Thesis for the degree of Doctor of Philosophy

Aspects of Wrapped Branes in String and M-Theory

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Abstract
This thesis consists of an introductory text together with five appended research papers. The Ariadne's thread through the whole thesis is various effects coming from high-dimensional p-branes in various subsectors of string and M-theory.

The low energy effective actions in string and M-theory consists of a classical supergravity together with quantum corrections. In particular the non-perturbative correction terms arise from instanton effects, which are interpreted as p-branes wrapping supersymmetric cycles. The general structure of the full effective action is the result of a complicated interplay between supersymmetry and U-duality. Requiring the action to be invariant under U-duality leads to mathematical functions called automorphic forms. Both perturbative and non-perturbative corrections seem to be captured by these functions. The U-duality groups can be found by analyzing the algebraic structures of the moduli space after toroidal compactification. Using this line of thinking, some simple examples of higher order derivative corrections in pure gravity are investigated.

Compactification on manifolds with special holonomy is also discussed in this thesis, with focus on the resulting moduli spaces. Certain quantum corrections to type IIA string theory compactified on a rigid Calabi-Yau threefold are analyzed.

Manifolds with special holonomy constitute target spaces of the topological subsectors in string and M-theory. The low energy effective action of these theories consists of a classical contribution from a form theory of gravity, which receives quantum corrections from branes wrapping supersymmetric cycles in the target space. In particular the dynamics of the M2- and M5-branes are discussed in the context of a topological version of M-theory.

This thesis consists of an introductory text and the following five appended research papers, henceforth referred to as Paper I-V:

I. L. Bao, V. Bengtsson, M. Cederwall and B. E. W. Nilsson, *Membranes for topological M-theory*, JHEP **0601** (2006) 150, [hep-th/0507077].

II. L. Bao, M. Cederwall and B. E. W. Nilsson, *A note on topological M5-branes and string-fivebrane duality*, JHEP **0806** (2008) 100, [hep-th/0603120].

III. L. Bao, M. Cederwall and B. E. W. Nilsson, *Aspects of higher curvature terms and U-duality*, Class. Quant. Grav. **25** (2008) 095001, [0706.1183 [hep-th]].

IV. L. Bao, J. Bielecki, M. Cederwall, B. E. W. Nilsson and D. Persson, *U-Duality and the Compactified Gauss-Bonnet Term*, JHEP **0807** (2008) 048, [0710.4907 [hep-th]].

V. L. Bao, A. Kleinschmidt, B. E. W. Nilsson, D. Persson and B. Pioline, *Instanton Corrections to the Universal Hypermultiplet and Automorphic Forms on SU(2,1)*, to appear.

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Contents

Introduction

Modern physics started its course almost a century ago with the birth of *quantum mechanics* and *general relativity*. In many ways these two theories can be considered as opposite poles. History has told us that these two pillars of theoretical physics seem to be incompatible with each other. However, there are reasons to hope that by choosing a clever language this can be rectified and the two theories made to live in perfect harmony with each other. Finding the appropriate framework to do so has been the ultimate quest of high energy physics for the last three decades.

Einstein's theory of general relativity couples gravitational motion to the geometry of spacetime. Gravitational systems are governed by equations of motion which retain their form under coordinate transformations. This theory works extremely well for heavy objects over large distance scales, as in astronomy for instance. Quantum mechanics, on the other hand, dictates that the energy cannot take arbitrarily small values. Rather, it is said to be quantized. Quantum mechanics is the framework to use when dealing with small physical systems like atoms.

The experimental discoveries of the *electromagnetic*, *weak* and *strong* forces pointed towards a general picture of the fundamental constituents of Nature. Both matter and forces are viewed as point-like particles, which are characterized by mass, spin, charge, etc.. Some of them obey Bose-Einstein statistics (*bosons*), others follow Fermi-Dirac statistics (*fermions*). In particular the forces are mediated through massless particles named *gauge bosons*. Since the gauge bosons move at the speed of light, the correct quantum theory describing these has to respect Lorentz symmetry. In this so called *relativistic quantum field theory*, elementary particles appear as states in the spectrum after quantization. The particle dynamics are then dictated by the scattering amplitudes, which are derived in terms of Feynman diagrams. The construction of the *Standard Model* containing the electromagnetic, weak and strong forces is so far the greatest success of quantum field theory. The Standard Model is a non-abelian gauge theory based on the Lie group $SU(3) \times SU(2) \times U(1)$. Although some of the Feynman diagrams seemed to give rise to divergences at first, it was later found that they can be canceled out by employing a clever renormalization scheme. Experimentally the Standard Model has been tested and seen to hold extremely well. The last major piece of the puzzle still missing is the experimental verification of the mechanism that gives mass to the elementary particles, which presumably happens via the Higgs mechanism.

The development of the Standard Model has been a process moving higher and higher up in energy scale, theoretically as well as experimentally. Extrapolating the three coupling constants in the theory to very high energies, it turns out that they intersect almost at one point. A new kind of symmetry which exchanges bosons with fermions then entered the stage. This *supersymmetry* made the Higgs sector of the Standard Model better behaved in the ultraviolet region, and as a side product the three forces became beautifully united. Hopefully the answer to whether or not supersymmetry really exists will be found in a not too distant future.

The idea of uniting the forces in Nature has been an enormously fruitful guide for theoretical physics during the twentieth century. It was this line of thinking that enabled the construction of the Standard Model. However, the theory describing the origin of the universe or interior of a black hole must contain both gauge theories and gravity. In other words, there exists a length scale, known as the *Planck length*

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616252(81) \times 10^{-35}\ \mathrm{m}, \tag{1}$$

where the spacetime itself is quantized. The ingredients in Eq. (1) are the three fundamental constants in Nature[1]: the reduced Planck constant $\hbar$, the gravitational constant G and the speed of light in vacuum c. Straightforwardly quantizing gravity leads to many problems that we do not know how to solve, e.g., it seems to be non-renormalizable.

The most successful attempt at quantizing gravity up to now is provided by *string theory*. The fundamental object in string theory is, as the name suggests, a string, which when moving around sweeps out a two-dimensional surface in spacetime named the *worldsheet*. The classical action is simply given by the area of the worldsheet. Quantizing this action, one finds not only gauge fields, but also a particle that can be interpreted as the graviton. One of the many beautiful properties of string theory is the fact that it contains only one free parameter — the Regge slope α'. Both the characteristic length scale ℓ_s and the tension $T_{\mathrm{F1,S}}$ of the fundamental string are expressed in terms of α' according to

$$\ell_s = \sqrt{\alpha'} \quad \text{and} \quad T_{\mathrm{F1},S} = \frac{1}{2\pi\alpha'}. \tag{2}$$

Moreover, the coupling constant appearing in *target space* is identified with the vacuum expectation value of the dilaton scalar field:

$$g_s = e^{<\phi>}. \tag{3}$$

[1]Throughout this thesis we will use natural units, i.e., $\hbar = c = 1$.

The fact that strings are one-dimensional resolves the problem of ultraviolet divergences in the scattering amplitudes. The major argument against string theory is that it requires a huge number of spacetime dimensions to be consistent. Requiring also invariance under supersymmetry constrains the dimension to be ten, which is still way to many compared to the four we observe. The basic idea of how to deal with this problem is *Kaluza-Klein compactification*, where the six superfluous spacelike dimensions are thought to be small. Although we cannot observe these compact dimensions directly, the four-dimensional physics is affected by their detailed structures. Much efforts have been made trying to understand the exact implications of various choices of internal manifolds.

Another puzzle in string theory for a long time was the fact that five self-consistent string theories with seemingly distinct properties were found. They are called type IIA, type IIB, type I, $O(32)$ heterotic and $E_8 \times E_8$ heterotic. In the low energy limit each of them is described by a corresponding *supergravity theory* together with perturbative as well as non-perturbative quantum corrections. This puzzle was eliminated by the discovery of dualities. *S-duality* exchanges weak and strong couplings, while *T-duality* relates certain string theories compactified on small radii with others compactified on large radii. These dualities collectively point towards an eleven-dimensional umbrella, which is named *M-theory* by E. Witten. For instance type IIA string theory is obtained from M-theory by circular compactification, in particular the IIA string coupling constant can be reinterpreted in terms of the compactification radius R_{11} and the Planck length,

$$g_s = \left(\frac{R_{11}}{\ell_p}\right)^{3/2} . \tag{4}$$

In the low energy limit, M-theory itself is described by an eleven-dimensional supergravity theory. As for the quantum theory, the fundamental object is believed to be a membrane. However, quantization of the membrane world-volume action has so far not been achieved in a satisfactory way.

Whatever the microscopic description of M-theory turns out to be, the various string theories should be thought of as perturbative descriptions of distinct corners of its parameter space. Dualities are the correct tool to use when relating these corners. Although string and M-theory are mathematically very beautiful, we shall not forget that the goal for physicists is to understand Nature. The discovery of higher-dimensional *p-branes* in string theory finally opened the door to semi-realistic gauge theories. Furthermore, during the last decade a new type of duality was discovered, which relates certain string configurations on some particular d-dimensional spacetime geometries to non-abelian gauge theories in one dimension less. As string and M-theory reveal more and more of their secrets, hopefully soon we will know whether or not this is the right track to take.

D-Branes

A special type of p-branes related to open strings with Dirichlet boundary conditions will play a central role in this thesis. Some basic properties of these so called *D-branes* are briefly reviewed here.

Dp-branes in string theory are $(p+1)$-dimensional objects on which open strings can end. They were first discovered as a consequence of the T-duality by J. Dai, R. G. Leigh and J. Polchinski [1], and independently by P. Hořava [2]. Later they were identified with BPS p-brane solutions of the ten-dimensional supergravity theories [3]. The presence of D-branes breaks the symmetries of the Minkowski space vacuum. In the vicinity of a Dp-brane the Lorentz symmetry is broken according to

$$SO(1,9) \to SO(1,p) \times SO(9-p), \tag{5}$$

while at least half of the supersymmetries are also broken.

Every massless gauge field in string theory is generated by an electric or magnetic p-brane source. Let A_n denote an n-form gauge field. Its field strength is then an $(n+1)$-form given by

$$F_{n+1} = dA_n. \tag{6}$$

Coupling terms containing the other gauge fields are for simplicity omitted on the right hand side. Due to $d^2 = 0$, field strengths defined as in Eq. (6) are invariant under the gauge transformations

$$\delta A_n = d\Lambda_{n-1}. \tag{7}$$

A $(p+1)$-form gauge field can be coupled to a p-brane via

$$S_{\text{int}} = e_p \int A_{p+1}, \tag{8}$$

where the pullback of A_{p+1} to the brane worldvolume is understood implicitly. The conventions we use are from Ref. [4]. The *electric* p-brane charge can be computed using Gauss's law

$$e_p = \int_{S^{D-p-2}} {*F_{p+2}}. \tag{9}$$

The integral (9) is computed over a sphere S^{D-p-2}, with D being the total number of spacetime dimensions. On the other hand, one can also define a *magnetic* charge according to

$$m_p = \int_{S^{p+2}} F_{p+2}. \tag{10}$$

Making the identification $F_{p+2} = *\tilde{F}_{D-p-2}$, we may reinterpret m_p as the electric charge of a dual $(D-p-4)$-brane

$$\tilde{S}_{\text{int}} = m_p \int \tilde{A}_{D-p-3}, \tag{11}$$

where $\tilde{A}_{D-p-3}$ is the gauge potential of $\tilde{F}_{D-p-2}$. Thus, p- and $(D-p-4)$-branes are dual to each other. In particular for $D = 10$ we find that p-branes are dual to $(6-p)$-branes, and it is motivated to set $e_{p-6} = m_p$. Moreover, the electric and magnetic charges have to satisfy the *Dirac quantization condition* [5, 6, 7]

$$e_p m_p = e_p e_{D-p-4} \in 2\pi\mathbb{Z}. \tag{12}$$

The Dp-branes are most naturally embedded in a target space, containing both spacetime and (odd) Grassmann coordinates, called superspace [8]. Theories formulated in superspace are manifestly target space supersymmetric. On the worldsheet local *kappa symmetry* is employed to ensure the matching between bosonic and fermionic degrees of freedom [9, 10, 11, 12, 13, 14]. The worldvolume theory of a single Dp-brane in type II string theories is governed by the *Dirac-Born-Infeld action* [15]

$$S_p = -T_{\mathrm{D}p} \int d^{p+1}\sigma \sqrt{-\det(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})}. \tag{13}$$

Here $G_{\alpha\beta}$ is the worldvolume pullback of the spacetime metric, while $F_{\alpha\beta}$ is the pullback of a combination with the Maxwell field strength and the Kalb-Ramond field. The worldvolume coordinates are denoted by σ^α. The action in Eq. (13) is non-linear, and expanding with respect to small $F_{\alpha\beta}$ leads to an infinite series of terms starting with the ordinary Maxwell action. The symbol $T_{\mathrm{D}p}$ stands for the tension of the p-brane. Using T-duality one can find the general expression

$$T_{\mathrm{D}p} = \frac{1}{g_s (2\pi)^p \alpha'^{\frac{p+1}{2}}}. \tag{14}$$

Let us emphasize the fact that the tension of a D-brane behaves as the inverse of the string coupling constant, i.e., $T_{\mathrm{D}p} \sim 1/g_s$. Later we will also encounter the so called NS5-branes, whose tension scales like $T_{\mathrm{NS5}} \sim 1/g_s^2$.

D-branes play an important role in string theory, since gauge theories arise naturally on the D-brane worldvolume [16]. This gives rise to new opportunities to find the Standard Model. From the gravitational point of view, D-branes living entirely in the compact dimensions provide a microscopic explanation for the thermodynamical properties of black holes [17]. In this thesis we will focus on D-branes wrapping supersymmetric cycles in a compact manifold. Two examples of such D-brane effects will be given. One is excitations of topological string theories on Calabi-Yau threefolds. The other is instanton corrections in the spacetime effective theory, arising when D-branes are completely wrapped on cycles in the internal manifold.

Outline

This thesis consists of four chapters, one appendix and five research papers. The conventions are self-consistent within each chapter, and they are kept as uniform as possible between the chapters.

Chapter 1 reviews the basics of the maximal supergravity theories in eleven and ten dimensions. The role of the S- and T-duality in string theory are described, and the U-duality conjecture is presented.

The theory of Kaluza-Klein compactification is reviewed in **Chapter 2**. The dimensional reduction on an n-dimensional torus is done explicitly, in particular the symmetry properties of the moduli space are analyzed in relation to the U-duality. Compactification on Calabi-Yau threefolds and G_2 manifolds is also discussed in this chapter. Moreover, this chapter also contains a brief account of the topological subsectors of string and M-theory residing on Calabi-Yau threefolds and G_2 manifolds.

Chapter 3 picks up where Chapter 1 ends and discusses the low energy effective actions of type II string theories beyond the supergravity level. It contains both perturbative and non-perturbative quantum corrections, organized as a double expansion in the Regge slope α' and the string coupling constant g_s. Both supersymmetry and U-duality turn out to be useful for finding the general structures of the correction terms.

The g_s expansion at each α'-level is encoded by mathematical functions called automorphic forms. Some mathematical backgrounds of automorphic forms is introduced in **Chapter 4**. The non-holomorphic Eisenstein series based on the discrete group $SL(2,\mathbb{Z})$ is worked through in detail as a guiding example. Various construction methods as well as the Fourier properties of it are presented. Generalization to Eisenstein series based on discrete subgroups of larger Lie groups is discussed. Moreover, construction of automorphic forms which transform under certain Lie groups is briefly mentioned. One of the constructions is based on p-adic numbers, the relevant properties of this mathematical field are given in **Appendix A**.

The appended research papers are grouped into two parts. The first part consists of Paper I and II and deals with the topological subsector of M-theory. The second part analyzes some symmetry structures of the quantum corrections in string theory effective actions. This part contains Paper III, IV and V.

Considering a target space with G_2 holonomy, the supersymmetric action of a membrane moving in this space is formulated in Paper I. The fact that this action is BRST-exact on-shell indicates that it is topological. It is suggested that this membrane is the fundamental object of the conjectured topological M-theory.

The action for a five-brane in topological M-theory is subsequently given in Paper II using the top-form formulation. After compactification on a circle, the M5-brane is identified with the NS5-brane in the topological A model. The Kodaira-Spencer equation appears as equation of motion for the three-form on the NS5-brane, which indicates a duality relation between the topological A and B models.

Paper III discusses symmetries of coset type for the gravitational $\mathcal{R}^2$, $\mathcal{R}^3$ and $\mathcal{R}^4$ corrections in the string effective action. This is achieved by dimensional reduction to three spacetime dimensions on n-torii. It is argued in this paper that requiring invariance under U-duality would require transforming automorphic forms.

The toroidal reduction of the Gauss-Bonnet combination is analyzed in detail in Paper IV. By investigating the dilaton exponents in the resulting action, the symmetry properties of this correction term are discussed. In particular focus is set on the "U-duality" symmetry $SL(n+1,\mathbb{R})$.

Paper V is also dealing with quantum corrections, although in another context. The system considered here is type IIA string theory compactified on a rigid Calabi-Yau threefold. The moduli space variables of this theory parameterizes the symmetric space $SU(2,1)/U(2)$. It is argued that the quantum corrections at the two-derivative level are captured by the non-holomorphic Eisenstein series based on the Picard modular group $SU(2,1;\mathbb{Z}[i])$. Physical interpretations are given for the various components of this Eisenstein series.

1

Supergravities and Dualities

This chapter is devoted to the subject of supergravity theories, which initially were considered as candidates for the unification of the Standard Model with Einstein's theory of general relativity. Nowadays they are understood as low energy limits of string and M-theory. In the limit of large string tension, or equivalently, when the Regge slope $\alpha' \to 0$ the massive particles become very heavy. It is then justified to approximate string theory with its low-energy effective supergravity. Even though supergravity theories only describe interactions between the massless modes, studying them has proven to be very fruitful. Most importantly they opened the door to the powerful non-perturbative tools called dualities, resulting in the second superstring revolution.

1.1 Higher-Dimensional Supergravites

A *supergravity theory*, originally proposed in Ref. [18], is the extension of gravity by supersymmetry. By definition it is invariant under local *super-Poincaré transformations*. Among other fields supergravity contains a massless spin-two graviton and its superpartner, the spin 3/2 *gravitino*. The number of gravitino fields is denoted by $\mathcal{N}$ and equals the number of copies of a supersymmetry. Supergravities can be formulated in many spacetime dimensions. However, constraining all the particle spins to be two or less, as is what has been observed in nature, it was shown in Ref. [19] that the maximal number of supercharges consistent with a single graviton is 32. This corresponds to an eleven-dimensional spacetime with Lorentzian metric[1]. In this section we will concentrate on supergravities in $D = 10$ and $D = 11$, since they are most

[1]Relaxing the Lorentzian metric constraint it is possible to have twelve dimensions with two of them being timelike, which is the background setup for the so called *F-theory* [20].

closely related to string and M-theory. The standard reference to supersymmetry and supergravity is the book by J. Wess and J. Bagger [21].

1.1.1 Eleven-Dimensional Supergravity

Ever since its discovery [22] *eleven-dimensional supergravity* has held a special place in high energy theoretical physics. This is the only supersymmetric theory in eleven dimensions. It contains one *supermultiplet*, transforming as a single representation of the supergroup $OSp(1|32)$. The field content of the supermultiplet consists of the elfbein ${E_M}^A$, the gravitino Ψ_M and a rank three gauge field $\mathcal{C}_{MNP}$. The index M is the curved spacetime index, while A is its tangent space equivalent. Since it contains the maximal number of supersymmetries permitted in eleven dimensions, this theory is called a *maximal supergravity*. The gravitino is a 32-component Majorana spinor, which transforms as a representation under $Spin(1,10)$.

