu\hat{S} - \nabla^\mu h_{\mu\nu})Y \end{aligned} \tag{3.251}$$

This equation can be rewritten by using

$$\Box\nabla_\nu H = [\nabla^\mu, \nabla_\nu]\nabla_\mu H + \nabla_\nu\Box H = -\nabla_\nu(\Delta + \frac{3}{2}m^2)H \tag{3.252}$$

and

$$\nabla^\mu\nabla_\nu\nabla^\rho h_{\mu\rho} = m^2\nabla_\nu G - \frac{3}{2}m^2\nabla^\mu h_{\mu\nu} \tag{3.253}$$

where (3.14) and (B.43) were used. These two equations inserted in (3.251) gives

$$\nabla^\mu h_{\mu\nu}(\Delta Y) + \nabla_\nu[(\Delta - \frac{3}{2}m^2)\hat{S} + (\Delta + \frac{3}{2}m^2)H + m^2G]Y - \frac{4}{3}\nabla_\nu\tilde{a}(\Delta Y) = 0 \tag{3.254}$$

where I have used (3.140).

The equation that results from the traceless part of (3.249), is obtained in exactly the same way as in section 3.3. The result is

$$(\Delta h^{TT}_{\mu\nu})Y + (\Delta Y)h^{TT}_{\mu\nu} + 3m^2h^{TT}_{\mu\nu}Y = 0 \tag{3.255}$$

I am now going to obtain the bosonic mass operators for the different fields, by using the definitions of the mass matrix M below. All modes that satisfy any of the equations in (3.239) are excluded for now, but will be analyzed later after the general analysis. The definition for spin 2 fields $h_{\mu\nu}$ are

$$\Delta_L h_{\mu\nu} + 2\nabla_{(\mu}\nabla^\rho h_{\nu)\rho} - \nabla_{(\mu}\nabla_{\nu)}h^\rho{}_\rho + 3m^2h_{\mu\nu} + M^2h_{\mu\nu} = 0, \tag{3.256}$$

while the definition for spin 1 fields B_μ are

$$\Delta B_\mu + \nabla_\mu\nabla^\rho B_\rho + M^2B_\mu = 0, \tag{3.257}$$

and the definition for spin 0 fields S are

$$\Delta S + M^2S = 0 \tag{3.258}$$

Finally, the definition for a transverse antisymmetric tensor field $a_{\mu_1\ldots\mu_p}$ is

$$\Delta a_{\mu_1\ldots\mu_p} + M^2a_{\mu_1\ldots\mu_p} = 0 \tag{3.259}$$

If we look at (3.258) we see that in this section I have not included an extra term $-m^2S$ as I did in the last section. The reason for this omission is that I have followed the definition used in [19]. One thing that might look a little odd as a result of this definition, is that a massless scalar will have $M^2 \neq 0$. Then the question is; how can it be a massless field if $M^2 \neq 0$. The answer is that the definition of a massless field is not that $M^2 = 0$, but in order to be massless the scalar field has to be in the same supermultiplet as the massless graviton (see chapter 4 for a discussion about supermultiplets).

I will start analyzing (3.255). If we compare this equation with the mass matrix definition (3.256) we see that the mass operator has to be Δ_0 for the spin 2 field $h_{\mu\nu}^{TT}$.

The mass operator for the scalar field S in (3.231) is $\Delta_L - 6m^2$, which is seen by comparing (3.231) with (3.258).

Comparing (3.218) with (3.259) gives that the mass operator for the antisymmetric tensor field $a_{\nu\rho}$ is Δ_1.

The towers of modes corresponding to any of these three fields are denoted 2^+, $0^{+(2)}$ or $Ant^{(1)}$ in table 3.3.

I am now going to obtain two mass operators from equations (3.224), (3.229) and (3.237). Inserting (3.237) into (3.224) and (3.229) gives two coupled equations, which are

$$\begin{aligned} (\Delta\tilde{a})Y + (\Delta Y)\tilde{a} + \frac{27}{8}m^2\hat{S}Y &= 0 \qquad (3.260) \\ (\Delta\hat{S})Y + (\Delta Y)\hat{S} + 18m^2\hat{S}Y + \frac{32}{3}(\Delta Y)\tilde{a} &= 0 \qquad (3.261) \end{aligned}$$

where I have used $a\theta = \tilde{a}Y$. Writing (3.260) and (3.261) in matrix form gives

$$Y\Delta\begin{pmatrix}\tilde{a}\\ \hat{S}\end{pmatrix} + \begin{pmatrix}\Delta & \frac{27}{8}m^2\\ \frac{32}{3}\Delta & \Delta + 18m^2\end{pmatrix}Y\begin{pmatrix}\tilde{a}\\ \hat{S}\end{pmatrix} = 0 \qquad (3.262)$$

If we compare this matrix equation with (3.258), we see that the mass operators correspond to the eigenvalues of the two by two matrix. Calculating the eigenvalues gives the following mass operators

$$\Delta_0 + 9m^2 \pm 6m\sqrt{\Delta_0 + \frac{9}{4}m^2} \qquad (3.263)$$

In table 3.3, I have called the towers of modes that correspond to either of these mass operators $0^{+(1)}$ or $0^{+(3)}$, where (1) refers to the minus sign and (3) to the plus sign.

Equations (3.347), (3.250) and (3.254) gives no new mass operators, but are satisfied.

Next, I will analyze (3.220) and (3.246), which are two coupled equations. As I showed earlier, it is possible to write $\frac{m}{2}\epsilon_{mnqr}Y^{nqr} = b_2\nabla_m Y$. It is also easy to show that the same LHS but with Y^{qr} instead of Y^{nqr} can be rewritten as

$$\frac{m}{2}\epsilon_{mnqr}Y^{qr} = 2k\nabla_{[m}Y_{n]} \qquad (3.264)$$

where k is a constant. This equation implies that

$$\frac{m}{2}\epsilon_{mnqr}\nabla^n Y^{qr} = k\Delta Y_m \qquad (3.265)$$

Inserting (3.265) into (3.246) and multiplying (3.220) (where I have raised the mn indices and let $m \to q$, $n \to r$ and $\rho \to \mu$) with $\frac{m}{2}\epsilon_{mnqr}$, gives

$$\begin{aligned}
(\Delta B_\mu)Y_n + (\Delta Y_m)B_\mu + 3m^2 B_\mu Y_n + 6(\Delta Y_n)\tilde{a}_\mu &= 0 \qquad (3.266)\\
(\Delta \tilde{a})Y_n + (\Delta Y_n)\tilde{a} + \frac{3}{2}m^2 B_\mu Y_n &= 0 \qquad (3.267)
\end{aligned}$$

Rewriting this equation in matrix form gives

$$Y_n \Delta \begin{pmatrix} B_\mu \\ \tilde{a}_\mu \end{pmatrix} + \begin{pmatrix} \Delta + 3m^2 & 6\Delta \\ \frac{3}{2}m^2 & \Delta \end{pmatrix} Y_n \begin{pmatrix} B_\mu \\ \tilde{a}_\mu \end{pmatrix} = 0 \qquad (3.268)$$

where $\tilde{a}_\mu = k a_\mu$. If we compare this matrix equation with (3.257), where the second term is zero because of (3.225) and (3.232), we see that the mass operators correspond to the eigenvalues of the two by two matrix. Calculating the eigenvalues gives the following mass operators

$$\Delta_1 + \frac{3}{2}m^2 \pm 3m\sqrt{\Delta_1 + \frac{1}{4}m^2} \qquad (3.269)$$

In table 3.3, I have called the towers of modes that correspond to either of these mass operators $1^{-(1)}$ or $1^{-(2)}$, where (1) refers to the minus sign and (2) to the plus sign.

Finally, I am going to analyze (3.222). This equation can be rewritten as

$$([-Q_x^2 + \Delta_y]a_{\mu\nu\rho} - 3mQ_x a_{\mu\nu\rho})Y = 0 \qquad (3.270)$$

where Q_x is defined as

$$Q_x a_3 = \mu a_3 \qquad (3.271)$$

where the index 3 means the a_3 is a 3-form and μ is the eigenvalue to Q_x. The index x (y) means that the operator acts in AdS_7 (M^4). Writing this equation in component form gives

$$\frac{1}{6}m\epsilon_{\mu\nu\rho\sigma_1\sigma_2\sigma_3\sigma_4}\nabla^{\sigma_1}a^{\sigma_2\sigma_3\sigma_4} = \mu a_{\mu\nu\rho} \qquad (3.272)$$

These equations implies that $\Delta a_{\mu\nu\rho} = -\mu^2 a_{\mu\nu\rho}$. This is shown in the same way as (3.158) in section 3.3 were shown.[23] Equation (3.270) can be rewritten as

$$\Big[Q_x + \Big(\frac{3}{2}m + \sqrt{\Delta_y + \frac{9}{4}m^2}\Big)\Big]\Big[Q_x + \Big(\frac{3}{2}m - \sqrt{\Delta_y + \frac{9}{4}m^2}\Big)\Big]a_{\mu\nu\rho}Y = 0 \qquad (3.273)$$

[23] The origin of the minus sign is the signature difference between AdS spacetime and the internal space, see appendix A.1.

Splitting this equation in two gives

$$\Big[Q_x + \Big(\underbrace{\frac{3}{2}m + \sqrt{\Delta_y + \frac{9}{4}m^2}}_{A}\Big)\Big]a_{\mu\nu\rho}Y \;\; = \;\; 0 \qquad (3.274)$$

$$\Big[Q_x + \Big(\underbrace{\frac{3}{2}m - \sqrt{\Delta_y + \frac{9}{4}m^2}}_{B}\Big)\Big]a_{\mu\nu\rho}Y \;\; = \;\; 0 \qquad (3.275)$$

Here we can read of the mass operators by squaring A and B, which gives the following mass operators

$$\Delta_0 + \frac{9}{2}m^2 \pm 3m\sqrt{\Delta_0 + \frac{9}{4}m^2} \qquad (3.276)$$

Modes corresponding to any of these mass operators are called $Ant^{(+),(-)}$ in table 3.3, where $(+)$ refers to the plus sign and $(-)$ to the minus sign.

Now it is time to analyze the modes that were excluded in the analysis above. I am starting with the $\Delta Y = 0$ case. The mode corresponding to $\Delta Y = 0$ can be included in 2^+, which is shown in the same way as in section 3.3.

Next, I will show that modes corresponding to $\Delta Y = 0$ can be included in $0^{+(3)}$ but not in $0^{+(1)}$. Inserting Y constant in (3.229), (3.250) and (3.254) gives

$$\Delta\hat{S} + 18m^2\hat{S} \;\; = \;\; 0 \qquad (3.277)$$

$$2\Delta H + \Delta\hat{S} + 2m^2G - 3m^2(7\hat{S} - H) \;\; = \;\; 0 \qquad (3.278)$$

$$m^2G + \Delta H + \Delta\hat{S} + \frac{3}{2}m^2(H - \hat{S}) \;\; = \;\; 0 \qquad (3.279)$$

Inserting Y constant in (3.206) (which gave (3.224)) gives nothing.[24] Eliminating m^2G from (3.278) and (3.279) gives (3.277).[25] This implies that there are only modes corresponding to $0^{+(3)}$ present in the spectrum when Y is constant, ($0^{+(3)}$ gives $M^2 = 18m^2$, if $\Delta Y = 0$).

Analyzing $Ant^{(+),(-)}$ when $\Delta Y = 0$ gives that modes corresponding to $\Delta Y = 0$ can be included only in $Ant^{(+)}$. This happens because the zero mode of $Ant^{(-)}$ can be gauged away because its field equation states that it is pure gauge.

Analyzing $1^{-(1),(2)}$ when $\Delta Y_m = 6m^2Y_m$ is analogous with the analyze of $1^{-(1),(2)}$ in section 3.3. The analyze gives that $\nabla^\mu B_\mu = 0$ can be imposed as an extra gauge condition, which means that modes corresponding to $\Delta Y_m = 6m^2Y_m$ can be included in $1^{-(1),(2)}$.

I am now going to show that modes corresponding to $\Delta Y = 4m^2Y$ can be included in $0^{+(3)}$ but not in $0^{+(1)}$. Inserting $\Delta Y = 4m^2Y$ in (3.224), (3.229), (3.250)

[24] This gives that the scalar field $\tilde{a}$ is not present in the spectrum when Y is constant.

[25] We can here set $H = 0$ as an extra gauge condition as we did in the analogous case in section 3.3.

and (3.254), with the extra gauge condition[26] $\hat{S} = 0$, gives

$$\Delta\tilde{a} + 4m^2\tilde{a} - \frac{9}{4}m^2 H = 0 \quad (3.280)$$

$$\frac{32}{3}\tilde{a} + H = 0 \quad (3.281)$$

$$2\Delta H + 7m^2 H + 2m^2 G - \frac{112}{3}m^2\tilde{a} = 0 \quad (3.282)$$

$$\Delta H + \frac{11}{2}m^2 H + m^2 G + \frac{8}{3}m^2\tilde{a} = 0 \quad (3.283)$$

Eliminating m^2G from (3.282) and (3.283) gives (3.281). Eliminating H from (3.280) and (3.281) gives

$$\Delta\tilde{a} + M^2\tilde{a} = 0 \quad (3.284)$$

where $M^2 = 28m^2$. $0^{+(3)}$ also gives $M^2 = 28m^2$, if $\Delta Y = 4m^2$.

The modes corresponding to $\Delta Y = 4m^2 Y$ can be included in both $Ant^{(+)}$ and $Ant^{(-)}$.

Finally, I will analyze what happens when M^4 have $b_2 \neq 0$, i.e, non-trivial solutions to $\Delta Y_{mn} = 0$. The implication of this is that the condition $\nabla^\mu a_\mu = 0$ is no longer derivable. This gives that (3.220) changes to

$$(\Delta a_\mu)Y_{mn} + \nabla_\mu\nabla^\rho a_\rho + \frac{3}{2}m\epsilon_{qsnm}\nabla^q Y^s B_\mu = 0 \quad (3.285)$$

If we now split a_μ in following way

$$a_\mu = a^T_\mu + \hat{a}_\mu, \quad \nabla^\mu a^T_\mu = 0 \quad (3.286)$$

Then inserting this equation in (3.285) gives two equations, where one of them can be combined with (3.246) to give $1^{-(1),(2)}$, and the other one is

$$\Delta\hat{a}_\mu + \nabla_\mu\nabla^\rho\hat{a}_\rho + M^2\hat{a}_\mu = 0 \quad (3.287)$$

where $M^2 = 0$. Comparing this equation with (3.257) gives that there is an extra mode with $M^2 = 0$, when $\Delta Y_{mn} = 0$ and $b_2 \neq 0$. This mode is called $1^{-(3)}$ in table 3.3.

The last thing I will do in this section is to check if the the mass operators obey the Breitenlohner-Freedman bound. All modes corresponding to spin 1 or spin 2 fields have $M^2 \geq 0$, which means that they satisfy the condition for classical stability. All scalar fields obey $M^2 \geq -\frac{9}{4}m^2$ (if we have the restriction that $\Delta_L \geq \frac{15}{4}m^2$),[27] which is the Breitenlohner-Freedman bound for scalar fields in AdS_7. The implication of this is that all fields satisfy the condition for classical stability if $\Delta_L \geq \frac{15}{4}m^2$.

[26] See section 3.3 for the reason to this extra gauge condition. The $\Delta Y = 4m^2 Y$ case here and the $\Delta Y = 7m^2 Y$ case in section 3.3 for the $0^{+(1),(3)}$ fields are both treated in exactly the same way.

[27] This restriction is satisfied, for example, by all supersymmetric solution, (when $M^4 = S^4$ is $\Delta_L \geq 16m^2$), see also the discussion at the end of section 3.3.

Spin	Mass operator
2^+	Δ_0
$1^{-(1),(2)}$	$\Delta_1 + \frac{3}{2}m^2 \pm 3m\sqrt{\Delta_1 + \frac{1}{4}m^2}$
$1^{-(3)}$	$M^2 = 0$
$0^{+(1),(3)}$	$\Delta_0 + 9m^2 \pm 6m\sqrt{\Delta_0 + \frac{9}{4}m^2}$
$0^{+(2)}$	$\Delta_L - 6m^2$
$Ant^{(1)}$	Δ_1
$Ant^{(+),(-)}$	$\Delta_0 + \frac{9}{2}m^2 \pm 3m\sqrt{\Delta_0 + \frac{9}{4}m^2}$

Table 3.3: Bosonic mass operators from the Freund-Rubin ansatz
In this table modes corresponding to $\Delta Y = 0$ and $\Delta Y = 4m^2 Y$ are not included in $0^{+(1)}$, modes corresponding to $\Delta Y = 0$ are not included in $Ant^{(-)}$, and the $1^{-(3)}$ modes only exist when $b_2 \neq 0$ and $\Delta Y_{mn} = 0$.

3.6 The $AdS_5 \times M^5$ bosonic mass operators

In this section, I will obtain the $AdS_5 \times M^5$ bosonic mass operators. First I split the D=10 indices into spacetime $(\mu, \nu, \ldots)$ and internal indices $(m, n, \ldots)$, and insert the Freund-Rubin ansatz (3.20) into the bosonic equations of motion (3.63), (3.69), (3.72) and (3.73), followed by substituting the various harmonic expansions of the D=10 fields into the bosonic equations of motion. Then the bosonic mass operators are obtained by diagonalizing the mass matrix for each type of field, mode by mode. Finally, some modes[28] that were excluded in the general analysis will be analyzed.

I will now drop the superscript "$\circ$", since it is superfluous at the linearized level, and split the indices into, $\mu = 0, 1, 2, 3, 4$ and $m = 5, 6, 7, 8, 9$.

Next, I insert the Freund-Rubin ansatz (3.20) and $\overset{\circ}{\Phi} = \overset{\circ}{A}_{MN} = \overset{\circ}{\hat{I}}_{MNPQ} = 0$ into the bosonic equations of motion.[29]

Equation (3.63) gives

$$\nabla^m \bar{p}_m + \nabla_\mu \bar{p}_\mu = 0 \tag{3.288}$$

Inserting (3.20) in (3.69) with $\mu\nu$ indices gives

$$\begin{aligned} & \Delta h_{\mu\nu} + 2\nabla_{(\mu}\nabla^\rho h_{\nu)\rho} + 2\nabla_{(\mu}\nabla^p h_{\nu)p} - \nabla_\mu\nabla_\nu h^\rho{}_\rho - \nabla_\mu\nabla_\nu h^p{}_p \\ = & \frac{1}{6} m \epsilon_{(\mu}{}^{\rho\sigma\gamma\epsilon} l_{\nu)\rho\sigma\gamma\epsilon} + 8m^2(g_{\mu\nu}h^\rho{}_\rho - h_{\mu\nu}) \end{aligned} \tag{3.289}$$

where I have used (B.7). Equation (3.63) with μn indices is

$$\begin{aligned} & \Delta h_{\mu n} + 2\nabla_{(\mu}\nabla^\rho h_{n)\rho} + 2\nabla_{(\mu}\nabla^p h_{n)p} - \nabla_\mu\nabla_n h^\rho{}_\rho - \nabla_\mu\nabla_n h^p{}_p \\ = & \frac{1}{12} m(\epsilon_\mu{}^{\rho\sigma\gamma\epsilon} l_{n\rho\sigma\gamma\epsilon} + \epsilon_n{}^{pqrs} l_{\mu pqrs}), \end{aligned} \tag{3.290}$$

[28]The massless graviton will be seen to be one of these modes.
[29]These fields are set to zero in order to obtain maximal spacetime symmetry, see the discussion before (3.17).

and (3.63) with mn indices is

$$\begin{aligned}
&\Delta h_{mn} + 2\nabla_{(m}\nabla^{\rho} h_{n)\rho} + 2\nabla_{(m}\nabla^{p} h_{n)p} - \nabla_m\nabla_n h^{\rho}{}_{\rho} - \nabla_m\nabla_n h^{p}{}_{p} \\
= \ &\frac{1}{6} m\epsilon_{(m}{}^{pqrs} l_{n)pqrs} + 8m^2(h_{mn} - g_{\mu\nu} h^{p}{}_{p}) \qquad (3.291)
\end{aligned}$$

Equation (3.72) with $\mu\nu$, μn and mn indices is

$$\nabla^{\rho}\bar{f}_{\mu\nu\rho} + \nabla^{p}\bar{f}_{\mu\nu p} = \frac{2i}{3} m\epsilon_{\mu\nu\rho\sigma\gamma}\bar{f}^{\rho\sigma\gamma} \qquad (3.292)$$

$$\nabla^{\rho}\bar{f}_{\mu n\rho} + \nabla^{p}\bar{f}_{\mu np} = 0 \qquad (3.293)$$

$$\nabla^{\rho}\bar{f}_{mn\rho} + \nabla^{p}\bar{f}_{mnp} = \frac{2i}{3} m\epsilon_{mnpqr}\bar{f}^{pqr} \qquad (3.294)$$

Inserting (3.20) in the RHS of (3.73) (which I will call $S_{\mu\nu\rho\sigma\tau}$) with $\mu\nu\rho\sigma\tau$ indices gives

$$\begin{aligned}
S_{\mu\nu\rho\sigma\tau} &= \frac{1}{5!}\epsilon_{\mu\nu\rho\sigma\tau s_1 s_2 s_3 s_4 s_5}\Big(2m(h^{p}{}_{p} + h^{\rho}{}_{\rho})\epsilon^{s_1 s_2 s_3 s_4 s_5} + l^{s_1 s_2 s_3 s_4 s_5} - 20 m h_t{}^{[s_1}\epsilon^{t s_2 s_3 s_4 s_5]}\Big) \\
&= 2m\epsilon_{\mu\nu\rho\sigma\tau}(h^{\rho}{}_{\rho} - h^{p}{}_{p}) + \frac{1}{5!}\epsilon_{\mu\nu\rho\sigma\tau s_1 s_2 s_3 s_4 s_5} l^{s_1 s_2 s_3 s_4 s_5} \qquad (3.295)
\end{aligned}$$

This means that (3.73) with $\mu\nu\rho\sigma\tau$ indices is

$$l_{\mu\nu\rho\sigma\tau} = 2m\epsilon_{\mu\nu\rho\sigma\tau}(h^{\rho}{}_{\rho} - h^{p}{}_{p}) + \frac{1}{5!}\epsilon_{\mu\nu\rho\sigma\tau s_1 s_2 s_3 s_4 s_5} l^{s_1 s_2 s_3 s_4 s_5} \qquad (3.296)$$

Equation (3.73) with $\mu\nu\rho\sigma r$ indices is

$$l_{\mu\nu\rho\sigma r} = -\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma\epsilon_1 r s_2 s_3 s_4 s_5} l^{\epsilon_1 s_2 s_3 s_4 s_5} + 4m\epsilon_{\mu\nu\rho\sigma\epsilon_1} h_r{}^{\epsilon_1} \qquad (3.297)$$

In these two equations have I used a variation of (3.87). Finally, (3.73) with $\mu\nu\rho qr$ indices is

$$l_{\mu\nu\rho qr} = \frac{2}{4!}\epsilon_{\mu\nu\rho\epsilon_1\epsilon_2 qr s_3 s_4 s_5} l^{\epsilon_1\epsilon_2 s_3 s_4 s_5} \qquad (3.298)$$

Equation (3.73) with $\mu\nu pqr$, $\mu npqr$ and $mnpqr$ indices are not included because these equations give the same information as equations (3.296)-(3.298).

I am now going to make a harmonic expansion of the fields $h_{MN}(x, y)$, $a_{MN}(x, y)$, $I_{MNPQ}(x, y)$ and $\phi(x, y)$. The expansions will take a simple form if the following gauge conditions are imposed

$$\nabla^m h_{m\nu} = 0 \qquad (3.299)$$

$$\nabla^m(h_{mn} - \frac{1}{5} g_{mn} h^{p}{}_{p}) = 0 \qquad (3.300)$$

$$\nabla^m a_{mN} = 0 \qquad (3.301)$$

$$\nabla^m I_{mNPQ} = 0 \qquad (3.302)$$

These gauge conditions give the following harmonic expansions of $h_{MN}(x,y)$, $a_{MN}(x,y)$, $I_{MNPQ}(x,y)$ and $\phi(x,y)$

$$\begin{aligned} h_{\mu\nu}(x,y) &= h_{\mu\nu}(x)Y(y) && (3.303)\\ h_{\mu n}(x,y) &= B_\mu(x)Y_n(y) && (3.304)\\ h_{mn}(x,y) &= S(x)X_{mn}(y)+\frac{1}{5}g_{mn}\hat{S}(x)Y(y) && (3.305)\end{aligned}$$

$$\begin{aligned} a_{\mu\nu}(x,y) &= a_{\mu\nu}(x)Y(y) && (3.306)\\ a_{\mu n}(x,y) &= a_\mu(x)Y_n(y) && (3.307)\\ a_{mn}(x,y) &= a(x)Y_{mn}(y) && (3.308)\end{aligned}$$

$$\begin{aligned} I_{\mu\nu\rho\sigma}(x,y) &= I_{\mu\nu\rho\sigma}(x)Y(y) && (3.309)\\ I_{\mu\nu\rho q}(x,y) &= I_{\mu\nu\rho}(x)Y_q(y) && (3.310)\\ I_{\mu\nu pq}(x,y) &= I_{\mu\nu}(x)Y_{pq}(y) && (3.311)\\ I_{\mu npq}(x,y) &= I_\mu(x)Y_{npq}(y) && (3.312)\\ I_{mnpq}(x,y) &= I(x)Y_{mnpq}(y) && (3.313)\end{aligned}$$

$$\phi(x,y) = \phi(x)Y(y) \qquad (3.314)$$

Here the p-forms $Y_{m_1\ldots m_p}(y)$ are transverse modes of Δ_p, which is the Hodge-de Rham operator on p-forms, see (B.37)-(B.40). Y is a spherical harmonic, Y_m is a vector spherical harmonic, Y_{mn}, Y_{mnp} and Y_{mnpq} are tensor spherical harmonics. X_{mn} are symmetric, transverse traceless modes of the Lichnerowicz operator Δ_L, see (B.23). I have suppressed all summation symbols and the corresponding isometry indices in the RHS of (3.303)-(3.314). From now on, the Lichnerowicz operator and the Hodge-de Rham operator are both denoted Δ. It will be clear from the context if Δ is the Hodge-de Rham operator acting on p-forms, or the Lichnerowicz operator acting on a symmetric, transverse traceless tensor. I will also drop the explicit coordinate dependence, because in the background metric (3.3) all operators are splitting up in a sum of one operator on AdS_5, and one on M^5. It can be seen from the gauge conditions (3.301)-(3.302) that the p-forms $Y_{m_1\ldots m_p}$ must be transverse, because

$$\nabla^m a_{mN}=0,\quad \nabla^m I_{mNPQ}=0 \Rightarrow \nabla^{m_1}Y_{m_1\ldots m_p}=0,\quad p=1,2,3,4 \qquad (3.315)$$

The field equation for ϕ is obtained by inserting (3.314) in (3.288), which gives

$$(\Delta\phi)Y+(\Delta Y)\phi=0 \qquad (3.316)$$

where I have used (B.37).

Next, I will analyze the field equations for a_{MN}. Inserting (3.306)-(3.308) into (3.293) gives

$$\begin{aligned} 0 &= 3\nabla^\rho\nabla_{[\mu}\bar{a}_{n\rho]}+3\nabla^p\nabla_{[\mu}\bar{a}_{np]} = -2\nabla^\rho\nabla_{[\mu}\bar{a}_{\rho]}\bar{Y}_n+\nabla^\rho\bar{a}_{\rho\mu}\nabla_n\bar{Y}\\ &- 2\nabla^p\nabla_{[n}\bar{Y}_{p]}\bar{a}_\mu = -(\Delta\bar{a}_\mu)\bar{Y}_n-(\Delta\bar{Y}_n)\bar{a}_\mu+\nabla^\rho\bar{a}_{\rho\mu}\nabla_n\bar{Y} \qquad (3.317)\end{aligned}$$

where I have used (B.42) and (3.322) below. This equation can be split in two equations due to orthogonality of the various mode functions. The two equations are

$$
\begin{aligned}
(\Delta \bar{a}_\mu)\bar{Y}_n + (\Delta \bar{Y}_n)\bar{a}_\mu &= 0 && (3.318)\\
\nabla^\rho \bar{a}_{\rho\mu} &= 0 && (3.319)
\end{aligned}
$$

Inserting (3.307)-(3.308) into (3.294) gives

$$
\begin{aligned}
0 &= 3\nabla^\rho\nabla_{[m}\bar{a}_{n\rho]} + 3\nabla^p\nabla_{[m}\bar{a}_{np]} - 2im\epsilon_{mnpqr}\nabla^p\bar{a}^{qr}\\
&= \Box\bar{a}\bar{Y}_{mn} + 2\nabla_{[m}\bar{Y}_{n]}\nabla^\rho\bar{a}_\rho + 3\nabla^p\nabla_{[m}\bar{Y}_{np]}\bar{a} - 2im\epsilon_{mnpqr}\nabla^p\bar{Y}^{qr}\bar{a}\\
&= -(\Delta\bar{a})\bar{Y}_{mn} - (\Delta\bar{Y}_{mn})\bar{a} - 2im\epsilon_{mnpqr}\nabla^p\bar{Y}^{qr}\bar{a} + 2\nabla_{[m}\bar{Y}_{n]}\nabla^\rho\bar{a}_\rho \qquad (3.320)
\end{aligned}
$$

where I have used (B.37) and (B.45). Splitting this equation in two gives

$$
\begin{aligned}
(\Delta\bar{a})\bar{Y}_{mn} + (\Delta\bar{Y}_{mn})\bar{a} + 2im\epsilon_{mnpqr}\nabla^p\bar{Y}^{qr}\bar{a} &= 0 && (3.321)\\
\nabla^\rho\bar{a}_\rho &= 0 && (3.322)
\end{aligned}
$$

Inserting (3.306)-(3.307) into (3.292) gives

$$
\begin{aligned}
0 &= 3\nabla^\rho\nabla_{[\mu}\bar{a}_{\nu\rho]} + 3\nabla^p\nabla_{[\mu}\bar{a}_{\nu p]} - 2im\epsilon_{\mu\nu\rho\sigma\gamma}\nabla^\rho\bar{a}^{\sigma\gamma}\\
&= 3\nabla^\rho\nabla_{[\mu}\bar{a}_{\nu\rho]}\bar{Y} + \Box\bar{Y}\bar{a}_{\mu\nu} - 2im\epsilon_{\mu\nu\rho\sigma\gamma}\nabla^\rho\bar{a}^{\sigma\gamma}\bar{Y}\\
&= -(\Delta\bar{Y})\bar{a}_{\mu\nu} - (\Delta\bar{a}_{\mu\nu})\bar{Y} - 2im\epsilon_{\mu\nu\rho\sigma\gamma}\nabla^\rho\bar{a}^{\sigma\gamma}\bar{Y} \qquad (3.323)
\end{aligned}
$$

where I have used (3.315), (B.37) and (B.45). This means that (3.292) is

$$
(\Delta\bar{Y})\bar{a}_{\mu\nu} + (\Delta\bar{a}_{\mu\nu})\bar{Y} + 2im\epsilon_{\mu\nu\rho\sigma\gamma}\nabla^\rho\bar{a}^{\sigma\gamma}\bar{Y} = 0 \qquad (3.324)
$$

Analogous to section 3.3 and 3.5 modes corresponding to $\Delta Y = 0$ or $\Delta Y_{mn} = 0$ must be excluded, and analyzed later after the general analysis.

Inserting (3.303)-(3.305) and (3.313) into (3.291) gives

$$
\begin{aligned}
&\Delta(SX_{mn} + \frac{1}{5}g_{mn}\hat{S}Y) - \frac{1}{5}(3\hat{S} + 5H)\nabla_m\nabla_n Y + 2\nabla^\rho B_\rho\nabla_{(m}Y_{n)}\\
= \; &\frac{5}{6}m\epsilon_{(m}{}^{pqrs}\nabla_{n)}Y_{pqrs}I + 8m^2(SX_{mn} - \frac{4}{5}g_{mn}\hat{S}Y) \qquad (3.325)
\end{aligned}
$$

where I have used $H = h^\nu{}_\nu$ and

$$
\nabla^p h_{np} = \frac{1}{5}\hat{S}\nabla_n Y \qquad (3.326)
$$

I will now take the trace of (3.325). The trace part will give one equation and the traceless part will give another one. The trace part gives

$$
(\Delta\hat{S})Y + (\Delta Y)H + \frac{8}{5}\hat{S}(\Delta Y) = -32m^2\hat{S}Y - \frac{5}{8}\tilde{I}(\Delta Y) \qquad (3.327)
$$

where I have used that X_{mn} is traceless, (3.315), $\tilde{I} = bI$ and

$$\frac{4}{3} m \epsilon_{mnpqr} Y^{npqr} = b \nabla_m Y \tag{3.328}$$

where b is a constant. The traceless part gives

$$(\Delta S) X_{mn} + (\Delta X_{mn}) S + 2 \nabla_{(m} Y_{n)} \nabla^\rho B_\rho = 8m^2 S X_{mn} \tag{3.329}$$

This equation can be split into two equations because X_{mn} and $\nabla_m Y_n$ are orthogonal.[30] The two equations are

$$\begin{aligned} (\Delta S) X_{mn} + (\Delta X_{mn}) S &= 8m^2 S X_{mn} & (3.330) \\ \nabla^\rho B_\rho &= 0 & (3.331) \end{aligned}$$

I will now put (3.327) and (3.330)-(3.331) back into (3.225) and take the divergence of the resulting equation. This will produce one new equation and a few new modes that will be excluded for now, and analyzed later. Inserting (3.327) and (3.330)-(3.331) into (3.325) gives

$$\frac{1}{5} g_{mn} (\Delta Y) \hat{S} - \nabla_m \nabla_n Y \Big(\frac{3}{5} \hat{S} + H \Big) = \frac{1}{5} g_{mn} (\Delta Y) \Big(H + \frac{8}{5} \hat{S} \Big) \tag{3.332}$$

where I have used (3.328) and a variation of (3.76). The divergence of this equation gives

$$(H + \frac{3}{5} \hat{S}) \Big(\Box (\nabla_n Y) + \frac{1}{5} \nabla_n (\Delta Y) \Big) = 0 \tag{3.333}$$

This equation can be rewritten with the help of

$$\Box (\nabla_n Y) = [\nabla^m, \nabla_n] \nabla_m Y + \nabla_n (\Box Y) = 4m^2 \nabla_n Y - \nabla_n (\Delta Y) \tag{3.334}$$

where (B.37) and (B.43) were used. Inserting this equation into (3.334) gives

$$(H + \frac{3}{5} \hat{S}) \nabla_n \Big(\Delta Y - 5m^2 Y \Big) = 0 \tag{3.335}$$

The implication of this equation is either that $\Delta Y = 5m^2 Y$ or

$$H + \frac{3}{5} \hat{S} = 0, \quad \Delta Y \neq 5m^2 Y \tag{3.336}$$

This means that we have more modes that have to be excluded, i.e. modes that satisfy $\Delta Y = 5m^2 Y$ or $\Delta Y = 0$, the latter because $Y = constant \to \nabla_n Y = 0$. Finally, modes that satisfy $\Delta Y_n = 8m^2 Y_n$ must also be excluded. This happens because when we take the divergence of (3.225), $\nabla^\rho B_\rho = 0$ is not possible to derive, if the following equation is satisfied

$$0 = 2 \nabla^m \nabla_{(m} Y_{n)} = \Box Y_n + \nabla^m \nabla_n Y_m = (-\Delta + 8m^2) Y_n \tag{3.337}$$

[30]This can be seen from (B.50) using that X_{mn} is transverse.

where (B.37) and (B.43) have been used. This means that we now have to exclude all modes that satisfy any of the following equations

$$\Delta Y = 0, \quad \Delta Y = 5m^2 Y, \quad \Delta Y_m = 8m^2 Y_m, \quad \Delta Y_{mn} = 0 \tag{3.338}$$

I am now going to analyze (3.290). Inserting (3.303)-(3.305), (3.309)-(3.310) and (3.312)-(3.313) gives the following LHS (which I will call $G_{\mu n}$)

$$G_{\mu n} = (\Delta B_\mu)Y_n + (\Delta Y_n)B_\mu - \nabla_\mu\Big(\frac{4}{5}\hat{S} + H\Big)\nabla_n Y + \nabla^\rho h_{\mu\rho}\nabla_n Y \tag{3.339}$$

where I have used (3.315) and (3.331). The RHS is

$$\begin{aligned} T_{\mu n} &= \frac{1}{12}m\Big[\epsilon_{\mu\rho\sigma\gamma\epsilon}(I^{\rho\sigma\gamma\epsilon}\nabla_n Y + 4\nabla^{[\rho}I^{\sigma\gamma\epsilon]}Y_n) + \epsilon_{npqrs}(Y^{pqrs}\nabla_\mu I + 4\nabla^{[p}Y^{qrs]}I_\mu)\Big] \\ &= \frac{1}{16}\nabla_\mu\theta\nabla_n Y + \frac{1}{3}m\epsilon_{\mu\rho\sigma\gamma\epsilon}\nabla^\rho I^{\sigma\gamma\epsilon}Y_n + \frac{1}{16}\nabla_\mu\tilde{I}\nabla_n Y + \frac{1}{4}\tilde{I}_\mu(\Delta Y_n) \end{aligned} \tag{3.340}$$

where I have used (3.328), $\tilde{I} = bI$, $\tilde{I}_\mu = \frac{1}{2}dI_\mu$,

$$\nabla_\mu\theta = \frac{4}{3}m\epsilon_{\mu\rho\sigma\gamma\epsilon}I^{\rho\sigma\gamma\epsilon} \tag{3.341}$$

and

$$\frac{4}{3}m\epsilon_{npqrs}Y^{qrs} = d\nabla_{[n}Y_{p]} \Rightarrow \frac{4}{3}m\epsilon_{npqrs}\nabla^p Y^{qrs} = \frac{1}{2}d(\Delta Y_n) \tag{3.342}$$

where (B.42) were used and d is a constant. Combining (3.339) and (3.340) gives

$$\begin{aligned} &(\Delta B_\mu)Y_n + (\Delta Y_n)B_\mu - \nabla_\mu\Big(\frac{4}{5}\hat{S} + H\Big)\nabla_n Y + \nabla^\rho h_{\mu\rho}\nabla_n Y \\ =\; &\frac{1}{16}\nabla_\mu\theta\nabla_n Y + \frac{1}{3}m\epsilon_{\mu\rho\sigma\gamma\epsilon}\nabla^\rho I^{\sigma\gamma\epsilon}Y_n + \frac{1}{16}\nabla_\mu\tilde{I}\nabla_n Y + \frac{1}{4}\tilde{I}_\mu(\Delta Y_n) \end{aligned} \tag{3.343}$$

This equation can be split into two equations, because Y_n and $\nabla_n Y$ are orthogonal. The two equations are

$$(\Delta B_\mu)Y_n + (\Delta Y_n)B_\mu = \frac{1}{3}m\epsilon_{\mu\rho\sigma\gamma\epsilon}\nabla^\rho I^{\sigma\gamma\epsilon} + \frac{1}{4}\tilde{I}_\mu(\Delta Y_n) \tag{3.344}$$

$$\nabla^\rho h_{\mu\rho} = \nabla_\mu\Big(H + \frac{4}{5}\hat{S} + \frac{1}{16}(\theta + \tilde{I})\Big) \tag{3.345}$$

Before I analyze the last equation of motion I will decompose $h_{\mu\nu}$. The decomposition of $h_{\mu\nu}$ is [12]

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \nabla_{(\mu}V^T_{\nu)} + (\nabla_\mu\nabla_\nu + \frac{1}{5}g_{\mu\nu}\Delta)\tilde{\phi} + \frac{1}{5}g_{\mu\nu}H \tag{3.346}$$

Here V^T_μ is transverse and $h^{TT}_{\mu\nu}$ is transverse and traceless. Using this equation and (3.345) makes it possible to show that $V^T_\mu = 0$, which is done in exactly the same way as in section 3.3.

Finally, it is time to analyze the last equation of motion (3.289). Inserting (3.303)-(3.305) and (3.309) gives

$$\begin{aligned} & (\Delta h_{\mu\nu})Y + (\Delta Y)h_{\mu\nu} + 2\nabla_{(\mu}\nabla^{\rho}h_{\nu)\rho}Y - \nabla_{\mu}\nabla_{\nu}(H+\hat{S}) \\ = \; & 8m^2(g_{\mu\nu}H - h_{\mu\nu})Y - \frac{1}{8}g_{\mu\nu}(\Delta\theta)Y \end{aligned} \tag{3.347}$$

where I have used (3.331), (3.315), (B.37) and

$$0 = 6g_{\mu[\tau}l_{\nu\rho\sigma\gamma\epsilon]} = 5g_{\mu[\tau}l_{|\nu|\rho\sigma\gamma\epsilon]} - g_{\mu\nu}l_{\tau\rho\sigma\gamma\epsilon} \tag{3.348}$$

which gives that

$$\frac{1}{6}m\epsilon_{(\mu}{}^{\rho\sigma\gamma\epsilon}l_{\nu)\rho\sigma\gamma\epsilon} = -\frac{1}{8}g_{\mu\nu}(\Delta\theta)Y \tag{3.349}$$

where I have used (3.341). This equation will give a total of three equations; one from the trace, one from the divergence and one from the traceless part. Taking the trace gives

$$2(\Delta H)Y + (\Delta Y)H + (\Delta\hat{S})Y + 2m^2GY = 32m^2HY - \frac{5}{8}(\Delta\theta)Y \tag{3.350}$$

where I have used $m^2G = \nabla^{\mu}\nabla^{\nu}h_{\mu\nu}$. Taking the divergence of (3.347) gives

$$\begin{aligned} & \nabla^{\mu}(\Delta h_{\mu\nu})Y + (\Delta Y)\nabla^{\mu}h_{\mu\nu} + \Box\nabla^{\rho}h_{\nu\rho}Y + \nabla^{\mu}\nabla_{\nu}\nabla^{\rho}h_{\mu\rho}Y - \Box\nabla_{\nu}(H+\hat{S})Y \\ = \; & 8m^2(\nabla_{\nu}H - \nabla^{\mu}h_{\mu\nu})Y - \frac{1}{8}\nabla_{\nu}(\Delta\theta)Y \end{aligned} \tag{3.351}$$

where I have used (3.341) and (B.43). This equation can be rewritten by using

$$\Box\nabla_{\nu}H = [\nabla^{\mu},\nabla_{\nu}]\nabla_{\mu}H + \nabla_{\nu}\Box H = -\nabla_{\nu}(\Delta + 4m^2)H \tag{3.352}$$

and

$$\nabla^{\mu}\nabla_{\nu}\nabla^{\rho}h_{\mu\rho} = m^2\nabla_{\nu}G - 4m^2\nabla^{\mu}h_{\mu\nu} \tag{3.353}$$

where (3.22) and (B.43) were used. These two equations inserted in (3.351) gives

$$\nabla^{\mu}h_{\mu\nu}(\Delta Y) + \nabla_{\nu}[(\Delta - 4m^2)H + (\Delta + 4m^2)\hat{S} + m^2G + \frac{1}{8}\Delta\theta]Y = 0 \tag{3.354}$$

where I have used (3.140).

The equation that results from the traceless part of (3.347), is obtained in exactly the same way as in section 3.3. The result is

$$(\Delta h^{TT}_{\mu\nu})Y + (\Delta Y)h^{TT}_{\mu\nu} + 8m^2h^{TT}_{\mu\nu}Y = 0 \tag{3.355}$$

Now it is time to analyze the field equations for G_{MNPQR}. Inserting (3.303), (3.305) and (3.313) in (3.296) gives

$$5\nabla_{[\mu}I_{\nu\rho\sigma\tau]}Y = 2m\epsilon_{\mu\nu\rho\sigma\tau}(H-\hat{S})Y + \frac{1}{4!}I\epsilon_{\mu\nu\rho\sigma\tau s_1s_2s_3s_4s_5}\nabla^{s_1}Y^{s_2s_3s_4s_5} \tag{3.356}$$

Using (3.328), (B.37) and $\tilde{I} = bI$ in the equation above, gives

$$m\nabla_{[\mu}I_{\nu\rho\sigma\tau]}Y = \frac{2}{5}m^2\epsilon_{\mu\nu\rho\sigma\tau}(H - \hat{S})Y - \frac{3}{4\cdot 5!}\epsilon_{\mu\nu\rho\sigma\tau}(\Delta Y)\tilde{I} \tag{3.357}$$

Inserting (3.304) and (3.312)-(3.313) into (3.297) gives

$$\begin{aligned} I_{\mu\nu\rho\sigma}\nabla_r Y + 4\nabla_{[\mu}I_{\nu\rho\sigma]}Y_r &= -\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma\epsilon_1 r s_2 s_3 s_4 s_5}(\nabla^{\epsilon_1}IY^{s_2s_3s_4s_5} + 4\nabla^{[s_2}Y^{s_3s_4s_5]}I^{\epsilon_1}) \\ &+ 4m\epsilon_{\mu\nu\rho\sigma\epsilon_1}B^{\epsilon_1}Y_r \end{aligned} \tag{3.358}$$

Using (3.328), (3.342), $\tilde{I}_\mu = \frac{1}{2}dI_\mu$ and $\tilde{I} = bI$ in the equation above, gives

$$mI_{\mu\nu\rho\sigma}\nabla_r Y + 4m\nabla_{[\mu}I_{\nu\rho\sigma]}Y_r = -\frac{3}{4\cdot 4!}\epsilon_{\mu\nu\rho\sigma\epsilon_1}(\nabla^{\epsilon_1}\tilde{I}\nabla_r Y + 4\tilde{I}^{\epsilon_1}(\Delta Y_r)) + 4m^2\epsilon_{\mu\nu\rho\sigma\epsilon_1}B^{\epsilon_1}Y_r \tag{3.359}$$

Splitting this equation into two equations because Y_r and $\nabla_r Y$ are orthogonal, gives

$$4m\nabla_{[\mu}I_{\nu\rho\sigma]}Y_r = -\frac{3}{4!}\epsilon_{\mu\nu\rho\sigma\epsilon_1}\tilde{I}^{\epsilon_1}(\Delta Y_r) + 4m^2\epsilon_{\mu\nu\rho\sigma\epsilon_1}B^{\epsilon_1}Y_r \tag{3.360}$$

$$mI_{\mu\nu\rho\sigma} = -\frac{3}{4\cdot 4!}\epsilon_{\mu\nu\rho\sigma\epsilon_1}\nabla^{\epsilon_1}\tilde{I} \tag{3.361}$$

The last equation can be rewritten as

$$\nabla_{\epsilon_1}\tilde{I} = \frac{4}{3}m\epsilon_{\epsilon_1\mu\nu\rho\sigma}I^{\mu\nu\rho\sigma} \tag{3.362}$$

Comparing this equation with (3.341) implies that

$$\nabla_\mu\theta = \nabla_\mu\tilde{I} \tag{3.363}$$

Before I analyze the last field equation, I will first combine (3.357) and (3.361) to obtain an equation for the scalar fields H, $\hat{S}$ and $\tilde{I}$. Doing this and multiplying with $\epsilon^{\mu\nu\rho\sigma\tau}$ gives

$$(\Delta\tilde{I})Y + (\Delta Y)\tilde{I} = 64m^2(H - \hat{S}) \tag{3.364}$$

Finally, I will analyze (3.298). Inserting (3.310)-(3.312) gives

$$3\nabla_{[\mu}I_{\nu\rho\sigma]}Y_{qr} + I_{\mu\nu\rho}2\nabla_{[q}Y_{r]} = \frac{2}{4!}\epsilon_{\mu\nu\rho\epsilon_1\epsilon_2 qrs_3s_4s_5}(I^{\epsilon_1\epsilon_2}3\nabla^{[s_3}Y^{s_4s_5]} + 2\nabla^{[\epsilon_1}I^{\epsilon_2]}Y^{s_3s_4s_5}) \tag{3.365}$$

Using (3.342) and $\tilde{I}_\mu = \frac{1}{2}dI_\mu$ in the equation above, gives

$$\begin{aligned} 3m\nabla_{[\mu}I_{\nu\rho\sigma]}Y_{qr} + 2mI_{\mu\nu\rho}\nabla_{[q}Y_{r]} &= \frac{6}{4!}m\epsilon_{\mu\nu\rho\epsilon_1\epsilon_2 qrs_3s_4s_5}I^{\epsilon_1\epsilon_2}\nabla^{s_3}Y^{s_4s_5} \\ &+ \frac{6}{4!}\epsilon_{\mu\nu\rho\epsilon_1\epsilon_2}\nabla^{\epsilon_1}\tilde{I}^{\epsilon_2}\nabla_{[q}Y_{r]} \end{aligned} \tag{3.366}$$

Splitting this equation into two equations because Y_{qr} and $\nabla_{[q}Y_{r]}$ are orthogonal, gives

$$\nabla_{[\mu}I_{\nu\rho\sigma]}Y_{qr} = \frac{2}{4!}\epsilon_{\mu\nu\rho\epsilon_1\epsilon_2 qrs_3s_4s_5}I^{\epsilon_1\epsilon_2}\nabla^{s_3}Y^{s_4s_5} \tag{3.367}$$

$$mI_{\mu\nu\rho} = +\frac{3}{4!}\epsilon_{\mu\nu\rho\epsilon_1\epsilon_2}\nabla^{\epsilon_1}\tilde{I}^{\epsilon_2} \tag{3.368}$$

I am now going to obtain the bosonic mass operators for the different fields, by using the definitions of the mass matrix M below. All modes that satisfy any of the equations in (3.338) are excluded for now, but will be analyzed later after the general analysis. The definition for spin 2 fields $h_{\mu\nu}$ are

$$\Delta_L h_{\mu\nu} + 2\nabla_{(\mu}\nabla^{\rho}h_{\nu)\rho} - \nabla_{(\mu}\nabla_{\nu)}h^{\rho}{}_{\rho} + 8m^2 h_{\mu\nu} + M^2 h_{\mu\nu} = 0, \qquad (3.369)$$

while the definition for spin 1 fields B_μ are

$$\Delta B_\mu + \nabla_\mu \nabla^\rho B_\rho + M^2 B_\mu = 0, \qquad (3.370)$$

and the definition for spin 0 fields S are

$$\Delta S + M^2 S = 0 \qquad (3.371)$$

Finally, the definition for a transverse antisymmetric tensor field $a_{\mu_1\ldots\mu_p}$ is

$$\Delta a_{\mu_1\ldots\mu_p} + M^2 a_{\mu_1\ldots\mu_p} = 0 \qquad (3.372)$$

In the definition of the mass matrix for scalar fields, I have used the same definition that was used in [20]. As in the previous section, using this definition gives that most massless scalar fields have $M^2 \neq 0$ (except $0^{+(1)}$ which have $M^2 = 0$). What is differnt here compared with the previous section, is that the massless scalar fields have different values of M^2, for example when $M^5 = S^5$ the three massless scalar fields[31] have $M^2 = -4, -3, 0m^2$. This might look a bit strange, but it is consistent.