The bosonic part of the eleven-dimensional supergravity action is

$$S_{11} = \frac{1}{2\kappa_{11}^2}\left[\int d^{11}x\sqrt{-G}R + \frac{1}{2}\int \mathcal{G}_4 \wedge *\mathcal{G}_4 + \frac{1}{6}\mathcal{G}_4 \wedge \mathcal{G}_4 \wedge \mathcal{C}_3\right], \tag{1.1}$$

where R is the curvature scalar defined using the metric $G_{MN} = \eta_{AB}{E_M}^A{E_N}^B$. The four-form field strength $\mathcal{G}_4 \equiv d\mathcal{C}_3$ is invariant under the gauge transformations

$$\mathcal{C}_3 \longmapsto \mathcal{C}_3' = \mathcal{C}_3 + d\Lambda_2 \tag{1.2}$$

and satisfies the Bianchi identity

$$d\mathcal{G}_4 = 0. \tag{1.3}$$

Einstein's equation together with

$$d * \mathcal{G}_4 + \frac{1}{2}\mathcal{G}_4 \wedge \mathcal{G}_4 = 0 \tag{1.4}$$

constitute the equations of motion. An alternative formulation can be found by introducing also a dual gauge field $\mathcal{C}_6$ and its corresponding field strength

$$\mathcal{G}_7 = d\mathcal{C}_6 + \frac{1}{2}\mathcal{C}_3 \wedge \mathcal{G}_4. \tag{1.5}$$

Requiring

$$* \mathcal{G}_4 = -\mathcal{G}_7, \tag{1.6}$$

Eq. (1.4) turns into the Bianchi identity of $\mathcal{G}_7$. The overall constant κ_{11} is related to the eleven-dimensional Newton's constant G_{11} and the 11-dimensional Planck length ℓ_p as

$$2\kappa_{11}^2 = 16\pi G_{11} = \frac{(2\pi\ell_p)^9}{2\pi}. \tag{1.7}$$

Using the supersymmetry variations

$$\begin{aligned}\delta E_M{}^A &= \bar{\varepsilon}\Gamma^A\Psi_M,\\ \delta\mathcal{C}_{MNP} &= -3\bar{\varepsilon}\Gamma_{[MN}\Psi_{P]},\\ \delta\Psi_M &= \nabla_M\varepsilon + \frac{1}{12}\left(\frac{1}{4!}\Gamma_M\mathcal{G}_{NPQR}\Gamma^{NPQR} - \frac{1}{2}\mathcal{G}_{MNPQ}\Gamma^{NPQ}\right)\varepsilon,\end{aligned} \tag{1.8}$$

we can obtain the full supersymmetric action. The variation $\delta\Psi_M$ given here is only to leading order in fermionic fields, additional terms which are quadratic in the fermionic fields have been dropped. The Dirac matrices are defined by $\Gamma_M = E_M{}^A\Gamma_A$, with Γ_A satisfying the Clifford algebra. Moreover, the covariant derivative appearing in Eq. (1.8) is given by

$$\nabla_M\varepsilon = \partial_M\varepsilon + \frac{1}{4}\omega_{MAB}\Gamma^{AB}\varepsilon, \tag{1.9}$$

where ω_{MAB} is the standard spin connection in tangent space.

In order to find bosonic solutions which also preserve some supersymmetries, the variation of the gravitino has to vanish:

$$\delta\Psi_M = \nabla_M\varepsilon + \frac{1}{12}\left(\frac{1}{4!}\Gamma_M\mathcal{G}_{NPQR}\Gamma^{NPQR} - \frac{1}{2}\mathcal{G}_{MNPQ}\Gamma^{NPQ}\right)\varepsilon = 0. \tag{1.10}$$

A spinor ε satisfying this equation is called a *Killing spinor*. More specifically for eleven-dimensional supergravity there are two stable maximally supersymmetric brane solutions, a 2-brane and a 5-brane, which are electrically and magnetically charged, respectively, with respect to the $\mathcal{C}_3$ field. The fact that they both saturate the Bogomolny-Prasad-Sommerfield (*BPS*) bound means that their masses are equal to their charges. These two solutions are precisely the long-wavelength limits of the M2- and M5-brane in M-theory with

$$T_{\mathrm{M2}} = 2\pi(2\pi\ell_p)^{-3} \quad \text{and} \quad T_{\mathrm{M5}} = 2\pi(2\pi\ell_p)^{-6} \tag{1.11}$$

being their tensions [4].

The uniqueness of eleven-dimensional supergravity caused much excitement when it was first introduced. Much of the hope of it being the Theory Of Everything died out when it was realized that $D = 11$ supergravity is non-chiral as well as non-renormalizable. However, it managed to come back to the forefront of physics when E. Witten pointed out the existence of eleven-dimensional M-theory. Instead of being a fundamental theory, $D = 11$ supergravity should be thought of as the classical limit of M-theory. The fact that it is not renormalizable is not an obstacle anymore since it is only an effective theory valid at low energies. Since then it has also been understood that four-dimensional chiral theories can be obtained from higher-dimensional non-chiral ones by compactifying on manifolds with suitable singularities [23, 24].

1.1.2 Type IIA Supergravity

The first hint towards M-theory is the construction of *type IIA supergravity.* This theory is obtained from eleven-dimensional supergravity by *dimensional reduction* [25]. Similar to how $D = 11$ supergravity is interpreted as the low energy limit of M-theory, type IIA supergravity is the low energy limit of type IIA superstring theory in ten dimensions [26].

Upon dimensional reduction on S^1, the eleven-dimensional metric gives rise to a ten-dimensional metric, a gauge field and a scalar (dilaton) in the following way

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + e^{2\sigma} A_\mu A_\nu & e^{2\sigma} A_\mu \\ e^{2\sigma} A_\nu & e^{2\sigma} \end{pmatrix}. \tag{1.12}$$

The conventions used here are the same as in Ref. [27]. As opposed to compactification, in dimensional reduction only the zero modes in the Fourier expansions of the various fields are kept. Similarly the three-form gauge field is decomposed into a three-form and a two-form

$$C_{\mu\nu\rho} = \mathcal{C}_{\mu\nu\rho} - (\mathcal{C}_{\nu\rho,10} A_\mu + \text{cyclic}), \quad B_{\mu\nu} = \mathcal{C}_{\mu\nu,10}. \tag{1.13}$$

The bosonic part of the dimensional reduced action can now be written as

$$\begin{aligned} S_{\text{IIA}} =& \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{\sigma} \left[R - \frac{1}{2 \cdot 4!} F_4^2 - \frac{1}{2 \cdot 3!} e^{-2\sigma} H_3^2 - \frac{1}{4} e^{2\sigma} F_2^2 \right] \\ &+ \frac{1}{4\kappa^2} \int B_2 \wedge dC_3 \wedge dC_3, \end{aligned} \tag{1.14}$$

where the field strengths are defined according to

$$F_2 = dA, \quad H_3 = dB_2, \quad F_4 = dC_3 - A \wedge H_3. \tag{1.15}$$

To bring the action to the standard *string frame* we need to rescale the metric $g_{\mu\nu} \to e^{-\sigma} g_{\mu\nu}$. The end result is

$$\begin{aligned} S_{\text{IIA,S}} =& \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12} H_3^2 \right) \right. \\ & \left. - \frac{1}{2 \cdot 4!} F_4^2 - \frac{1}{4} F_2^2 \right] + \frac{1}{4\kappa^2} \int B_2 \wedge dC_3 \wedge dC_3, \end{aligned} \tag{1.16}$$

with $\phi \equiv 3\sigma/2$. Later in Section 3.1 we will see that the factor $e^{-2\phi}$ in front of the curvature scalar originates from a spherical string worldsheet. Sometimes it is useful to express the type IIA supergravity action without this dilaton factor, which is known as the *Einstein frame.* This can be achieved by yet another Weyl rescaling of the metric, $g_{\mu\nu} \to e^{\phi/2} g_{\mu\nu}$, yielding

$$
\begin{aligned}
S_{\mathrm{IIA,E}} =& \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[\left(R - \frac{1}{2}(\nabla\phi)^2 - \frac{e^{-\phi}}{12} H_3^2 \right) \right. \\
& \left. - \frac{e^{\phi/2}}{2 \cdot 4!} F_4^2 - \frac{e^{3\phi/2}}{4} F_2^2 \right] + \frac{1}{4\kappa^2} \int B_2 \wedge dC_3 \wedge dC_3 .
\end{aligned}
\tag{1.17}
$$

Notice that compared to Eq. (1.16) new couplings between the dilaton and the R-R fields have also appeared.

Decomposing the gravitino in eleven dimensions into representations of $Spin(1,9)$, we obtain a Majorana gravitino (ψ_α^μ) and a Majorana dilatino (λ_α). Using the Γ_{11} matrix each of these can be decomposed again into a pair of Majorana-Weyl spinors of opposite chirality. Together with the graviton ($g_{\mu\nu}$), the antisymmetric tensor ($B_{\mu\nu}$), the dilaton (ϕ), the vector (A_μ) and the antisymmetric three tensor ($C_{\mu\nu\rho}$), they form a single supermultiplet of $\mathcal{N} = (1,1)$ supersymmetry. All the supersymmetry transformations can be found in Ref. [25], in particular transformations of the fermionic fields in the Einstein frame are given by

$$
\begin{aligned}
\delta\lambda =& \frac{1}{2\sqrt{2}} \nabla_\mu \phi \Gamma^\mu \Gamma^{11} \varepsilon + \frac{3}{16\sqrt{2}} e^{3\phi/4} F^{(2)}_{\mu\nu} \Gamma^{\mu\nu} \varepsilon + \frac{i}{24\sqrt{2}} e^{-\phi/2} H_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \varepsilon \\
& - \frac{i}{192\sqrt{2}} e^{\phi/4} F^{(4)}_{\mu\nu\rho\sigma} \Gamma^{\mu\nu\rho\sigma} \varepsilon , \\
\delta\psi^\mu =& \nabla^\mu \varepsilon + \frac{1}{64} e^{3\phi/4} F^{(2)}_{\nu\rho} \left(\Gamma^{\mu\nu\rho} - 14 g^{\mu\nu} \Gamma^\rho \right) \Gamma^{11} \varepsilon \\
& + \frac{1}{96} e^{-\phi/2} H_{\nu\rho\sigma} \left(\Gamma^{\mu\nu\rho\sigma} - 9 g^{\mu\nu} \Gamma^{\rho\sigma} \right) \Gamma^{11} \varepsilon \\
& + \frac{i}{256} e^{\phi/4} F^{(4)}_{\nu\rho\sigma\tau} \left(\Gamma^{\mu\nu\rho\sigma\tau} - \frac{20}{3} g^{\mu\nu} \Gamma^{\rho\sigma\tau} \right) \Gamma^{11} \varepsilon .
\end{aligned}
\tag{1.18}
$$

The covariant derivative is defined as $\nabla_\mu \varepsilon = \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} \right) \varepsilon$. The full action of type IIA supergravity is obtained by acting on the bosonic part in Eq. (1.17) with supersymmetry transformations.

The type IIA string coupling constant is defined in terms of the vacuum expectation value of the dilaton

$$g_s = e^{<\phi>} . \tag{1.19}$$

As a result of the dimensional reduction (1.12), the string length scale is related to the Planck constant via

$$\ell_p = g_s^{1/3} \ell_s , \tag{1.20}$$

with $\ell_s = \sqrt{\alpha'}$. At the same time Newton's constant in ten and eleven dimensions are related as

$$G_{11} = 2\pi R_{11} G_{10} , \tag{1.21}$$

where the radius of the compact circle is then found to be $R_{11} = g_s \ell_s$. Using

$$16\pi G_{10} = \frac{1}{2\pi}(2\pi\ell_s)^8(g_s)^2 \tag{1.22}$$

one can thus show that κ appearing in Eq. (1.17) should be defined as

$$2\kappa^2 = \frac{1}{2\pi}(2\pi\ell_s)^8. \tag{1.23}$$

Moreover, we find that the string coupling constant satisfies

$$g_s = \left(\frac{R_{11}}{\ell_p}\right)^{3/2}. \tag{1.24}$$

Just like the physical fields, most of the branes contained in IIA supergravity also have eleven-dimensional origins [28, 29]. The M2-brane wrapped on the compactified circle is a IIA fundamental string F1, with tension given by, in the string frame,

$$T_{\mathrm{F1,S}} = 2\pi R_{11} T_{\mathrm{M2}} = \frac{1}{2\pi\ell_s^2}. \tag{1.25}$$

On the other hand, an M2-brane not wrapping around the compactified circle is a D2-brane. Similarly the M5-brane gives rise to a D4- or an NS5-brane. The origin of the D0- and D6-branes are slightly harder to guess. The former corresponds to the lowest Kaluza-Klein momentum mode along the compactified circle. The latter is the magnetic dual of the D0-brane, and its physical interpretation is a Kaluza-Klein monopole. The presence of a D8-brane would however lead to a mass deformation of the IIA supergravity. Since no eleven-dimensional lift of *massive IIA supergravity* is yet known, the origin of the D8-brane is not as well understood as the other branes.

Once type IIA supergravity is formulated we can revert the argument. By going to the strong coupling limit, we would have rediscovered its eleven-dimensional origin [30, 29].

1.1.3 Type IIB Supergravity

Besides type IIA supergravity, there exists one more maximal supergravity in ten dimensions. This theory is called *type IIB supergravity* and describes the massless limit of the type IIB superstring [31, 32, 26]. The supermultiplet of type IIB supergravity contains the graviton ($g_{\mu\nu}$), two scalars (ϕ, C_0), two antisymmetric tensors (B_2, C_2), one "self-dual" four-form (C_4), two Majorana-Weyl gravitini of the same chirality (or one Weyl gravitino ψ^μ) and two Majorana-Weyl dilatini of the same chirality (or one Weyl dilatino λ). The metric, ϕ and B_2 belong to the NS-NS sector, while C_0, C_2 and C_4 belong to the R-R sector. Since all the fermions are of the same chirality, type IIB supergravity is chiral and said to have $\mathcal{N} = (2,0)$ supersymmetry.

The property that the field strength of the four-form C_4 is *self-dual* is impossible to obtain from a covariant action. One can thus either work entirely at the level of equations of motion, or one can write down an action which yields all other equations except for the self-duality and then impose this condition by hand. It is the latter approach we are going to utilize[2].

One important feature of this theory is the existence of a global $SL(2,\mathbb{R})$ invariance. Elements of this matrix group

$$SL(2,\mathbb{R}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1; a, b, c, d \in \mathbb{R} \right\} \tag{1.26}$$

act by fractional linear transformations on the scalars

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}, \quad \text{with} \quad \tau = C_0 + ie^{-\phi}, \tag{1.27}$$

and linearly on the two-forms

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \longmapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}. \tag{1.28}$$

The bosonic part of the action in the Einstein frame can now be written as

$$\begin{aligned} S_{\text{IIB,E}} =& \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[R - \frac{1}{2} \frac{\partial\tau\partial\bar{\tau}}{\Im(\tau)^2} - \frac{1}{12} |G_3|^2 - \frac{1}{2 \cdot 5!} F_5^2 \right] \\ & + \frac{1}{2i} \int C_4 \wedge G_3 \wedge \bar{G}_3, \end{aligned} \tag{1.29}$$

where

$$H_3 = dB_2, \quad F_3 = dC_2, \quad G_3 = i \frac{F_3 + \tau H_3}{\sqrt{\Im(\tau)}}, \quad F_5 = dC_4 - C_2 \wedge H_3. \tag{1.30}$$

The notation $\Im(x)$ is referring to the imaginary part of x. In addition we have to impose the self-duality condition

$$F_5 = *F_5. \tag{1.31}$$

Straightforward computation shows that both the action and the self-duality condition are invariant under $SL(2,\mathbb{R})$. The choice of Einstein frame has made this quite transparent. In fact the invariance of the scalar sector can be made

[2]There is also a manifestly covariant formulation, by extending the theory with an auxiliary scalar field together with an extra gauge symmetry [33].

manifest once one observes that the moduli space, parameterized by ϕ and C_0, is isomorphic to the symmetric space $SL(2,\mathbb{R})/SO(2)$. More about this matter later when we discuss S-duality.

The supersymmetry transformations can be found in [31, 32], for instance the transformations of the fermions are give by

$$\begin{aligned}\delta\lambda =& -\frac{1}{2}e^{\phi}\Gamma^{\mu}\varepsilon^{*}\partial_{\mu}\tau - \frac{i}{24}e^{\phi/2}\Gamma^{\mu\nu\rho}\varepsilon G^{(3)}_{\mu\nu\rho},\\ \delta\psi^{\mu} =& \nabla^{\mu}\varepsilon - \frac{i}{1920}\Gamma^{\mu_1\ldots\mu_5}\Gamma^{\mu}\varepsilon F^{(5)}_{\mu_1\ldots\mu_5} + \frac{1}{96}\left(\Gamma^{\mu\nu\rho\sigma} - 9g^{\mu\nu}\Gamma^{\rho\sigma}\right)\varepsilon^{*}G^{(3)}_{\nu\rho\sigma},\end{aligned} \tag{1.32}$$

where $\varepsilon = \varepsilon_L + i\varepsilon_R$ and $\varepsilon^* = \varepsilon_L - i\varepsilon_R$ define the complexified version of a left and a right Majorana-Weyl spinor. Acting recursively on the bosonic action with the supersymmetry transformations we will find the full supersymmetric action. The detailed computation can be found in Refs. [31, 32].

The brane content of IIB supergravity includes odd-dimensional D(−1)-, D1-, D3-, D5- and D7-branes, which act as sources for the R-R gauge fields. The D(−1)- and D7-branes are coupled electrically and magnetically to the C_0 potential, respectively. Similarly C_2 is coupled to D1 and D5, while C_4 is coupled to the self-dual D3. In addition there are electric and magnetic sources for the B_2 field, namely the fundamental string F1 and the NS5-brane, respectively.

1.1.4 The Democratic Formulation

By extending the R-R fields with their Hodge duals, the authors of Ref. [34] managed to formulate both type IIA and IIB supergravity in a uniform way. The field content in this formulation becomes

$$\begin{aligned}\text{IIA}: &\quad \{g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(1)}, C^{(3)}, C^{(5)}, C^{(7)}, C^{(9)}, \psi_{\mu}, \lambda\},\\ \text{IIB}: &\quad \{g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(0)}, C^{(2)}, C^{(4)}, C^{(6)}, C^{(8)}, \psi_{\mu}, \lambda\}.\end{aligned} \tag{1.33}$$

The extra degrees of freedom will later be removed by self-duality constraints. Type IIA contains fermions of both chiralities, while the opposite is valid for IIB with $\Gamma^{11}\psi_{\mu} = \psi_{\mu}$ and $\Gamma^{11}\lambda = -\lambda$.

The notations will be hugely simplified if we define a collective gauge potential

$$\mathbf{C} = \sum_{n=1,\frac{1}{2}}^{5,\frac{9}{2}} C^{(2n-1)}, \tag{1.34}$$

where the sums run over the integers in IIA and half-integers $\frac{1}{2}\ldots\frac{9}{2}$ in IIB. The field strengths are then given by

$$H = dB \quad \text{and} \quad \mathbf{G} = d\mathbf{C} - H \wedge \mathbf{C} + G^{(0)}e^{B}, \tag{1.35}$$

with $\mathbf{G} = \sum_{n=0,\frac{1}{2}}^{5,\frac{9}{2}} G^{(2n)}$ being a collective field strength. $G^{(0)}$ is the constant mass parameter of IIA supergravity, while it vanishes in IIB supergravity. Notice that Eq. (1.35) should be read off order by order in form degrees, in particular the last term in $\mathbf{G}$ is only present in IIB. The corresponding Bianchi identities are given by

$$dH = 0 \quad \text{and} \quad d\mathbf{G} = H \wedge \mathbf{G}. \tag{1.36}$$

We are now ready to present the bosonic action:

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[e^{-2\phi} \left(R - 4(\partial\phi)^2 + \frac{1}{2} H \cdot H \right) + \frac{1}{4} \sum_{n=0,\frac{1}{2}}^{5,\frac{9}{2}} G^{(2n)} \cdot G^{(2n)} \right]. \tag{1.37}$$

As already mentioned, to remove the extra degrees of freedom the gauge potentials have to obey self-duality constraints

$$G^{(2n)} = (-1)^{[n]} * G^{(10-2n)} + (fermi), \tag{1.38}$$

where $[n]$ refers the integer part of n.

This formulation can also be applied to massive type IIA supergravity, where one also adds a nine-form $C^{(9)}$. The dual of its field strength satisfies $G^{(0)} = m$, with m being the Romans mass. This theory was first constructed in Ref. [35] as a deformed version of ordinary IIA supergravity. Though its classical eleven-dimensional lift is so far not known, the theory should be contained in M-theory.

As a side comment, a similar idea of grouping even and odd differential forms has also been employed in the context of generalized complex structures [36], although the reason behind it is of another character. There the geometry of a manifold is described by differential forms, with even and odd forms being mapped to Weyl spinors of different chiralities.

1.2 S-duality

The $SL(2,\mathbb{R})$ invariance of type IIB supergravity expressed in Einstein frame is a perfect example of a phenomenon known as *S-duality*. Since the coupling constant in that theory is defined as $g_s = e^{<\phi>}$, physically the operation $\tau \to -\frac{1}{\tau}$ with τ being defined as

$$\tau = C_0 + ie^{-\phi} \tag{1.39}$$

corresponds to an inversion of the coupling constant. In other words, strong coupling physics maps to the weak coupling regime.

This kind of duality was first discovered as a duality between electric and magnetic quantities in Maxwell's equations. Later it was generalized to $\mathcal{N} = 4$ super Yang-Mills theory under the name of *Montonen-Olive duality* [37]. The most general Lagrangian of $\mathcal{N} = 4$ SYM has the following form

$$\mathcal{L}_{\mathrm{SYM}} = \frac{1}{g^2}\mathrm{Tr}(F \wedge *F) + \frac{\theta}{8\pi^2}\mathrm{Tr}(F \wedge F). \tag{1.40}$$

The second term is topological, and thus does not have any significance for the classical equations of motion. However, after quantization the story changes, since now the quantum states are characterized also by the θ angle. Furthermore, defining a modular parameter as

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{\mathrm{YM}}^2}, \tag{1.41}$$

the quantized theory is invariant under the modular group $SL(2, \mathbb{Z})$ by fractional transformations [38]. Similar behavior has also been studied in $\mathcal{N} = 2$ Seiberg-Witten gauge theories [39, 40].

Similar to the super Yang-Mills theory, quantizing IIB supergravity will break the continuous $SL(2, \mathbb{R})$ to a symmetry of the modular group $SL(2, \mathbb{Z})$ [41]. An intuitive understanding of this can be achieved by studying the BPS states in the theory. A BPS state is supersymmetric and saturates certain equality relations between its mass and charges. If the maximal number of supersymmetries are preserved we simply call it BPS, if only half of the supersymmetries are preserved we call it half-BPS, etc.. One property that makes BPS states interesting is that they are protected by supersymmetry. That is, as long as the supersymmetry is unbroken they are stable under rescaling of the coupling constant, leading to many scaling independent properties. The only occasion this fails is when another representation becomes degenerate with the BPS multiplet, then a mechanism similar to the Higgs mechanism might take place.

The fact that the $SL(2, \mathbb{R})$ symmetry rotates the doublet (B_2, C_2) makes the states coupled to these potentials suitable for study. It turns out that one can form a bound state of p F-strings and q D-strings. By the Dirac quantization argument the tensions of these so called (p, q) *strings* must take discrete values. Unlike the gauge potentials, the tensions are rotated by the discrete $SL(2, \mathbb{Z})$ group. Starting from the tension of a fundamental string we can thus find the tension of an arbitrary (p, q) string by modular transformations, which in the Einstein frame is given by

$$T_{(p,q)} = \frac{|p + q\tau|}{\sqrt{\Im(\tau)}} T_{\mathrm{F1,S}}, \quad p \text{ and } q \text{ co-prime.} \tag{1.42}$$

Here $T_{\text{F1,S}} = \frac{1}{2\pi l_s^2}$ denotes the tension of a fundamental string in the string frame. The F- and D-strings correspond precisely to the special cases $(1,0)$ and $(0,1)$, respectively:

$$T_{\text{F1,E}} = \sqrt{g_s} T_{\text{F1,S}}, \quad T_{\text{D1,E}} = \frac{1}{\sqrt{g_s}} T_{\text{F1,S}}. \tag{1.43}$$

Notice that the D-string tension formula is valid only when $\Re(\tau) = 0$. Since both F1 and D1 are 1/2-BPS states, the formula (1.42) is indeed valid for all couplings. At weak string coupling ($g_s \ll 1$), the D-strings are too heavy to be observed. The situation becomes the opposite at strong coupling. The S-duality, which manifests itself as the modular group, exchanges the roles of F- and D-strings. Lastly, a junction of three (p,q) strings requires charge conservation: $\sum_i p^{(i)} = \sum_i q^{(i)} = 0$ for $i = 1, 2, 3$.

Under the modular group, the D3-brane transforms as a singlet. Therefore S-duality does not pose any additional constrain on how (p,q) strings can end on a D3-brane. The D5- and NS5-branes can also be grouped into a stable (p,q) five-brane, which is the magnetic dual of the (p,q) string. The fluctuations of the D5-brane are described by F-strings attached to it, with the same relation being true also for NS5-brane and D-strings. The (p,q) five-brane has similar modular properties as the (p,q) string. The $SL(2,\mathbb{Z})$ transformations on the D7-branes are however more complicated.

In order to understand the S-duality at a deeper level we need first to introduce a new concept called T-duality.

1.3 T-duality

T-duality is a symmetry of string theory which arises as a consequence of compactification on an n-torus T^n. Before stating the symmetry group in the general case, let us first illustrate the phenomenon using the simplest example, the bosonic closed string compactified on a circle with radius R.

The notion of circular compactification simply means that the string worldsheet along the compactified direction in the target space should have a periodic boundary condition

$$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi w R, \quad w \in \mathbb{Z}, \tag{1.44}$$

where we have assumed the 25th space direction to be compact. The remaining spacetime coordinates are assumed for simplicity to be Minkowski. Here τ and σ are the standard worldsheet parameters. The discrete number w, called the *winding number*, denotes the number of times the string winds around the compact direction. The oscillator expansion in the compact direction then becomes

$$X^{25}(\tau, \sigma) = x^{25} + 2\alpha' p^{25} \tau + 2wR\sigma + (\textit{oscillators}). \tag{1.45}$$

Since X^{25} is compact, the momentum of the center of mass along this direction, p^{25}, must be quantized

$$p^{25} = \frac{n}{R}, \qquad n \in \mathbb{Z}. \tag{1.46}$$

Dividing the expansion into left- and right-movers

$$X^{25}(\tau, \sigma) = X^{25}_{\mathrm{L}}(\tau + \sigma) + X^{25}_{\mathrm{R}}(\tau - \sigma), \tag{1.47}$$

we may define

$$P_{\mathrm{L}} = n\frac{\alpha'}{R} + wR \quad \text{and} \quad P_{\mathrm{R}} = n\frac{\alpha'}{R} - wR. \tag{1.48}$$

It is now apparent that the mass squared [4]

$$\alpha' M^2 = \alpha' \left[\left(\frac{n}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 \right] + 2N_{\mathrm{L}} + 2N_{\mathrm{R}} - 4 \tag{1.49}$$

as well as the oscillator number matching condition

$$N_{\mathrm{L}} - N_{\mathrm{R}} = nw \tag{1.50}$$

are invariant under the transformation

$$R \leftrightarrow \frac{\alpha'}{R}, \qquad n \leftrightarrow w. \tag{1.51}$$

The compact coordinate itself will transform as

$$X^{25}_{\mathrm{L}} \to X^{25}_{\mathrm{L}} \quad \text{and} \quad X^{25}_{\mathrm{R}} \to -X^{25}_{\mathrm{R}}, \tag{1.52}$$

and similar results are obtained for the respective currents. Not only the spectrum matches perfectly, also the interactions respect this so called T-duality. What T-duality really implies is that string theory compactified on a circle with radius R is equivalent to compactification on another circle with radius α'/R, provided that the winding number and the momentum are interchanged at the same time. Note that the fact that the string can wind around the compact dimension is crucial for this duality to exist, and thus T-duality can never be a property of a compactified point-particle theory.

The duality transformation in Eq. (1.51) is not a coincidence, and the reason can be understood as follows. The pair $(P_{\mathrm{L}}, P_{\mathrm{R}})$ from Eq. (1.48) can be considered as vectors in a space endowed with the metric $\begin{pmatrix} \frac{1}{2\alpha'} & 0 \\ 0 & -\frac{1}{2\alpha'} \end{pmatrix}$. A natural choice for the basis vectors of this space is

$$\vec{e}_1 = (R, -R) \quad \text{and} \quad \vec{e}_2 = \left(\frac{\alpha'}{R}, \frac{\alpha'}{R}\right), \tag{1.53}$$

resulting in the following metric of scalar products

$$\xi = \begin{pmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.54}$$

Since the vectors $(P_{\rm L}, P_{\rm R})$ are discrete quantities, they define an integer lattice with Lorentzian metric. Being the unique two-dimensional Lorentzian lattice which is also even and unimodular, this lattice is known in the literature as $\Pi_{1;1}$:

$$\Pi_{1;1} = \left\{ m^i \vec{e}_i \mid m^i \in \mathbb{Z}; i = 1, 2 \right\}, \tag{1.55}$$

see Refs. [42, 43]. The symmetry group that leaves this lattice invariant is

$$O(1,1;\mathbb{Z}) = \left\{ x \in GL(2,\mathbb{Z}) \middle| x^T \xi x = \xi \right\}. \tag{1.56}$$

Explicitly solving the equation $x^T \xi x = \xi$ shows that $O(1,1;\mathbb{Z}) \cong \mathbb{Z}_2$, where the only non-trivial solution precisely correspond to the exchange of momentum and winding number.

Generalization to compactification of the superstring on an arbitrary n-torus T^n is straightforward. The momenta and winding numbers then describe the even self-dual lattice

$$\underbrace{\Pi_{1;1} \oplus \ldots \oplus \Pi_{1;1}}_{n \text{ times}}. \tag{1.57}$$

The symmetry group of this lattice is the infinite discrete group $O(n,n;\mathbb{Z})$, which is defined by

$$O(n,n;\mathbb{Z}) = \left\{ x \in GL(2n,\mathbb{Z}) \middle| x^T \xi x = \xi \right\} \tag{1.58}$$

with

$$\xi = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} \tag{1.59}$$

being the invariant metric. This is the most general result of T-duality [44]. One thing worth noticing is that T-duality, seen as a symmetry of M-theory, only acts on momentum excitations and winding modes, in other words, it is a perturbative symmetry. For instance the one-loop partition function has been shown to respect T-duality [44]. More generally, it is valid order by order in the g_s expansion.

Extending to open strings, it can be shown that under T-duality Neumann boundary conditions turn into Dirichlet ones. This property naturally lead to the concept *Dirichlet-branes* or *D-branes*, which are defined as hypersurfaces on which an open string can end. The D-branes themselves can then also undergo T-dualization. Later in the context of Calabi-Yau compactification a special type of T-duality has been extensively studied under the name *mirror symmetry* [45].

1.4 U-duality

Now we are fully equipped to return to the $SL(2,\mathbb{Z})$ symmetry in IIB superstring theory. Compactifying the IIB theory on a circle with radius R_B shows that it is actually equivalent to the IIA theory compactified on a circle with radius $R_A = \alpha'/R_B$. The fact that IIA and IIB theories are related via a T-duality indicates that they have a common higher-dimensional origin. One can now go on comparing IIB compactified on a circle with M-theory compactified on a two-torus. Study of BPS states, i.e., (p,q) string, D-branes, M-branes, etc., shows the matching again works perfectly well. Refs. [4, 27] provide some flavors of the kind of computation involved. The most interesting observation is, however, the identification

$$\tau_{\mathrm{M}} = \tau_{\mathrm{B}}, \tag{1.60}$$

where τ_{M} is the complex structure modulus of the two-torus on which M-theory is compactified, and τ_{B} is the complex scalar of IIB theory defined in Eq. (1.26). This relation tells us the weak-strong coupling symmetry in IIB, which manifests itself as $SL(2,\mathbb{Z})$ acting on τ_{B}, can be interpreted as modular transformations on the M-theory two-torus [46, 47, 48]. S-duality has now received a geometric explanation! Though the relations are proven after compactification to nine dimensions, the symmetry should work even in the decompactification limit $R_B \to \infty$.

We have just glimpsed at the interplay between S- and T-duality, both of them being symmetries of M-theory. Compactify now string theory on an $(n-1)$-torus. As already been argued in Section 1.3, the perturbative T-duality symmetry group becomes $O(n-1, n-1; \mathbb{Z})$. When the determinant is 1 the T-transformation maps IIA and IIB to themselves, while if the determinant is -1 the T-transformation maps IIA $\leftrightarrow$ IIB. This indicates that IIA and IIB are different sectors of one common underlying theory — the eleven-dimensional M-theory. We can also switch to the M-theory point of view, where instead compactification on an n-torus has been performed. This leads to the S-duality group $SL(n,\mathbb{Z})$, which is part of the diffeomorphism group yielding conformally equivalent n-torii. Together, S- and T-duality intertwine in a non-trivial way to generate the so called *U-duality*. All the U-duality groups are summarized in Table 1.1. Curiously they all belong to the E-series of exceptional Lie groups [49].

The first hint towards U-duality came from toroidal compactification of eleven-dimensional supergravity. Due to the simple geometry of T^n, toroidal compactification preserves all supersymmetries, therefore the resulting supergravity theories are all maximal supersymmetric. By studying the scalar sector E. Cremmer and B. Julia were able to generalize the coset construction of the IIB moduli space [50]. They showed that in $(11-n)$ dimensions the scalars of the compactified supergravity parameterize the symmetric space

Dimension	U-duality	Global Symmetry	Local Symmetry
11	**1**	**1**	**1**
10, IIA	**1**	SO(1,1;$\mathbb{R}$)/$\mathbb{Z}_2$	**1**
10, IIB	SL(2,$\mathbb{Z}$)	SL(2,$\mathbb{R}$)	SO(2)
9	SL(2,$\mathbb{Z}$)$\times\mathbb{Z}_2$	SL(2,$\mathbb{R}$)$\times$O(1,1;$\mathbb{R}$)	SO(2)
8	SL(3,$\mathbb{Z}$)$\times$SL(2,$\mathbb{Z}$)	SL(3,$\mathbb{R}$)$\times$SL(2,$\mathbb{R}$)	U(2)
7	SL(5,$\mathbb{Z}$)	SL(5,$\mathbb{R}$)	USp(4)
6	O(5,5;$\mathbb{Z}$)	O(5,5;$\mathbb{R}$)	USp(4)$\times$USp(4)
5	$\mathrm{E}_{6(6)}(\mathbb{Z})$	$\mathrm{E}_{6(6)}(\mathbb{R})$	USp(8)
4	$\mathrm{E}_{7(7)}(\mathbb{Z})$	$\mathrm{E}_{7(7)}(\mathbb{R})$	SU(8)
3	$\mathrm{E}_{8(8)}(\mathbb{Z})$	$\mathrm{E}_{8(8)}(\mathbb{R})$	Spin(16)

Table 1.1: *Symmetries of M-theory under toroidal compactification.*

$E_{n(n)}/\mathcal{K}(E_{n(n)})$. The Lie algebra of $E_{n(n)}$ is the split real form of the complex Lie algebra $\mathfrak{e}_n$, see for instance Ref. [51], and $\mathcal{K}(*)$ denotes the maximal compact subgroup of its argument. In Table 1.1 $E_{n(n)}$ and $\mathcal{K}(E_{n(n)})$ are labeled as global and local symmetries, respectively. Indeed, the maximal compact subgroups are the generalizations of local Lorentz symmetry. Moreover, the scalar Lagrangian is a non-linear sigma model on this coset, details of this construction will be given in Section 2.1. The rest of the form fields elegantly fit the picture by transforming as representations of $E_{n(n)}$. Compactifying all the way to three spacetime dimensions, the entire theory will be described by scalar fields only.

Going beyond supergravities, it is conjectured that the continuous $E_{n(n)}(\mathbb{R})$ symmetry will be broken into the discrete $E_{n(n)}(\mathbb{Z})$, which contains the S- and T-duality groups as subgroups [52, 53]. The moduli space of scalars is then described by the double quotient

$$\mathcal{M}_{E_{n(n)}} = E_{n(n)}(\mathbb{Z})\backslash E_{n(n)}/\mathcal{K}(E_{n(n)}). \tag{1.61}$$

Similarly the BPS branes are mapped among themselves under U-duality. This property is very useful since less well understood branes can be mapped to well-studied ones. But more importantly dualities provide a window to the difficult non-perturbative effects in string theory [54, 55, 56, 57]. A more recent example is [58, 59], where a chain of dualities has been employed to explore the instanton effects from the hypermultiplets in $D = 4$ $\mathcal{N} = 2$ string compactifications.

Since toroidal compactification preserves the largest amount of symmetry, it is believed that U-duality is a true symmetry of M-theory. Even if at low energies it is non-linearly realized, at the Planck scale it might be linearly realized. Evidence for U-duality has mainly come from higher order derivative corrections in M-theory. The most seminal example is Ref. [60], where the authors successfully predicted the existence of an infinite sum of instanton contributions in the R^4 corrections to the type IIB superstring effective action. More about the higher order derivative expansion in α' can be found in Chapter 3.

Having established U-duality, it is evident that functions with simple transformation properties under $E_{n(n)}(\mathbb{Z})$ are of special interest, as they can be used to build up the effective action. In mathematics, functions on a moduli space $G/\mathcal{K}(G)$ which transform with a $\mathcal{K}(G)$ factor under the discrete group $G(\mathbb{Z})$ are called *automorphic forms*. The so called Eisenstein series used in Ref. [60] is precisely of this type. In three dimensions the moduli space contains all the bosonic degrees of freedom in the maximal supergravity. Let alone the difficulties of defining the $E_{8(8)}(\mathbb{Z})$ group [53, 61], this is a particular interesting case for construction of automorphic forms. Chapter 4 will explore deeper into the realm of automorphic forms.

Looking at Table 1.1 it is tempting to continue the compactification process below three dimensions. By studying maximal $\mathcal{N} = 16$ supergravity in two dimensions, it has been shown that the classical equations of motion exhibit affine E_9 symmetry [62, 63, 64]. The reason for this symmetry enhancement can be traced down to the integrability of the Lax pair of linear equations, which is an equivalent formulation of the equations of motion. Further down in one time dimension, it was proven that at spacelike singularities the dynamics of gravitational theories can be described as billiard motion inside the fundamental Weyl chamber of the hyperbolic Kac-Moody algebra $\mathfrak{e}_{10}$ [65]. The authors of Ref. [66] then argued that the corresponding Kac-Moody group E_{10} should be a true symmetry of M-theory. As the infinite-dimensional structure of E_{10} makes things very complicated, so far only equivalence between truncated $E_{10}/\mathcal{K}(E_{10})$ coset models and certain parts of supergravity theories has been shown. Another feature of E_{10} in its favor is the fact that its self-dual root lattice precisely coincides with the lattice of the vertex operator algebra of string theory states [67]. For those brave enough to continue further down the U-duality group chain, the role of the Lorentzian Kac-Moody group E_{11} in relation to M-theory has also been discussed [68, 69].