I will start analyzing (3.355). If we compare this equation with the mass matrix definition (3.369) we see that the mass operator has to be Δ_0 for the spin 2 field $h^{TT}_{\mu\nu}$.

Comparing (3.318) with (3.370) were the second term is zero because of (3.322), gives that the mass operator for the complex spin 1 field $\bar{a}_\mu$ is Δ_1.

In table 3.4, I have called modes corresponding to any of these mass operators 2^+ or 1^+.

The mass operator for the complex scalar field ϕ in (3.316) is Δ_0, which is seen by comparing (3.316) with (3.371). Modes corresponding to this mass operator are called $0^{+(2)}$ in table 3.4.

Comparing (3.330) with (3.371) gives that the mass operator for the scalar field S is $\Delta_L - 8m^2$. Modes corresponding to this mass operator are called $0^{+(4)}$ in table 3.4.

I am now going to analyze (3.327), (3.336) and (3.364). Inserting (3.336) into (3.327) and (3.364) gives two coupled equations, which are

$$(\Delta\hat{S})Y + (\Delta Y)\hat{S} + \frac{5}{8}(\Delta Y)\tilde{I} + 32m^2\hat{S}Y = 0 \qquad (3.373)$$

$$(\Delta\tilde{I})Y + (\Delta Y)\tilde{I} + \frac{512}{5}m^2\hat{S}Y = 0 \qquad (3.374)$$

[31] See section 4.3.

Writing these equations in matrix form gives

$$Y\Delta\begin{pmatrix}\hat{S}\\ \tilde{I}\end{pmatrix}+\begin{pmatrix}\Delta+32m^2 & \frac{5}{8}\Delta\\ \frac{512}{5}m^2 & \Delta\end{pmatrix}Y\begin{pmatrix}\hat{S}\\ \tilde{I}\end{pmatrix}=0 \tag{3.375}$$

If we compare this matrix equation with (3.371), we see that the mass operators correspond to the eigenvalues of the two by two matrix. Calculating the eigenvalues gives the following mass operators

$$\Delta_0+16m^2\pm 8m\sqrt{\Delta_0+4m^2} \tag{3.376}$$

In table 3.4, I have called the towers of modes that correspond to either of these mass operators $0^{+(1)}$ or $0^{+(3)}$, where (1) refers to the minus sign and (3) to the plus sign.

Equations (3.345), (3.350) and (3.354) gives no new mass operators, but are satisfied.

Next, I will analyze the complex scalar field $\bar{a}$ in (3.321). Equation (3.321) can be rewritten as

$$(\Delta\bar{a})\bar{Y}_{mn}+(\Delta\bar{Y}_{mn})\bar{a}+4mQ\bar{Y}_{mn}\bar{a}=0 \tag{3.377}$$

where the operator Q is defined as

$$QY_2=\mu Y_2 \tag{3.378}$$

Here μ is the eigenvalue to Q, and the index on Y_2 means that Y_2 is a 2-form. The operator Q written in component form is

$$\frac{1}{2}i\epsilon_{mnpqr}\nabla^pY^{qr}=\mu Y_{mn} \tag{3.379}$$

These equations implies that $\Delta_2=Q^2$, which is shown in the following way

$$Q^2Y_{mn}=-\frac{1}{4}\epsilon_{mnpqr}\nabla^p\epsilon^{qrstu}\nabla_sY_{tu}=-3\nabla^p\nabla_{[p}Y_{mn]}=\Delta Y_{mn} \tag{3.380}$$

where I have used (B.45). Identifying (3.379)-(3.380) in (3.377) makes it easy to obtain the mass operator, which is

$$Q^2+4mQ \tag{3.381}$$

Modes corresponding to this mass operator in table 3.4 are called $0^{-(1)}$ and $0^{-(2)}$, where (1) refers to the negative part of the spectrum of Q and (2) to the positive part.

I will now analyze (3.367). Multiplying with $\epsilon^{\mu\nu\rho\sigma\tau}$ gives

$$\epsilon_{\sigma\tau\mu\nu\rho}\nabla^\mu I^{\nu\rho}Y_{qr}+\epsilon_{qrstu}\nabla^sY^{tu}I_{\sigma\tau}=0 \tag{3.382}$$

This equation can be rewritten as

$$(iQ_x+Q_y)I_{\sigma\tau}Y_{qr}=0 \tag{3.383}$$

where I have used (3.379) and

$$Q_x I_2 = \mu' I_2 \to \frac{1}{2}\epsilon_{\sigma\tau\mu\nu\rho}\nabla^\mu I^{\nu\rho} = \mu' I_{\sigma\tau} \tag{3.384}$$

This equation implies that $\Delta_{2_x} = Q_x^2$, which is shown in the following way

$$Q_x^2 I_{\mu\nu} = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma\tau}\nabla^\rho\epsilon^{\sigma\tau\gamma\epsilon\xi}\nabla_\gamma I_{\epsilon\xi} = -3\nabla^\rho\nabla_{[\mu}I_{\nu\rho]} = \Delta_x I_{\mu\nu} \tag{3.385}$$

where I have used (B.45). The mass operator for the antisymmetric tensor field is obtained by squaring Q_y, which gives that Q^2 is the correct mass operator. Modes corresponding to this mass operator in table 3.4 are called $Ant^{(1),(2)}$, where (1) refers to the negative part of the spectrum of Q and (2) to the positive part.

Next, I analyze (3.324). This equation can be rewritten as

$$([Q_x^2 + \Delta_y]\bar{a}_{\mu\nu} + 4imQ_x\bar{a}_{\mu\nu})\bar{Y} = 0 \tag{3.386}$$

Rewriting this equation by first solving for Q_x gives

$$\left[Q_x + i\left(2m + \sqrt{\Delta_y + 4m^2}\right)\right]\left[Q_x + i\left(2m - \sqrt{\Delta_y + 4m^2}\right)\right]\bar{a}_{\mu\nu}\bar{Y} = 0 \tag{3.387}$$

Finally, splitting this equation in two gives

$$\left[Q_x + i\Big(\underbrace{2m + \sqrt{\Delta_y + 4m^2}}_{A}\Big)\right]\bar{a}_{\mu\nu}\bar{Y} \;=\; 0 \tag{3.388}$$

$$\left[Q_x + i\Big(\underbrace{2m - \sqrt{\Delta_y + 4m^2}}_{B}\Big)\right]\bar{a}_{\mu\nu}\bar{Y} \;=\; 0 \tag{3.389}$$

Here we can read of the mass operators for the complex antisymmetric tensor field $\bar{a}_{\mu\nu}$ by squaring A and B, which gives

$$\Delta_0 + 8m^2 \pm 4m\sqrt{\Delta_0 + 4m^2} \tag{3.390}$$

Modes corresponding to any of these mass operators are called $Ant^{(+),(-)}$ in table 3.4, where $(+)$ refers to the plus sign and $(-)$ to the minus sign.

Finally, I am going to analyze (3.344), (3.360) and (3.368). Inserting (3.368) into (3.360), multiplying with $\epsilon^{\mu\nu\rho\sigma\epsilon_3}$ and lowering ϵ_3, gives

$$(\Delta\tilde{I}_{\epsilon_3} + \nabla_{\epsilon_3}\nabla^\mu\tilde{I}_\mu)Y_r + (\Delta Y_r)\tilde{I}_{\epsilon_3} - 32m^2 B_{\epsilon_3}Y_r = 0 \tag{3.391}$$

where I have used (B.6)-(B.7). Inserting (3.368) into (3.344) and letting $\mu \to \epsilon_3$ and $n \to r$ gives

$$(\Delta B_{\epsilon_3})Y_r + (\Delta Y_r)B_{\epsilon_3} + \frac{1}{4}(\Delta\tilde{I}_{\epsilon_3} + \nabla_{\epsilon_3}\nabla^\mu\tilde{I}_\mu)Y_r - \frac{1}{4}\tilde{I}_{\epsilon_3}(\Delta Y_r) = 0 \tag{3.392}$$

where I have used (B.7). The vector field $\tilde{I}_\mu$ can be decomposed in two parts, which are

$$\tilde{I}_\mu = \tilde{I}^T_\mu + \nabla_\mu \tilde{I}', \quad \nabla^\mu \tilde{I}^T_\mu = 0 \tag{3.393}$$

where $\tilde{I}^T_\mu$ is transverse. Inserting this equation into (3.391) and (3.392) gives that $\nabla_\mu \tilde{I}' = 0$ and two coupled equations for the transverse fields $\tilde{I}^T_{\epsilon_3}$ and B_{ϵ_3} (Equation (3.331) implies that B_{ϵ_3} is transverse). The two equations are

$$\begin{aligned} (\Delta\tilde{I}_{\epsilon_3})Y_r + (\Delta Y_r)\tilde{I}_{\epsilon_3} - 32m^2 B_{\epsilon_3} Y_r &= 0 & (3.394)\\ (\Delta B_{\epsilon_3})Y_r + (\Delta Y_r)B_{\epsilon_3} + \frac{1}{4}(\Delta\tilde{I}_{\epsilon_3})Y_r - \frac{1}{4}\tilde{I}_{\epsilon_3}(\Delta Y_r) &= 0 & (3.395) \end{aligned}$$

Using (3.394) in (3.395) gives the following system of coupled equations in matrix form

$$Y_m \Delta \begin{pmatrix} \tilde{I}_{\epsilon_3} \\ B_{\epsilon_3} \end{pmatrix} + \begin{pmatrix} \Delta & -32m^2 \\ -\frac{1}{2}\Delta & \Delta + 8m^2 \end{pmatrix} Y_m \begin{pmatrix} \tilde{I}_{\epsilon_3} \\ B_{\epsilon_3} \end{pmatrix} = 0 \tag{3.396}$$

If we compare this matrix equation with (3.370), where the second term is zero because the fields are transverse, we see that the eigenvalues of the two by two matrix gives the mass operators of the spin 1 fields. Calculating the eigenvalues gives the following mass operators

$$\Delta_1 + 4m^2 \pm 4m\sqrt{\Delta_1 + m^2} \tag{3.397}$$

In table 3.4, I have called the towers of modes that correspond to either of these mass operators $1^{-(1)}$ or $1^{-(2)}$, where (1) refers to the minus sign and (2) to the plus sign.

Now it is time analyze the modes that were excluded in the analysis above. I am starting with the $\Delta Y = 0$ case. The mode corresponding to $\Delta Y = 0$ can be included in 2^+, which is shown in the same way as in section 3.3.

Next, I will show that modes corresponding to $\Delta Y = 0$ can be included in $0^{+(3)}$ but not in $0^{+(1)}$. Inserting Y constant in (3.327), (3.364), (3.350) and (3.354), and imposing $H = 0$ as an extra gauge condition,[32]gives

$$\begin{aligned} \Delta\hat{S} + 32m^2\hat{S} &= 0 & (3.398)\\ \Delta\tilde{I} + 64m^2\hat{S} &= 0 & (3.399)\\ \Delta\hat{S} + 2m^2 G + \frac{5}{8}\Delta\tilde{I} &= 0 & (3.400)\\ m^2 G + \Delta\hat{S} + 4m^2\hat{S} + \frac{1}{8}\Delta\tilde{I} &= 0 & (3.401) \end{aligned}$$

Eliminating $m^2 G$ from (3.400) and (3.401) gives

$$\Delta\hat{S} + 8m^2\hat{S} - \frac{3}{8}\Delta\tilde{I} = 0 \tag{3.402}$$

Inserting (3.399) into (3.402) gives (3.398), which implies that there is only one field left when Y is constant. The implication of this is that modes corresponding to

[32]For an explanation of this gauge condition, see the analogous case in section 3.3.

$\Delta Y = 0$ can be included only in $0^{+(3)}$ but not in $0^{+(1)}$, ($0^{+(3)}$ gives $M^2 = 32m^2$, if $\Delta Y = 0$).

Analyzing $Ant^{(+),(-)}$ when $\Delta Y = 0$ gives that modes corresponding to $\Delta Y = 0$ can be included only in $Ant^{(+)}$. This happens because the zero mode of $Ant^{(-)}$ can be gauged away because its field equation states that it is pure gauge. This case is very similar to the $Ant^{(+),(-)}$ case in AdS_7.

Analyzing $1^{-(1),(2)}$ when $\Delta Y_m = 8m^2 Y_m$ is analogous with the analyze of $1^{-(1),(2)}$ in section 3.3. The analyze gives that $\nabla^\mu B_\mu = 0$ can be imposed as an extra gauge condition, which means that modes corresponding to $\Delta Y_m = 8m^2 Y_m$ can be included in $1^{-(1),(2)}$.

Modes corresponding to $\Delta Y_{mn} = 0$ can be included without any problems in both $0^{-(1),(2)}$ and $Ant^{(1),(2)}$.

I am now going to show that modes corresponding to $\Delta Y = 5m^2 Y$ can be included in $0^{+(3)}$ but not in $0^{+(1)}$. Inserting $\Delta Y = 5m^2 Y$ in (3.327), (3.364), (3.350) and (3.354), with the extra gauge condition[33] $\hat{S} = 0$, gives

$$H + \frac{5}{8}\tilde{I} = 0 \tag{3.403}$$

$$\Delta\tilde{I} + 5m^2\tilde{I} - 64m^2 H = 0 \tag{3.404}$$

$$2\Delta H - 27m^2 H + 2m^2 G - \frac{5}{8}\Delta\tilde{I} = 0 \tag{3.405}$$

$$\Delta H + m^2 H + m^2 G + \frac{1}{8}\Delta\tilde{I} + \frac{5}{8}m^2\tilde{I} = 0 \tag{3.406}$$

Combining (3.403) and (3.404) gives

$$\Delta\tilde{I} + M^2\tilde{I} = 0 \tag{3.407}$$

where $M^2 = 45m^2$. Eliminating $m^2 G$ from (3.405) and (3.406) and combining the result with (3.404) also gives (3.406). The implication of this is that modes corresponding to $\Delta Y = 5m^2 Y$ can be included in $0^{+(3)}$ but not in $0^{+(1)}$, ($0^{+(3)}$ gives $M^2 = 45m^2$, if $\Delta Y = 5m^2 Y$).

The modes corresponding to $\Delta Y = 5m^2 Y$ can be included in both $Ant^{(+)}$ and $Ant^{(-)}$.

The last thing I will do in this section is to check if the the mass operators obey the Breitenlohner-Freedman bound. All modes corresponding to spin 1 or spin 2 fields have $M^2 \geq 0$, which means that they satisfy the condition for classical stability. All scalar fields obey $M^2 \geq -4m^2$ (if we have the restriction that $\Delta_L \geq 4m^2$),[34] which is the Breitenlohner-Freedman bound for scalar fields in AdS_5. The implication of this is that all fields satisfy the condition for classical stability if $\Delta_L \geq 4m^2$.

[33] See section 3.3 for the reason to this extra gauge condition. The $\Delta Y = 5m^2 Y$ case here and the $\Delta Y = 7m^2 Y$ case in section 3.3 for the $0^{+(1),(3)}$ fields are both treated in exactly the same way.

[34] This restriction is satisfied, for example, by all supersymmetric solution, (when $M^5 = S^5$ is $\Delta_L \geq 20m^2$), see also the discussion in the end of section 3.3.

Spin	Mass operator
2^+	Δ_0
$1^{-(1),(2)}$	$\Delta_1 + 4m^2 \pm 4m\sqrt{\Delta_1 + m^2}$
1^+	Δ_1
$0^{+(1),(3)}$	$\Delta_0 + 16m^2 \pm 8m\sqrt{\Delta_0 + 4m^2}$
$0^{+(2)}$	Δ_0
$0^{+(4)}$	$\Delta_L - 8m^2$
$0^{-(1),(2)}$	$Q^2 + 4mQ$
$Ant^{(1),(2)}$	Q^2
$Ant^{(+),(-)}$	$\Delta_0 + 8m^2 \pm 4m\sqrt{\Delta_0 + 4m^2}$

Table 3.4: Bosonic mass operators from the Freund-Rubin ansatz
In this table modes corresponding to $\Delta Y = 0$ and $\Delta Y = 5m^2 Y$ are not included in $0^{+(1)}$, and modes corresponding to $\Delta Y = 0$ are not included in $Ant^{(-)}$.

4

Mass spectrum

In this chapter, I compute the mass spectrum for various compactifications of 11D supergravity and type IIB 10D supergravity, by using appendix C and the mass operators obtained in chapter 3. I will also obtain the massless and the first two massive supermultiplets from the mass spectrum, (except the $Minkowski_4 \times (K3 \times T^3)$ case where I will obtain only the massless supermultiplet).

The different compactifications that will be analyzed are: $AdS_4 \times S^7$, $AdS_7 \times S^4$ and $AdS_5 \times S^5$ in section 4.1-4.3 and $Minkowski_4 \times T^7$ and $Minkowski_4 \times (K3 \times T^3)$ in section 4.4-4.5.

4.1 The $AdS_4 \times S^7$ mass spectrum

In this section, I derive the $AdS_4 \times S^7$ mass spectrum and show that it corresponds to $N = 8$ supermultiplets (I will show this for the first 3 levels $n = 0, 1, 2$ in table 4.1). M^2 for the various fields is obtained by using table 3.1 and the eigenvalues of the Hodge-de Rham and the Lichnerowicz operators obtained in appendix C: See table 4.1 for the complete spectrum.

The symmetry of the vacuum for $AdS_4 \times S^7$ is described by the supergroup $OSp(4|8)$, which have $SO(3,2) \times SO(8)$ as bosonic subgroup. Here $SO(3,2)$ is the isometry group of AdS_4[1] and $SO(8)$ is the isometry group of S^7. That $SO(8)$ is the isometry group of S^7 implies that $SO(8)$ is the gauge group in this compactification. This happens because the mass operators are invariant under $SO(8)$.

Now, I will demonstrate that the $AdS_4 \times S^7$ mass spectrum can be written as $N = 8$ supermultiplets. The multiplicity of a $N = 8$ supermultiplet in 4 dimensions, is shown in table 4.2 for massless and massive supermultiplets.

[1] We also know that $SO(3,2)$ is the conformal group in 3 dimensions. This identification will be very important in the next chapter.

Spin	$SO(8)$ rep	M^2	$n=0$	$n=1$	$n=2$
2^+	$(n,0,0,0)$	$(n+3)^2-9$	1	8_v	35_v
$(\frac{3}{2})^{(1)}$	$(n,0,0,1)$	n^2	8_s	56_c	224_{vs}
$(\frac{3}{2})^{(2)}$	$(n-1,0,1,0)$	$(n+6)^2$	-	8_c	56_s
$1^{-(1)}$	$(n,1,0,0)$	$(n+1)^2-1$	28	160_v	567_v
$1^{-(2)}$	$(n-2,1,0,0)$	$(n+5)^2-1$	-	-	28
1^+	$(n-1,0,1,1)$	$(n+3)^2-1$	-	56_v	350
$(\frac{1}{2})^{(1)}$	$(n+1,0,1,0)$	n^2	56_s	224_{vc}	672_{vc}
$(\frac{1}{2})^{(2)}$	$(n-1,1,1,0)$	$(n+2)^2$	-	160_c	840_s
$(\frac{1}{2})^{(3)}$	$(n-2,1,0,1)$	$(n+4)^2$	-	-	160_s
$(\frac{1}{2})^{(4)}$	$(n-2,0,0,1)$	$(n+6)^2$	-	-	8_s
$0^{+(1)}$	$(n+2,0,0,0)$	$(n-1)^2-1$	35_v	112_v	294_v
$0^{+(3)}$	$(n-2,0,0,0)$	$(n+7)^2-1$	-	-	1
$0^{+(2)}$	$(n-2,2,0,0)$	$(n+3)^2-1$	-	-	300
$0^{-(1)}$	$(n,0,2,0)$	$(n+1)^2-1$	35_c	224_{cv}	$840'_s$
$0^{-(2)}$	$(n-2,0,0,2)$	$(n+5)^2-1$	-	-	35_s

Table 4.1: The $AdS_4 \times S^7$ mass spectrum

In order to have the complete spectrum, I have taken the fermionic part of the mass spectrum from [12]. The $SO(8)$ representations are also taken from [12]. $(1,0,0,0) = 8_v$, $(0,0,1,0) = 8_c$ and $(0,0,0,1) = 8_s$, where v refers to the vector representation, s to the spinor representation and c to the conjugate spinor representation. That the three different representations have the same dimension is easily seen from the $SO(8)$ dynkin diagram. The last 3 columns shows the $SO(8)$ representations of the 3 first levels in the mass spectrum, i.e, the massless level ($n=0$) and the two first massive levels ($n=1,2$). M^2 is in units of m^2 in this table and table 4.2-4.3.

Spin	Massless		Massive	
	Mult.	d.o.f	Mult.	d.o.f
2	1	2	1	5
$\frac{3}{2}$	8	2	8	4
1	28	2	27	3
$\frac{1}{2}$	56	2	48	2
0	70	1	42	1

Table 4.2: Multiplicity and d.o.f (degrees of freedom) for a $N=8$ supermultiplet in 4 dimensions

The first column under "Massless" and "Massive" gives the multiplicity, while the second column gives the number of d.o.f. The number of d.o.f. were obtained by using table 2 in [12]. The multiplicity for the massive supermultiplets is the same as the multiplicity for a $N=8$ massless supermultiplet in 5 dimensions, see table 4.6 were $Ant2$ can be seen as spin 1 fields.

To obtain the massless supermultiplet, I will use the supercharges Q_I on the highest state, which is the spin 2 graviton singlet, to obtain the second highest state, which are the spin $\frac{3}{2}$ gravitinos. Repeating this procedure all the way down to the lowest state, which are the scalars, gives the complete massless supermultiplet.[2] If now we analyze the representation theory for the supercharges we see that only completely antisymmetric states are included in the massless supermultiplet.

The super charges transform under 8_s in $SO(8)$. Now using the supercharges on the spin 2 state and so on gives[3]

$$\begin{array}{ll} 2 & 1 \\ \frac{3}{2} & 1 \times_a 8_s = 8_s \\ 1 & 8_s \times_a 8_s = 28 \\ \frac{1}{2} & 28 \times_a 8_s = 56_s \\ 0 & 56_s \times_a 8_s = 35_v + 35_c \end{array} \tag{4.1}$$

The index a implies an antisymmetric product.

The massive supermultiplets are also obtained by using the supercharges. In this case there are central charges included in the algebra. These central charges are responsible for the fact that the massive supermultiplets are short multiplets with spin range 0-2, and not 0-4, which is the case for long massive supermultiplets. Analyzing the representation theory for the supercharges gives the following massive supermultiplets, (I give only the first two massive supermultiplets)

$$\begin{array}{ll} 2 & 8_v \\ \frac{3}{2} & 8_c + 56_c \\ 1 & 56_v + 160_v \\ \frac{1}{2} & 160_c + 224_{vc} \\ 0 & 112_v + 224_{cv} \end{array} \tag{4.2}$$

and

$$\begin{array}{ll} 2 & 35_v \\ \frac{3}{2} & 56_s + 224_{vs} \\ 1 & 28 + 350 + 567_v \\ \frac{1}{2} & 8_s + 160_s + 672_{vc} + 840_s \\ 0 & 1 + 35_s + 294_v + 300 + 840'_s \end{array} \tag{4.3}$$

These two supermultiplets have the same multiplicity as the massive part of table 4.2.

[2] What I am doing here is very similar to ordinary quantum mechanics when one start with the highest (lowest) weight state and obtains all the other states by using the step down (step up) operators.

[3] In these calculations I have used table 37 in [22]

If we now compare (4.1), (4.2) and (4.3) with the last 3 columns in table 4.1. We see that $n = 0$ gives the massless supermultiplet (4.1), $n = 1$ gives the first massive supermultiplet (4.2) and $n = 2$ gives the second massive supermultiplet (4.3).