1.5 Web of Dualities

In this thesis we are focusing on the links between M-theory, type IIA and type IIB superstring theories. There are in fact five self-consistent string theories in ten dimensions: type IIA, type IIB, type I, $O(32)$ heterotic and $E_8 \times E_8$ heterotic. Each of them has a supergravity as the low energy limit. Together with M-theory, all of these are related by dualities:

i. As already been discussed thoroughly, after circular compactification to nine dimensions *type IIA* and *IIB* strings are related via T-duality [44]. Moreover, IIB string theory is self-dual under S-duality [70].

ii. Compactifying *type IIA* string on K3 is dual to $E_8 \times E_8$ *heterotic* string compactified on T^4 [70, 57]. At low energies both theories are given by the six-dimensional $\mathcal{N} = (1,1)$ supergravity. This duality is non-perturbative since the coupling constants are related via $g_s^{\text{het}} \leftrightarrow \frac{1}{g_s^{\text{IIA}}}$. Compactifying both sides further on a two-torus provides an explanation for the Montonen-Olive duality in four-dimensional gauge theory.

iii. Another duality between *type IIA* and *heterotic* string is provided by compactifying IIA on K3$\times T^2$ and heterotic theory on certain elliptically fibred Calabi-Yau threefolds [71, 72], respectively. In four dimensions they correspond to $\mathcal{N} = 2$ supersymmetric gauge theories.

iv. Similar to the type II strings, $E_8 \times E_8$ and $O(32)$ *heterotic* strings are opposite sides of a T-duality [44], after compactification on a circle.

v. $O(32)$ *heterotic* string is S-dual to $O(32)$ *type I* string [73, 70]. In particular, both of these theories have $D = 10$ $\mathcal{N} = 1$ supergravity as their low energy limit.

vi. In the previous text we have already argued that *type IIA* string theory can be obtained by circular compactification of *M-theory*.

vii. Starting from the $E_8 \times E_8$ *heterotic* string, via a chain of dualities passing $O(32)$ heterotic, type I and type IIA, it can be shown that $E_8 \times E_8$ heterotic string theory is dual to *M-theory* compactified on the one-dimensional orbifold $T^1/\mathbb{Z}_2$ [74, 75].

There is now convincing evidence that the various string theories are in fact describing different perturbative regions of an eleven-dimensional poorly understood non-perturbative theory, which usually is also referred to as M-theory [41, 29]. We need to study all the string theories to get a more complete picture of this underlying theory.

2

Compactification and Geometry

Though superstring theories very elegantly put gravity and non-abelian gauge theories on equal footing, one major concern is how to make contact with the four-dimensional real world. The most successful way to resolve this problem is based on old ideas by T. Kaluza and O. Klein, where the extra dimensions are thought to be curled-up. These internal dimensions are simply too small to be observed at energy scales accessible to current experiments. Nevertheless, the topology of the extra dimensions is directly affecting the four-dimensional physics. Since it has been suggested that supersymmetry might be broken at lower energy than the compactification scale, the most interesting candidates for the internal manifold are those preserving some supersymmetry. Examples of such compact manifolds are the n-torus, Calabi-Yau n-folds and manifolds with G_2 holonomy. Compactification of the maximal supersymmetric theory on each of these three cases will be discussed in this chapter. Another use for compactification has already been mentioned in Section 1.5. Dualities that relate the different string theories appear after appropriate compactifications.

Many of the basic concepts in geometry are discussed and used here, however for systematic treatments of Riemannian and complex manifolds the reader is advised to read for instance Refs. [26, 76, 77]. Good reviews of Kaluza-Klein compactification can be found in Refs. [78, 79, 80].

2.1 Torus Compactification

The original aim of the *Kaluza-Klein compactification* program was to rewrite the four-dimensional gravity coupled to a Maxwell gauge field as a pure gravity theory in five spacetime dimensions [81, 82]. From the five-dimensional viewpoint, the geometry of the spacetime consists of four infinite spacetime dimensions together with one compact circle as the fifth dimension. In fact, the compactification procedure was already employed in Subsec. 1.1.2, where we obtained type IIA supergravity from eleven-dimensional supergravity. For completeness we will review it again, before generalizing the compact circle to an arbitrary n-torus.

2.1.1 Compactification on a Circle

Let us consider compactification of a D-dimensional field theory on a circle with radius R. Since the circular dimension is periodic, all the fields can be Fourier expanded in the compact coordinate. As an example a massless D-dimensional scalar field can be written as

$$\hat{\phi}(x^M) = \sum_{l=0}^{\infty} \phi^{(l)}(x^\mu) e^{ilx^{D-1}/R}, \tag{2.1}$$

where $x^M = (x^\mu, x^{D-1})$ are the curved spacetime coordinates. The non-compact external coordinates are denoted x^μ with $\mu = 0, \dots, (D-2)$; and the compact direction x^{D-1} takes its value on the interval $[0, 2\pi R[$. For notational simplicity all the D-dimensional quantities will have a hat. Fourier transforming the Klein-Gordon equation

$$\hat{\partial}^M \hat{\partial}_M \hat{\phi} = 0 \tag{2.2}$$

we find

$$\left(-E^2 + p_1^2 + \cdots + p_{D-2}^2 - \frac{l^2}{R^2} \right) \phi^{(l)}(x^\mu) e^{ilx^{D-1}/R} = 0. \tag{2.3}$$

The mass of the Kaluza-Klein excitation $\phi^{(l)}$ is thus $m_{(l)} = \frac{l}{R}$. If the radius of the circle is taken to be very small, the masses of the excitations will become very large, and at the level of effective actions we can neglect all the massive excitations. This limit of compactification is called *dimensional reduction.*

When dimensionally reducing gravity a convenient Ansatz for the vielbein is

$$\hat{e}_M{}^A = e^{\frac{1}{2}s\phi} \begin{pmatrix} e_\mu{}^\alpha & e^{-\frac{1}{2}f\phi}\mathcal{A}_\mu \\ 0 & e^{-\frac{1}{2}f\phi} \end{pmatrix}, \tag{2.4}$$

where $s = \sqrt{\frac{2}{(D-2)(D-3)}}$ and $f = (D-2)s$ take purely numerical values. The index $A = (\alpha, D-1)$ is the tangent space counterpart to the curved spacetime index $M = (\mu, D-1)$. Gravity in D dimensions will thus give rise to a $(D-1)$-dimensional metric $g_{\mu\nu}$, a scalar field ϕ and a graviphoton $\mathcal{A}_\mu$. A *Weyl rescaling* factor $e^{\frac{1}{2}s\phi}$ has been included in Eq. (2.4) to ensure that we end up in the *Einstein frame* for the dimensionally reduced action. Using the vielbein one-form

$$\hat{e}^A = dx^M \hat{e}_M{}^A = \left(e^{\frac{1}{2}s\phi} e^\alpha, e^{\frac{1}{2}(s-f)\phi} \left[dx^{D-1} + dx^\mu \mathcal{A}_\mu \right] \right) = \left(\hat{e}^\alpha, \hat{e}^{D-1} \right) \tag{2.5}$$

we can define the anholonomy

$$\hat{d}\hat{e}^A = \frac{1}{2}\hat{e}^B \wedge \hat{e}^C \hat{\Omega}_{BC}{}^A. \tag{2.6}$$

Since all the fields only depend on the external coordinates x^μ, the derivative one-form takes the form $\hat{d} = d = dx^\mu \partial_\mu$. Setting the torsion to zero leads to the following spin connection

$$\hat{\omega}_{ABC} = \frac{1}{2}\left(\hat{\Omega}_{ABC} - \hat{\Omega}_{BCA} + \hat{\Omega}_{CAB}\right). \tag{2.7}$$

The curvature two-form can then be found as

$$\hat{R}^A{}_B = \frac{1}{2}\hat{e}^C \wedge \hat{e}^D \hat{R}_{CD}{}^A{}_B = d\hat{\omega}^A{}_B + \hat{\omega}^A{}_C \wedge \hat{\omega}^C{}_B, \tag{2.8}$$

where $\hat{\omega}^A{}_B = \hat{e}^C \hat{\omega}_C{}^A{}_B$. The Ricci tensor $\hat{R}_{AB}$ and curvature scalar $\hat{R}$ are obtained in the standard way by contracting the curvature tensor $\hat{R}_{AB}{}^C{}_D$ with the metric

$$\hat{g}_{MN} = \eta_{AB}\hat{e}_M{}^A \hat{e}_N{}^B = e^{s\phi}\begin{pmatrix} g_{\mu\nu} + e^{-f\phi}\mathcal{A}_\mu \mathcal{A}_\nu & e^{-f\phi}\mathcal{A}_\mu \\ e^{-f\phi}\mathcal{A}_\nu & e^{-f\phi} \end{pmatrix}. \tag{2.9}$$

In particular the curvature scalar becomes

$$\hat{R} = e^{-s\phi}\left\{R - \frac{1}{4}e^{-f\phi}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} - \frac{1}{2}(\partial\phi)^2 - sD_\alpha\partial^\alpha\phi\right\}, \tag{2.10}$$

with $\mathcal{F}_{\alpha\beta} = 2e^\mu{}_{[\alpha}e^\nu{}_{\beta]}\partial_\mu \mathcal{A}_\nu$ being the field strength of the graviphoton. Multiplying with the volume measure $\hat{e} = e^{s\phi}\sqrt{|g|}$ we find that the dimensionally reduced Einstein-Hilbert Lagrangian indeed is in the Einstein frame

$$\tilde{\mathcal{L}}^{(D-1)}_{\text{EH}} = \hat{e}\hat{R} = \sqrt{|g|}\left\{R - \frac{1}{4}e^{-f\phi}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} - \frac{1}{2}(\partial\phi)^2 - sD_\alpha\partial^\alpha\phi\right\}. \tag{2.11}$$

Due to the relation

$$\sqrt{|g|}D_\alpha X^\alpha = \partial_\mu(\sqrt{|g|}X^\mu) \tag{2.12}$$

the last term in Eq. (2.11) is a total derivative. After integration, total derivatives only give rise to boundary terms, therefore we will omit them from now on.

Similar to the vielbein, any field strength

$$\hat{F}_p = \frac{1}{p!}\hat{e}^{M_1} \wedge \cdots \wedge \hat{e}^{M_p}\hat{F}_{M_1 \dots M_p} \tag{2.13}$$

can be decomposed into $(D-1)$-dimensional fields according to

$$\begin{aligned}\hat{F}_p \equiv & \frac{1}{p!} e^{\mu_1} \wedge \cdots \wedge e^{\mu_p} F_{\mu_1 \ldots \mu_p} \\ & + \frac{1}{(p-1)!} e^{\mu_1} \wedge \cdots \wedge e^{\mu_{p-1}} \wedge \left(dx^{D-1} + \mathcal{A} \right) F_{\mu_1 \ldots \mu_{p-1}, D-1},\end{aligned} \tag{2.14}$$

where $F_{\mu_1 \ldots \mu_p}$ and $F_{\mu_1 \ldots \mu_{p-1}, D-1}$ only depend on the external coordinates x^μ. Components like $F_{\mu_1 \ldots \mu_{p-2}, (D-1), (D-1)}$ vanish due to the antisymmetric property of $F_{M_1 \ldots M_p}$. Also, the gauge potentials are dimensionally reduced in the same fashion. The relations between the field strengths and the gauge potentials in $(D-1)$ dimensions are in general quite complicated.

Having rewritten the action in terms of the dimensionally reduced fields, we can find the equations of motion by varying the action with respect to the metric and the gauge potentials. Sometimes it is simpler to directly reduce the D-dimensional equations of motion, since they are linear combinations of the $(D-1)$-dimensional equations of motion. Similarly, the Bianchi identities in $(D-1)$ dimensions can also be found by reducing the corresponding D-dimensional ones.

The dimensionally reduced expressions we have obtained are not yet in the final form. We still need to incorporate the Hodge dualization in $(D-1)$ dimensions:

$$(*t) := \frac{1}{p!(D-1-p)!} \frac{1}{\sqrt{|g|}} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_{D-1}} \varepsilon_{\mu_{p+1} \ldots \mu_{D-1} \nu_1 \ldots \nu_p} t^{\nu_1 \ldots \nu_p}, \tag{2.15}$$

where t is a $(D-1)$-dimensional p-form. Using the Hodge dualization all the field strengths with form degree $p > \left[\frac{D-1}{2}\right]$ can be dualized into forms with degree $p < \left[\frac{D-1}{2}\right]$. To illustrate how it works let us consider pure gravity dimensionally reduced from four to three dimensions. The Lagrangian in this case takes the form

$$\tilde{\mathcal{L}}^{(3)}_{\mathrm{EH}} = \sqrt{|g|} \left\{ R - \frac{1}{4} e^{-2\phi} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} - \frac{1}{2} (\partial\phi)^2 \right\}. \tag{2.16}$$

Since we are in three dimensions, it is possible to dualize the field strength $\mathcal{F}_{\mu\nu}$ to a vector $\mathcal{F}_\mu$. The dualization is achieved by first adding a Lagrange multiplier, which is proportional to the Bianchi identity for $\mathcal{F}_{\alpha\beta}$

$$\delta \tilde{\mathcal{L}}^{(3)}_{\mathrm{EH}} = -\sqrt{|g|} \frac{1}{2} \chi \varepsilon^{\mu\nu\rho} \partial_\mu \mathcal{F}_{\nu\rho}, \tag{2.17}$$

to the Lagrangian and then vary $(\tilde{\mathcal{L}}^{(3)}_{\mathrm{EH}} + \delta \tilde{\mathcal{L}}^{(3)}_{\mathrm{EH}})$ with respect to $\mathcal{F}_{\mu\nu}$. The result we obtain is precisely the Hodge dual

$$\mathcal{F}_{\alpha\beta} = \frac{1}{\sqrt{|g|}} e^{2\phi} \varepsilon_{\alpha\beta\gamma} \partial^\gamma \chi, \tag{2.18}$$

where χ is some arbitrary scalar field. Putting Eq. (2.18) into $(\tilde{\mathcal{L}}^{(3)}_{\mathrm{EH}} + \delta\tilde{\mathcal{L}}^{(3)}_{\mathrm{EH}})$ we find at last

$$\mathcal{L}^{(3)}_{\mathrm{EH}} = \sqrt{|g|}\left\{R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\right\}. \tag{2.19}$$

Later on we will discuss the symmetry properties of this Lagrangian. Notice that the reduced Lagrangian before dualization is always marked with a tilde.

2.1.2 Generalization to n-Torus

The n-dimensional torus (or n-*torus*) is a compact manifold defined as

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}. \tag{2.20}$$

Geometrically this manifold is isomorphic to the space $\mathbb{R}^n/\mathbb{Z}^n$, where $\mathbb{Z}^n$ corresponds to integral shifts in n dimensions. The Euler characteristic of the n-torus is vanishing

$$\chi_{\mathrm{e}}(T^n) = 0, \tag{2.21}$$

while its fundamental group $\pi_1(T^n)$ is a free abelian group of rank n.

The Two-Torus

As an illustration, let us consider the two-torus $T^2 = S^1 \times S^1$. The standard construction in physics is to start with the complex plane and then mod out a two-dimensional lattice:

$$T^2 = \mathbb{C}/\Lambda_{(w_1,w_2)}, \tag{2.22}$$

where

$$\Lambda_{(w_1,w_2)} = \{mw_1 + nw_2 \mid m, n \in \mathbb{Z}\}. \tag{2.23}$$

In other words, the complex coordinate z of this space is doubly periodic

$$z \sim z + w_1 \quad \text{and} \quad z \sim z + w_2, \tag{2.24}$$

with the periods w_1 and w_2 satisfying $\frac{w_1}{w_2} \notin \mathbb{R}$. Using the coordinate transformation $z \to z/w_2$, the pair of periods (w_1, w_2) may be normalized to $(w_1/w_2, 1)$.

The normalized period

$$\tau := \frac{w_1}{w_2} \tag{2.25}$$

describes the shape of the two-torus and is called the *complex structure moduli*. We can set $\Im(\tau) > 0$ without loosing generality. Transforming the complex structure according to

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1 \tag{2.26}$$

using integer parameters a, b, c and d, we obtain a new two-torus which is conformally equivalent to the original one. The group generated by all such transformations is called the *modular group* $PSL(2,\mathbb{Z})$. The moduli space of all inequivalent two-torii is thus

$$\mathcal{M}_{PSL(2,\mathbb{R})} = H/PSL(2,\mathbb{Z}), \tag{2.27}$$

where H is the complex upper half-plane. More properties of the modular group can be found in Subsec. 4.1.1.

In addition to the complex structure moduli, one can also define a *Kähler structure moduli* ρ

$$\rho := w_1 w_2 \tag{2.28}$$

describing the size of the two-torus. Fixing these two parameters completely determines the two-torus. A T-duality on the w_2 circle sends w_2 to $1/w_2$, exchanging the roles of the complex structure and Kähler structure. This is the simplest example of *mirror symmetry*.

Dimensional Reduction on n-Torii

Dimensional reduction from D dimensions on a one-dimensional circle is easily generalized to a product of n circles. This can be done by circular compactifying dimension by dimension.

Using Eq. (2.4) recursively n times the vielbein becomes [79, 80]

$$\hat{e}_M{}^A = e^{\frac{1}{2}\vec{s}\cdot\vec{\phi}} \begin{pmatrix} e_\mu{}^\alpha & e^{-\frac{1}{2}\vec{f}^a\cdot\vec{\phi}}\mathcal{A}_\mu{}^m u_m{}^a \\ 0 & e^{-\frac{1}{2}\vec{f}^a\cdot\vec{\phi}} u_m{}^a \end{pmatrix}, \tag{2.29}$$

where $\vec{\phi} = (\phi_1, \ldots, \phi_n)$ is a vector containing n Kaluza-Klein scalar fields. These are multiplied with the coefficients

$$\begin{aligned} \vec{s} &= (s_1, \ldots, s_n)\,, \\ \vec{f^a} &= (s_1, \ldots, s_{a-1}, (D-2-n+a)s_a, \underbrace{0, \ldots, 0}_{n-a}), \end{aligned} \tag{2.30}$$

with

$$s_a = \sqrt{\frac{2}{(D-2-n+a)(D-3-n+a)}}. \tag{2.31}$$

The index conventions are such that $M = (\mu, m)$ and $A = (\alpha, a)$ denote the curved and flat spacetime indices, respectively. All the D-dimensional objects are marked with a hat.