I am now going to check that the bosons and the fermions in each of the three first supermultiplets have the same number of d.o.f. Using table 4.2 gives that there are 128 bosonic d.o.f. and 128 fermionic d.o.f. in the massless supermultiplet. Table 4.2 also gives that there are 1024 ($8 \cdot 128$) bosonic d.o.f. and 1024 fermionic d.o.f. in the first massive supermultiplet, and 4480 ($35 \cdot 128$) bosonic d.o.f. and 4480 fermionic d.o.f. in the second massive supermultiplet. This result was of curse expected since one knows that $N = 8$ supermultiplets are supersymmetric, and supersymmetry demands that the number of d.o.f. for bosonic and fermionic fields are the same.

If we now look at table 4.1 we see that all the modes obey the Breitenlohner-Freedman stability criterion, with equality only for the $0^{+(1)}$ scalars in the 112_v representation of $SO(8)$. We also see that different massive states in massive multiplets have different mass. This is one difference compared to Minkowski spacetime, because in Minkowski spacetime all states in a given supermultiplet must have the same mass (this is required by supersymmetry). This is another example of the problem with the definition of mass in AdS spacetime.

Before ending this section, I will discuss the singleton supermultiplet [23], which consists of 8 scalars and 8 spinors. This is the smallest supermultiplet admitted by the supergroup $OSp(4|8)$. The reason that this supermultiplet is not included in the $AdS_4 \times S^7$ mass spectrum, is because the 8_v scalars corresponding to $n = -1$ in $0^{+(1)}$ could be gauged away in section 3.3, (see the $\Delta Y = 7m^2 Y$ special case).[4] The same happens with the 8_s spinors corresponding to $n = -1$ in $(\frac{1}{2})^{(1)}$. What actually happened were that these fields could be gauged away in the bulk of AdS spacetime but not on the boundary: See the discussion in [23]. This implies that the singletons do not live in the bulk of AdS spacetime but live on the boundary of AdS. If we go to Minkowski spacetime there is no supermultiplet analogous to the singleton supermultiplet.

4.2 The $AdS_7 \times S^4$ mass spectrum

In this section, I repeat the above analysis for the case of the $AdS_7 \times S^4$ mass spectrum and show that this mass spectrum corresponds to $N = 4$ supermultiplets (I will show this for the first 3 levels $n = 0, 1, 2$ in table 4.3). M^2 for the various fields is obtained by using table 3.3 and the eigenvalues of the Hodge-de Rham and the Lichnerowicz operators obtained in appendix C: see table 4.3 for the complete spectrum.

The symmetry of the vacuum for $AdS_7 \times S^4$ is described by the super group $OSp(6, 2|4)$, which have $SO(6, 2) \times SO(5)$ as bosonic subgroup. Here $SO(6, 2)$ is the isometry group of AdS_7[5] and $SO(5)$ is the isometry group of S^4. That $SO(5)$ is the

[4] This is the conformal mode in the Δ_0 spectrum [12].

[5] We also know that $SO(6, 2)$ is the conformal group in 6 dimensions. This identification will be very important in the next chapter.

Spin	$SO(5)$ rep	M^2	$n=0$	$n=1$	$n=2$
2^+	$(0,n)$	$n(n+3)$	1	5	14
$(\frac{3}{2})^{(1)}$	$(1,n)$	$(n+\frac{5}{4})^2$	4	16	40
$(\frac{3}{2})^{(2)}$	$(1,n-1)$	$(n+\frac{7}{4})^2$	-	4	16
$1^{-(1)}$	$(2,n)$	$(n+1)^2-1$	10	35	81
$1^{-(2)}$	$(2,n-2)$	$(n+2)^2-1$	-	-	10
$(\frac{1}{2})^{(1)}$	$(1,n+1)$	$(n+\frac{3}{4})^2$	16	40	80
$(\frac{1}{2})^{(2)}$	$(3,n-1)$	$(n+\frac{5}{4})^2$	-	20	64
$(\frac{1}{2})^{(3)}$	$(3,n-2)$	$(n+\frac{7}{4})^2$	-	-	20
$(\frac{1}{2})^{(4)}$	$(1,n-2)$	$(n+\frac{9}{4})^2$	-	-	4
$0^{+(1)}$	$(0,n+2)$	$(n-1)(n+2)$	14	30	55
$0^{+(3)}$	$(0,n-2)$	n^2+5n+4	-	-	1
$0^{+(2)}$	$(4,n-2)$	$n(n+3)$	-	-	35
$Ant^{(1)}$	$(2,n-1)$	$n(n+3)+2$	-	10	35
Ant^{+}	$(0,n-1)$	$(n+2)^2$	-	1	5
Ant^{-}	$(0,n+1)$	$(n+1)^2$	5	14	30

Table 4.3: The $AdS^7 \times S^4$ mass spectrum

In order to have the complete spectrum, I have taken the fermionic part of the mass spectrum from [19]. The $SO(5)$ representations are also taken from [19]. The last 3 colums shows the $SO(5)$ representations of the 3 first levels in the mass spectrum, i.e, the massless level ($n=0$) and the two first massive levels ($n=1,2$).

isometry group of S^4 implies that $SO(5)$ is the gauge group in this compactification. This happens because the mass operators are invariant under $SO(5)$.

In this section, I will not obtain the three first supermultiplets by using the supercharges as I did in the last section. What I will do instead is to check that the mass spectrum gives the correct supermultiplets with the correct number of d.o.f. If we compare the $n=0$ column in table 4.3 with column 2 in table 4.4, we easily see that the correct massless supermultiplet is obtained. Also calculating the d.o.f. for the bosons and fermions, gives that both the bosons and fermions have 128 d.o.f. each.[6] Calculating the d.o.f. for the bosons and the fermions when $n=1,2$ in table 4.3, using table 4.4, gives: 640 ($5\cdot 128$) d.o.f. each, when $n=1$, and 1792 ($14\cdot 128$) d.o.f. each, when $n=2$. This implies that $n=1,2$ in table 4.3 gives the correct two first massive supermultiplets.

Also in this compactification I have obtained short supermultiplets, which are obtained because of the central charges in the representation algebra, see [19].

If we now look at table 4.3 we see that all the modes obey the Breitenlohner-Freedman stability criterion, with equality for no modes.[7]

[6] The d.o.f. of Ant^- are included in the bosons d.o.f.

[7] The $0^{+(1)}$ scalars in the 14 representation of $SO(5)$ are the one closest to the Breitenlohner-Freedman boundary ($-2 > -\frac{9}{4}$).

Spin	Massless		Massive
	Mult.	d.o.f	d.o.f
2	1	14	20
$\frac{3}{2}$	4	16	20
1	10	5	6
$\frac{1}{2}$	16	4	4
0	14	1	1
$Ant3$	5	10	10
$Ant2$	-	10	15

Table 4.4: Multiplicity and d.o.f (degrees of freedom) for a $N = 4$ supermultiplet in 7 dimensions
The first column under "Massless" gives the multiplicity, while the second column give the number of d.o.f. In the massive case I have only included the d.o.f because the multiplicity varies for different massive supermultiplets. The number of d.o.f were obtained by using table 2 in [12]. $Ant2$ means an antisymmetric 2-form, while $Ant3$ means an antisymmetric 3-form.

Analogous to the last section, there is a smaller supermultiplet than the massless supermultiplet also in the $AdS_7 \times S^4$ compactification. This supermultiplet is called the doubleton, which consists of 5 scalars ($n = -1$ in $0^{+(1)}$), 8 spinors ($n = -1$ in $(\frac{1}{2})^{(1)}$) and 1 antisymmetric 3-form field ($n = -1$ in Ant^-). Analogous to the singeltons, these doubleton fields live on the boundary of AdS_7 and not in the bulk. The scalars originates from the $\Delta Y = 4m^2 Y$ modes that could be gauged away from $0^{+(1)}$ and the antisymmetric 3-form field originates from the $\Delta Y = 0$ mode that could be gauged away from Ant^-.

4.3 The $AdS_5 \times S^5$ mass spectrum

I will now obtain the $AdS_5 \times S^5$ mass spectrum and show that this mass spectrum corresponds to $N = 8$ supermultiplets (I will show this for the first 3 levels $n = 0, 1, 2$ in table 4.5). M^2 for the various fields is obtained by using table 3.4 and the eigenvalues of the Hodge-de Rham and the Lichnerowicz operators obtained in appendix C: See table 4.5 for the complete spectrum.

The symmetry of the vacuum for $AdS_5 \times S^5$ is described by the super group $SU(2,2|4)$, which have $SO(4,2) \times SO(6)$ as bosonic subgroup. Here $SO(4,2)$ is the isometry group of AdS_5[8] and $SO(6)$ is the isometry group of S^5. That $SO(6)$ is the isometry group of S^5 implies that $SO(6)$ is the gauge group in this compactification. This happens because the mass operators are invariant under $SO(6)$, which is isomorphic to $SU(4)$.

[8]We also know that $SO(4,2)$ is the conformal group in 4 dimensions. This identification will be very important in the next chapter.

Spin	M^2	$n=0$	$n=1$	$n=2$
2^+	$n(n+4)$	1	6	20
$(\frac{3}{2})^{(1)}$	n^2	4_c	20_c	60_c
$(\frac{3}{2})^{(2)}$	$(n+4)^2$	-	4_c	20_c
$1^{-(1)}$	$(n+1)^2-1$	15	64	175
$1^{-(2)}$	$(n+3)^2-1$	-	-	15
1^+	$n(n+4)+3$	-	15_c	64_c
$(\frac{1}{2})^{(1)}$	$(n+\frac{1}{2})^2$	20_c	60_c	$140'_c$
$(\frac{1}{2})^{(2)}$	$(n+\frac{3}{2})^2$	4_c	20_c	60_c
$(\frac{1}{2})^{(3)}$	$(n+\frac{3}{2})^2$	-	36_c	140_c
$(\frac{1}{2})^{(4)}$	$(n+\frac{5}{2})^2$	-	4_c	20_c
$(\frac{1}{2})^{(5)}$	$(n+\frac{5}{2})^2$	-	-	36_c
$(\frac{1}{2})^{(6)}$	$(n+\frac{7}{2})^2$	-	-	4_c
$0^{+(1)}$	$(n-2)(n+2)$	$20'$	50	105
$0^{+(3)}$	$(n+2)(n+6)$	-	-	1
$0^{+(2)}$	$n(n+4)$	1_c	6_c	$20'_c$
$0^{+(4)}$	$n(n+4)$	-	-	84
$0^{-(1)}$	$(n-1)(n+3)$	10_c	45_c	126_c
$0^{-(2)}$	$(n+1)(n+5)$	-	-	10_c
$Ant^{(1)}$	$(n+2)^2$	-	10	45
$Ant^{(2)}$	$(n+2)^2$	-	$\bar{10}$	$\bar{45}$
Ant^+	$(n+3)^2$	-	1_c	6_c
Ant^-	$(n+1)^2$	6_c	$20'_c$	50_c

Table 4.5: The $AdS^5 \times S^5$ mass spectrum

In order to have the complete spectrum, I have taken the fermionic masses from [20] and the $SU(4)$ representations from [19]. The last 3 columns shows the $SU(4)$ representations of the 3 first levels in the mass spectrum, i.e, the massless level ($n=0$) and the two first massive levels ($n=1,2$). The label c in some of the representations means that it is a complex representations, which implies that the dimension of the representations doubles, i.e. for example $\dim(6_c)=12$.

Spin	Massless		Massive
	Mult.	d.o.f	d.o.f
2	1	5	9
$\frac{3}{2}$	8	4	6
1	15	3	4
$\frac{1}{2}$	48	2	2
0	42	1	1
$Ant2$	12	3	3

Table 4.6: Multiplicity and d.o.f (degrees of freedom) for a $N = 8$ supermultiplet in 5 dimensions
The first column under massless gives the multiplicity, while the second column gives the number of d.o.f. In the massive case I have only included the d.o.f because the multiplicity varies for different massive supermultiplets. The number of d.o.f. were obtained by using table 2 in [12]. Here $Ant2$ means an antisymmetric 2-form. In this table in the massless case, it is possible to think of the $Ant2$ fields as spin 1 fields, since they have the same number of d.o.f. as spin 1 fields. If one now moves the twelve $Ant2$ fields to the spin 1 row, it is easy to see that one has obtained the same supermultiplet structure as in the massive AdS_4 case.

In this section, I am not going to obtain the three first supermultiplets by using the super charges as I did in section 4.1. I will only check that the mass spectrum in table 4.5 gives the correct supermultiplets with the correct number of d.o.f. If we compare the $n = 0$ column in table 4.5 with column 2 in table 4.6, we easily see that the correct massless supermultiplet is obtained. Also calculating the d.o.f. for the bosons and fermions, gives that both the bosons and fermions have 128 d.o.f. each. Calculating the d.o.f. for the bosons and the fermions when $n = 1, 2$ in table 4.5, using table 4.6, gives: 768 $(6 \cdot 128)$ d.o.f. each, when $n = 1$, and 2560 $(20 \cdot 128)$ d.o.f. each, when $n = 2$. This implies that $n = 1, 2$ in table 4.5 gives the correct two first massive supermultiplets.

Also in this compactification I have obtained short supermultiplets, which are obtained as a consequence of the central charges in the representation algebra.

If we now look at table 4.5 we see that all the modes obey the Breitenlohner-Freedman stability criterion, with equality only for the $0^{+(1)}$ scalars in the $20'$ representation of $SU(4)$. As I mentioned in section 3.6, the massless scalars $0^{+(1)}$, $0^{+(2)}$ and $0^{-(1)}$ have different "mass" ($-2m^2$, $0m^2$ and $-3m^2$). The reason this happens can be understood if one obtains the massless spectrum algebraically with group theory, see [24]. This is an example of how difficult it is to define what mass is in AdS spacetime.

Analogous to the last two sections, there is a smaller supermultiplet than the massless supermultiplet also in the $AdS_5 \times S^5$ compactification. This supermultiplet is called the doubleton, which consists of 6 real scalars ($n = -1$ in $0^{+(1)}$), 4 complex spinors ($n = -1$ in $(\frac{1}{2})^{(1)}$) and 1 complex antisymmetric 2-form field

Spin	$M^2 = 0$	$M^2 = 1$	$M^2 = 2$
2^+	1	7	21
$(\frac{3}{2})^{(1)}$	8	56	168
$1^{-(1)}$	7	42	126
$1^{-(2)}$	-	42	126
1^+	21	105	315
$(\frac{1}{2})^{(1)}$	56	336	1008
$0^{+(1)}$	1	7	21
$0^{+(2)}$	27	140	420
$0^{+(3)}$	7	-	-
$0^{+(4)}$	-	7	21
$0^{-(1)}$	35	140	420
Ant	1	-	-

Table 4.7: The $Minkowski_4 \times T^7$ mass spectrum
The second to the fourth column gives the number of the various modes in the first 3 levels. The mass is in units of r^{-2}, r is the radius of T^7. The massive fermion modes are included in order to have the complete mass spectrum.

($n = -1$ in Ant^-). Analogous to the singeltons, these doubleton fields live on the boundary of AdS_5 and not in the bulk. The scalars comes from the $\Delta Y = 5m^2 Y$ modes that could be gauged away from $0^{+(1)}$ and the complex antisymmetric 2-form field comes from the $\Delta Y = 0$ mode that could be gauged away from Ant^-.

4.4 The $Minkowski_4 \times T^7$ mass spectrum

Next, I will obtain the $Minkowski_4 \times T^7$ mass spectrum and show that the mass spectrum corresponds to $N = 8$ massless and massive supermultiplets. M^2 for the various bosonic fields is obtained by using table 3.2 and the eigenvalues of the Hodge-de Rham and the Lichnerowicz operators obtained in appendix C. I have also used that the number of zero modes for the different operators are

$$\Delta_0 \to 1, \quad \Delta_1 \to 7, \quad \Delta_2 \to 21, \quad \Delta_L \to 27, \tag{4.4}$$

and that the number of modes coming from the various operators in a specific massive level are

$$\Delta_0 \to 1, \quad \Delta_1 \to 6, \quad \Delta_2 \to 15, \quad \Delta_L \to 20, \quad \Delta_3 \to 20 \tag{4.5}$$

multiplied with the degeneracy for that level. See table 4.7 for the complete spectrum. The gauge group for this compactification is $[U(1)]^7$, which is the isometry group of T^7.

I will now show an alternative way to obtain the complete massless supermultiplet and not just the bosonic part.

The massless supermultiplet can easily be obtained by first realizing that the spherical harmonics and the tensor spherical harmonics are independent of the angles when $n_i = 0, i = 1, \ldots, 7$. This makes it possible to obtain the complete massless supermultiplet directly from the following fields: h_{MN}, a_{MNP} and Ψ^A_M, where $A = 1, \ldots, 32$ is the spinor index.

The first field h_{MN} gives: $h_{\mu\nu}$, which is the graviton, $h_{\mu n}$ gives 7 vectors B_μ (because $n = 4, \ldots, 10$) and h_{mn} gives 28^+ scalars ϕ (because a symmetric 2-indices tensor have 28 independent components when $m = 4, \ldots, 10$).

The antisymmetric tensor a_{MNP} gives: $a_{\mu\nu\rho}$ (this field have no d.o.f), $a_{\mu\nu p}$ gives 7 antisymmetric tensor fields $a_{\mu\nu}$, which in 4 dimensions can be dualised to 7^+ scalars $\tilde{a}$, $a_{\mu np}$ gives 21 vectors a_μ (because a antisymmetric 2-indices tensor have 21 independent components when $m = 4, \ldots, 10$) and a_{mnp} gives 35^- pseudoscalars a (because a antisymmetric 3-indices tensor have 35 independent components when $m = 4, \ldots, 10$).

The fermionic field Ψ^A_M gives: $\Psi^a_{\mu\alpha}$ $a = 1, \ldots, 8, \alpha = 1, 2, 3, 4$, where a and α are spinor indices, gives 8 spin $\frac{3}{2}$ gravitinos $\psi_{\mu\alpha}$ and $\Psi^a_{m\alpha}$ gives 56 spin $\frac{1}{2}$ spinors χ_α (because $7 \cdot 8 = 56$, where 7 comes from the m index and 8 from the a index).

The massless supermultiplet obtained here is seen in column 2 in table 4.7, where $h_{\mu\nu}$ correspond to 2^+, $\psi_{\mu\alpha}$ to $(\frac{3}{2})^{(1)}$, B_μ to $1^{-(1)}$, a_μ to 1^+, χ_α to $(\frac{1}{2})^{(1)}$, ϕ to $0^{+(1)}$ and $0^{+(2)}$, $\tilde{a}$ to $0^{+(3)}$, a to $0^{-(1)}$ and $a_{\mu\nu\rho}$ to Ant.

Calculating the number of d.o.f gives 128 bosonic and 128 fermionic d.o.f for the massless level and 896+896 for the first massive level and 2688+2688 for the second massive level. This result was of course expected since I have obtained $N = 8$ supermultiplets.

4.5 The massless $Minkowski_4 \times (K3 \times T^3)$ spectrum

Finally, I am going to obtain the massless spectrum[9] of the compactification of 11D supergravity on $K3 \times T^3$, and show that this spectrum corresponds to an $N = 4$ supermultiplet $(1, 4, 6, 4, 1 + 1)$[10] coupled to 22 $N = 4$ spin 1 matter multiplets $(1, 4, 3 + 3)$,[11] see table 4.8 for the spectrum. The metric for the Ricci flat space $K3$ is not known but there is an existence proof. What is known about $K3$ is that it has 58 parameters, a self-dual Riemann tensor and no continuous symmetries. It is also known to have Euler number $\chi = 24$, Hirzebruch signature $\tau = 16$ and the following betti numbers

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = 22, \quad b_3 = 0, \quad b_4 = 1 \tag{4.6}$$

[9] All the results in this section, except the results for the a_{MNP} field and the T^3 betti numbers, are taken from [12, 25].

[10] The number of spin 2 to spin 0 particles are written from left to right. In the column for the scalars means 1+1 that there is 1 scalar and 1 pseudoscalar.

[11] The number of spin 1 to spin 0 particles are written from left to right.

D=11	d=4	Spin	$M^2 = 0$
h_{MN}	$h_{\mu\nu}$	2	1
	B_μ	1	3
	ϕ	0	64
Ψ_M^A	$\psi_{\mu\alpha}$	3/2	4
	χ_α	1/2	92
a_{MNP}	$a_{\mu\nu\rho}$	-	1
	$\tilde{a}$	0	3
	a_μ	1	25
	a	0	67

Table 4.8: The $Minkowski_4 \times (K3 \times T^3)$ massless mass spectrum

As in the last section, the $Minkowski_4 \times (K3 \times T^3)$ spectrum will be obtained from the fields h_{MN}, a_{MNP} and Ψ_M^A, but in a different way.

The field h_{MN} gives: one graviton $h_{\mu\nu}$, since Δ_0 have one zero mode, $h_{\mu n}$ gives 3 vectors B_μ, since T^3 have 3 killing vectors and $K3$ have none, h_{mn} gives 64^+ scalars ϕ, since $K3$ have 58 parameters and T^3 have 6.

To obtain the fields from a_{MNP} I have to use the formula for betti numbers of a product space $M = M' \times M''$, which is [12]

$$b_p = \sum_{r=0}^{p} b'_r b''_{p-r} \tag{4.7}$$

The space T^3 have betti numbers b_p=1, 3, 3, 1, where $p = 0, 1, 2, 3$. I can now obtain the fields given by a_{MNP} by using that the numbers of fields from $a_{\mu\nu\rho}$, $a_{\mu\nu p}$, $a_{\mu np}$ and a_{mnp} are given by the betti numbers b_0, b_1, b_2 and b_3. This implies that: $a_{\mu\nu\rho}$ gives 1 field $a_{\mu\nu\rho}$ (this field has no d.o.f.), $a_{\mu\nu p}$ gives 3 antisymmetric tensor fields $a_{\mu\nu}$, which in 4 dimensions can be dualised to 3^+ scalars $\tilde{a}$, $a_{\mu np}$ gives 25 vectors a_μ and a_{mnp} gives 67^- pseudoscalars a.

The field Ψ_M^A gives 4 spin $\frac{3}{2}$ gravitinos $\psi_{\mu\alpha}$, from 4 killing spinors on $K3 \times T^3$ (2 on $K3$×2 on T^3), and 92 spin $\frac{1}{2}$ spinors χ_α, from 40 on $K3 \times 2$ killing spinors on T^3 plus 6 on $T^3 \times 2$ killing spinors on $K3$.

By looking at the spectrum in table 4.8 we see that it corresponds to an $N = 4$ supermultiplet $(1, 4, 6, 4, 1 + 1)$ coupled to 22 $N = 4$ spin 1 matter multiplets $(1, 4, 3 + 3)$.

Calculating the number of d.o.f. for this compactification gives 192 bosonic and 192 fermionic d.o.f.

The reason that I have not obtained the massive spectrum, is because $K3$ has no symmetries, which makes it impossible to obtain the massive spectrum by group theory. The only thing one can say about the massive states, is that they must fall into $N = 4$ massive supermultiplets. For a more complete investigation of this compactification see [25].

5

The AdS_{d+1}/CFT_d correspondence

The AdS_{d+1}/CFT_d correspondence (sometimes called the Maldacena conjecture) was first proposed by Maldacena [26] and later more rigorously defined by Gubser, Klebanov and Polyakov [27] and Witten [28]. The correspondence describes a duality relation between a CFT (conformal field theory) living on the d-dimensional boundary (which is compactified $Minkowski_d$) of AdS_{d+1}, and a compactified string or M-theory with vacuum $AdS_{d+1} \times M^{D-(d+1)}$. The implication of this duality is that the string or M-theory in $AdS_{d+1} \times M^{D-(d+1)}$ describes the same physics as the corresponding CFT in $Minkowski_d$. It also implies that if we have a conformal field theory we can obtain the corresponding string theory by using the AdS_{d+1}/CFT_d correspondence, or vice versa. To do this is in general very difficult, and no one has succeeded so far. What one has managed to do, however, is to use the correspondence in the large N-limit (this will be explained below) where one can use supergravity as a valid approximation of string or M-theory.

One of the things physicists hope to achive with this correspondance is to be able to describe QCD (quantum chromodynamics) with a string theory. The reason for this, is that then it would be possible to describe non-perturbative physics in QCD with perturbative string theory, which would be a tremendous achievement, since it is usually easier to perform calculations perturbatively then non-perturbatively.

In this chapter, I am first going to give an introduction to the AdS_{d+1}/CFT_d correspondence in 5.1. Then in 5.2, I will obtain the scaling dimensions for the operators corresponding to scalars in the dual supergravity theories on $AdS_{4,5,7} \times S^{7,5,4}$, by using the mass spectra obtained in chapter 4.1-4.3. Finally, deformations of conformal field theories or the dual supergravity theories will be analyzed in 5.3.

5.1 Introduction to the AdS_{d+1}/CFT_d correspondence

In this section, I will try to give convincing arguments why the AdS_{d+1}/CFT_d correspondence is valid, and I will also give a short introduction to some of the consequences of the correspondence.

I will begin by showing that it is very likely that type IIB string theory compactified on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super-Yang-Mills theory in $Minkowski_4$. Consider N parallell D3-branes that are located very close to each other. These D3-branes are extended along a (3+1)-dimensional plane in (9+1)-dimensional spacetime. If we now consider string theory in this background with open and closed strings, where the closed strings describe perturbatively the excitations of empty space and the open strings ending on the D3-branes describe perturbatively the excitations of the D3-branes. If the system has low energy (energies lower then the string scale $1/l_s$, were l_s is the string length) then it is possible to write an effective Lagrangian describing the interactions, since only the massless string states can be excited. Here the massless open string states give an $\mathcal{N} = 4$ vector supermultiplet in (3+1)-dimensions, which have $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory[1] as the low energy effective Lagrangian. This theory has a $U(N)$ gauge group since the excitations of one D3-brane give a $U(1)$ gauge group, which implies that N parallell D3-branes stacked together give a $U(N)$ gauge group. The massless closed string states on the other hand give a gravity supermultiplet in (9+1)-dimensions, with type IIB supergravity as the low energy effective Lagrangian. The complete effective action of the massless modes have the following form

$$S = S_{bulk} + S_{brane} + S_{int} \tag{5.1}$$

where S_{bulk} is the supergravity action plus some higher derivative corrections, S_{brane} is the $\mathcal{N} = 4$ $U(N)$ SYM (super-Yang-Mills) action plus some higher derivative corrections, and S_{int} describes the interaction between the bulk modes and the brane modes. We now let $l_s \to 0$ ($\alpha' \to 0$) and keep the energy and all the dimensionless parameters fixed, including the string coupling constant g_s and N. In this limit the coupling $\kappa \sim g_s\alpha'^2 \to 0$, where κ is the κ in $g_{MN} = \eta_{MN} + \kappa h_{MN}$, i.e., κ gives the strength of the fluctuations. This implies that the interacting Lagrangian relating the bulk and the brane vanishes. It also implies that the action S_{brane} only contains pure $\mathcal{N} = 4$ $U(N)$ SYM, since the higher derivative terms in the brane action vanishes in the limit above. This pure $\mathcal{N} = 4$ $U(N)$ SYM theory is a super conformal field theory. The final implication of the limit above, is that the supergravity theory becomes a free theory in the bulk. The conclusion so far is that we have a $\mathcal{N} = 4$ $U(N)$ SYM theory on the branes that is not interacting with the free supergravity theory in the bulk.

Now will look at the same system from a different point of view. As I have shown in chapter 2.3, the D3-brane metric and field strength can be obtained from the type IIB supergravity equations of motion. Equation (2.85) and (2.89) gives the following metric for N parallel D3-branes stacked together

$$ds^2 = f^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu x^\nu + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \quad f = 1 + \frac{k'}{r^4} \tag{5.2}$$

[1]The N in $U(N)$ is from the number of D3-branes, while the $\mathcal{N}$ in $\mathcal{N} = 4$ SYM is the number of supersymmetries.

where k' can be shown to be [29]: $k' = kN = 4\pi g_s \alpha'^2 N$. If we now analyze this equation we see that the g_{tt} part of the metric is not constant. This implies that the relation between the energy E measured by an observer at infinity ($r = \infty$) and the energy E_p measured by an observer at constant r, are

$$E = f^{-\frac{1}{4}} E_p \tag{5.3}$$

The implication of this, is that the closer an object is brought to the horizon ($r = 0$) of the brane, the lesser energy is measured by an observer at infinity ($E = f^{-\frac{1}{4}} E_p \to 0$, if $r \to 0$). We now take the low energy limit in the background (5.2). Then there are two kinds of massless excitations (seen by an observer at infinity) which are decoupled from each other. The first kind are excitations with any wavelength that are brought closer and closer to the horizon, while the other kind are excitations with high wavelength propagating in the bulk. What we now have is a low energy theory that consists of two decoupled pieces, one free bulk supergravity and the near horizon geometry of the D3-branes. Here the near horizon geometry of the D3-branes are $AdS_5 \times S^5$, which is seen by approximating f in (5.2) with $f \sim \frac{k'}{r^4}$, when $r \ll k'^{\frac{1}{4}}$. This changes (5.2) to

$$ds^2 = \frac{r^2}{k'^{\frac{1}{2}}} \eta_{\mu\nu} dx^\mu x^\nu + \frac{k'^{\frac{1}{2}}}{r^2} dr^2 + k'^{\frac{1}{2}} d\Omega_5^2 \tag{5.4}$$

where $k'^{\frac{1}{2}}$ is the square of the radius of AdS_5 and S^5.

What we have learned so far is that both from the point of view of the supergravity description of the brane (the latter example), and the field theory description of open strings living on the brane (the former example), there are two decoupled theories in the low energy limit. In both cases one of the decoupled theories is supergravity in flat space. But the description of the other theory looks very different in the two cases. The former is a pure $\mathcal{N} = 4$ $U(N)$ SYM in (3+1)-dimensions, and the latter is type IIB supergravity on $AdS_5 \times S^5$. Since both these theories are supposed to describe the same physics (but from a different point of view) it is natural to make the following conjecture (the Maldacena conjecture) [26]:

$\mathcal{N} = 4$ $U(N)$ SYM in (3+1)-dimensions is the same as (or dual to) type IIB superstring theory on $AdS_5 \times S^5$.

The discussion above makes it plausible that the conjecture is true, but it is not a proof.

Now I will discuss what kind of approximations one can make if one wants to calculate perturbatively. It can be shown (see [29]) that one can trust perturbative calculations in $\mathcal{N} = 4$ $U(N)$ SYM theory when

$$g_{YM}^2 N \sim g_s N \sim \frac{k}{l_s^4} \ll 1 \tag{5.5}$$

where g_{YM} is the Yang-Mills coupling and $g_{YM}^2 N$ is the 't Hooft coupling. On the other hand, the supergravity approximation of superstring theory becomes reliable when the radius of curvature $k'^{\frac{1}{4}}$ of AdS_5 and S^5 becomes large compared to the string length l_s, which is seen from [29]

$$\frac{k}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1 \tag{5.6}$$

The implication of (5.5)-(5.6)[2] is that the AdS_{d+1}/CFT_d correspondence is a correspondence between a weakly coupled string theory and a strongly coupled field theory, or vice versa. This means that in the limit when supergravity is a good approximation of superstring theory, it is possibly to obtain non-perturbative information about the strongly coupled field theory. This is one feature that makes the correspondence very useful, but also very hard to prove.

We can see from (5.6) that N must be very large in order for the supergravity approximation of superstring theory to be valid. When I talked about the large N-limit in the introduction, what I meant was that $\lambda = g_{YM}^2 N$ was fixed but large and $N \to \infty$. The reason to take the limit $N \to \infty$ and not N large but finite, is becauce in this limit it is possible to check if the correspondence is valid. For example, in the limit above chiral fields[3] in the strongly coupled (3+1)-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM[4] are dual to Kaluza-Klein harmonics on $AdS_5 \times S^5$. In this case irrelevant, marginal, and relevant operators in the field theory correspond to massive, massless, and "tachyonic" modes in the supergravity theory. The "tachyonic" modes have $M^2 < 0$, but M^2 is larger then the Breitenlohner-Freedman bound. The stringy excitations of $AdS_5 \times S^5$ corresponds to operators whose scaling dimensions diverge for $N \to \infty$, when λ is large (i.e. non-chiral operators). Witten showed in [28] that the spectrum of Kaluza-Klein excitations on $AdS_5 \times S^5$, as I have computed in chapter 4.3, can be matched precisely with certain operators in the field theory. For example, the massless graviton field corresponds to the energy-momentum tensor in the field theory and the massless $SU(4)$ gauge fields corresponds to the global $SU(4)$ currents in the field theory. This matching of operators with excitations (modes) in the supergravity theory gives a strong indication that the conjecture is valid (at least in the large N-limit).

Another fact that gives the correspondence credibility is that both the $\mathcal{N} = 4$ $SU(N)$ SYM theory and the supergravity theory have the same supergroup $SU(2,2|4)$, with bosonic subgroup $SO(4,2) \times SO(6)$. As was mentioned in chapter 4, $SO(4,2)$ is the AdS_5 isometry group and $SO(6)$ is the S^5 isometry group. For the field theory $SO(4,2)$ is the conformal group and $SO(6) \simeq SU(4)$ is the R-symmetry group. This means that the theories have the same global symmetries.

Above, in the description of the correspondence between $\mathcal{N} = 4$ $SU(N)$ SYM and supergravity on $AdS_5 \times S^5$, the gauge group in the field theory had to be $SU(N)$ and not $U(N)$. This happens because in order for the supergravity theory to be dual to a $U(N)$ field theory, the doubleton modes on the boundary must be included in the

[2] In these equations I have assumed that $g_s < 1$. If $g_s > 1$ then we can perform an $SL(2,\mathbb{Z})$ duality transformation and get conditions similer to (5.5)-(5.6) but with $g_s \to \frac{1}{g_s}$.

[3] i.e, fields in small representations of the supersymmetry algebra.

[4] Why the gauge group is $SU(N)$ and not $U(N)$ is explained below.

supergravity spectrum. This is the case because an $U(N)$ gauge theory is essentially equivalent to a free $U(1)$ vector multiplet times a $SU(N)$ gauge theory, where the $U(1)$ vector multiplet is the doubleton. But the doubleton is not included in type IIB supergravity on $AdS_5 \times S^5$, which implies that the correspondence in the case above is between a $SU(N)$ field theory and supergravity.

The AdS_{d+1}/CFT_d correspondence obeys the holographic principle.[5] What this means is that the conformal field theory on the d-dimensional boundary of AdS_{d+1} can be used to describe physics in $(d+1)$-dimensional AdS_{d+1} spacetime. This implies that the d.o.f. in the boundary theory can be exactly matched with the d.o.f. in the bulk theory. This is analogous to the entropy of a black hole, which is proportional to the surface area of the black hole (and not to the volume). In [30], it was found that if one analyze how the information in the boundary field theory corresponds to the information in the bulk supergravity theory, one obtains that long distance effects in the bulk theory correspond to short distance effects in the boundary theory, and vice versa. This is called the UV-IR correspondence.

Now, I will discuss how scalars in the supergravity theory in $(d+1)$-dimensions are related to operators in the conformal field theory in d-dimansions. In the discussion below I will use the following form of the AdS_{d+1} metric,

$$ds^2 = \frac{R^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu), \quad \mu = 0, \ldots, d-1. \tag{5.7}$$

where the boundary is at $z = 0$ and R is the radius. In the bulk theory there can be two kinds of solutions to the bulk scalar wave equation

$$(\frac{1}{m^2}\Box - m'^2)\phi(z, x) = 0 \tag{5.8}$$

where m is the inverse radius of AdS_{d+1}. The first solution are propagating normalizable modes and the second are non-normalizable modes. The former modes $\phi^+(z, x)$, describe physical low energy excitations of spacetime. These modes can be shown to behave as z^{Δ_+} near the boundary $(z = 0)$ of AdS_{d+1}, where Δ_+ is the conformal weight of the corresponding operator. Here Δ_+ is the plus solution of the equation $\Delta(\Delta - d) = m'^2$, which has the following solutions

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m'^2}, \quad m'^2 = \frac{M^2}{m^2} \tag{5.9}$$

We see here that this equation gives a relation between the scaling dimension of an operator in the field theory, and M^2 of the corresponding scalar in the supergravity theory.

The non-normalizable modes $\phi^-(z, x)$ behave as $z^{\Delta_-}\phi_0(x)$ near the boundary $(z = 0)$ of AdS_{d+1}, where Δ_- is the minus solution in (5.9) and $\phi_0(x)$ is the boundary (sourse) field that corresponds to $\phi(z, x)$.

The normalizable modes can be shown to correspond to states in the field theory, and non-normalizable modes to sources, which implies that the non-normalizable

[5] The name holographic is from the analogy with an ordinary hologram.

modes act as backgrounds in the supergravity theory. As a consequence of this, if we set the non-normalizable modes to zero then we can study states of the original unmodified conformal field theory. But if we instead turn on the non-normalizable modes, then we have introduced a source in the conformal field theory, which deformes the theory (i.e, the scalar fixpoints are changed, see 5.3 for an explanation of scalar fixpoints). For a more thorough investigation of how scalars and the corresponding operators are related see the next section and [31, 32].

As a conclusion of this section, I will have a short discussion about which field theories are dual to M-theory (11D supergravity in the low energy limit) on $AdS_4 \times S^7$ or $AdS_7 \times S^4$.

M-theory on $AdS_4 \times S^7$ is conjectured to be dual to the $\mathcal{N} = 8$ $d = 3$ super conformal field theory with supergroup $OSp(4|8)$, while M-theory on $AdS_7 \times S^4$ is conjectured to be dual to the $\mathcal{N} = (0,2)$ $d = 6$ super conformal field theory with supergroup $OSp(6,2|4)$. These conjectures are not as well tested as the conjecture in AdS_5 but are belived to be correct. The reason that the conjecture in AdS_5 is better tested, is becauce the $\mathcal{N} = 4$ SYM theory is much more well understood then the field theories in $d = 3$ and $d = 6$.

5.2 Calculation of scaling dimension for operators dual to scalars

In this section, I am going to obtain the scaling dimension Δ for operators in the field theory that are dual to scalars in the supergravity theory on $AdS_{4,5,7} \times S^{7,5,4}$. I will also show that the lowest energy E_0 for the scalars is $E_0 = \Delta$.[6]

I will start by obtaining E_0 in the $AdS_4 \times S^7$ case, where there are 5 towers of scalars, $0^{+(1)}$, $0^{+(2)}$, $0^{+(3)}$, $0^{-(1)}$ and $0^{-(2)}$. M^2 for the various towers are shown in table 4.1. The lowest energy E_0 for scalars in these towers is obtained by using [12]

$$E_0 = \frac{3}{2} \pm \sqrt{1 + \frac{M^2}{m^2}}, \quad E_0 \geq \frac{1}{2} \tag{5.10}$$

We see here that as long as $E_0 \geq \frac{1}{2}$, then both the plus and the minus solutions are valid. How one knows which solution to choose will be shown below. Inserting M^2 for the various towers of modes in equation (5.10) gives

$$\begin{aligned}
0^{+(1)} &: \quad E_0 = \frac{3}{2} \pm \frac{|n-1|}{2}, \quad n = 0, 1, 2, \dots \\
0^{+(2)} &: \quad E_0 = \frac{n+6}{2}, \quad n = 2, 3, 4, \dots \\
0^{+(3)} &: \quad E_0 = \frac{n+10}{2}, \quad n = 2, 3, 4, \dots \\
0^{-(1)} &: \quad E_0 = \frac{3}{2} \pm \frac{n+1}{2}, \quad n = 0, 1, 2, \dots
\end{aligned}$$

[6]Here E_0 is dimensionless, to obtain the correct dimension multiply with m, which is the inverse radius of S^7.

$$0^{-(2)} \quad : \qquad E_0 = \frac{n+8}{2}, \quad n = 2,3,4,\dots \tag{5.11}$$

We see here that for the $0^{+(1)}$ modes with $n = 0,1,2,3$ and the $0^{-(1)}$ modes with $n = 0,1$ there are two possible solutions. I am now going to show which solutions are the correct ones (I will do it for $n = 0,1$). In order to obtain the correct solutions we have to see which representations of the $N = 8$ supersymmetry algebra the scalars are in. In the case of $N = 1$, all linear unitary irreducible representations of AdS supersymmetry have been classified by Heidenreich [33]. They are

$$\text{Class1}: \quad D\Big(\frac{1}{2},0\Big) \oplus D\Big(1,\frac{1}{2}\Big); \tag{5.12}$$

$$\text{Class2}: \quad D(E_0,0) \oplus D\Big(E_0+\frac{1}{2},\frac{1}{2}\Big) \oplus D(E_0+1,0), \quad E_0 > \frac{1}{2}; \tag{5.13}$$

$$\text{Class3}: \quad D(s+1,s) \oplus D\Big(s+\frac{3}{2},s+\frac{1}{2}\Big), \quad s = \frac{1}{2},1,\frac{3}{2},\dots; \tag{5.14}$$

$$\begin{aligned}\text{Class4}: \quad & D(E_0,s) \oplus D\Big(E_0+\frac{1}{2},s+\frac{1}{2}\Big) \oplus D\Big(E_0+\frac{1}{2},s-\frac{1}{2}\Big) \\ & \oplus D(E_0+1,s).\end{aligned} \tag{5.15}$$

Here E_0 is the lowest energy and s is the spin. Class 1 is the singleton multiplet and will not be used here. Class 2 is the Wess-Zumino supermultiplet, which is massless for $E_0 = 1$ and massive otherwise. Class 3 is the massless gauge supermultiplet with spins s and $s+\frac{1}{2}$, with $s \geq \frac{1}{2}$. Finally, class 4 is the massive higher spin supermultiplet.

Now we could decompose the $N = 8$ representations into $N = 1$ representations, in order to find out the correct E_0 for $0^{+(1)}$ and $0^{-(1)}$. But this is not necessary; actually it is enough to make a branching of the $SO(8)$ representations into $SO(7)$ representations, in order to see the general structure (see below). Starting with the massless ($n = 0$) supermultiplet gives[7]

$$\begin{array}{ll} 2 & 1 \to 1 \\ \frac{3}{2} & 8_s \to 1+7 \\ 1 & 28 \to 7+21 \\ \frac{1}{2} & 56_s \to 21+35 \\ 0 & 35_v + 35_c \to 35+35 \end{array} \tag{5.16}$$

Here we see that 1 in spin 2 and 1 in spin 3/2 must be in the class 3 representation, 7 in spin 3/2 and 7 in spin 1 in class 3, and that also 21 in spin 1 and 21 in spin 1/2 are in class 3. The scalars 35 and 35, and 35 in spin 1/2 on the other hand are in class 2 with $E_0 = 1$. This means that the scalars have $E_0 = 1$ and $E_0 = 2$ respectively. Here $E_0 = 1$ is the lowest energy for $0^{+(1)}$ since these are towers of scalar modes, and $E_0 = 2$ is the lowest energy for $0^{-(1)}$ since these are towers of pseudoscalar modes. The conclusion of this discussion is that in the massless level the minus solution is

[7]The branching of $SO(8) \to SO(7)$ is obtained from table 36 in [22].

included in $0^{+(1)}$, and for $0^{-(1)}$ the plus solution is included. Continuing with the $n = 1$ level gives

$$\begin{array}{ll} 2 & 8_v \to 8 \\ \frac{3}{2} & 8_c + 56_c \to 8 + 8 + 48 \\ 1 & 56_v + 160_v \to 8 + 48 + 48 + 112 \\ \frac{1}{2} & 160_c + 224_{vc} \to 48 + 112 + 112 + 112' \\ 0 & 224_{cv} + 112_v \to 112 + 112' + 112' \end{array} \tag{5.17}$$

Here are 8 in spin 2, 8 in spin 1 and both 8:s in spin 3/2 in the class 4 representation, with $E_0 = 3$ and $s = \frac{3}{2}$. We can also see that the 48:s in spin 3/2, 1 and 1/2 are in class 4 with $E_0 = \frac{5}{2}$ and $s = 1$, and that the 112:s in spin 1, 1/2 and 0 are in class 4 with $E_0 = 2$ and $s = \frac{1}{2}$. Finally, the $112'$:s in spin 1/2 and 0 are in the class 2 representation with $E_0 = \frac{3}{2}$. The conclusion of this is that $E_0 = \frac{5}{2}$ is included in $0^{-(1)}$, which is the plus solution. For $0^{+(1)}$ $E_0 = \frac{3}{2}$ for both the plus and the minus solutions.

If one now repeats the discussion above for $n = 2, 3$ one will find that only the plus solutions can be included in $0^{+(1)}$, which implies that the lowest energy E_0 for $0^{+(1)}$ and $0^{-(1)}$ are

$$\begin{array}{lcll} 0^{+(1)} & : & E_0 = \dfrac{n+2}{2}, & n = 0, 1, 2, \dots \\ 0^{-(2)} & : & E_0 = \dfrac{n+4}{2}, & n = 0, 1, 2, \dots \end{array} \tag{5.18}$$

Now it is time to obtain the scaling dimensions for operators corresponding to scalars in the dual theory, by showing that $\Delta = E_0$. To show this, I will use a formula for Δ that was obtained by Breitenloher and Freedman in [18]. The formula and the mass definition for a scalar field ϕ are

$$\Delta_\pm = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 - 4\alpha}, \quad \alpha \leq \frac{9}{4}, \quad \Delta_\pm > \frac{1}{2}, \tag{5.19}$$

and

$$-\Box\phi - \alpha\tilde{m}^2\phi = 0. \tag{5.20}$$

where $\alpha = 2$ gives a massless scalar field. The mass definition that is used in chapter 3.1 is

$$-\Box\phi + (M^2 - 8m^2)\phi = 0 \tag{5.21}$$

If we compare (5.20) and (5.21) and use that $\alpha = 2$ and $M^2 = 0$ gives massless scalar fields, we see that $M^2 = 8m^2 - 4\alpha m^2$. Inserting this relation into (5.10) gives

$$E_0 = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 - 4\alpha} \tag{5.22}$$

Comparing this equation with (5.19) gives that the scaling dimension $\Delta = E_0$. This result is summarized in table 5.1.

Field	Scaling dimension $\Delta(=E_0)$
$0^{+(1)}$	$(n+2)/2, \quad n=0,1,\ldots$
$0^{+(2)}$	$(n+6)/2, \quad n=2,3,\ldots$
$0^{+(3)}$	$(n+10)/2, \quad n=2,3,\ldots$
$0^{-(1)}$	$(n+4)/2, \quad n=0,1,\ldots$
$0^{+(1)}$	$(n+8)/2, \quad n=2,3,\ldots$

Table 5.1: The scaling dimensions for operators corresponding to scalars in the dual supergravity theory on $AdS_4 \times S^7$

Field	Scaling dimension $\Delta(=E_0)$
$0^{+(1)}$	$2(n+2), \quad n=0,1,\ldots$
$0^{+(2)}$	$2(n+3), \quad n=2,3,\ldots$
$0^{+(3)}$	$2(n+4), \quad n=2,3,\ldots$

Table 5.2: The scaling dimensions for operators corresponding to scalars in the dual supergravity theory on $AdS_7 \times S^4$

Now the careful reader probably wonder why I did not use equation (5.9) to obtain the scaling dimensions. There is a simple reason for this. If I would have used (5.9) with $m'^2 = M^2/m^2$, I would have obtained the wrong result. To get the correct result, one first have to scale m'^2 with 1/4, i.e, $m'^2 \to \frac{1}{4}m'^2$ and then shift with $-8m'^2$. The shift has to be made since there is a shift with $-8m^2$ in (5.21), and the scaling originate from the difference in the definition of radius ($r = m^{-1}$) in AdS_4 and S^7. The Riemann tensors that have been used for AdS_4 and S^7 in the earlier chapters are

$$R_{\mu\nu\rho\sigma} = 4m^2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{5.23}$$

and

$$R_{mnpq} = m^2(g_{mp}g_{nq} - g_{mq}g_{np}), \tag{5.24}$$

where the 4 in (5.23) is the reason for the scaling of m'^2. We see from the discussion above that it is very important to keep track of the different definitions of mass in AdS.

I will now obtain the scaling dimensions for operators corresponding to scalars in $AdS_7 \times S^4$. I will show that $E_0 = \Delta$ also in this case. The lowest energy E_0 for the three scalar towers can be fund in table 5.2, where I have used

$$E_0 = 3 \pm \frac{1}{2}\sqrt{36 + 16\frac{M^2}{m^2}}, \quad E_0 \geq 2 \tag{5.25}$$

Here it can be shown that only the plus solutions are included (I have not done that myself, but the derivation is analogous to the $AdS_4 \times S^7$ case).

I am now going to obtain the scaling dimensions by showing that $\Delta = E_0$. To show this, I will use a formula for Δ that was obtained by Breitenloher and Freedman

Field	Scaling dimension $\Delta(=E_0)$
$0^{+(1)}$	$n+2, \quad n=0,1,\ldots$
$0^{+(2)}$	$n+4, \quad n=0,1,\ldots$
$0^{+(3)}$	$n+6, \quad n=2,3,\ldots$
$0^{+(4)}$	$n+4, \quad n=2,3,\ldots$
$0^{-(1)}$	$n+3, \quad n=0,1,\ldots$
$0^{-(2)}$	$n+5, \quad n=2,3,\ldots$

Table 5.3: The scaling dimensions for operators corresponding to scalars in the dual supergravity theory on $AdS_5 \times S^5$

in [18]. The formula for a scalar field ϕ is

$$\Delta_{\pm} = 3 \pm \frac{1}{2}\sqrt{36-4\alpha}, \quad \alpha \leq 9, \quad \Delta_{\pm} > 2, \tag{5.26}$$

with mass defined as in (5.20). If we compare (5.20) with the mass definition in chapter 3.3, we see that $M^2 = -\frac{1}{4}\alpha m^2$. Inserting this relation into (5.25) gives

$$E_0 = 3 \pm \frac{1}{2}\sqrt{36-4\alpha} \tag{5.27}$$

Comparing this equation with (5.26) gives that the scaling dimension $\Delta = E_0$.