The internal vielbein $\tilde{e}_m{}^a := e^{-\frac{1}{2}\vec{f}^a\cdot\vec{\phi}}u_m{}^a$ is the Borel representative of the coset $GL(n,\mathbb{R})/SO(n)$, where $e^{-\frac{1}{2}\vec{f}^a\cdot\vec{\phi}}$ is the contribution from the Cartan generators and

$$u_m{}^a = [(1-\mathcal{A}_{(0)})^{-1}]_m{}^a \tag{2.32}$$

corresponds to the positive root generators. The scalar fields $\mathcal{A}_{(0)}$ come from dimensionally reducing the Kaluza-Klein vectors. The Maurer-Cartan form can be constructed as follows:

$$G_\gamma{}^{ab} := u^{ma}\partial_\gamma u_m{}^b = e^{-\frac{1}{2}(\vec{f}^a-\vec{f}^b)\cdot\vec{\phi}}\left(P_\gamma{}^{ab} + Q_\gamma{}^{ab}\right), \tag{2.33}$$

where $G_\gamma{}^{ab}$ is non-zero only when $a<b$. P_γ viewed as a matrix is symmetric and traceless, i.e., it takes its values in the Lie algebra $sl(n,\mathbb{R})$. Moreover, P_γ is also transforming as a representation of $SO(n)$. The antisymmetric Q_γ, on the other hand, is a gauge connection for the local $SO(n)$ symmetry. Thus, in addition to the equations of motion and Bianchi identities, the system also has to satisfy the *Maurer-Cartan equations*:

$$\begin{aligned}
&D_{[\gamma}P_{\delta]ab} + Q_{[\gamma|a|}{}^cP_{\delta]cb} + P_{[\gamma|a|}{}^cQ_{\delta]cb} - \frac{1}{2}(\vec{f}^a-\vec{f}^b)\cdot(\partial_{[\gamma}\vec{\phi})Q_{\delta]ab} = 0,\\
&D_{[\gamma}Q_{\delta]ab} + Q_{[\gamma|a|}{}^cQ_{\delta]cb} + P_{[\gamma|a|}{}^cP_{\delta]cb} - \frac{1}{2}(\vec{f}^a-\vec{f}^b)\cdot(\partial_{[\gamma}\vec{\phi})P_{\delta]ab} = 0.
\end{aligned} \tag{2.34}$$

Notice that there is no summation over the indices a and b. The covariant derivative appearing in Eq. (2.34) is an ordinary tangent space one containing the spin connection, $D=\partial+\omega$.

Following the recipe given in Subsec. 2.1.1, we can compute the spin connection from the vielbein, and then obtain the curvature tensor. Details of this computation can be found in Paper IV. The resulting Einstein-Hilbert Lagrangian is

$$\tilde{\mathcal{L}}_{\text{EH}}^{(D-n)} = \sqrt{|g|}\left\{R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{4}e^{-\vec{f}^c\cdot\vec{\phi}}\mathcal{F}_{\alpha\beta c}\mathcal{F}^{\alpha\beta c} - P_{\alpha bc}P^{\alpha bc}\right\}, \tag{2.35}$$

with $\mathcal{F}_{\alpha\beta c} = 2u_{ma}e^\mu{}_{[\alpha}e^\nu{}_{\beta]}\partial_\mu\mathcal{A}_\nu{}^m$. We have again omitted the term which is a total derivative. Similar to the gravity sector, the form fields can also by reduced one dimension at a time. The equations of motion are found by varying the reduced action, while the Bianchi identities are obtained by direct dimensional reduction of the corresponding D-dimensional ones. Just like in the circular compactification, it is sometimes convenient to Hodge dualize the form-fields. In particular in $D-n=3$ dimensions all the bosonic degrees of freedom can be dualized into scalar fields. Analyzing also the fermionic fields shows that compactification on n-torus will preserve all the supersymmetries.

The dimensional reduction we have described here works when we are only interested in the classical theory. However, if we want to investigate quantum effects also the Kaluza-Klein excitations we have ignored so far become important. As an example we can consider the one-loop $\mathcal{R}^4$ correction in the low energy effective action of M-theory. Upon compactification to lower dimensions, the Kaluza-Klein modes of the fields along the compact directions are interpreted as the windings of Euclidean D-particle world-lines [83]. The massless fields reproduce the one-loop string correction, while winding the D-particle world-lines around the compact circle gives rise to the tree-level string correction. Moreover, the massive D-particles are responsible for the non-perturbative D-instanton effects. Higher derivative corrections in string theory will be treated more thoroughly in Chapter 3.

2.1.3 Coset Symmetry

Dimensional reduction on an n-torus always leads to a simple moduli space, i.e, a space of scalar fields with coset symmetry. The most clean example of this phenomenon is the reduction of D-dimensional pure gravity. As we have already explained in Subsec. 2.1.2, the symmetric part of the Maurer-Cartan form P_α is an element of $sl(n,\mathbb{R})$. The Einstein-Hilbert Lagrangian (2.35) has thus manifest $SL(n,\mathbb{R})$ symmetry, where P_α enters in terms of the quadratic Casimir invariant. However, dimensionally reducing all the way to three dimensions, we may dualize the Kaluza-Klein vector according to Eq. (2.18) to obtain

$$\begin{aligned}\mathcal{L}^{(3)}_{\text{EH}} =&\sqrt{|g|}\Bigg\{R-\frac{1}{2}(\partial\vec{\phi})^2\\&-\frac{1}{2}\sum_{c=1}^{n}e^{\vec{f}^c\cdot\vec{\phi}}(\partial\vec{\chi})^2-\frac{1}{2}\sum_{\substack{b,c=1\\b<c}}^{n}e^{(\vec{f}^b-\vec{f}^c)\cdot\vec{\phi}}G_{\alpha bc}G^{\alpha bc}\Bigg\}.\end{aligned}\tag{2.36}$$

This operation will enhance the symmetry of the action to a global $SL(n+1,\mathbb{R})$ together with a local $SO(n+1)$. Compared to Eq. (2.35) we have also rewritten $P_{\alpha bc}P^{\alpha bc}$ in terms $G_{\alpha bc}$.

The first indication of $SL(n+1,\mathbb{R})$ symmetry is provided by the dilaton exponents in Eq. (2.36)

$$\vec{f}^a-\vec{f}^b\quad(b>a)\qquad\text{and}\qquad\vec{f}^a.\tag{2.37}$$

These vectors span a linear space endowed with the scalar product

$$\vec{f}^a\cdot\vec{f}^b=2\delta^{ab}+\frac{2}{D-n-2},\tag{2.38}$$

see Eq. (2.30) and (2.31). Choosing $\{\vec{f}^1-\vec{f}^2,\ldots,\vec{f}^{n-1}-\vec{f}^n,\vec{f}^n\}$ to be the set of basis vectors, one can show that the matrix of scalar products among the basis vectors is precisely the Cartan matrix of the Lie algebra $sl(n+1,\mathbb{R})$. The basis vectors can thus be identified with the simple root vectors of $sl(n+1,\mathbb{R})$. Notice that the simple root vector $\vec{f}^n$ is obtained only after the dualization, which is consistent with the fact that dualization enlarges the symmetry group.

To show that the symmetry enhancement does occur, let us consider the reduction of four-dimensional pure gravity on a circle example again. The Lagrangian after Hodge dualization, given in Eq. (2.19),

$$\mathcal{L}^{(3)}_{\mathrm{EH}} = \sqrt{|g|}\left\{R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\right\} \tag{2.39}$$

can indeed be shown to contain the the coset symmetry $SL(2,\mathbb{R})/SO(2)$. Let

$$E \doteq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad H \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad F \doteq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.40}$$

be the generators of the Lie algebra $sl(2,\mathbb{R})$. The Borel representative of the coset $SL(2,\mathbb{R})/SO(2)$ is defined as

$$\mathcal{V} := e^{\frac{\phi}{2}H}e^{\chi E} \doteq \begin{pmatrix} e^{\frac{\phi}{2}} & e^{\frac{\phi}{2}}\chi \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix}, \tag{2.41}$$

which transforms according to

$$\mathcal{V} \longmapsto k\mathcal{V}g, \quad g \in SL(2,\mathbb{R}) \;\text{ and }\; k \in SO(2). \tag{2.42}$$

$SL(2,\mathbb{R})$ is the global symmetry of this system, while the local $SO(2)$ symmetry is used to bring the coset representative $\mathcal{V}$ back to the upper triangular form. The Hermitian and anti-Hermitian parts of the Maurer-Cartan form are then defined as

$$\begin{aligned} \mathcal{P}_\alpha &:= \frac{1}{2}\left\{\mathcal{V}\partial_\alpha\mathcal{V}^{-1} + (\mathcal{V}\partial_\alpha\mathcal{V}^{-1})^\dagger\right\}, \\ \mathcal{Q}_\alpha &:= \frac{1}{2}\left\{\mathcal{V}\partial_\alpha\mathcal{V}^{-1} - (\mathcal{V}\partial_\alpha\mathcal{V}^{-1})^\dagger\right\}, \end{aligned} \tag{2.43}$$

respectively. Explicit computation shows that the Lagrangian

$$\mathcal{L}^{(3)}_{\mathrm{EH}} = \sqrt{|g|}\left\{R - \frac{1}{2}\mathrm{Tr}\left(\mathcal{P}_\alpha\mathcal{P}^\alpha\right)\right\} \tag{2.44}$$

is manifestly invariant under $SL(2,\mathbb{R})/SO(2)$, since under coset transformations $\mathcal{P}_\alpha \longmapsto k\mathcal{P}_\alpha k^{-1}$.

The analysis above is more or less unaltered for the reduction of pure gravity to three dimensions on an n-torus. Instead of having n Kaluza-Klein scalar fields as in Subsec. 2.1.2, we need to extract the determinant of the internal vielbein

$$\tilde{e}_m{}^a = e^{-\frac{D-2}{n}\varphi}\varepsilon_m{}^a, \tag{2.45}$$

with $\varepsilon_m{}^a$ being an element of $SL(n,\mathbb{R})$. Defining the Borel representative of $SL(n+1,\mathbb{R})/SO(n+1)$ accordingly, it can be shown that the Lagrangian takes the form of Eq. (2.44).

Dimensionally reduced pure gravity is not the only example where the moduli space has coset symmetry. The most interesting example in physics is perhaps the toroidal reduction of eleven-dimensional supergravity. There, the moduli variables parameterize the $E_{n(n)}/\mathcal{K}(E_{n(n)})$ coset space, with $\mathcal{K}(E_{n(n)})$ denoting the maximal compact subgroup of $E_{n(n)}$. The global exceptional $E_{n(n)}$ groups are the results of symmetry enhancements from dualizing the form-fields, which already start to occur at dimensions higher than three. The observation of coset symmetry in reduced eleven-dimensional supergravity is one of the corner stones of the U-duality conjecture, see Section 1.4.

One might wonder to what extent this coset symmetry survives when quantum corrections are included in the low energy effective action of M-theory. Paper III examines the symmetry structures of $\mathcal{R}^3$ and $\mathcal{R}^4$ corrections by toroidal reducing to three dimensions. A more complete reduction was carried out for the $\mathcal{R}^2$ correction in Paper IV, where an investigation of the dilaton exponents was also given. Similar analysis of dilaton exponents of compactified higher order derivative corrections can be found in Refs. [84, 85, 86, 87]. The answer seems to be automorphic forms based on the U-duality groups $E_{n(n)}(\mathbb{Z})$. Chapters 3 and 4 will be dealing with this subject.

2.2 Calabi-Yau Compactification

The simple geometry of the n-torus makes it possible to compute the compactified quantities explicitly, however the large amount of symmetry also makes the compactification procedure preserve all the supersymmetries. Therefore we cannot use a torus as the internal compact manifold to arrive at realistic four-dimensional physics. Instead, we have to search for manifolds that break supersymmetry in a controlled way. The most promising candidate is the so called *Calabi-Yau n-fold*.

2.2.1 Calabi-Yau Manifold

A Calabi-Yau manifold M with n complex dimensions is a *Kähler* manifold with vanishing *first integral Chern class* [26, 76]

$$c_1(M) = 0. \tag{2.46}$$

An equivalent definition is that M is Kähler and admits a holomorphic n-form that is nowhere vanishing. For a compact Calabi-Yau manifold, the local *holonomy group* is contained in $SU(n)$. The fact that such manifolds have a Kähler metric with vanishing Ricci curvature was first conjectured by Calabi, and later proven by Yau.

The topology of a Calabi-Yau manifold is characterized by the *Hodge numbers* $h^{p,q}$, which count the number of $\Delta_{\bar{\partial}}$-harmonic (p,q)-forms. The harmonic (p,q)-forms are in one-to-one correspondence to the generators of the Dolbeault cohomology group $H^{p,q}_{\bar{\partial}}(M)$. However, all the Hodge numbers are not independent. Complex conjugation yields the relation

$$h^{p,q} = h^{q,p}, \tag{2.47}$$

while Poincaré duality leads to

$$h^{p,q} = h^{n-q,n-p}. \tag{2.48}$$

In addition one can show that

$$h^{p,0} = h^{n-p,0}, \tag{2.49}$$

and if we restrict us to simply-connected Calabi-Yau manifolds then $h^{1,0} = h^{0,1} = 0$. Lastly, $h^{0,0} = 1$ for any compact connected Kähler manifold. Taking all these relations into account, the complete cohomological information is contained in the *Hodge diamond*. For instance for $n = 3$ the Hodge diamond takes the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.50}$$

showing that only $h^{1,1}$ and $h^{2,1}$ are independent Hodge numbers. The dimension of the kth de Rham cohomology $H^k(M)$ is called the *Betti number*, which is related to the Hodge numbers via

$$b_k = \sum_{p=0}^{k} h^{p,k-p}. \tag{2.51}$$

The *Euler characteristic* is then given by

$$\chi_e = \sum_{k=0}^{2n} (-1)^k b_k. \tag{2.52}$$

For a Calabi-Yau three-fold it becomes $\chi_e(\mathrm{CY}_3) = 2(h^{1,1} - h^{2,1})$.

We may deform a Calabi-Yau manifold continuously without changing its topological properties. These deformations are characterized by the expectation values of the massless scalar fields, called *moduli fields*. Examples of moduli fields have already been given in Subsec. 2.1.2, when we discussed the two-torus. In that case, different choices of the complex structure moduli τ and Kähler structure moduli ρ correspond to torii with different shapes and sizes. However, altering the values of the moduli fields still results in a two-torus.

Examples

Some explicit examples of Calabi-Yau manifolds are listed below. In principle examples of compact Calabi-Yau n-folds can be constructed as submanifolds of the complex projective space $\mathbb{CP}^{n+1}$ for all $n > 1$.

i. The complex plane $\mathbb{C}$ is a non-compact one-dimensional Calabi-Yau manifold, while the two-torus T^2 is a compact such.

ii. For $n = 2$, $\mathbb{C} \times \mathbb{C}$ and $\mathbb{C} \times T^2$ are naturally non-compact examples. The only compact Calabi-Yau two-folds are T^4 and K3, where the latter is a submanifold of $\mathbb{CP}^3$. Requiring also the holonomy to be $SU(2)$ leaves K3 as the only candidate. Since $SU(2)$ is isomorphic to $Sp(1)$, K3 is also hyperkähler[1]. The Hodge numbers of K3 are

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{2.53}$$

and its Euler characteristic is $\chi_e(\mathrm{K3}) = 24$. K3 surfaces are useful in string theory to show the duality between type IIA and heterotic strings.

iii. There are many Calabi-Yau three-folds, in fact the classification of these is still an open problem. One example is the quintic hypersurface in $\mathbb{CP}^3$, which is described by

$$G(z^1, \ldots, z^5) = 0, \qquad z^a \in \mathbb{C}^5 \tag{2.54}$$

with the polynomial $G(z^1, \ldots, z^5)$ satisfying

$$G(\lambda z^1, \ldots, \lambda z^5) = \lambda^5 G(z^1, \ldots, z^5). \tag{2.55}$$

[1] A hyperkähler manifold is a Riemannian manifold of $4k$ dimensions whose holonomy group is contained in $Sp(k)$.

This manifold has the Hodge numbers

$$h^{1,1} = 1 \quad \text{and} \quad h^{2,1} = 101, \tag{2.56}$$

yielding the Euler characteristic $\chi_e = -200$. The fact that Calabi-Yau three-folds break 3/4 of the supersymmetries makes them interesting for string compactifications.

iv. Calabi-Yau four-folds break 7/8 of the supersymmetries. These are useful for F-theory compactifications [20].

2.2.2 Calabi-Yau Three-fold

Calabi-Yau three-fold (CY_3) is the most promising candidate for the internal compact directions if we want to extract real world physics from string theory. Before discussing compactification of any particular string models, let us first study some general properties of this manifold.

The Moduli Space

The moduli space of CY_3 is found by deforming the metric continuously while keeping the topological properties, the entire moduli space refers to an infinite family of Calabi-Yau manifolds [88]. The δg_{ab} and $\delta g_{\bar{a}\bar{b}}$ deformations form the *complex structure moduli space* $\mathcal{M}_C$ of complex dimension $h^{2,1}$. The name comes from the fact that this kind of transformations deforms the complex structure $J^i{}_j$, which is defined for the CY_3 viewed as a real six-dimensional manifold

$$J^i{}_k J^k{}_j = -\delta^i_j, \quad N^k{}_{ij} = J^l{}_i(\partial_l J^k{}_j - \partial_j J^k{}_l) - J^l{}_j(\partial_l J^k{}_i - \partial_i J^k{}_l) = 0. \tag{2.57}$$

$N^k{}_{ij}$ is called the Nijenhuis tensor. The complex structure moduli space itself is a Kähler manifold, with the Kähler potential given by

$$\mathcal{K}_C = -\ln\left(i\int \Omega \wedge \bar{\Omega}\right). \tag{2.58}$$

The $(3,0)$-form Ω, being holomorphic and covariantly constant, is uniquely defined. Introducing A^I and B_I as a basis for the Calabi-Yau three-cycles with their intersections satisfying

$$A^I \cap B_J = -B_J \cap A^I = \delta^I_J, \quad A^I \cap A^J = B_I \cap B_J = 0, \tag{2.59}$$

we can define the coordinates X^I on $\mathcal{M}_C$ according to

$$X^I = \int_{A^I} \Omega, \quad I = 0, \ldots, h^{2,1}. \tag{2.60}$$

Integrating over the B cycles

$$F_I = \int_{B_I} \Omega \tag{2.61}$$

then yields functions of X^I, which can be shown to be governed by a holomorphic function F:

$$F_I = \frac{\partial F}{\partial X^I}. \tag{2.62}$$

The *prepotential* F is homogeneous of degree two

$$F(\lambda X) = \lambda^2 F(X). \tag{2.63}$$

Since the Calabi-Yau volume does not belong to $\mathcal{M}_C$, F will exact in α' when we later consider string theory compactifications. The Kähler potential can now be rewritten as

$$e^{-\mathcal{K}_C} = -i \sum_{I=0}^{h^{2,1}} \left(X^I \bar{F}_I - \bar{X}^I F_I \right). \tag{2.64}$$

A Kähler manifold whose Kähler potential only depends on a single holomorphic prepotential is studied using the mathematics of *special geometry*.

Let us now turn to the $\delta g_{a\bar{b}}$ deformations of the CY_3 metric. These will alter the Kähler structure, whose associated Kähler form is defined as

$$J = i g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}, \tag{2.65}$$

where $g_{a\bar{b}}$ is the CY metric. The corresponding moduli space is therefore called the *Kähler structure moduli space* $\mathcal{M}_K$. The number of independent such deformations is $h^{1,1}$. However, in string theory we can complexify the Kähler form by adding the NS-NS two-form B

$$\mathcal{J} = B + iJ. \tag{2.66}$$

The complexified Kähler moduli space then has real dimension $2h^{1,1}$. For the rest of this section $\mathcal{M}_K$ will always be complexified. The Kähler moduli space is also a Kähler manifold. The Kähler potential takes the form

$$\mathcal{K}_K = -\ln \left(\frac{4}{3} \int J \wedge J \wedge J \right) = -\ln (8V), \tag{2.67}$$

where V is the volume of the CY_3. The Kähler potential in this case can also be written in terms of a holomorphic prepotential. In the context of string compactifications this prepotential receives non-perturbative α' corrections.

In summary the moduli space of a CY_3 is factorized (at least locally) into two parts

$$\mathcal{M} = \mathcal{M}_C \times \mathcal{M}_K. \tag{2.68}$$

The variables of the moduli space are the massless scalar fields coming from the compactification procedure. For instance a ten-dimensional p-form will after compactification yield b_p moduli fields, where b_p denotes the Betti numbers of the Calabi-Yau manifold. One major problem with string compactifications is that the vacuum expectation values of the moduli space parameters are not fixed by the compactification itself, leading to an arbitrariness of the four-dimensional physics. By including non-zero background tensor fields the moduli space can be stabilized. However, flux compactification results, in general, in many possible vacua [89]. This is the short résumé of the string theory landscape problem [90].

Mirror Symmetry

Recall from Subsec. 2.1.2 that the two moduli variables of a two-torus can be interchanged by T-duality. A similar symmetry exists also for the Calabi-Yau three-fold [91]. The conjectured *mirror symmetry* relates any CY_3 M with another CY_3 $\tilde{M}$ such that their Dolbeault cohomology groups satisfy

$$H^{p,q}(M) = H^{3-p,q}(\tilde{M}). \tag{2.69}$$

Thus, mirror symmetry exchanges the complex structure and Kähler structure moduli spaces [92]. The only exception is when M has $h^{2,1}(M) = 0$, since a CY_3 cannot have vanishing $h^{1,1}(\tilde{M})$. Eq. (2.69) leads to $h^{1,1}(M) = h^{2,1}(\tilde{M})$ and vice versa for the dimensions of the cohomology groups. The Euler characteristics then satisfy $\chi_e(M) = -\chi_e(\tilde{M})$. Similar to the two-torus example, mirror symmetry between Calabi-Yau three-folds can also be understood in terms of T-duality [45]. The key observation is that any CY_3 can be thought of as a T^3 fibration over some three-dimensional base manifold B. Mirror symmetry is then, roughly speaking, simultaneous T-dualization on all three directions of the T^3.

Here we have merely provided some flavors of the huge research area concerning mirror symmetry, [93] is a nice reference for deeper readings.

2.2.3 Compactification of Type II Strings

Compactification on a Calabi-Yau manifold cannot be made as explicit as the toroidal compactification, in most cases we do not even know any global metric of the CY. Nevertheless, we can learn many general properties of such compactification without specifying the exact compactification Ansatz. The fact that Calabi-Yau three-folds have $SU(3)$ as the holonomy group allows only 1/4 of the supersymmetries to survive after compactifying on them [94]. Thus, compactification of the heterotic string on a CY_3 leads to an $\mathcal{N}=1$ theory in four dimensions, which is appealing from phenomenological point of view. However, following the main theme of this thesis we will only analyze compactification of the type II string theories.

$D=4$, $\mathcal{N}=2$ Supermultiplets

Compactifying type II string theories on a CY_3 leads to four-dimensional theories with $\mathcal{N}=2$ supersymmetry. Three types of massless supermultiplets appear in the resulting theories, all of them containing four bosonic and four fermionic degrees of freedom:

i. A *vector multiplet* consists of one vector, two gauginos and two scalars. The maximum helicity of this multiplet is thus 1.

ii. A *hypermultiplet* has maximum helicity 1/2. It contains two spin 1/2 fields and four scalars.

iii. One single *supergravity multiplet* exists, which consists of one graviton, two gravitini and one graviphoton. This multiplet has maximum helicity 2.

Type IIA on CY_3

Let us first consider type IIA theory compactified on a Calabi-Yau three-fold M. The index convention is such that $M=(\mu,i,\bar{\imath})$. The ten-dimensional metric gives rise to a four-dimensional metric $G_{\mu\nu}$, $h^{1,1}$ real scalars $G_{i\bar{\jmath}}$ and $h^{2,1}$ complex scalars G_{ij}. The NS two-form becomes $h^{1,1}$ real scalars $B_{i\bar{\jmath}}$ together with a four-dimensional antisymmetric tensor $B_{\mu\nu}$, which can be dualized into a real scalar. The ten-dimensional scalar remains a real scalar ϕ in four dimensions. In the R-R sector, the three-form gives rise to $h^{1,1}$ vectors $C_{\mu i\bar{\jmath}}$ and $(2h^{2,1}+2)$ real scalars ($C_{ij\bar{k}}$ and C_{ijk}). And lastly, the vector ends up as a four-dimensional vector C_μ. All the fields organize themselves nicely into the following supermultiplets:

gravity multiplet:	$G_{\mu\nu}$, C_μ, Ψ_μ, $\tilde{\Psi}_\mu$
universal hypermultiplet:	ϕ, $B_{\mu\nu}$, C_{ijk}, fermions
$h^{1,1}$ vector multiplets:	$G_{i\bar{\jmath}}$, $B_{i\bar{\jmath}}$, $C_{\mu i\bar{\jmath}}$, fermions
$h^{2,1}$ hypermultiplets:	G_{ij}, $C_{ij\bar{k}}$, fermions.

Table 2.1: *Field contents of type IIA string theory compactified on CY_3.*

Counting the number of real scalar fields we find the dimension of the moduli space to be $2h^{1,1} + (4h^{2,1} + 4)$. Using supersymmetric arguments it can be shown that at the tree-level neutral couplings between vector multiplets and hypermultiplets are forbidden [95]. The moduli space thus factorizes as

$$\mathcal{M}_M = \mathcal{M}_V \times \mathcal{M}_H. \tag{2.70}$$

The moduli space of vector multiplets $\mathcal{M}_V$ is a special Kähler manifold with dimension $2h^{1,1}$, at the classical level it can be identified with the moduli space of complexified Kähler structures $\mathcal{M}_K$. In Subsec. 2.2.2 it was argued that this space receives non-perturbative α' corrections from the worldsheet instantons. $\mathcal{M}_H$, on the other hand, has dimension $4(h^{2,1}+1)$ and is a so called quaternionic Kähler manifold[2]. The classical hypermultiplet moduli space is contains the CY_3 complex structure moduli space together with the moduli space coming from the universal hypermultiplet. Since the dilaton ϕ belongs to a hypermultiplet, $\mathcal{M}_H$ will receive non-perturbative corrections in the string coupling g_s.

Most often Calabi-Yau compactification does not lead to moduli space with coset symmetry. One exception is when compactifying type IIA string theory on a *rigid Calabi-Yau manifold*, which has $h^{2,1} = 0$. There is then only one hypermultiplet, namely the universal hypermultiplet. The hypermultiplet moduli fields parameterize the coset space $SU(2,1)/U(2)$ [96]. PAPER V discusses possible g_s corrections to this universal hypermultiplet sector.

Type IIB on CY_3

Compactification of type IIB theory on CY_3 is very similar to the type IIA case. The NS-NS sector is identical, while the ten-dimensional R-R sector now consists of even form potentials. The axion gives an axion C in four dimensions. The two-form gives rise to $h^{1,1}$ real scalars $C_{i\bar{j}}$ and a four-dimensional two-form $C_{\mu\nu}$, the latter can be dualized into another scalar field. The self-dual four-form leads to $h^{1,1}$ real scalars $C_{\mu\nu i\bar{j}}$ and $(h^{2,1} + 1)$ vectors ($C_{\mu ij\bar{k}}$ and $C_{\mu ijk}$). $C_{\mu ijk}$ corresponds to the unique $(3,0)$-form of a Calabi-Yau manifold. In summary we have:

gravity multiplet:	$G_{\mu\nu}$, $C_{\mu ijk}$, Ψ_μ, $\tilde{\Psi}_\mu$
universal hypermultiplet:	ϕ, C, $B_{\mu\nu}$, $C_{\mu\nu}$, fermions
$h^{2,1}$ vector multiplets:	G_{ij}, $C_{\mu ij\bar{k}}$, fermions
$h^{1,1}$ hypermultiplets:	$G_{i\bar{j}}$, $B_{i\bar{j}}$, $C_{i\bar{j}}$, $C_{\mu\nu i\bar{j}}$, fermions.

Table 2.2: *Field contents of type IIB string theory compactified on CY_3.*

[2] A quaternionic Kähler manifold is a Riemannian manifold of $4k$ dimensions which has its holonomy group contained in $\frac{Sp(k)\times Sp(1)}{\mathbb{Z}_2}$.

The dimension of the moduli space is $2h^{2,1} + 4(h^{1,1} + 1)$. It can again be decomposed into a vector multiplet and a hypermultiplet subspace as in Eq. (2.70). The vector multiplet moduli space $\mathcal{M}_V$ has dimension $2h^{2,1}$ and it is still special Kähler. However, it now parameterizes the complex structure moduli space of the CY_3. The hypermultiplet moduli space $\mathcal{M}_H$ is quaternionic Kähler and $4(h^{1,1} + 1)$-dimensional. It describes Kähler structure deformations. Since both the CY volume and the dilaton belong to $\mathcal{M}_H$, the hypermultiplet moduli space receives perturbative as well as non-perturbative α' and g_s corrections. The tree-level geometry of $\mathcal{M}_V$, on the other hand, is exact.

A consequence of the mirror symmetry is that type IIA string theory compactified on M is equivalent to type IIB string theory compactified on $\tilde{M}$. More explicitly, the respective moduli spaces can be identified

$$\mathcal{M}_V(M) = \mathcal{M}_H(\tilde{M}) \quad \text{and} \quad \mathcal{M}_H(M) = \mathcal{M}_V(\tilde{M}). \tag{2.71}$$

Eq. (2.71) is a statement with deep impact. It tells us that the prepotential of the IIB complex structure moduli space, which is classically exact, is mapped to the prepotential of the IIA Kähler structure moduli space, which contains quantum α' corrections. Comparison of the BPS solutions which wrap supersymmetric cycles on the both sides of the mirror symmetry serves as a consistent check for the symmetry itself.

2.2.4 Non-perturbative Instanton Effects

Up to now we have only analyzed the massless sector after compactification. The reason the massive Kaluza-Klein modes are ignored is they are very heavy and can be integrated out in the low energy limit. It was found out that the Calabi-Yau moduli space contains singularities, near which p-branes wrapping *supersymmetric cycles* become very light. These contribute as non-perturbative *instanton effects* in the low energy effective action [97, 98]. We will emphasize again that this picture of the low energy effective action being an infinite expansion in the string coupling constant, containing perturbative loop and non-perturbative instanton corrections, is only correct near the moduli space singularities. The low energy quantum theory away from the singularities requires a strong coupling description.

One simple example of moduli space singularities is the *conifold singularity* [88]. Let us recall the coordinate parametrization of the complex structure moduli space of a CY from Subsec. 2.2.2. When one of the coordinates, for example

$$X^1 = \int_{A^1} \Omega \tag{2.72}$$

is zero the moduli space metric will become singular. Near the singularity the metric takes the form

$$G_{1\bar{1}} = \frac{\partial^2 \mathcal{K}_C}{\partial X^1 \partial \bar{X}^{\bar{1}}}, \tag{2.73}$$

with

$$\mathcal{K}_C \sim \ln\left(|X^1|^2 \ln |X^1|^2\right). \tag{2.74}$$

The singularity at $X^1 = 0$ is of conifold type. It is a real singularity, since the curvature scalar diverges at this point. The corresponding cycle A^1 is called a vanishing cycle.

As we have already mentioned, near moduli space singularities Euclidean p-branes wrapping supersymmetric cycles lead to instanton corrections. A supersymmetric cycle of a Calabi-Yau manifold is a cycle which preserves some supersymmetry when a brane is wrapped around it. For CY_3 there are two types of cycles which allow a non-zero covariant constant spinor, both of them characterized by the two p-forms which are invariant under the $SU(3)$ holonomy group:

i. A *holomorphic two-cycle* has real dimension two and satisfies

$$\bar{\partial} x^a = 0 \quad \text{and} \quad \partial x^{\bar{a}} = 0. \tag{2.75}$$

The volume of a holomorphic submanifold is measured by the Kähler form.

ii. The volume of a *special Lagrangian submanifold* W, on the other hand, is measured by the covariant constant $(3,0)$-form. At the same time the pullback of the Kähler form to W vanishes. A special Lagrangian submanifold has real dimension three. Ref. [99] nicely reviews the properties of this kind of submanifolds.

Identifying any of these submanifolds with a p-brane worldsheet will put constraints on the worldsheet. Configurations satisfying these constraints will be stable, moreover their volumes are minimized.

From the ten-dimensional point of view, the compact CY manifold is entirely spacelike. Branes wrapping supersymmetric cycles inside the CY are therefore Euclidean. These are precisely the classical instanton configurations, i.e., events in spacetime. In general, the fundamental-string instantons lead to non-perturbative α' corrections, while the D-branes and the NS5-branes give rise to non-perturbative g_s corrections. To be more concrete, IIA string theory compactified on a CY contains F1 worldsheet instantons in $\mathcal{M}_V$, while in $\mathcal{M}_H$ there are D2- and NS5-brane instantons. For the IIB theory $\mathcal{M}_V$ is exact, and $\mathcal{M}_H$ contains worldsheet F1 as well as D(-1)-, D1-, D3-, D5- and NS5-brane instantons.

The quantum corrections to the moduli spaces of CY compactified type II string theories have been investigated in a series of papers [97, 100, 101, 102, 103, 104, 58]. Paper V addresses this question when the Calabi-Yau manifold is rigid.

2.3 G_2 Manifold

For compactification of M-theory to four dimensions we need seven-dimensional manifolds which only break supersymmetry partially. The manifold $\mathrm{CY}_3 \times S^1$ is certainly the simplest choice. The natural generalizations of $\mathrm{CY}_3 \times S^1$ are manifolds with G_2 holonomy. Compactification of M-theory on such manifolds breaks 7/8 of the supersymmetries, resulting in theories with $\mathcal{N} = 1$ in $D = 4$.

The group G_2 is an exceptional simple Lie group, properties of it can be found for instance in Ref. [51]. It has rank 2 and contains in total 14 root vectors. The Cartan matrix takes the form

$$A_{G_2} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}. \tag{2.76}$$

The compact real form of G_2 is a subgroup of $Spin(7)$, which is the covering of the Lorentz group of a Euclidean seven-dimensional manifold. We can thus rewrite all the $Spin(7)$ representations in terms of G_2 representations [105], see Table 2.3. The singlet **1** from the spinor decomposition is the one and only covariant constant spinor

$$\nabla_M \xi = 0, \tag{2.77}$$

with M being the eleven-dimensional spacetime index.

Adjoint representation of $Spin(7)$:	$\mathbf{21} \to \mathbf{14} + \mathbf{7}$
Vector representation of $Spin(7)$:	$\mathbf{7} \to \mathbf{7}$
Spinor representation of $Spin(7)$:	$\mathbf{8} \to \mathbf{7} + \mathbf{1}$

Table 2.3: *$Spin(7)$ representations decomposed into G_2 representations.*

A seven-dimensional manifold with G_2 holonomy is referred shortly as a G_2 *manifold*. It is orientable and Ricci flat. Similar to the Calabi-Yau case, a G_2 manifold is characterized by a covariant constant real three-form Φ called the *associative calibration* [106, 105]. The associative calibration is both closed and co-closed

$$d\Phi = 0 \quad \text{and} \quad d * \Phi = 0. \tag{2.78}$$

The Hodge dual four-form $*\Phi$ is named the *coassociative calibration*. These forms are attached to supersymmetric three- and four-cycles, whose volumes satisfy [107]

$$\mathrm{Vol}(S_3) = \int_{S_3} f^*\Phi \quad \text{and} \quad \mathrm{Vol}(S_4) = \int_{S_4} \tilde{f}^*(*\Phi). \tag{2.79}$$

f^* and $\tilde{f}^*$ denote pullbacks to the respective submanifold. The volume condition of the three-cycle S_3 is equivalent to the projection equation [4]

$$P_-\epsilon = \frac{1}{2}\left(1 - \frac{i}{6}\varepsilon^{ijk}\partial_i X^M \partial_j X^N \partial_k X^P \Gamma_{MNP}\right)\epsilon = 0, \tag{2.80}$$

where X^M is the eleven-dimensional spacetime coordinate, and $i, j, \ldots$ are the local indices on S_3. The spinor ϵ is constant and lives in seven dimensions. Similarly the four-cycle S_4 can be found by solving

$$\left(1 - \frac{i}{4!}\varepsilon^{ijkl}\partial_i X^M \partial_j X^N \partial_k X^P \partial_l X^Q \Gamma_{MNPQ}\right)\epsilon = 0. \tag{2.81}$$

Branes wrapping the supersymmetric cycles will be stable, their volumes are then minimized. Moreover, both cycles break 1/2 of the supersymmetries of the objects winding around them. Smooth G_2 manifolds can be constructed by starting from seven-dimensional orbifolds and then resolving the singularities.

Calabi-Yau and G_2 manifolds are both examples of manifolds with *special holonomy* [106]. An n-dimensional Riemannian manifold M with special holonomy is characterized by the fact that its holonomy group satisfies $\mathrm{Hol}(M) \subset Spin(n)$. Furthermore, it exhibits a covariant constant spinor as well as invariant forms known as *calibrations*. The eight-dimensional $Spin(7)$ manifold belongs also to this category. This manifold breaks 15/16 of the supersymmetries. Thus, M-theory compactified on $Spin(7)$ manifolds yields $\mathcal{N} = 1$ supersymmetry in three dimensions.

2.4 The Topological Sector

One peculiar feature of manifolds with special holonomy is that they allow for classical theories of gravity where the metric is not a fundamental quantity. Instead, the metric can be constructed in terms of the form-fields. Such theories are known as *form theories of gravity*. The well-known reformulation of three-dimensional pure gravity in terms of Chern-Simons gauge theory [108] serves as the prototype. Here we will focus on form gravity theories in six and seven dimensions, and then briefly comment on their relations to topological subsectors of M-theory.

2.4.1 Six Dimensions

In six dimensions there exists two different form theories of gravity. One is the Kähler gravity theory [109] and the other is the Kodaira-Spencer gravity theory [110]. Both of them can be defined on Calabi-Yau three-folds.

Kähler Gravity Theory

The *Kähler gravity theory* describes variations of the complexified Kähler structure on a CY$_3$ M. The action of this theory is

$$S_{\text{Kähler}} = \int_M \left(\frac{1}{2} K \frac{1}{d^{c\dagger}} dK + \frac{1}{3} K \wedge K \wedge K \right), \tag{2.82}$$

where K is a variation of the complexified Kähler form and $d^c = \partial - \bar{\partial}$. The equations of motion are

$$dK + d^{c\dagger}(K \wedge K) = 0. \tag{2.83}$$

The action in Eq. (2.82) is invariant under the gauge transformations

$$\delta_\alpha K = d\alpha - d^{c\dagger}(K \wedge \alpha), \tag{2.84}$$

with the one-form α satisfying $d^{c\dagger}\alpha = 0$.

We can decompose K into massless and massive modes

$$K = K_0 + d^{c\dagger}\gamma, \tag{2.85}$$

where the dynamical degrees of freedom consist of the massive modes $\gamma \in \Omega^3(M)$, while the massless Kähler moduli $K_0 \in H^{1,1}(M, \mathbb{C})$ are constant. The action (2.82) can then be rewritten as

$$S_{\text{Kähler}} = \int_M \left(\frac{1}{2} d\gamma \wedge d^{c\dagger}\gamma + \frac{1}{3} K \wedge K \wedge K \right). \tag{2.86}$$

without the non-local term.