We see that also in this case we have to make a scaling of m'^2, if we want to obtain the correct result from equation (5.9). The scaling that has to be made is: $m'^2 \to 4m'^2$. The reason for this, is the difference of radius in AdS_7 and S^4. Here the Riemann tensor for S^4 is defined as in (5.24), and for AdS_7 as in (5.23) except that $4m^2$ is changed to $\frac{1}{4}m^2$.

Next, it is time to obtain the scaling dimensions for operators corresponding to scalars in $AdS_5 \times S^5$, and show that $E_0 = \Delta$. The lowest energy E_0 for the six scalar towers can be fund in table 5.3, where I have used

$$E_0 = 2 \pm \sqrt{4 + \frac{M^2}{m^2}}, \quad E_0 \geq 1, \tag{5.28}$$

where only the plus solutions are included.

Here I will obtain the scaling dimensions by showing that $\Delta = E_0$. To do this, I use the 5 dimensional version of (5.19), which is

$$\Delta_{\pm} = 2 \pm \sqrt{4-\alpha}, \quad \alpha \leq 4, \quad \Delta_{\pm} > 1, \tag{5.29}$$

with mass defined as in (5.20). If we compare (5.20) with the mass definition in chapter 3.4, we see that $M^2 = -\alpha m^2$. Inserting this relation into (5.29) gives

$$E_0 = 2 \pm \sqrt{4-\alpha} \tag{5.30}$$

which implies that the scaling dimension $\Delta = E_0$.

We see here that in this case we do not have to make a scaling if we want to obtain the correct result from equation (5.9). The reason for this, is that the radius in AdS_5 and S^5 are the same. This can be seen from that the Riemann tensor for S^5 is defined as in (5.24), and for AdS^5 as in (5.23) except that $4m^2$ is changed to m^2.

The final thing I will do in this section is to list the scalars and their representations that corresponds to relevant and marginal operators[8]. An operator is relevant if $\Delta < d$ and marginal if $\Delta = d$, where d is the dimension of the field theory. The field theory (with $d = 3$) that corresponds to 11D supergravity on $AdS_4 \times S^7$ has 6 relevant and 2 marginal operators, which corresponds to the following scalars

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$0^{+(1)}$	R, 35_v	R, 112_v	R, 294_v	R, $672'_v$	M, 1386_v	-
$0^{-(1)}$	R, 35_c	R, 224_{cv}	M, $840'_s$	-	-	-

Here R is for relevant and M for marginal. In the $AdS_7 \times S^4$ case ($d = 6$), there are 1 relevant and 1 marginal operator, which corresponds to the following scalars

	$n=0$	$n=1$	$n=2$
$0^{+(1)}$	R, 14	M, 30	-

Finally, in the $AdS_5 \times S^5$ case ($d = 4$), there are 3 relevant and 3 marginal operators, which corresponds to the following scalars

	$n=0$	$n=1$	$n=2$	$n=3$
$0^{+(1)}$	R, 20	R, 50	M, 105	-
$0^{+(2)}$	M, 1_c	-	-	-
$0^{-(1)}$	R, 10_c	M, 45_c	-	-

where the index c denotes complex representation.

5.3 Deformations of supergravity theories and RG-flow

In this section, there will be a discussion on how deformations of supergravity theories gives a renormalization group flow (RG-flow) in the dual conformal field theory. This is illustrated by an example where $\mathcal{N} = 4$ SYM in 4-dimensions in the UV (ultraviolet) flows to an $\mathcal{N} = 1$ super conformal field theory in the IR (infrared). I will also discuss an example where $AdS_4 \times S^7$ is deformed to $AdS_4 \times \hat{S}^7$, where $\hat{S}^7$ is the squashed 7-sphere with isometry $SO(5) \times SO(3)$. In this example I find some unexpected results which I will try to explain.

What I would like to show in this section is that if we deform a supergravity theory on $AdS_{d+1} \times M^{D-d-1}$ (1), i.e. give a non-zero VEV to a scalar field (turning

[8]These are the important operators in deformations of field theories, see next section.

on a non-normalizable mode) with $E_0 < d$, this will give a kink solution that interpolates between the vacuum above (1) and a supergravity theory compactified on $AdS_{d+1} \times M'^{D-d-1}$ (2), where M'^{D-d-1} has a smaller isometry group then M^{D-d-1}. On the field theory side this correspond to a renormalization group flow from the superconformal field theory in the UV (dual to (1)) to a different (super $\mathcal{N} > 0$ or not super $\mathcal{N} = 0$) conformal field theory in the IR (dual to (2)). We see here that the condition $E_0 < d$ implies that a relevant operator in the field theory has been turned on. This is called a relevant deformation (or relevant perturbation). It can also be shown that there is a c-theorem which states that the central charge c is monotonically decreasing in the flow from UV to IR, i.e. $c_{IR}/c_{UV} < 1$. This ratio will be closer to one if the new isometry group is close to the old one or if there is a lot of supersymmetry left in the IR solution. What all this means will hopefully become clear after the example below.

5.3.1 $\mathcal{N} = 4$ SYM RG-flow to $\mathcal{N} = 1$ SCFT

In this example which is an incomplete review of [34, 35], I will study deformations of type IIB supergravity on $AdS_5 \times S^5$. As we know from the previous sections this theory is dual to $\mathcal{N} = 4$ SYM theory in 4-dimensions in the large N limit, i.e. $\lambda = g_{YM}^2 N \gg 1$ and $N \to \infty$. What I will do here is to give non-zero VEV to scalars with $E_0 < 4$ in the massless supermultiplet (the scalars are in $20'$ and 10_c). This is equivalent to turning on a bosonic (scalar) mass term with $\Delta = 2$ and a fermionic mass term with $\Delta = 3$ in the $\mathcal{N} = 4$ SYM theory. To be more precise: Turning on the scalar (in $20'$) dual to the boson mass term brakes the $SO(6)$ symmetry down to $SO(4) \times SO(2)$, while turning on the scalar (in 10_c) dual to the fermion mass term brakes $SO(6)$ down to $SU(3)$, and if we turn on both these scalars (which we do in this example) the $SO(6)$ symmetry is broken down to $SU(2) \times U(1)$.

The conformal field theory in the IR is an $\mathcal{N} = 1$ theory with supergroup $SU(2,2|1)$ and bosonic subgroup $SO(4,2) \times SU(2) \times U(1)$. This means that the supersymmetry is broken from $\mathcal{N} = 4$ down to $\mathcal{N} = 1$ and $SU(4)$ is broken down to $SU(2) \times U(1)$. The supergravity theory dual to the IR field theory has not yet been constructed. But this is not so bad because the $\mathcal{N} = 2$ gauged supergravity theory in 5 dimensions which is believed to be a consistent truncation of the unknown theory can be used as long as the deformation is made on scalars in the massless supermultiplet. The reason for this is that the $\mathcal{N} = 2$ gauged supergravity theory only gives the massless spectrum of the full theory. In the UV region the $\mathcal{N} = 8$ gauged supergravity theory in 5 dimensions is believed to be a consistent truncation of type IIB supergravity compactified on $AdS_5 \times S^5$. This is not completely proved, but I will assume that it is correct. The $\mathcal{N} = 8$ gauged supergravity theory gives the massless spectrum of the type IIB supergravity theory on $AdS_5 \times S^5$. That a theory is a consistent truncation of another theory means that the equations of motions of the higher dimensional theory are exactly satisfied if the equations of motions of the truncated (or lower dimensional) theory are satisfied. From now on when I use the word supergravity, I mean gauged supergravity.

Critical point	Unbroken gauge symmetry	Cosmological constant Λ	Unbroken supersymmetry	Central charge c_{IR}/c_{UV}
1	$SO(6)$	$-\frac{3}{4}g^2$	$N = 8$	1
2	$SO(5)$	$-\frac{3^{5/3}}{8}g^2$	$N = 0$	$\frac{2\sqrt{2}}{3} \sim 0.9428$
3	$SU(3)$	$-\frac{27}{32}g^2$	$N = 0$	$\frac{16\sqrt{2}}{27} \sim 0.8381$
4	$SU(2) \times U(1) \times U(1)$	$-\frac{3}{8}(\frac{25}{2})^{1/3}g^2$	$N = 0$	$\frac{4}{5} = 0.8$
5	$SU(2) \times U(1)$	$-\frac{2^{4/3}}{3}g^2$	$N = 2$	$\frac{27}{32} \sim 0.8438$

Table 5.4: The critical points to the scalar potential V

The unbroken gauge symmetry is that of the supergravity, and corresponds to the R-symmetry in the field theory. The unbroken supersymmetry is that of the supergravity theory, and should be halved to get the supersymmetry in the corresponding field theory. In this thesis the fix points 1 and 5 are the interesting ones.

The AdS_5/CFT_4 correspondence implies that the $\mathcal{N} = 8$ gauged supergravity supermultiplet is dual to the $\mathcal{N} = 4$ SYM energy-momentum tensor supermultiplet, where the supergravity scalars $1 \oplus 1 \oplus 10 \oplus \bar{10} \oplus 20'$ correspond, respectively, to the gauge coupling g_{YM}, the θ-angle, the fermion bilinear operators and the scalar bilinear operators.

The Lagrangian of gauged $\mathcal{N} = 8$ supergravity has a scalar potential V which is invariant under $SO(6) = SU(4)$. The fix points of the potential V give rise to various AdS_5 solutions of gauged supergravity. We will later see that the fix point with $SU(2) \times U(1)$ symmetry and one-quarter supersymmetry describes the IR fix point in the RG-flow (5 in table 5.4). All the fix points with at least $SU(2)$ symmetry can be seen in table 5.4, which is copied from [35].

The scalar potential V that were used to obtain the fix points is [34]

$$\begin{aligned} V &= -\frac{g^2}{4}\Big[\rho^{-4}(1-\cos^2(2\phi)(\sinh^2(\varphi_1)-\sinh^2(\varphi_2))^2) \\ &\quad +\rho^2(\cosh(2\varphi_1)+\cosh(2\varphi_2)) + \frac{1}{16}\rho^8(2+2\sin^2(2\phi) \\ &\quad -2\sin^2(2\phi)\cosh(2(\varphi_1-\varphi_2)) - \cosh(4\varphi_1) - \cosh(4\varphi_2))\Big] \end{aligned} \tag{5.31}$$

where $\rho = e^{\frac{\varphi_3}{\sqrt{6}}}$ and g is the coupling constant. Here where we only are interested in fix points 1 and 5, it can be shown that it is consistent to set $\varphi_2 = 0$ and $\phi = 0$, where $\varphi_2 = 0$ is required in order to preserve the $U(1)$ symmetry and $\phi = 0$ is required in order to preserve some supersymmetry in the IR fix point. Inserting $\varphi_2 = \phi = 0$ in (5.31) gives

$$V = -\frac{g^2}{4}\Big[\rho^{-4}(1-\sinh^4(\varphi_1)) + \rho^2(1+\cosh(2\varphi_1)) + \frac{1}{16}\rho^8(1-\cosh(4\varphi_1))\Big] \tag{5.32}$$

From this equation the fix points for 1 and 5 in table 5.4 can be obtained. These are

$$1: \qquad \varphi_1 = \varphi_3 = 0 \tag{5.33}$$

$$5: \qquad \varphi_1 = \pm\frac{1}{2}\ln(3), \quad \varphi_3 = \frac{1}{\sqrt{6}}\ln(2) \tag{5.34}$$

Inserting (5.33) and (5.34) into (5.32) gives

$$V_1 = -\frac{3}{4}g^2, \quad V_5 = -\frac{2^{4/3}}{3}g^2 \tag{5.35}$$

The potential V in (5.32) can be written in the following way

$$V = \frac{g^2}{8}\Big(\Big|\frac{\partial W}{\partial \varphi_1}\Big|^2 + \Big|\frac{\partial W}{\partial \varphi_3}\Big|^2\Big) - \frac{g^2}{3}|W|^2 \tag{5.36}$$

where

$$W = \frac{\rho^{-2}}{4}\Big[\cosh(2\varphi_1)(\rho^6 - 2) - (3\rho^6 + 2)\Big], \tag{5.37}$$

is the superpotential. Why we write the potential as a function of the superpotential will be explained below. The superpotential has fix points as in (5.33) and (5.34). The difference between the W and V is that W only has fix points if the supergravity solutions at the fix points are supersymmetric [34], i.e. $N > 0$. This implies that if we write down the superpotential that corresponds to (5.31), then this superpotential would only have fix points for case 1 and 5 in table 5.4, see [34] for this superpotential.

As a first check that the $\mathcal{N} = 1$ superconformal field theory is dual to $\mathcal{N} = 2$ gauged supergravity, I will compare c_{IR}/c_{UV} on the field theory side and on the supergravity side. The field theory side gives [34]

$$\frac{c_{IR}}{c_{UV}} = \frac{27}{32}, \tag{5.38}$$

while the supergravity side gives

$$\frac{c_{IR}}{c_{UV}} = \Big(\frac{V_5}{V_1}\Big)^{-3/2} = \Big(\frac{9}{4 \cdot 2^{4/3}}\Big)^{3/2} = \frac{27}{32} \tag{5.39}$$

where I have used (5.35). We see that we obtain the same result in the two cases, which is one indication that the $\mathcal{N} = 1$ conformal field theory is dual to $\mathcal{N} = 2$ gauged supergravity.

Before I construct the kink solution that interpolates between the AdS_5 geometries at the two critical points 1 and 5, I will discuss the c-theorem. The kink ansatz in 5-dimensions is

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu - dr^2, \quad \mu = 0,1,2,3, \tag{5.40}$$

where I have used the mostly minus metric convention, which has the consequence that the AdS spacetime has a positive cosmological constant Λ.[9] If $A(r) = \frac{r}{l}$ then (5.40) becomes AdS_5 with Ricci tensor $R_{\hat{\mu}\hat{\nu}} = \frac{4}{l^2}g_{\hat{\mu}\hat{\nu}}$, $\hat{\mu} = 0,1,2,3,4$. The kink solution below interpolates between an $r \to \infty$ "ultraviolet" AdS_5 region, where

[9] This means that Λ in table 5.4 has no minus sign with this convention.

$A(r)$ is linear, and an $r \to -\infty$ "infrared" region, where $A(r)$ is again linear, but with a larger positive slope. Here $r = \infty$ is the boundary of AdS_5 and $r = -\infty$ is the horizon. If one now can show that $A''(r) < 0$ then it is possible to derive the following c-theorem

$$c(r) = \frac{c_0}{|A'|^3}, \quad c_0 = \text{constant}. \tag{5.41}$$

where $c(r)$ is monotonically decreasing when r decreases. This c-theorem and $A''(r) < 0$ were shown in section 4 of [34], where they used Einstein's equations and the weak energy condition. The constant c_0 in (5.41) is defined as

$$c_0 = \frac{g^3 N^2}{32}, \tag{5.42}$$

when we have $\mathcal{N} = 8$ gauged supergravity (at $r \to \infty$). With this definition $c_{UV} = \frac{N^2}{4}$ and $c_{IR} = \frac{27}{128} N^2$ (see (5.46) below), which gives the same value as (5.38)-(5.39) if we compute c_{IR}/c_{UV}. It is also important to know that (5.41) is only stationary at conformal fix points of the flow, i.e. fix points where the supergravity theory corresponds to a conformal field theory.

To summarize: We have found a c-theorem which applies to theories where a gravity dual can be found in which Einstein's equations hold, and in which the gravity approximation applies. We must also have the large N-limit of course.

I am now going to obtain the kink solution that interpolates between the AdS_5 geometries at the two critical points 1 and 5, where I will use (5.40) as the interpolating metric. For this solution we want to preserve $N = 1$ supersymmetry on the branes (i.e, $N = 1$ supersymmetry on the field theory side) all a long the flow, which means that the kink itself must be supersymmetric. This implies that the variations of the spin 1/2 and spin 3/2 fields must be zero. Solving $\delta\psi_{\mu a} = 0$ and $\delta\chi_{abc} = 0$, where a is a spinor index, gives the following equations of motion for the scalars $\varphi_1(r)$, $\varphi_3(r)$ and the function $A(r)$,

$$\frac{d\varphi_j}{dr} = \frac{g}{2}\frac{\partial W}{\partial \varphi_j}, \quad j = 1, 3, \tag{5.43}$$

$$\frac{dA}{dr} = -\frac{g}{3} W \tag{5.44}$$

where W is the superpotential (5.37). Here we see the reason for obtaining the superpotential. It is possible to obtain the kink solution without the superpotential by using the gravitational and the scalar equations of motions. But then the kink solution is harder to obtain because one would have to solve a system of coupled second order differential equations.

Now back to the c-theorem. Inserting (5.44) into (5.41) and using (5.42) gives

$$c(r) = \frac{27}{32}\frac{N^2}{|W|^3} \tag{5.45}$$

If we now insert (5.33)-(5.34) into (5.37) and use that result in (5.45) we get

$$c_{UV} = \frac{N^2}{4}, \quad c_{IR} = \frac{27}{128} N^2 \tag{5.46}$$

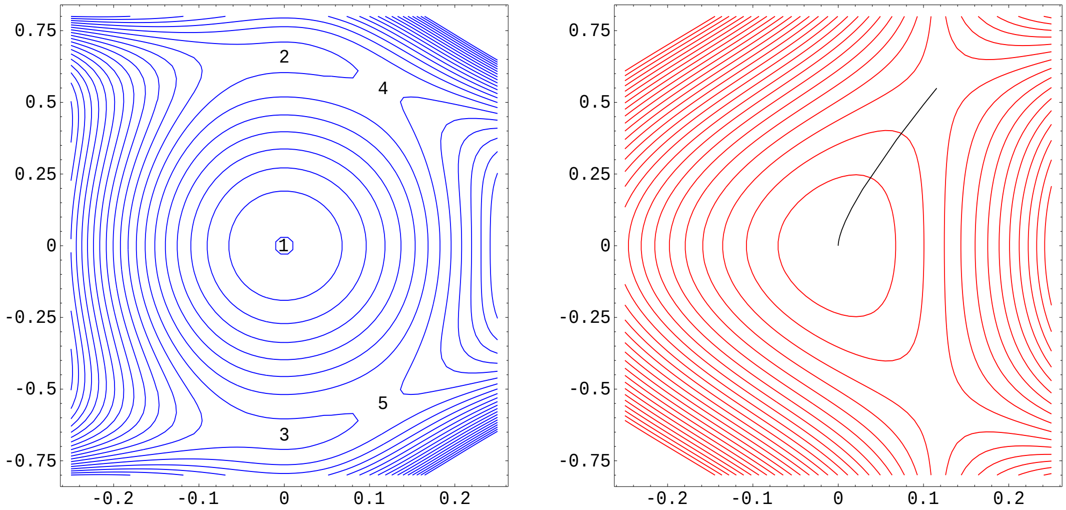

Figure 5.1: The contour map of V (on the left) and W (on the right), with φ_1 on the vertical axis and $\alpha = \frac{1}{\sqrt{6}}\varphi_3$ on the horizontal axis. The five labeled points are the only extrema of V in this plane. V has vanishing first derivatives in all directions orthogonal to the plane. A numerical solution of the steepest descent equations is shown superimposed on the contour plot of W. This figure is taken from [34].

That $A''(r) < 0$ is easily seen if we take the derivative with respect to r in (5.44) and use the chain rule and (5.43). This gives

$$A'' = -\frac{g^2}{6}\sum_{j=1,3}\left|\frac{\partial W}{\partial \varphi_j}\right|^2 = -\frac{2}{3}\sum_{j=1,3}\left|\frac{\partial \varphi_j}{\partial r}\right|^2 \leq 0 \tag{5.47}$$

If we now look at figure 5.1, we see the contour maps of V and W on the $(\alpha = \frac{1}{\sqrt{6}}\varphi_3, \varphi_1)$ parameter space. The map of V shows five extrema. Point 1 is the maximally supersymmetric point, which is a local maximum of both V and W. Points 2 and 3 are $\mathbb{Z}_2$ equivalent $SU(3)$ invariant points, i.e, case 3 in table 5.4. In the φ_1 direction these points sit at a quadratic minimum of V, and in the φ_3 direction they sit on a cubic inflection point. We also see that these fix points are not fix points for W, since they are not supersymmetric. Points 4 and 5 are $\mathbb{Z}_2$ equivalent $\mathcal{N} = 2$ supersymmetric points. These points are generic quadratic saddle points of both V and W, and the contours surrounding them are locally hyperbolas. The superkink which we study follows a path from the central maximum of W down the "ridge" to the saddle point 4. The plotted kink in figure 1 is a plotted numerical solution, because the authors of [34] were unable to find an analytic solution to the steepest decent equations (5.43)-(5.44). For a discussion of the numerical solution and a analytical treatment of the end points, see below.

If we now analyze the mass matrix M_{ij} for the scalar fields φ_j we would find [34]: In order to depart from the UV fix point ($r = \infty$) the operator that drives the flow must be a relevant operator (i.e. $\Delta < 4$), and to approach the IR fix point ($r = -\infty$) the operator that drives the flow there, must be an irrelevant operator (this operator has $\Delta = 3 + \sqrt{7} > 4$). We also see that the supergravity scalar fields φ_1 and φ_3 are dual to operators of dimension 3 and 2, respectively; the fermion and the scalar mass term in the $\mathcal{N} = 4$ SYM theory.

I will now analyze the flow in the neighborhood of $\varphi_j = 0$, by performing a series expansion of W and use the expansion in (5.43), ($g = 2$ in the calculations below). The series expansion of W is

$$W = -\frac{3}{2} - \frac{\varphi_1^2}{2} - \varphi_3^2 + \sqrt{\frac{8}{3}}\varphi_1^2\varphi_3 + \cdots \tag{5.48}$$

Using this expansion of W in (5.43) gives

$$\varphi_1(r) \sim a_0 e^{-r}, \quad \varphi_3(r) \sim \sqrt{\frac{8}{3}} a_0^2 r e^{-2r} + a_1 e^{-2r}, \quad \text{as } r \to \infty \tag{5.49}$$

We see here that roughly speaking, two powers of φ_1 are worth one power of φ_3 in the expansion of W. We also see that the steepest descent path has a parabolic form near the UV fix point. In (5.49) a_0 is an integration constant, which up to factors of order unity, equals the fermion mass resulting from the term $\frac{1}{2}m\mathrm{Tr}\Phi_3^2$ in the superpotential W below. The constant in the first term of φ_3 is determined by supersymmetry and is, up to factors of order unity, the square of the scalar mass resulting from the term $\frac{1}{2}m\mathrm{Tr}\Phi_3^2$ in the superpotential below. The constant a_1 can not be determined by supersymmetry, and were not calculated analytically in [34]. For a numerical evaluation, see below.

The superpotential written with super fields Φ_j is

$$W = \mathrm{Tr}\Phi_3[\Phi_1, \Phi_2] + \frac{1}{2}m\mathrm{Tr}\Phi_3^2 \tag{5.50}$$

on the boundary. Here the first term is required by $N = 4$ supersymmetry, and the second term brakes $N = 4$ down to $N = 1$.

Close to the IR fix point (4 in figure 5.1) the scalar fields are behaving like

$$\varphi_1(r) \sim \frac{1}{2}\ln(3) - b_0 e^{\lambda r}, \quad \varphi_3(r) \sim \frac{1}{\sqrt{6}}\ln(2) - \frac{\sqrt{7}-1}{\sqrt{6}} b_0 e^{\lambda r}, \quad \text{as } r \to -\infty \tag{5.51}$$

where

$$\lambda = \frac{2^{5/3}}{3}(\sqrt{7} - 1) \tag{5.52}$$

The constant b_0 is not independent of a_0 and a_1, but there are two combinations of the three constants that are invariant. These are $b_0 a_0^\lambda$ and $\frac{a_1}{a_0^2} + \sqrt{\frac{8}{3}}\ln(a_0)$. Solving these invariants numerically gives

$$b_0 a_0^\lambda \approx 0.1493, \quad \frac{a_1}{a_0^2} + \sqrt{\frac{8}{3}}\ln(a_0) \approx -1.4694. \tag{5.53}$$

Here minus the logarithm of the first number is a measure of the width of the kink, and the second number represent the choice of vacuum state which leads to infrared physics with non-vanishing central charge c.

What is left to do now is to obtain the correspondence between the operators in the $\mathcal{N} = 1$ $SCFT$ and the modes in the $\mathcal{N} = 2$ gauged supergravity theory and check that the result is correct. For a derivation of this see [34]. The result is a perfect map between modes on the supergravity side and operators on the field theory side.

Conclusion: I have shown above that it is very likely that the (super)-kink solution interpolating between the AdS_5 geometries at the two fix points 1 and 4 in figure 5.1, correspond to a RG-flow from $\mathcal{N} = 4$ SYM in the UV to a $\mathcal{N} = 1$ $SCFT$ in the IR with $SU(2) \times U(1)$ symmetry.

For a much more extensive discussion of the content in this subsection, see [34] by Freedman, Gubser, Pilch and Warner, or a shorter version [35] by Warner.