In Ref. [107] it was argued that the Kähler gravity is identical to the action

$$V_S(J) = \frac{1}{6} \int_M J \wedge J \wedge J \tag{2.87}$$

introduced by N. Hitchin [111]. For a CY$_3$ the two-form J is simply the Kähler form. The identification with $S_{\text{Kähler}}$ is achieved by constructing a stable[3] four-form

$$\sigma = \frac{1}{2} J \wedge J = \sigma_0 + d\gamma, \tag{2.88}$$

[3] A p-form X_p is *stable* if all forms in a neighborhood of X_p are equivalent to X_p by a local $GL(n)$ action.

where gauge field γ is the same three-form as we have introduced earlier. Extremizing the action (2.87) with respect to γ one finds

$$dJ = 0. \tag{2.89}$$

Kodaira-Spencer Gravity Theory

The action of the *Kodaira-Spencer gravity theory* has the following form

$$S_{\mathrm{KS}} = \frac{1}{2}\int_M A' \frac{1}{\partial}\bar{\partial}A' + \frac{1}{6}\int_M (A\wedge A)'\wedge A', \tag{2.90}$$

where A is a vector-valued one-form. The notation $Y' := (Y \cdot \Omega_0)$ stands for the product of some form Y with the background holomorphic $(3,0)$-form resulting in a three-form. For the non-local term $\frac{1}{\partial}\bar{\partial}A'$ to be well-defined, we need to constrain A according to

$$\partial A' = 0. \tag{2.91}$$

The field A is variation of the holomorphic $(3,0)$-form Ω

$$\Omega = \Omega_0 + A' + (A\wedge A)' + (A\wedge A\wedge A)', \tag{2.92}$$

the Kodaira-Spencer gravity theory thus describes deformations of the complex structure of a CY_3.

Decomposing A' into massless modes and massive modes

$$A' = A_0' + \partial\kappa, \tag{2.93}$$

the Kodaira-Spencer action becomes

$$S_{\mathrm{KS}} = \frac{1}{2}\int_M \partial\kappa\wedge\bar{\partial}\kappa + \frac{1}{6}\int_M (A\wedge A)'\wedge A'. \tag{2.94}$$

The massless modes $A_0' \in H^{2,1}(M,\mathbb{C})$ are static, while the massive ones $\kappa \in H^{1,1}(M,\mathbb{C})$ are dynamical. Varying the action with respect to κ we find the equations of motion

$$\bar{\partial}A' + \partial(A\wedge A)' = 0. \tag{2.95}$$

The equations of motion (2.95) together with the constraint (2.91) require the holomorphic $(3,0)$-form Ω to be closed on-shell

$$d\Omega = 0. \tag{2.96}$$

The action (2.94) contains a shift symmetry

$$\kappa \to \kappa + \zeta, \quad \text{with} \quad \partial\zeta = 0, \tag{2.97}$$

which can be used to remove the anti-holomorphic part of κ, i.e.,

$$\bar{\partial}\kappa = 0. \tag{2.98}$$

For this reason the Kodaira-Spencer theory is said to be chiral.

A closely related theory has been given in Ref. [111], where the action is directly written in terms of the holomorphic $(3,0)$-form Ω

$$V_H(\rho) = -\frac{i}{4}\int_M \Omega \wedge \bar{\Omega} = \frac{1}{2}\int_M \hat{\rho}(\rho) \wedge \rho. \tag{2.99}$$

The three-form ρ, defined as the real part of Ω, is a stable form, while $\hat{\rho}(\rho)$ is the imaginary part of Ω:

$$\Omega = \rho + \hat{\rho}(\rho). \tag{2.100}$$

Letting ρ vary in a fixed cohomology class

$$\rho = \rho_0 + d\beta \tag{2.101}$$

we may find the equations of motion

$$d\rho = 0 \quad \text{and} \quad d\hat{\rho} = 0. \tag{2.102}$$

Again, we obtain the integrability condition for Ω from Eq. (2.96). This theory is, however, not chiral. Arguments were presented in Ref. [107] that this form theory of gravity is a sum of the Kodaira-Spencer theory together with its conjugate.

Both V_S and V_H can be defined simultaneously on a Calabi-Yau threefold. The following conditions then have to be obeyed in addition:

$$\begin{aligned} J \wedge \rho &= 0, \\ 2V_S(J) &= V_H(\rho). \end{aligned} \tag{2.103}$$

The Topological A Model

Quantizing the Kähler gravity leads to the *topological A model*, which roughly speaking is the topological subsector of type IIA superstring theory on a CY_3. It contains two types of objects: fundamental strings and D2-branes.

Fundamental strings wrap the two-dimensional holomorphic cycles. Their scattering amplitudes depend only on the Kähler form of the CY_3 target space. Moreover, there are quantum mechanical instanton corrections encoded by the so called *Gromov-Witten invariants*. The low energy effective theory of the closed strings is precisely the Kähler gravity.

The D2-branes, on the other hand, wrap the special Lagrangian three-cycles. They are charged under the field γ from the Kähler gravity. Open strings ending on a stack of N D2-branes describe a $U(N)$ Chern-Simons theory. Fundamental topological strings ending on D2-branes is a stable configuration only if the Kähler form J and the holomorphic $(3,0)$-form Ω satisfy

$$J \wedge \Omega = 0. \qquad (2.104)$$

The Topological B Model

The Kodaira-Spencer gravity is the low energy effective theory of the closed strings in the *topological B model*. The B model describes the topological subsector of type IIB string theory.

This theory contains fundamental strings, D(−1)-, D1-, D3- and D5-branes, all of them wrapping holomorphic cycles. The scattering amplitudes of the fundamental strings depends, in contrast to the A model, only on the complex structure. Furthermore, they do not receive any worldsheet instanton corrections. The D5-brane wraps a connected six-dimensional submanifold, its low energy physics is governed by a holomorphic Chern-Simons theory. The effective theories of the other D-branes can be obtained by dimensional reducing this holomorphic Chern-Simons theory. The D1-brane is charged under the field κ from the Kodaira-Spencer gravity.

Mirror symmetry is believed to be an exact duality at the quantum level between the A and B model on different Calabi-Yau threefolds. Some general features of the mirror symmetry can be found in Subsec. 2.2.2. Also it has been conjectured that the A and B model are S-dual to each other on the same CY_3 [112, 113].

Topological string theories are simplified versions of string theories, where gravity does not pose any independent degrees of freedom. Because of its simplicity many quantities, such as the partition function, may be computed explicitly in topological string theories. However, they turn out to provide many useful insights for the ten-dimensional string theories, Ref. [114] explores some of these links.

2.4.2 Seven Dimensions

In seven dimensions it is possible to construct form theories of gravity on G_2 manifolds. The story is quite similar to what happens in six dimensions. Quantizing this system has been suggested to yield a topological version of M-theory.

Form Gravity in $D = 7$

A real three-form Φ on a G_2 manifold Y can be used to form a symmetric tensor

$$B_{ij} = -\frac{1}{144}\epsilon^{k_1 \dots k_7}\Phi_{ik_1k_2}\Phi_{jk_3k_4}\Phi_{k_5k_6k_7}, \tag{2.105}$$

with $i, j, \dots = 1, \dots, 7$. Defining the metric as

$$g_{ij} = (\det B)^{-1/9} B_{ij}, \tag{2.106}$$

we may compute the volume of Y according to

$$V_7(\Phi) = \int_Y \sqrt{g_\Phi} = \int_Y (\det B)^{1/9}. \tag{2.107}$$

The metric in Eq. (2.106) provides also a Hodge operator, which we denote by $*_\Phi$. The action can now be rewritten as

$$V_7(\Phi) = \int_Y \Phi \wedge *_\Phi \Phi. \tag{2.108}$$

Assume $\Phi = \Phi_0 + dB$ with $d\Phi_0 = 0$. Varying the action with respect to the two-form gauge potential B we will obtain the equations of motion

$$d\Phi = 0 \quad \text{and} \quad d *_\Phi \Phi = 0. \tag{2.109}$$

These equations dictate that Φ has to be the associative calibration.

Alternatively, one can start with a four-form, going through a similar procedure one finds that it has to be identified with the coassociative calibration. The resulting action is identical to $V_7(\Phi)$.

For the special G_2 manifolds $Y = M \times \mathbb{R}$ and $Y = M \times S^1$, where M is a CY_3 we can decompose the associative and coassociative calibration as follows

$$\Phi = \Re(\Omega) + J \wedge dt \quad \text{and} \quad *_\Phi \Phi = \Im(\Omega) \wedge dt + \frac{1}{2} J \wedge J. \tag{2.110}$$

Recall J is the Kähler form of the CY_3, while Ω is the holomorphic $(3,0)$-form. The seventh dimension is parameterized by the coordinate t. The equations of motion in Eq. (2.109) dimensionally reduce to

$$dJ = 0 \quad \text{and} \quad d\Omega = 0. \tag{2.111}$$

The action itself becomes

$$V_7(\Phi) = 3V_S(J) + 2V_H(\rho). \tag{2.112}$$

Topological M-theory

The conjectured mirror symmetry and S-duality between topological A and B model suggests the existence of a *topological M-theory* on G_2 manifolds. The topological M-theory on $M \times S^1$ is then identified with the A model on the Calabi-Yau threefold M. The classical limit of the low energy effective theory would be the form theory of gravity we have described. The fact that the form gravity in seven dimensions reduces to a combinations of Kähler and Kodaira-Spencer gravities is also hinting towards a unification.

In Refs. [115, 116, 107] the following definition of topological M-theory was given: Topological M-theory on a G_2 manifold Y is defined to be equivalent to A model topological strings on $Y/U(1)$. D2-branes wrapping special Lagrangian submanifolds in $Y/U(1)$ should be lifted to points at which the circle fibration degenerates. These are the Kaluza-Klein monopoles.

The fundamental object in topological M-theory is the membrane, which is the seven-dimensional lift of the A model fundamental string. Thus, membranes wrapping associative three-cycles will give rise to quantum corrections in the low energy effective theory [117]. Viewing topological M-theory as the topological subsector of eleven-dimensional M-theory, some properties of the topological membrane was discussed in Paper I. Since membranes and strings are dual to each other in seven dimensions, one can also study topological G_2 strings [118]. Topological M5-branes living on G_2 manifolds were analyzed in Paper II, where comments were also made regarding their relation to the topological string models.

3

String Effective Actions

When we introduced the supergravity theories in Chapter 1 we explained that they should be viewed as classical low energy limits of the superstring theories. More explicitly, they describe the classical dynamics of the massless particles in string theory. By computing scattering amplitudes of these massless particles it was soon realized that the full effective field theory action of string theory is a simultaneous expansion in two parameters. One is the Regge slope α' and the other is the string coupling constant g_s. Though the scattering amplitudes are computationally very complicated, it is sometimes possible to find shortcuts using the symmetries of the theory. It is believed that both supersymmetry and U-duality should survive the quantization process, and therefore also be symmetries of the low energy effective action. In particular by requiring invariance under U-duality it turns out that one can even move away from the perturbative regime. In this chapter a short review of the general structure of the superstring effective action will be given.

3.1 Scattering Amplitudes

The *effective field theory action* of a quantum theory describes the dynamics of the massless particles. The general structure of such an action consists of a classical part and its quantum corrections. In the case of string theory, the classical part is a supergravity theory described in Section 1.1. The most direct method to determine the quantum corrections is by computing scattering amplitudes of the relevant particles. The idea of this section is to list a few important results, while for explicit computations and techniques of this vast research area the reader is asked to turn to the references given. We will be setting our focus mostly on the type II theories, but also occasionally mention some results for M-theory. One of the early observations of superstring theory, which also played a role making it favorable as a force unifying theory, is the fact that its perturbation theory is finite to all loop orders [119, 120, 121]. Thus, no renormalization of the scattering amplitudes is needed.

The first correction term to the conventional Einstein-Hilbert action is contained in the *tree-level four-graviton* scattering amplitude. Using a different argument, it was shown by the authors of Ref. [122] that the leading order correction in the Einstein frame has the following form

$$\Delta S \propto \frac{\alpha'^3}{2\kappa^2} \int d^{10}x \sqrt{-g} \mathcal{R}^4, \tag{3.1}$$

with $\alpha' = \ell_s^2$ being the Regge slope (or the inverse string tension). The relation between κ and the string length scale ℓ_s can be found in Subsec. 1.1.2. The notation $\mathcal{R}^4$ is a short hand for a quartic order contraction of the Weyl curvature tensor. In this particular case

$$\mathcal{R}^4 = t_8^{a_1 \dots a_8} t_8^{b_1 \dots b_8} R_{a_1 a_2 b_1 b_2} R_{a_3 a_4 b_3 b_4} R_{a_5 a_6 b_5 b_6} R_{a_7 a_8 b_7 b_8}, \tag{3.2}$$

where the contraction of t_8 with four two-tensors M_{ab} is defined as

$$\begin{aligned} t_8 M^4 =& 8\mathrm{Tr}(M_1 M_2 M_3 M_4) + 8\mathrm{Tr}(M_1 M_3 M_2 M_4) + 8\mathrm{Tr}(M_1 M_3 M_4 M_2) \\ & - 2\mathrm{Tr}(M_1 M_2)\mathrm{Tr}(M_3 M_4) - 2\mathrm{Tr}(M_2 M_3)\mathrm{Tr}(M_4 M_1) \\ & - 2\mathrm{Tr}(M_1 M_3)\mathrm{Tr}(M_2 M_4). \end{aligned} \tag{3.3}$$

Eq. (3.1) shows that quantum corrections in the effective action are clearly higher order in the number of derivatives. Since $[\alpha'] = L^2$, ΔS is indeed dimensionless. The reason we did not specify which of the type II theories this amplitude belongs to is because the perturbative contributions to $\partial^{2n}\mathcal{R}^4$ corrections are the same in the IIA and IIB theories for $n \leq 4$ [123]. The enormous complexity of scattering calculations is tested by the fact that the full *tree-level four-particle* amplitude, including the fermionic terms, to all orders in α' has been worked out only recently in Ref. [124].

The four-particle loop computations have traditionally been carried out using the *Ramond-Neveu-Schwarz* (RNS) or the *Green-Schwarz* (GS) formalisms. For instance the *one-loop four-graviton* amplitude was shown to contain terms of the form $\mathcal{R}^4$ and $\mathcal{D}^6\mathcal{R}^4$ [125, 126], with $\mathcal{D}$ representing some combination of derivatives and fields with the same dimension as a spacetime derivative. Working a bit harder, the *two-loop four-graviton* amplitude is also calculable in the RNS formulation [127]. Among other things it contains terms of the form $\mathcal{D}^4\mathcal{R}^4$ and $\mathcal{D}^6\mathcal{R}^4$. The correct Regge slope order can always be restored afterwards by dimensional analysis.

Since the discovery of the *pure spinor* formalism [128], the computation of scattering amplitudes has become considerably more efficient. The tree-level, one-loop and two-loop four-graviton scattering amplitudes were recomputed in this formalism [129, 130], and proven to be identical to the results from the RNS formulation [131, 132]. A prescription presented in Ref. [133] has made arbitrary multiloop calculations in principle accessible. The resulting expressions in pure spinor superfields are very elegant, in fact both the one-loop and two-loop kinematic factors can be expressed in terms of the tree-level one [134]. The difficult part lies in the expansion in component fields. Recently progress has been made on the *tree-level* and *one-loop five-particle* scattering amplitudes [135, 136, 137], showing that terms of the form $\mathcal{D}^{2n}\mathcal{R}^5$ exist.

To summarize the results from the scattering amplitude computations we have the following perturbative expansion of the low energy effective action:

$$\begin{aligned}\mathcal{S}_{\mathrm{S}} =&\alpha'^{-4}\int d^{10}x\sqrt{-g}e^{-2\phi}\left\{\left[(R+\dots)+\left(\alpha'^3\mathcal{R}^4+\alpha'^4\mathcal{D}^2\mathcal{R}^4\dots\right)\right.\right.\\ &\left.+\left(\alpha'^4\mathcal{R}^5+\alpha'^5\mathcal{D}^2\mathcal{R}^5+\dots\right)+\dots\right]\\ &+e^{2\phi}\left[\left(\alpha'^3\mathcal{R}^4+\alpha'^6\mathcal{D}^6\mathcal{R}^4+\dots\right)+\left(\alpha'^6\mathcal{D}^4\mathcal{R}^5+\dots\right)\right]\\ &\left.+e^{4\phi}\left[\alpha'^5\mathcal{D}^4\mathcal{R}^4+\alpha'^6\mathcal{D}^6\mathcal{R}^4+\dots\right]+\dots\right\}.\end{aligned} \tag{3.4}$$

The form-fields and fermionic terms are omitted for simplicity. Notice also that we have changed to the string frame, where there is an overall dilaton factor $e^{-2\phi}$. Clearly Eq. (3.4) is a double sum in α' and e^{ϕ}:

$$\mathcal{S}_{\mathrm{S}}=\alpha'^{-4}\int d^{10}x\sqrt{-g}e^{-2\phi}\sum_{n=0}^{\infty}\sum_{g=0}^{\infty}\alpha'^n e^{2g\phi}L_{(n,g)}, \tag{3.5}$$

where g is the genus of the string worldsheet. Since $g_s=e^{<\phi>}$, the general jargon is that $\mathcal{S}$ is an expansion in the *Regge slope* and *string coupling*. Although the entire perturbative effective action can be constructed systematically in this way, the computations are very tedious and time consuming. One additional complication is posed by the relation $[\mathcal{D},\mathcal{D}]\sim\mathcal{R}$. Using $\mathcal{D}^2\mathcal{R}^5$ as an illustration, even though there is no contribution to this term from the one-loop five-graviton amplitude [136], to really exclude the one-loop contribution we must also compute the one-loop six-graviton amplitude.

We would like to mention that for comparison particle scattering amplitudes in M-theory have been computed as well, so far up to two-loops [83, 138, 139, 140]. The M-theory amplitudes are directly linked to the IIA amplitudes by compactification, which can then be mapped to IIB using T-duality in nine dimensions. In Chapter 1 we have explained that U-duality is expected to be a symmetry of string theory, therefore these operations are justified. It turns out that only very few terms have an eleven-dimensional lift in the decoupling limit. For instance the one-loop contribution to the type II $\mathcal{R}^4$ correction can be traced to a one-loop four-particle amplitude in M-theory.

The rest, e.g., the tree-level $\mathcal{R}^4$ or the entire $\mathcal{D}^4\mathcal{R}^4$ are purely artifacts of the compactification process. More generally, it was argued in Ref. [139] that only terms with same dimensions as

$$\hat{\mathcal{R}}^{3k+1}, \quad k \in \mathbb{N} \tag{3.6}$$

are allowed in M-theory, where $\hat{\mathcal{R}}$ denotes the eleven-dimensional curvature tensor. As an interesting side remark, the orders $(3k+1)$ can be related to certain weights of the Kac-Moody group E_{10} [84], and thus seem to support the E_{10} conjecture we mentioned in Section 1.4. However, a caveat to this symmetry analysis is given in Paper IV.

As scattering computations became more and more sophisticated, complementing methods based on symmetry properties was developed in parallel. These symmetries are precisely those discussed in Chapter 1, namely supersymmetry and U-duality. In fact, we have already mentioned the usefulness of T-duality. The subsequent two sections will be devoted to various symmetry results. From now on we will focus on the type IIB theory. A better way to organize the perturbative part of the effective action is then

$$\begin{aligned}
\mathcal{S}_{\text{IIB,S}} =&\alpha'^{-4}\int d^{10}x\sqrt{-g}e^{-2\phi}\left\{[R+\dots]\right.\\
&+\alpha'^3\left[\left(c_{(3,0)}+c_{(3,1)}e^{2\phi}\right)\mathcal{R}^4+\dots\right]\\
&+\alpha'^4\left[\mathcal{D}^2\mathcal{R}^4+\mathcal{R}^5+\dots\right]\\
&+\alpha'^5\left[\left(c_{(5,0)}+c_{(5,2)}e^{4\phi}\right)\mathcal{D}^4\mathcal{R}^4+\mathcal{D}^2\mathcal{R}^5+\dots\right]\\
&+\alpha'^6\left[\left(c_{(6,0)}+c_{(6,1)}e^{2\phi}+c_{(6,2)}e^{4\phi}+c_{(6,3)}e^{6\phi}\right)\mathcal{D}^6\mathcal{R}^4\right.\\
&\left.\left.+\left(\tilde{c}_{(6,0)}+\tilde{c}_{(6,2)}e^{4\phi}\right)\mathcal{D}^4\mathcal{R}^5+\dots\right]+\dots\right\},
\end{aligned} \tag{3.7}$$

where $c_{(n,g)}$ are purely numerical coefficients.