5.3.2 Investigation of a possible superpotential in the 7-sphere and squashed 7-sphere case

I am now going to discuss a case similar to the case in the last subsection, but with $d = 3$. The two supergravity theories that the kink solution interpolate between are: 11D supergravity on $AdS_4 \times S^7$ (1) and 11D supergravity on $AdS_4 \times \hat{S}^7$ (2), where $\hat{S}^7$ is the squashed 7-sphere. Here (1) has gauge group $SO(8)$ and $N = 8$ supersymmetry, and (2) $SO(5) \times SO(3)$ gauge group and $N = 1, 0$ supersymmetry (1 for left oriented and 0 for right oriented squashed 7-sphere). The Lagrangian for the scalars that drives the flow are [36]

$$L = \sqrt{-g}\Big(R - \frac{1}{2}(\partial\tilde{u})^2 - \frac{1}{2}(\partial\tilde{v})^2 - V(\tilde{u},\tilde{v})\Big), \tag{5.54}$$

where the potential V is

$$V(\tilde{u},\tilde{v}) = e^{-\frac{9\tilde{u}}{\sqrt{63}}}\Big(- 6e^{\frac{4\tilde{v}}{\sqrt{42}}} - 48e^{-\frac{3\tilde{v}}{\sqrt{42}}} + 12e^{-\frac{10\tilde{v}}{\sqrt{42}}} + 2Q^2 e^{-\frac{12\tilde{u}}{\sqrt{63}}}\Big) \tag{5.55}$$

and Q is a constant. The fix points to this potential are

$$(1): \qquad \tilde{u} = \frac{\sqrt{63}}{12}\ln(3^{-2}Q^2), \quad \tilde{v} = 0, \tag{5.56}$$

$$(2): \qquad \tilde{u} = \frac{\sqrt{63}}{12}\ln(3^{-4}5^{10/7}Q^2), \quad \tilde{v} = \frac{\sqrt{42}}{7}\ln(5) \tag{5.57}$$

where fix point (1) correspond to case (1), and (2) to case (2). Calculating V for these fix points (with $Q = 3$) gives $V_1 = -24$ and $V_2 \approx -20.1$. If we now use that $c_{IR}/c_{UV} < 1$ and (5.39), we see that the field theory dual to the supergravity theory in fix point (1) is in the IR, and fix point (2) gives a field theory in the UV. This means that there is a relevant operator that drives the flow from the UV fix point (2), to the IR fix point (1) where there is an irrelevant operator that drives the flow.

The irrelevant operator is in the 300 of $0^{+(2)}$ and has $\Delta = 4$, while the relevant operator has $\Delta = 4/3, 5/3$ for $N = 1, 0$, see [36].

Now I will try to find a superpotential W that is related to V in the following way

$$V = 4\Big[4\Big(\frac{\partial W}{\partial \tilde{u}}\Big)^2 + 4\Big(\frac{\partial W}{\partial \tilde{v}}\Big)^2 - 3W^2\Big] \tag{5.58}$$

where the coupling constant is $g = \sqrt{2}$. I will use the following ansatz for W

$$W(\tilde{u}, \tilde{v}) = e^{-\frac{9\tilde{u}}{2\sqrt{63}}}\Big(Ae^{\frac{2\tilde{v}}{\sqrt{42}}} + Be^{-\frac{5\tilde{v}}{\sqrt{42}}}\Big) + Ce^{-\frac{21\tilde{u}}{2\sqrt{63}}} \tag{5.59}$$

Inserting this ansatz into (5.58) gives

$$v(\tilde{u}, \tilde{v}) = e^{-\frac{9\tilde{u}}{\sqrt{63}}}\Big(-\frac{16}{3}A^2e^{\frac{4\tilde{v}}{\sqrt{42}}} + \frac{8}{3}B^2e^{-\frac{10\tilde{v}}{\sqrt{42}}} - \frac{448}{21}ABe^{-\frac{3\tilde{v}}{\sqrt{42}}}\Big) + 16C^2e^{-\frac{21\tilde{u}}{\sqrt{63}}} \tag{5.60}$$

Comparing this equation with (5.55) implies

$$A = \pm\frac{3}{\sqrt{8}}, \quad B = \pm\frac{3}{\sqrt{2}}, \quad C = \pm\frac{1}{\sqrt{8}}|Q|, \quad AB > 0 \tag{5.61}$$

Inserting this into the ansatz (5.59) gives[10]

$$W(\tilde{u}, \tilde{v}) = \frac{1}{\sqrt{8}}e^{-\frac{9\tilde{u}}{2\sqrt{63}}}\Big(3e^{\frac{2\tilde{v}}{\sqrt{42}}} + 6e^{-\frac{5\tilde{v}}{\sqrt{42}}} \pm |Q|e^{-\frac{12\tilde{u}}{2\sqrt{63}}}\Big) \tag{5.62}$$

The fix points for this superpotential when we have the $-|Q|$ case are

$$\tilde{u} = \frac{\sqrt{63}}{12}\ln(3^{-4}5^{10/7}Q^2), \quad \tilde{v} = \frac{\sqrt{42}}{7}\ln(5), \tag{5.63}$$

and there are no fix points in the $+|Q|$ case. These results are very surprising. Since we know that fix point (1) is supersymmetric, we would have expected that this point were a fix point for the super potential W. But this is not the case for either $-|Q|$ or $+|Q|$. That point (2) is only fix point for $-|Q|$ and not for $+|Q|$ was more expected, since the supergravity theory in this fix point is supersymmetric only when the squashed 7-sphere is left oriented but not for right orientation. Now the question is, why do we get this strange result. Well, it is quite obvious that something is wrong, but what is wrong?

1. It is possible that W is not the correct superpotential. This could be the case since I never showed that W in (5.62) is the only solution to equation (5.58).

2. Another possibility is that there is something wrong with the potential V, since it looks like it is impossible to find a superpotential with two fix points. If this is true, then it was not a consistent truncation to only include the two scalars $\tilde{u}$ and $\tilde{v}$ in V, and set the other scalars to zero.

3. The potential V is completely wrong.

[10]We can also have $-W$ as super potential, but this case gives nothing different in the discussion below.

4. V and W are correct.

Conclusion: If 1 is correct then one must try to find the correct superpotential. But if 2 is correct and it was not a consistent truncation, then there are a few possibilities: A. It was an almost consistent truncation which led to a very small error, and the kink solution that can be obtained by using Einstein's equations and the scalar equations are a good approximation to the exact solution, (for this solution see [36]). B. It was far from a consistent truncation, which implies that the kink solution and at least some of the conclusions in [36] are wrong. If 3 is correct then we must find the right potential before we can obtain any useful results. Finally, if 4 is correct then there is no RG-flow at all, which means that the derivation of the RG-flow in [36] was wrong.

For a discussion about this case where they never use the super potential, see [36].

6

Conclusions

The p-brane solutions that were obtained in chapter 2 are very important in the understanding of string and M-theory. Especially, the D3-brane (and other D-branes) is important in the understanding of the various S, T and U-dualities in string and M-theory (see figure 1.1). The p-branes are also important in the formulation of the AdS/CFT correspondence, which was introduced in chapter 5. For example, the connection between the world volume field theory and the near horizon geometry ($AdS_5 \times S^5$) of N parallel D3-branes led Maldacena to formulate his conjecture [26].

If we go back to the near horizon geometry of p-branes. We saw in chapter 3 that each of the three p-branes that were obtained in chapter 2 have a near horizon geometry which is equivalent to the vacuum of a spontaneously compactified supergravity theory, a result which is instrumental in the formulation of the AdS/CFT correspondence. The mass operators and mass spectra that were obtained in chapter 3 and 4 are also very important in the investigation of the AdS/CFT correspondence. Especially when checking the correspondence between chiral operators in the field theory against modes in the supergravity theory.

In this thesis most of the results are accepted and believed to be correct. The big exception to this is of course the final section in chapter 5, where the investigation of the round and squashed 7-spheres led to some unexpected results. The study of this case will be continued after this thesis is written. I feel very confident that it will be possible to find a solution to the problem with the superpotential.

There are also a lot of other consequences of the correspondence that are important to examine. For example, formulating a string theory description of QCD would be a great success. It is also important to use the correspondence to obtain non-perturbative results in M-theory, which might lead to a non-perturbative formulation of M-theory in the future. I think that this is probably the most important task for string theory physicists in the future. Succeeding with this will hopefully lead to discoveries that can be confirmed by experiment. It will also probably lead to a way to single out the true vacuum from all possible perturbative vacua,[1] which would be a great success for M-theory. I personally believe that doing research in string/M-theory will be very interesting in the future.

Finally, I hope the reader have enjoyed reading this thesis.

[1]A vacuum that results from a compactification of a perturbative string theory.

A

Conventions

In this appendix I will summarize the various conventions used in this thesis.

A.1 Indices

Type of space	World indices	Coordinates	Signature
D=11 or D=10 spacetime	$M, N, P, \ldots$	z^M	$(-,+,+,\ldots,+)$
$AdS_{4,5,7}$ spacetime	$\mu, \nu, \rho, \ldots$	x^μ	$(-,+,+,\ldots,+)$
Internal space $M^{7,5,4}$	$m, n, p, \ldots$	y^m	$(+,+,+,\ldots,+)$

A.2 Notation

$$A_{[M_1\ldots M_p]} = \frac{1}{p!}(A_{M_1\ldots M_p} + \text{antisymmetric permutations}) \tag{A.1}$$

$$A_{(M_1\ldots M_p)} = \frac{1}{p!}(A_{M_1\ldots M_p} + \text{symmetric permutations}) \tag{A.2}$$

$$\{X, Y\} = XY + YX \tag{A.3}$$

$$[X, Y] = XY - YX \tag{A.4}$$

B

Tensors

In this appendix I am going to define the Levi-Civita tensor $\epsilon_{M_1 \ldots M_n}$, and tensor density $\varepsilon_{M_1 \ldots M_n}$. I will also define the Riemann tensor $R^M{}_{NPQ}$ and derive some of its properties. Finally, there will be a discussion about the Hodge-de Rham operator.

B.1 The Levi-Civita tensor and tensor density

The Levi-Cevita tensor $\epsilon_{M_1 \ldots M_n}$ and tensor density $\varepsilon_{M_1 \ldots M_n}$ in n dimensions are completely antisymmetric, for example for n=5 are they defined as

$$\epsilon_{01234} = \sqrt{|g|} \tag{B.1}$$

$$\epsilon^{01234} = \pm \frac{1}{\sqrt{|g|}} \tag{B.2}$$

$$\varepsilon_{01234} = 1 \tag{B.3}$$

$$\varepsilon^{01234} = \pm 1 \tag{B.4}$$

where plus means $(+,+,+,+,+)$ signature, and minus means $(-,+,+,+,+)$ signature, and g is the determinant of the metric.

The following relations were very useful in this thesis [37]:

$$\epsilon^{M_1 \ldots M_n} \epsilon_{M_1 \ldots M_n} = \pm n! \tag{B.5}$$

$$\epsilon^{M_1 \ldots M_{n-1} M'_n} \epsilon_{M_1 \ldots M_{n-1} M_n} = \pm (n-1)! \delta^{M'_n}_{M_n} \tag{B.6}$$

$$\epsilon^{M_1 \ldots M_k M'_{k+1} \ldots M'_n} \epsilon_{M_1 \ldots M_k M_{k+1} \ldots M_n} = \pm k!(n-k)! \delta^{M'_{k+1} \ldots M'_n}_{[M_{k+1} \ldots M_n]} \tag{B.7}$$

where plus means $(+,+,\ldots,+)$ signature, and minus means $(-,+,\ldots,+)$ signature, and the generalized Kronecker delta is

$$\delta^{M'_{k+1} \ldots M'_n}_{[M_{k+1} \ldots M_n]} = \delta^{M'_{k+1}}_{[M_{k+1}} \ldots \delta^{M'_n}_{M_n]} \tag{B.8}$$

Note that (B.5)-(B.7) are also valid for the tensor density $\varepsilon_{M_1 \ldots M_n}$.

B.2 The Riemann tensor

The Riemann tensor has the following definition

$$R^M{}_{NPQ} = \partial_P \Gamma^M_{NQ} - \partial_Q \Gamma^M_{NP} + \Gamma^R_{NQ}\Gamma^M_{RP} - \Gamma^R_{NP}\Gamma^M_{RQ} \tag{B.9}$$

Here the affine connection Γ^P_{MN} is defined as

$$\Gamma^P_{MN} = \frac{1}{2} g^{PQ}[\partial_M g_{QN} + \partial_N g_{QM} - \partial_Q g_{MN}] \tag{B.10}$$

One can see here that Γ^P_{MN} is symmetric in the two lower indices, which happens because the metric is symmetric.

The Riemann tensor has the following symmetry properties

$$R_{MNPQ} = -R_{NMPQ} = -R_{MNQP} = R_{PQMN} \tag{B.11}$$

The Ricci tensor is defined as

$$R_{MN} = R^Q{}_{MQN} = g^{PQ} R_{PMQN}, \tag{B.12}$$

and the Ricci scalar is defined as

$$R = R^M{}_M = g^{MN} R_{MN} \tag{B.13}$$

B.3 Variation of R_{MN}

Variation of R_{MN} gives

$$\begin{aligned} \delta R_{MN} &= \partial_P \delta\Gamma^P_{MN} - \partial_N \delta\Gamma^P_{MP} + \delta\Gamma^R_{MN}\Gamma^P_{RP} + \Gamma^R_{MN}\delta\Gamma^P_{RP} \\ &- \delta\Gamma^R_{MP}\Gamma^P_{RN} - \Gamma^R_{MP}\delta\Gamma^P_{RN} \\ &= \nabla_P(\delta\Gamma^P_{MN}) - \nabla_N(\delta\Gamma^P_{MP}) \end{aligned} \tag{B.14}$$

where I have used that

$$\nabla_P(\delta\Gamma^P_{MN}) = \partial_P\delta\Gamma^P_{MN} - \Gamma^R_{MP}\delta\Gamma^P_{RN} - \Gamma^R_{NP}\delta\Gamma^P_{MR} + \Gamma^P_{PR}\delta\Gamma^R_{MN} \tag{B.15}$$

$$\nabla_N(\delta\Gamma^P_{MP}) = \partial_N\delta\Gamma^P_{MP} - \Gamma^R_{MN}\delta\Gamma^P_{RP} \tag{B.16}$$

Before I calculate the variation of Γ^P_{MN}, I will prove the following relation

$$\delta g^{PQ} = -g^{PR} g^{SQ} \delta g_{SR} \tag{B.17}$$

Proof:

$$0 = \delta(\delta^P_R) = \delta(g^{PS} g_{SR}) = \delta g^{PS} g_{SR} + g^{PS}\delta g_{SR} \tag{B.18}$$

Multiplying with g^{RQ} and using $g^{RQ}g_{SR} = \delta^Q_S$ gives (B.17). The variation of Γ^P_{MN} is

$$\begin{aligned}
\delta\Gamma^P_{MN} &= \frac{1}{2}\delta g^{PQ}[\partial_M g_{QN} + \partial_N g_{QM} - \partial_Q g_{MN}] \\
&+ \frac{1}{2}g^{PQ}[\partial_M \delta g_{QN} + \partial_N \delta g_{QM} - \partial_Q \delta g_{MN}] \\
&= -\delta g_{RS} g^{PR}\Gamma^S_{MN} + \frac{1}{2}g^{PQ}[\partial_M \delta g_{QN} + \partial_N \delta g_{QM} - \partial_Q \delta g_{MN}] \\
&= \frac{1}{2}g^{PQ}[\nabla_M \delta g_{QN} + \nabla_N \delta g_{QM} - \nabla_Q \delta g_{MN}] \qquad \text{(B.19)}
\end{aligned}$$

where I have used (B.10), (B.17) and

$$\begin{aligned}
\nabla_M \delta g_{QN} &= \partial_M \delta g_{QN} - \Gamma^S_{MQ}\delta g_{SN} - \Gamma^S_{MN}\delta g_{SQ} \\
\nabla_N \delta g_{QM} &= \partial_N \delta g_{QM} - \Gamma^S_{NQ}\delta g_{SM} - \Gamma^S_{NM}\delta g_{SQ} \\
\nabla_Q \delta g_{MN} &= \partial_Q \delta g_{MN} - \Gamma^S_{QM}\delta g_{SN} - \Gamma^S_{QN}\delta g_{SM} \qquad \text{(B.20)}
\end{aligned}$$

Inserting (B.20) in the last line of (B.19) gives

$$\begin{aligned}
\delta\Gamma^P_{MN} &= \frac{1}{2}g^{PQ}[\nabla_M \delta g_{QN} + \nabla_N \delta g_{QM} - \nabla_Q \delta g_{MN}] \\
&= \frac{1}{2}g^{PQ}[\partial_M \delta g_{QN} + \partial_N \delta g_{QM} - \partial_Q \delta g_{MN} - 2\Gamma^S_{MN}\delta g_{SQ}] \\
&= \delta\Gamma^P_{MN} \qquad \text{(B.21)}
\end{aligned}$$

which means that (B.19) is correct. Here I have compared line 2 in (B.21) with line 3 in (B.19). Inserting (B.19) into (B.14) gives

$$\delta R_{MN} = \frac{1}{2}\Big[-\Box\delta g_{MN} + \nabla^P\nabla_M \delta g_{PN} + \nabla^P\nabla_N \delta g_{MP} - \nabla_N\nabla_M \delta g_P{}^P\Big] \qquad \text{(B.22)}$$

where I have used the metric postulate $\nabla_N g^{PQ} = 0$ and $\Box = \nabla^P\nabla_P$.

I am now going to rewrite δR_{MN} with the help of the Lichnerowicz operator Δ_L, which have the following definition

$$\Delta_L h_{MN} = -\Box h_{MN} - 2R_{MPNQ}h^{PQ} + 2R_{(M}{}^P h_{N)P} \qquad \text{(B.23)}$$

I want to write δR_{MN} in the following way

$$\delta R_{MN} = \frac{1}{2}\Delta_L \delta g_{MN} + \Big(\text{terms like } \nabla^P\nabla_N \delta g_{MP}\Big) \qquad \text{(B.24)}$$

Next, I will calculate the exact form of (B.24) using

$$\begin{aligned}
\frac{1}{2}\Delta_L \delta g_{MN} &= -\frac{1}{2}\Box\delta g_{MN} - R_{MPNQ}\delta g^{PQ} + R_{(M}{}^P \delta g_{N)P} \\
&= \frac{1}{2}\Big(-\Box\delta g_{MN} - \nabla_M\nabla^P \delta g_{NP} + \nabla^P\nabla_M \delta g_{NP} \\
&- \nabla_N\nabla^P \delta g_{MP} + \nabla^P\nabla_N \delta g_{MP}\Big) \qquad \text{(B.25)}
\end{aligned}$$

where I have used [38]

$$\begin{aligned} 2\nabla_{[M}\nabla_{P]}\delta g_N{}^P &= R_{MPN}{}^Q\delta g_Q{}^P - R_M{}^P\delta g_{NP}, \\ 2\nabla_{[N}\nabla_{P]}\delta g_M{}^P &= R_{NPM}{}^Q\delta g_Q{}^P - R_N{}^P\delta g_{MP}, \end{aligned} \tag{B.26}$$

and

$$\nabla_{[M}\nabla_{P]}\delta g_N{}^P + \nabla_{[N}\nabla_{P]}\delta g_M{}^P = R_{MPNQ}\delta g^{PQ} - R^P_{(M}\delta g_{N)P} \tag{B.27}$$

Comparing (B.25) with (B.22) gives

$$\delta R_{MN} = \frac{1}{2}\Delta_L\delta g_{MN} + \nabla_{(M}\nabla^P\delta g_{N)P} - \frac{1}{2}\nabla_M\nabla_N\delta g_P{}^P \tag{B.28}$$

B.4 The Hodge-de Rham operator Δ_p

The Hodge-de Rham operator Δ_p acting on p-forms, is defined as[1]

$$\Delta = d\delta + \delta d \tag{B.29}$$

where d is the exterior derivative, which maps p-forms into (p+1)-forms, and δ is its adjoint, which maps p-forms into (p−1)-forms.

A p-form Y with components $Y_{m_1\ldots m_p}$ in the coordinate basis dy^m, is defined as

$$Y = \frac{1}{p!}Y_{m_1\ldots m_p}dy^{m_1}\wedge\cdots\wedge dy^{m_p} \tag{B.30}$$

where

$$dy^{m_1}\wedge\cdots\wedge dy^{m_p} = dy^{[m_1}\otimes\cdots\otimes dy^{m_p]} \tag{B.31}$$

This means that a p-form is completely antisymmetric. The exterior derivative d and its adjoint δ, are defined as

$$(dY)_{m_1\ldots m_{p+1}} = (p+1)\partial_{[m_1}Y_{m_2\ldots m_{p+1}]} \tag{B.32}$$

$$(\delta Y)_{m_1\ldots m_{p-1}} = -\nabla^n Y_{nm_1\ldots m_{p-1}} \tag{B.33}$$

A form obeying $dY = 0$ is called closed, a form obeying $\delta Y = 0$ is called coclosed, and a form obeying $\Delta Y = 0$ is called harmonic. If the form Y can be written as $d\alpha$ it is called exact, while if it can be written as $\delta\beta$ it is called coexact. The operator Δ is a manifestly non-negative operator, which is seen from

$$(Y,\Delta Y) = (dY,dY) + (\delta Y,\delta Y) \geq 0 \Rightarrow \Delta \geq 0 \tag{B.34}$$

where $(Y,dY) = (\delta Y,Y)$ and the norm (Y,Y) is defined as

$$(Y,Y) = \int\sqrt{-g}d^Dy Y_{m_1\ldots m_p}Y^{m_1\ldots m_p} \tag{B.35}$$

[1] In this section, the results (B.29)-(B.40) are taken from [12].

where the integration is over the d-dimensional Einstein space, (d=4,5,7 in this thesis). (B.34) is only zero if Δ is acting on a form Y that is closed and coclosed. A p-form Y can be uniquely decomposed into its exact, coexact and harmonic parts. The decomposition is

$$Y = d\alpha + \delta\beta + Y_H \tag{B.36}$$

The three parts of the equation are mutually orthogonal with respect to the norm (B.35). The number of closed but not exact p-forms on a manifold is given by the p:th betti number b_p, which means that there are b_p zero modes of Δ_p acting on p-forms on the manifold.

The Hodge-de Rham operator acting on 0-3-forms in component form, is defined as

$$\Delta_0 Y = -\Box Y \tag{B.37}$$
$$\Delta_1 Y_m = -\Box Y_m + R_m{}^n Y_n \tag{B.38}$$
$$\Delta_2 Y_{mn} = -\Box Y_{mn} - 2R_{mpnq}Y^{pq} - 2R^p{}_{[m}Y_{n]p} \tag{B.39}$$
$$\Delta_3 Y_{mnp} = -\Box Y_{mnp} - 6R^q{}_{[mn}{}^r Y_{p]rq} + 3R_{[m}{}^r Y_{np]r} \tag{B.40}$$

I am now going to calculate some relations that were useful in derivation of the bosonic mass operators for $AdS_{4,5,7}$.

$$\begin{aligned} 2\nabla^n\nabla_{[m}Y_{n]} &= \nabla^n\nabla_m Y_n - \Box Y_m = \nabla_m\nabla^n Y_n + R_m{}^n Y_n - \Box Y_m \\ &= \Delta_1 Y_m + \nabla_m\nabla^n Y_n \end{aligned} \tag{B.41}$$

which gives

$$2\nabla^n\nabla_{[m}Y_{n]} = \Delta_1 Y_m \tag{B.42}$$

if Y_m is a transverse mode of Δ_1, ($\nabla^n Y_n = 0$). In (B.41)

$$[\nabla^n, \nabla_m]Y_n = R_m{}^n Y_n \tag{B.43}$$

were used. Next,

$$\begin{aligned} 3\nabla^q\nabla_{[q}Y_{mn]} &= \Box Y_{mn} + \nabla^q\nabla_m Y_{nq} + \nabla^q\nabla_n Y_{qm} = \Box Y_{mn} + 2R_{mpnq}Y^{pq} \\ &+ 2R^p{}_{[m}Y_{n]p} - 2\nabla_{[m}\nabla^q Y_{n]q} \\ &= -\Delta_2 Y_{mn} - 2\nabla_{[m}\nabla^q Y_{n]q} \end{aligned} \tag{B.44}$$

which gives

$$3\nabla^q\nabla_{[q}Y_{mn]} = -\Delta_2 Y_{mn} \tag{B.45}$$

if Y_{mn} is a traverse mode of Δ_2, ($\nabla^m Y_{mn}$). In (B.44)

$$[\nabla^q, \nabla_m]Y_{nq} = R_{mpnq}Y^{pq} + R_m{}^p Y_{np} \tag{B.46}$$

were used. Finally,

$$\begin{aligned} 4\nabla^q\nabla_{[q}Y_{mnp]} &= \Box Y_{mnp} - \nabla^q\nabla_m Y_{npq} + \nabla^q\nabla_n Y_{pqm} - \nabla^q\nabla_p Y_{qmn} \\ &= \Box Y_{mnp} + 6R^q{}_{[mn}{}^r Y_{p]rq} - 3R_{[m}{}^r Y_{np]r} - 3\nabla_{[m}\nabla^q Y_{np]q} \\ &= -\Delta_3 Y_{mnp} - 3\nabla_{[m}\nabla^q Y_{np]q} \end{aligned} \tag{B.47}$$

which gives

$$4\nabla^q\nabla_{[q}Y_{mnp]} = -\Delta_3 Y_{mnp} \tag{B.48}$$

if Y_{mnp} is a transverse mode of Δ_3, ($\nabla^m Y_{mnp} = 0$). In (B.47) I have used

$$[\nabla^q, \nabla_m]Y_{npq} = R^q{}_{mn}{}^r Y_{rpq} + R^q{}_{mp}{}^r Y_{nrq} + R_m{}^r Y_{npr} \tag{B.49}$$

I will now show that $Y_{m_1\ldots m_p}$ and $\nabla_{m_1}Y_{m_2\ldots m_p}$ are orthogonal. The norm (B.35) gives

$$\begin{aligned}
(\nabla_{m_1}Y_{m_2\ldots m_p}, Y_{m_1\ldots m_p}) &= \int \nabla_{m_1}Y_{m_2\ldots m_p}Y_{m'_1\ldots m'_p}g^{m_1m'_1}\cdots g^{m_pm'_p}\sqrt{-g}d^dy \\
&= -\int \nabla^{m_1}Y_{m_1\ldots m_p}Y^{m_2\ldots m_p}\sqrt{-g}d^dy = 0
\end{aligned} \tag{B.50}$$

where I have made a partial integration and used that the surface term is zero on a compact manifold, and that $Y_{m_1\ldots m_p}$ are transverse (see (3.98)). This also means that $\Delta Y_{m_1\ldots m_p}$ and $\nabla_{m_1}Y_{m_2\ldots m_p}$ are orthogonal, since Δ commutes with ∇_{m_1}, which is easily seen from (B.29).

C

Eigenvalues of the Hodge-de Rham and the Lichnerowicz operators

In this appendix, I am going to obtain the eigenvalues of the Hodge-de Rham and the Lichnerowicz operators on S^n $(n = 4, 5, 7)$ in section C.1, and on T^7 in section C.2.

C.1 Eigenvalues of the Hodge-de Rham and the Lichnerowicz operators on $S^{4,5,7}$

In this section, I will obtain the eigenvalues of the Hodge-de Rham and the Lichnerowicz operators on S^n, where $n = 4, 5, 7$. These operators are invariant under $SO(n+1)$, which is the isometry group of S^n. The eigenvalues will be obtained by using the method of embedding the sphere S^n in flat space R^{n+1}, and using that

$$\Box_{n+1}\phi_k(y) = (\partial^2_{y_1} + \partial^2_{y_2} + \cdots + \partial^2_{y_{n+1}})\phi_k(y) = 0 \tag{C.1}$$

where

$$\phi_k(y) = C_{m_1\cdots m_k} y^{m_1} \cdots y^{m_k} \tag{C.2}$$

Here $C_{m_1\cdots m_k}$ is symmetric and traceless over any two indices. This can be seen from the following example

$$\Box_{n+1}\phi_2 = \partial_m\partial^m y^{m_1} y^{m_2} C_{m_1 m_2} = 2\delta^{m_1 m_2} C_{m_1 m_2} = 0 \tag{C.3}$$

To begin with, I will derive the eigenvalues for the Δ_0 operator. Writing $\Box_{n+1}$ in spherical coordinates gives

$$\Box_{n+1} = \frac{1}{r^n}\partial_r(r^n\partial_r) + \frac{1}{m^2 r^n}\Box_{S^n} \tag{C.4}$$

where $r = \sqrt{\delta_{mn} y^m y^n}$ is the radial coordinate, $\theta_1, \ldots, \theta_n$ are the angles and m^2 is inserted in order to quantize the eigenvalues in units of m^2, see below. Before I use (C.4) I will rewrite (C.2) in spherical coordinates, which gives

$$\phi_k(r, \theta_1, \ldots, \theta_n) = C_{m_1 \cdots m_k} \Psi^{m_1 \cdots m_k} r^k = \Psi_k(\theta_1, \ldots, \theta_n) r^k \tag{C.5}$$

where Ψ is a function of the angles (i.e. a spherical harmonic). Using (C.4) in (C.1) gives

$$0 = \Box_{n+1} \phi_k = \frac{1}{r^n} \partial_r (r^n \partial_r \Psi_k r^k) + \frac{1}{m^2 r^n} \Box_{S^n} \Psi_k r^k = \frac{r^k}{r^2} \Big(k(k+n-1) + \frac{1}{m^2} \Box_{S^n} \Big) \Psi_k \tag{C.6}$$

This equation implies that

$$\Delta_0 \Psi_k = -\Box_{S^n} \Psi_k = [k(k+n-1)] m^2 \Psi_k \tag{C.7}$$

which gives that the eigenvalue of Δ_0 on S^n is $[k(k+n-1)]m^2$, $k = 0, 1, 2 \ldots$.

Now it is time to derive the eigenvalues of Δ_1 on S^n, by calculating the eigenvalues of $\Box_{S^n}$ and using (B.38). As before, I will use

$$\Box_{n+1} Y_m = 0 \tag{C.8}$$

where

$$Y_m(r, \theta_i) = r^k \Psi'_m(\theta_i), \quad i = 1, \ldots, n \tag{C.9}$$

Here I have suppressed the label k on Y_m and $m = r, 1, 2, \ldots, n$. This equation gives that

$$Y_i = e_i^m Y_m = r^{k+1} \Psi_i(\theta_i) \tag{C.10}$$

where $e_i^m = r E_i^m(\theta_i)$ is a vielbein. The vector Y_m is required to obey [39]

$$x^m Y_m = \partial^m Y_m = 0 \tag{C.11}$$

in order to be a vector spherical harmonic on S^n. Before I can obtain the eigenvalues I have to calculate the affine connection Γ^p_{mn}.

The definition of spherical coordinates in n+1 dimensions are

$$\begin{aligned}
x_1 &= r \sin\theta_n \cdots \sin\theta_2 \cos\theta_1 \\
x_2 &= r \sin\theta_n \cdots \sin\theta_1 \\
x_3 &= r \sin\theta_n \cdots \sin\theta_3 \cos\theta_2 \\
x_4 &= r \sin\theta_n \cdots \sin\theta_4 \cos\theta_3 \\
&\vdots \qquad \vdots \\
x_{n+1} &= r \cos\theta_n
\end{aligned} \tag{C.12}$$

This definition gives the following diagonal metric

$$g_{rr} = 1, \quad g_{ii} = r^2 \sin^2\theta_n \cdots \sin^2\theta_{i+1}, \quad g_{ij} = 0, i \neq j \tag{C.13}$$

I will now calculate Γ^p_{mn}, having at least one r index. The definition (B.10) gives

$$
\begin{aligned}
\Gamma^r_{rr} = \frac{1}{2}g^{rq}[\partial_r g_{qr} + \partial_r g_{qr} - \partial_q g_{rr}] &= 0 && \text{(C.14)}\\
\Gamma^i_{rr} = \frac{1}{2}g^{iq}[\partial_r g_{qr} + \partial_r g_{qr} - \partial_q g_{rr}] &= 0 && \text{(C.15)}\\
\Gamma^r_{ri} = \Gamma^r_{ir} = \frac{1}{2}g^{rq}[\partial_i g_{qr} + \partial_r g_{qi} - \partial_q g_{ir}] &= 0 && \text{(C.16)}\\
\Gamma^j_{ri} = \Gamma^j_{ir} = \frac{1}{2}g^{jq}[\partial_i g_{qr} + \partial_r g_{qi} - \partial_q g_{ir}] &= \frac{1}{r}\delta^j_i && \text{(C.17)}\\
\Gamma^r_{ji} = \Gamma^r_{ij} = \frac{1}{2}g^{rq}[\partial_i g_{qj} + \partial_j g_{qi} - \partial_q g_{ij}] &= -\frac{1}{r}g_{ij} && \text{(C.18)}
\end{aligned}
$$

where I have used (C.13).

Now its time to use (C.8), (C.10) and (C.13)-(C.18) to obtain the eigenvalues of $\Box_{S^n}$, and then use (B.38) to obtain the eigenvalues of Δ_1. Equation (C.8) gives

$$
0 = \Box_{n+1}Y_i(r,\theta_i) = f(r,\partial_r)Y_i + \frac{1}{m^2r^2}(\Box_{S^n})Y_i \tag{C.19}
$$

where $f(r,\partial_r)Y_i$ is

$$
\begin{aligned}
f(r,\partial_r)Y_i &= \partial_r\partial_r Y_i - \partial_r(\Gamma^j_{ri}Y_j) - g^{jk}\Gamma^r_{jk}\partial_r Y_i - \Gamma^j_{ri}\partial_r Y_j\\
&+ g^{jk}\Gamma^r_{jk}\Gamma^{k'}_{ri}Y_{k'} + \Gamma^j_{ri}\Gamma^k_{rj}Y_k + g^{jk}\Gamma^r_{ki}\Gamma^{k'}_{jr}Y_{k'}\\
&= \frac{1}{r^2}\Big[k(k+n-1) - 1\Big]Y_i && \text{(C.20)}
\end{aligned}
$$

where I have used (C.10) and (C.13)-(C.18). This equation implies that

$$
\Box_{S^n}\Psi_i = -[k(k+n-1) - 1]m^2\Psi_i \tag{C.21}
$$

Using (B.38), (3.9), (3.23) and (3.15) gives the following eigenvalues of Δ_1 on S^7, S^5 and S^4

$$
\begin{aligned}
S^7: &\quad [k(k+6)+5]m^2 && \text{(C.22)}\\
S^5: &\quad [k(k+4)+3]m^2 && \text{(C.23)}\\
S^4: &\quad [k(k+3)+2]m^2 && \text{(C.24)}
\end{aligned}
$$

where $k = 1, 2, 3, \ldots$.

Next, I will obtain the eigenvalues of Δ_2, Δ_L and Q, (Q is only obtained when n=5). The tensor Y_{mn} is defined as

$$
Y_{mn}(r,\theta_i) = r^k\Psi'_{mn}(\theta_i), \quad i = 1,\ldots,n \tag{C.25}
$$

Here I have suppressed the label k on Y_{mn} and $m = r, 1, 2, \ldots, n$. This equation gives that

$$
Y_{ij} = e^m_i e^n_j Y_{mn} = r^{k+2}\Psi_{ij}(\theta_i) \tag{C.26}
$$

where $e_i^m = rE_i^m(\theta_i)$ is a vielbein. Equation (C.8) gives

$$0 = \Box_{n+1} Y_{ij}(r, \theta_i) = f(r, \partial_r) Y_{ij} + \frac{1}{m^2 r^2} (\Box_{S^n}) Y_{ij} \tag{C.27}$$

where $f(r, \partial_r)Y_{ij}$ is

$$\begin{aligned} f(r,\partial_r)Y_{ij} &= \partial_r\partial_r Y_{ij} - g^{kl}\Gamma^r_{lk}\partial_r Y_{ij} - \Gamma^k_{ri}\partial_r Y_{kj} - \Gamma^k_{rj}\partial_r Y_{ik} - \partial_r\Gamma^k_{ri}Y_{kj} \\ &+ g^{kl}\Gamma^r_{lk}\Gamma^{k'}_{ri}Y_{k'j} + \Gamma^l_{ri}\Gamma^k_{rl}Y_{kj} + g^{kl}\Gamma^r_{li}\Gamma^{k'}_{kr}Y_{k'j} + \Gamma^l_{rj}\Gamma^k_{ri}Y_{kl} - \partial_r\Gamma^k_{rj}Y_{ik} \\ &+ g^{kl}\Gamma^r_{lk}\Gamma^{k'}_{rj}Y_{ik'} + \Gamma^l_{rj}\Gamma^k_{rl}Y_{ik} + g^{kl}\Gamma^r_{lj}\Gamma^{k'}_{kr}Y_{ik'} + \Gamma^l_{ri}\Gamma^k_{rj}Y_{kl} \\ &= \frac{1}{r^2}\Big[k(k+n-1)-2\Big]Y_{ij} \end{aligned} \tag{C.28}$$

where I have used (C.10), (C.13)-(C.18) and (C.26). This equation implies that

$$\Box_{S^n}\Psi_{ij} = -[k(k+n-1)-2]m^2\Psi_{ij} \tag{C.29}$$

Using (B.39), (3.9), (3.23), (3.15), (C.28) and[1]

$$R_{ijkl} = m^2(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{C.30}$$

for a sphere S^n, gives the following eigenvalues of Δ_2 on S^7, S^5 and S^4

$$S^7: \quad [k(k+6)+8]m^2 \tag{C.31}$$
$$S^5: \quad [k(k+4)+4]m^2 \tag{C.32}$$
$$S^4: \quad [k(k+3)+2]m^2 \tag{C.33}$$

where $k = 1, 2, \ldots$.

Equation (C.32) and (3.380) gives that the eigenvalues to Q on S^5 is

$$\pm\sqrt{[k(k+4)+4]m^2} = \pm[k+2]m, \quad k = 1, 2, \ldots \tag{C.34}$$

Using (B.23), (3.9), (3.23), (3.15) , (C.28) and (C.30), gives the following eigenvalues of Δ_L on S^7, S^5 and S^4

$$S^7: \quad [k(k+6)+12]m^2 \tag{C.35}$$
$$S^5: \quad [k(k+4)+8]m^2 \tag{C.36}$$
$$S^4: \quad [k(k+3)+6]m^2 \tag{C.37}$$

where $k = 2, 3, \ldots$.

To obtain the eigenvalues of Q on S^7, I first have to obtain the eigenvalue of Δ_3 on S^7. The tensor Y_{mnp} is defined as

$$Y_{mnp}(r, \theta_i) = r^k \Psi'_{mnp}(\theta_i), \quad i = 1, \ldots, n \tag{C.38}$$

[1] Here one can see the reason for the different prefactors in the Freund-Rubin ansätze (3.6), (3.12) and (3.20). With the chosen prefactors one can see that (C.30) gives (3.9), (3.15) and (3.23) if one contracts the indises i and k in (C.30). This gives that m in this appendix and m in chapter 3 are the same.

Here I have suppressed the label k on Y_{mnp} and $m = r, 1, 2, \ldots, n$. This equation gives that

$$Y_{ijk} = e_i^m e_j^n e_k^p Y_{mnp} = r^{k+3}\Psi_{ijk}(\theta_i) \tag{C.39}$$

where $e_i^m = rE_i^m(\theta_i)$ is a vielbein. Equation (C.8) gives

$$0 = \Box_{n+1} Y_{ijk}(r, \theta_i) = f(r, \partial_r)Y_{ijk} + \frac{1}{m^2 r^2}(\Box_{S^n})Y_{ijk} \tag{C.40}$$

where $f(r, \partial_r)Y_{ijk}$ is

$$\begin{aligned} f(r,\partial_r)Y_{ijk} &= \partial_r\partial_r Y_{ij} - g^{k'l}\Gamma^r_{lk'}\partial_r Y_{ijk} - \Gamma^l_{ri}\partial_r Y_{ljk} - \Gamma^l_{rj}\partial_r Y_{ilk} - \Gamma^l_{rk}\partial_r Y_{ijl} + \Big(- \partial_r \Gamma^l_{ri} Y_{ljk} \\ &+ g^{k'l}\Gamma^r_{lk'}\Gamma^{l'}_{ri}Y_{l'jk} + \Gamma^{k'}_{ri}\Gamma^l_{rk'}Y_{ljk} + g^{k'l}\Gamma^r_{li}\Gamma^{l'}_{k'r}Y_{l'jk} + \Gamma^l_{rj}\Gamma^{l'}_{ri}Y_{l'lk} + \Gamma^l_{rk}\Gamma^{l'}_{ri}Y_{l'jl} \\ &+ 2\cdot(6 \text{ terms that give the same contribution as the previous 6 terms})\Big) \\ &= \frac{1}{r^2}\Big[k(k+n-1) - 3\Big]Y_{ijk} \end{aligned} \tag{C.41}$$

where I have used (C.10), (C.13)-(C.18) and (C.39). This equation implies that

$$\Box_{S^n}\Psi_{ijk} = -[k(k+n-1) - 3]m^2\Psi_{ijk} \tag{C.42}$$

Using (B.40), (3.9), (C.30) and (C.39) gives the following eigenvalues of Δ_3 on S^7

$$[k(k+6) + 9]m^2 \tag{C.43}$$

Using this equation and (3.158) gives that

$$\pm[k+3]m, \quad k = 1, 2, \ldots \tag{C.44}$$

is the eigenvalue of Q on S^7.

C.2 Eigenvalues of the Hodge-de Rham and the Lichnerowicz operators on T^7

Here I will derive the eigenvalues of the Hodge-de Rham and the Lichnerowicz operators on T^7.

On T^7 is the Riemann tensor $R_{mnpq} = 0$. This implies that

$$\Delta_p Y_{m_1\cdots m_p} = -\Box Y_{m_1\cdots m_p} \tag{C.45}$$

To start with I will obtain the eigenvalues of $\Delta_0 = -\Box$. A spherical harmonic on T^7 can be written as

$$Y(\theta_i) = e^{i\vec{n}\vec{\theta}} = e^{in^i\theta_i} \tag{C.46}$$

where $\vec{n}$ is a 7-vector containing positive integers. Using (C.45) with $p = 0$ and (C.46) gives

$$\Delta_0 Y = -\Box Y = -\frac{1}{r^2}\partial_i\partial^i e^{in^j\theta_j} = \vec{n}^2 r^{-2} Y \tag{C.47}$$

where I have quantized the eigenvalues in units of the square of the inverse radius r^{-1}. The first four values of M^2 and the degeneracy are

M^2 in units of r^{-2}	$\vec{n}$	Degeneration
0	$(0,\ldots,0)$	1
1	$(0,\ldots,0,1,0\ldots,0)$	7
2	$(0,0,1,0,0,1,0)$	21
3	$(1,0,1,0,0,1,0)$	35

I will now obtain the eigenvalues of Δ_1. The vector spherical harmonic Y_i is defined as

$$Y_i(\vec{\theta}) = \sum_{\vec{n}} a_i^{(\vec{n})} e^{i\vec{n}\vec{\theta}} \tag{C.48}$$

We also know that Y_i is transverse, i.e, $\partial^i Y_i = 0$. Using (C.45), (C.48) and that Y_i is transverse, gives that the eigenvalues of Δ_1 are the same as the eigenvalues of Δ_0 and also have the same degeneration.

The eigenvalues of Δ_2, Δ_3 and Δ_L are obtained in the same way as the eigenvalues of Δ_1, with the result that they have the same eigenvalues and degeneration as Δ_0.

D

AdS and the (super)-conformal group

This appendix contains a short discussion on the (super)-conformal group and its algebra. I will also give the definition of AdS spacetime and obtain its metric.

D.1 The conformal group and its algebra

The conformal group is the group of transformations which preserve the form of the metric up to an arbitrary scale factor, i.e, $g_{\mu\nu}(x) \rightarrow \hat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$, where $\mu = 0, \ldots, d-1$. The conformal group of Minkowski spacetime is generated by the Poincaré transformations, the scale transformation

$$x^\mu \rightarrow \lambda x^\mu \tag{D.1}$$

and the special conformal transformations

$$x^\mu \rightarrow \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2} \tag{D.2}$$

where a^μ is a constant vector.

I will now write down the algebra of the conformal group, which has the following generators: $M_{\mu\nu}$ for Lorentz transformations ($M_{\mu\nu}$ is antisymmetric), P_μ for translations, D for scaling transformations (D.1) and K_μ for special conformal transformations (D.2). All these generators annihilate the vacuum of the conformal theory. The conformal algebra is

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho} M_{\nu\sigma} \pm permutations, \quad [M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), \\
&[M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho} K\nu - \eta_{\nu\rho} K_\mu), \quad [P_\mu, K_\nu] = 2i(M_{\mu\nu} - \eta_{\mu\nu} D), \\
&[M_{\mu\nu}, D] = 0, \quad [D, P_\mu] = -iP_\mu, \quad [D, K_\mu] = iK_\mu
\end{aligned} \tag{D.3}$$

with all other commutators vanishing. We can now redefine the generators as

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu), \quad J_{\mu(d+1)} = \frac{1}{2}(K_\mu + P_\mu), \quad J_{(d+1)d} = D \tag{D.4}$$

where J_{ab} ($a = 0, \ldots, d+1$) are the generators for the group $SO(d,2)$. Then we easily see that the algebra of the conformal group is isomorphic to the algebra of the group $SO(d,2)$ (which is the isometry group of AdS_{d+1}).

For the conformal group the interesting representations (for physical applications) are involving operators (or fields) which are eigenfunctions of the scaling operator D with eigenvalue $-i\Delta$, where Δ is the scaling dimension of the field. This means that under the scaling transformation (D.1) an operator (or field) $\phi(x)$ is transforming as $\phi(x) \to \phi'(x) = \lambda^\Delta \phi(\lambda x)$. The commutation relations (D.3) imply that the operator K_μ lowers the dimensions of the field, while the operator P_μ raise it. In unitary field theories there is a lower bound on the dimension of fields, for example; for scalar fields it is $\Delta \geq (d-2)/2$. This implies that each representation of the conformal group must have some operator of lowest dimension, which must be annihilated by K_μ (at $x = 0$). These operators are called primary operators.

The conformal group can be extended to the superconformal group if we include supersymmetry. The new generators in the algebra are: the supersymmetry generators Q, the fermionic generators S (one for each supersymmetry generator Q) and sometimes the R-symmetry generators R. Schematically (suppressing all indices), the commutation relations of the superconformal algebra include (D.3) and

$$\begin{aligned} &[D,Q] = -\frac{i}{2}Q, \quad [D,S] = \frac{i}{2}S, \quad [K,Q] \simeq S, \quad [P,S] \simeq Q \\ &\{Q,Q\} \simeq P, \quad \{S,S\} \simeq K, \quad \{Q,S\} \simeq M + D + R \end{aligned} \tag{D.5}$$

The reason that I have not written the exact commutation relations is because they are different for different dimensions and different R-symmetry groups. It can be shown that superconformal theories exist only for $d \leq 6$. For a more extensive introduction to the conformal group see for example [40].

D.2 AdS spacetime

In this section, I will discuss the AdS metric. If we start with a flat $(p+3)$-dimensional space with metric

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2, \tag{D.6}$$

then a $(p+2)$-dimensional AdS space can be represented as the hyperboloid

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = R^2 \tag{D.7}$$

where R is the radios of AdS_{p+2}. From this we see that AdS_{p+2} have the isometry $SO(2, p+1)$. If we now insert

$$\begin{aligned} X_0 &= R\cosh\rho\cdot\cos\tau, \quad X_{p+2} = R\cosh\rho\cdot\sin\tau, \\ X_i &= R\sinh\rho\cdot\Omega_i \quad (i = 1,\ldots,p+1; \sum_i \Omega_i^2 = 1) \end{aligned} \tag{D.8}$$

(which solves (D.7)) into (D.6), then we obtain the metric on AdS_{p+2}, which is

$$ds^2 = R^2(-\cosh^2\rho\cdot d\tau^2 + d\rho^2 + \sin^2\rho\cdot d\Omega^2) \tag{D.9}$$

where $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$. The coordinates (τ, ρ, Ω_i) are called global coordinates, since they cover the entire hyperboloid once. The hyperboloid has the topology of $S^1 \times \mathbb{R}^{p+1}$, since the metric behaves near $\rho = 0$ as $ds^2 \simeq R^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega^2)$. Here S^1 represents closed timelike curves in the τ direction. To obtain a causal spacetime, we simply unwrap the circle S^1 (i.e, take $-\infty < \tau < \infty$) and obtain the universal covering of the hyperboloid without closed timelike curves. Earlier in this thesis when I referred to AdS, I was referring to its universal cover.

I will now write the metric in Poincaré coordinates $(u, t, \vec{x})$, where $(0 < u, \vec{x} \in \mathbb{R}^p)$ and t is the time coordinate. These coordinates are defined as

$$\begin{aligned} X_0 &= \frac{1}{2u}\Big(1 + u^2(R^2 + \vec{x}^2 - t^2)\Big), \quad X_{p+1} = \frac{1}{2u}\Big(1 - u^2(R^2 - \vec{x}^2 + t^2)\Big), \\ X^{p+2} &= Rut, \quad X_i = Rux_i \quad i = 1,\ldots,p. \end{aligned} \tag{D.10}$$

where R is the radius. Substituting these coordinates into (D.6) gives a new metric for AdS_{p+2}, which is

$$ds^2 = R^2\Big(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2)\Big) = \frac{R^2}{z^2}\Big(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\Big), \quad \text{for } u = 1/z \tag{D.11}$$

Finally, calculating the Riemann and the Ricci tensors for AdS_{p+2}, gives

$$R_{\mu\nu\rho\sigma} = \frac{1}{R^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \Rightarrow R_{\mu\nu} = (p+1)\frac{1}{R^2}g_{\mu\nu} \tag{D.12}$$

For a more extensive introduction to the AdS spacetime see for example [29].

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