3.2 U-duality Completion

If U-duality indeed is a symmetry of M-theory, it is natural to expect that the low energy effective actions are invariant under U-duality transformations. The successful $SL(2,\mathbb{Z})$ completion of the $\mathcal{R}^4$ correction in type IIB theory suggests that this might be a correct path to take.

Let us follow Ref. [60] and work through the type IIB example. From scattering amplitude computations the perturbative $\mathcal{R}^4$ correction is found to be

$$S^{(3)^*}_{\text{IIB,E,pert}} = \alpha'^{-4}\int d^{10}x\sqrt{-g}\alpha'^3\left(2\zeta(3)e^{-3\phi/2}+4\zeta(2)e^{\phi/2}\right)\mathcal{R}^4, \tag{3.8}$$

where the two terms correspond to tree-level and one-loop contributions, respectively. $\zeta(s)$ is the Riemann zeta function defined as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Eq. (3.8) is written in the Einstein frame where U-duality is most apparent. The U-duality group of type IIB string theory is the modular group $SL(2,\mathbb{Z})$, see Table 1.1. It acts on the dilaton axion $\tau = \tau_1 + i\tau_2 = C_0 + ie^{-\phi}$ according to

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}, \tag{3.9}$$

where the parameters a, b, c, d are integer numbers satisfying $ad - bc = 1$. While the $\mathcal{R}^4$ combination itself does not transform under modular transformations, its dilaton coefficients in Eq. (3.8) obviously do. The idea is to add more terms inside the parenthesis to make the total expression modular invariant. It turns out that the function

$$\begin{aligned} \mathcal{E}^{(0,0)}(\tau, \bar{\tau}) =& 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} \\ &+ 4\pi\sqrt{\tau_2}\sum_{N\neq 0} \mu_{-2}(N) N K_1(2\pi|N|\tau_2) e^{2\pi i N \tau_1} \end{aligned} \tag{3.10}$$

does the trick. The sum above runs over all non-zero integers, $K_1(x)$ is the modified Bessel function and the so called *instanton measure* $\mu_{-2}(N)$ is defined as a sum over all divisors of N

$$\mu_{-2}(N) := \sum_{n|N} n^{-2}. \tag{3.11}$$

Eq. (3.10) is the $SL(2,\mathbb{Z})$ *non-holomorphic Eisenstein series*, in Subsec. 4.2.3 we show that it can be rewritten in a more compact form

$$\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau, \bar{\tau}) = \sum_{(m,n)}{}' \frac{\tau_2^s}{|m + n\tau|^{2s}}, \tag{3.12}$$

with $s = 3/2$. The verification that Eq. (3.12) is invariant under the modular transformations (3.9) is now straightforward.

Let us compare Eq. (3.10) with (3.12). The axion independent part of Eq. (3.12) corresponds precisely to the perturbative terms we have found from scattering amplitudes. These are called "constant terms". The additional new part is argued to contain non-perturbative contributions coming from the D(-1)-instantons. In type IIB string theory the D(-1) brane is the electric source for the R-R scalar C_0, since it is localized in both space and time it qualifies as an instanton. To see that the infinite sum in Eq. (3.10) is an instanton effect we can expand the Bessel function when its argument approaches infinity

$$K_1(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left[1 + \mathcal{O}(1/x) + \ldots\right]. \tag{3.13}$$

In the weak coupling limit the Eisenstein series then takes the form

$$\begin{aligned}\mathcal{E}^{(0,0)}(\tau,\bar{\tau}) =& 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} \\ &+ 2\pi \sum_{N\neq 0} \mu_{-2}(N)\sqrt{N}e^{-S_{\text{inst}}(\tau)}\left[1+\mathcal{O}\left(\frac{1}{|N|\tau_2}\right)+\ldots\right],\end{aligned} \tag{3.14}$$

where

$$S_{\text{inst}}(\tau) = 2\pi|N|e^{-\phi} - 2\pi i N C_0. \tag{3.15}$$

The real part of Eq. (3.15) indeed coincides with the Euclidean D(-1) instanton action found in Ref. [141]. The fact that there is also an imaginary part is similar to what happens in the case of Yang-Mills instantons when the theta angle is non-zero. Physically the instanton contributions come from one single instanton carrying N units of charge, since several separate instantons would contribute to corrections with higher α' orders. The degeneracy of an instanton with charge N is given by the instanton measure $\mu_{-2}(N)$. Moreover, the sub-leading terms in the Bessel function expansion describe the perturbative excitations around the instanton background. Notice that the instanton action in Eq. (3.15) is invariant under discrete displacements of the axion field C_0, indicating that after adding the instanton effects only a discrete symmetry group remains.

The fact that the string effective action must contain more than just perturbative contributions was realized long before any explicit U-duality completion was deviced. To see this let us fix the α' level in the double expansion (3.5) and consider the large genus limit. In Ref. [142] it was argued that in this limit, due to the growth of the torus volume the perturbative genus expansion of the action behaves as

$$\lim_{g\to\infty} g_s^{2g} L_{(n,g)} \sim g_s^{2g} a^{-2g}(2g)!, \tag{3.16}$$

with some constant a. Summing the right hand side of Eq. (3.16) over all possible genus is divergent. However, this sum can be regularized by adding terms that are suppressed by e^{-1/g_s}. Soon after the discovery of D-brane instantons, which have tensions scaling as $T \sim e^{-1/g_s}$, one realized that this provided a physical explanation for this regularization. Turning back to the type IIB theory, the additional contributions indeed come from instantons. Also, the non-holomorphic Eisenstein series $\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau,\bar{\tau})$ converges only when $\Re(s) > 1$, and the series in Eq. (3.10) does satisfy the convergence condition.

Having accepted the role of the instantons, we can look at the brane content of type IIB string theory. The only possible candidate for instanton there is the D(-1)-brane. Using T-duality in nine Euclidean dimensions it can be mapped to a type IIA D0-particle wrapping m times around the compact Euclidean time direction. The action for a threshold bound state of n such D0-particles can be found independently, after T-dualization it becomes [60]

$$S^{(m,n)} = 2\pi|mn|e^{-\phi} - 2\pi i mn C_0. \tag{3.17}$$

Comparison with Eq. (3.15) shows that these two actions are identical provided that we set $N = mn$. This indicates again that the instanton contributions in Eq. (3.10) are correct.

Based on the above arguments, the authors of Ref. [60] made the conjecture that the exact $\mathcal{R}^4$ correction in type IIB string theory is

$$S^{(3)^*}_{\mathrm{IIB,E}} = \alpha'^{-4} \int d^{10}x\sqrt{-g}\alpha'^3 \mathcal{E}^{(0,0)}(\tau,\bar{\tau})\mathcal{R}^4, \tag{3.18}$$

where $\mathcal{E}^{(0,0)}(\tau,\bar{\tau}) = \mathcal{E}^{SL(2,\mathbb{Z})}_{3/2}(\tau,\bar{\tau})$. The most interesting prediction resulting from this conjecture is that there are no perturbative contributions beyond one-loop. The implications of such perturbative non-renormalization theorems were first discussed in Ref. [138], and later a proof was given in Ref. [143]. Explicit two-loop computations have verified that there is indeed no two-loop contribution.

Inspired by how well the U-duality completion worked for the $\mathcal{R}^4$ correction, other higher order derivative corrections in IIB string theory were investigated using *automorphic lifts*. The general procedure is that one starts with some perturbative contributions to a particular correction, and then one obtains a U-duality invariant action by replacing the genus dependent coefficients with appropriate $SL(2,\mathbb{Z})$ Eisenstein series. Combining compactified M-theory scattering amplitudes with automorphic lifts, structures of $\mathcal{D}^{2n}\mathcal{R}^4$ corrections ($n \geq 0$) have been studied [144, 145, 146, 147, 148]. Furthermore, it was argued that $\mathcal{D}^{2n}\mathcal{R}^4$ does not receive perturbative corrections above n string loops [123, 145], proof for the cases $0 < n < 6$ was given in Ref. [123]. As a side note, recently non-renormalization theorems in type IIB string theory have attracted attention as a gateway to investigating the ultraviolet finiteness of perturbative $D = 4$ $\mathcal{N} = 8$ maximal supergravity amplitudes.

3.3 Supersymmetry

Similar to the situation in supergravity, the complete low energy effective action can, at least in theory, be obtained by supersymmetric completion of the computed scattering amplitudes. A systematic analysis of the full supersymmetry is however easier said than done [149]. As the number of correction terms grows in the action, the supersymmetry transformations also need to be deformed

$$\delta = \delta_0 + \sum_{n=0}^{\infty} \alpha'^n \delta_n, \tag{3.19}$$

where we have suppressed the expansion in g_s. The invariance of the action then requires

$$\left(\delta_0 + \sum_{n=0}^{\infty} \alpha'^n \delta_n\right) \left(S_0 + \sum_{m=0}^{\infty} \alpha'^m S_m\right) = 0 \tag{3.20}$$

to be fulfilled order by order in α'. Finding the full non-linear supersymmetry transformations is thus as difficult as finding the action itself, not to speak about using it to find the action. Nevertheless, we shall not loose our faith in supersymmetry entirely, since in fact the linearized version of it can be quite useful.

The n-particle scattering amplitude in the pure spinor formalism contains automatically all n-field terms related by linearized supersymmetry δ_0. The tree-level four-particle scattering computation in Ref. [124] shows an example of how it works. But we can do better and go beyond the n-field truncation.

In the weak coupling limit, by considering a superfield which describes linearized fluctuations, one can relate terms with different numbers of fields at the same α' level. The grand example is again the α'^3 corrections in the type IIB action [150, 151, 152]. In this case the linear superfield, describing fluctuations Δ around a flat background with a constant dilaton axion τ, is

$$\Phi = \tau + \Delta(\theta, g_{\mu\nu}, \psi_\mu, \lambda, \dots), \tag{3.21}$$

where θ^A is the Grassmann variable, $g_{\mu\nu}$ is the metric, ψ_μ^A is the gravitino, λ^A is the dilatino and the ellipsis denotes the rest of the physical fields. The superfield shall satisfy the holomorphic constraint

$$\bar{D}\Phi = 0 \tag{3.22}$$

and the on-shell condition

$$D^4\Phi = \bar{D}^4\bar{\Phi}, \tag{3.23}$$

with D being the linear fermionic derivative in the superspace

$$D_A = \frac{\partial}{\partial\theta_A} + 2i(\Gamma^\mu\bar{\theta})_A\partial_\mu, \quad \bar{D}_A = -\frac{\partial}{\partial\bar{\theta}^A}. \tag{3.24}$$

Consider now the following linearized supersymmetric on-shell action

$$\tilde{S} = \Re\left(\alpha'^{-4}\int d^{10}x d^{16}\theta\sqrt{-g}\alpha'^3 e^{4\phi}F[\Phi]\right), \tag{3.25}$$

where the chiral superfield $F[\Phi]$ is some arbitrary function of Φ. By integrating out the Grassmann variables in Eq. (3.25) and then identifying the coefficient in front of the $\mathcal{R}^4$ term with the Eisenstein series $\mathcal{E}^{(0,0)}$, we finally arrive at the expression

$$\begin{aligned} S^{(3)}_{\text{IIB,E}} =& \alpha'^{-4} \int d^{10}x\sqrt{-g}\alpha'^3 \left\{ \mathcal{E}^{(12,-12)}(\tau,\bar{\tau})\lambda^{16} + \mathcal{E}^{(11,-11)}(\tau,\bar{\tau})\bar{\psi}\lambda^{15} \right. \\ & \left. + \cdots + \mathcal{E}^{(0,0)}(\tau,\bar{\tau})\mathcal{R}^4 + \ldots \right\}. \end{aligned} \tag{3.26}$$

The function $\mathcal{E}^{(0,0)}$ is the non-holomorphic Eisenstein series we just described in Section 3.2. The fermionic integrals in Eq. (3.25) require the other coefficients $\mathcal{E}^{(k,-k)}$ to satisfy

$$\mathcal{E}^{(k,-k)} \propto e^{-k\phi} \frac{\partial^k}{\partial \tau^k} \mathcal{E}^{(0,0)}. \tag{3.27}$$

We should emphasize that linearized supersymmetry works in the limit $g_s \to 0$, and therefore only guarantees that Eq. (3.27) holds for the leading instanton terms. By invoking U-duality, relation (3.27) can be extended to the entire Eisenstein series. Supersymmetry and U-duality really work hand in hand. The functions $\mathcal{E}^{(k,-k)}$ are called *generalized non-holomorphic Eisenstein series* in mathematics, constructions of these will be discussed in Section 4.3. For now we just need to know that under modular transformations a function of this kind receives an $SO(2)$ phase, which precisely compensates for the corresponding phase coming from the field combination it is multiplied with. The entire action (3.26) is thus invariant under the modular group. Notice that all the terms in Eq. (3.26) are by construction at the α'^3 level, to verify its correctness we will have to compute all the n-point scattering amplitudes up to sixteen particles.

3.4 Beyond Type IIB String Theory

So far in this chapter we have mostly concentrated on higher order derivative corrections in type IIB superstring theory. We have learned that requiring invariance under U-duality leads to Eisenstein series based on the modular group $SL(2,\mathbb{Z})$. However, automorphic lifts can be useful in more situations, the most direct generalization is toroidal compactification of M-theory to dimensions lower than ten [54, 153, 53, 154, 155, 156, 157, 87]. In Section 1.4 we explained how the U-duality group $E_{n(n)}(\mathbb{Z})$ grows when compactifying on n-torii with larger and larger dimensions. Similar to the IIB case, the low energy effective action in these cases also exhibit a double expansion in Regge slope and string coupling. Part of the correction terms can be found by dimensionally reducing the corresponding terms in ten dimensions, in the same way as how the one-loop $\mathcal{R}^4$ term in IIB is related to the one-loop four-graviton term in M-theory. Due to the compactification procedure, the curvature tensor itself might receive quantum corrections. Also, we may find some contributions by studying the supersymmetry properties. The exact effective action can then be obtained by requiring invariance under $E_{n(n)}(\mathbb{Z})$ order by order in α'. It is the Eisenstein series based on $E_{n(n)}(\mathbb{Z})$ which encodes the information on the expansion in g_s. Most often the Eisenstein series is also transforming as some representation of the maximal compact subgroup of $E_{n(n)}(\mathbb{R})$, which is the generalized form of the $\mathcal{E}^{(k,-k)}$ function.

Higher order derivative corrections have been discussed in heterotic string theory as well, where α' corrections occur already at the four-derivative $\mathcal{R}^2$ level. The S-duality between heterotic string on T^6 and type IIA on K3$\times T^2$ turns out to be a handy tool, see for instance Refs. [158, 159, 57]. Possibilities for automorphic lifts of the dimensionally reduced $\mathcal{R}^2$, $\mathcal{R}^3$ and $\mathcal{R}^4$ corrections are discussed in Paper III. The toroidal reduction of the $\mathcal{R}^2$ term has been carried out more carefully in Paper IV, which also contains a symmetry analysis of this term.

In fact we can discuss subsectors of M-theory in general. If the classical moduli space of such a theory describes a coset space $G/\mathcal{K}(G)$, with $\mathcal{K}(G)$ being the maximal compact subgroup of G, the quantum corrections of the moduli space should be written in terms of Eisenstein series based on $G(\mathbb{Z})$. One example that has been explicitly worked out is the Einstein-Liouville gravity compactified from four to three spacetime dimensions [160]. Another example is the scalar sector of type IIA string theory compactified on a rigid Calabi-Yau threefold, where the moduli space receives quantum corrections already at two-derivative level. It is argued in Paper V that the corrections are given by an invariant non-holomorphic Eisenstein series based on the Picard modular group $SU(2,1;\mathbb{Z}[i])$.

Having motivated the importance of the Eisenstein series or automorphic forms in string theory, Chapter 4 will be dedicated to discussions of these from a mathematical point of view. The heuristic structure is the same as the $SL(2,\mathbb{Z})$ Eisenstein series, with perturbative and non-perturbative contributions. However, for a larger symmetry group more types of instantons will contribute and thus makes the story far more complicated.

4

Automorphic Forms

Believing U-duality to be a true symmetry of M-theory leads naturally to mathematical functions called automorphic forms. In the low energy effective theory, all the quantum corrections are encoded into these functions. Studying the automorphic forms will thus provide us a powerful tool towards understanding the quantum nature of string theory.

The rich configuration of the automorphic forms has made this area of mathematics a meeting ground for complex analysis, number theory and algebraic geometry. It is of course a mission signed impossible trying to expose the full glory of such an active research field in a few pages. The intention here is to give the basic ideas and highlight a few difficulties. Most of our understanding of the automorphic forms comes from a special case called modular forms, going through examples of these carefully will yield indispensable intuition for the general cases.

4.1 Modular Forms

Starting with the complex upper half-plane $H = \{\tau \in \mathbb{C} \,|\, \Im(\tau) > 0\}$, modular forms are analytic functions on this space endowed with certain transformation properties under the discrete group $SL(2, \mathbb{Z})/\{\pm 1\}$. The conventions here will mainly follow Refs. [161] and [162], which contains a more systematic introduction to the subject for the interested reader.

4.1.1 The Modular Group

On the complex upper half-plane H there exists a natural conformal transformation

$$\tau \longmapsto \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}, \tag{4.1}$$

with real parameters $\{a, b, c, d\}$ satisfying $ad - bc = 1$. Letting the parameters constitute a 2×2 matrix, it is nothing but an element of the real form $SL(2, \mathbb{R})$:

$$SL(2,\mathbb{R}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1; a, b, c, d \in \mathbb{R} \right\}. \tag{4.2}$$

Since the element $-1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ leaves H invariant, only the subgroup $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm 1\}$ acts faithfully. Choosing the parameters integer valued, one defines the *modular group* as $SL(2,\mathbb{Z})/\{\pm 1\}$. From now on the notation $SL(2,\mathbb{Z})$ will be used when referring to the modular group, the same will also apply to the more general cases later on. The generators of the Lie algebra A_1 are taken to be

$$E \doteq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad H \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad F \doteq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.3}$$

It is now possible to show that the entire modular group can be generated by two elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{4.4}$$

obeying

$$S^2 = 1 \quad \text{and} \quad (ST)^3 = 1. \tag{4.5}$$

In terms of transformations on the upper half-plane they translate into

$$S \cdot \tau = -\frac{1}{\tau} \quad \text{and} \quad T \cdot \tau = \tau + 1. \tag{4.6}$$

Defining

$$\tau = \chi + ie^{-\phi}, \tag{4.7}$$

where ϕ is the dilaton and χ is the axion. The transformation S corresponds precisely to the S-duality in type IIB string theory. Using the modular group all points outside of $D = \{\tau \in H \big| |\tau| \geq 1; |\Re(\tau)| \leq \frac{1}{2}\}$ can be mapped into D. The region D is called the *fundamental domain*, and contains three singular points: i, ∞ and $e^{i\pi/3}$.

The transformation in Eq. (4.1) provides an identification of H with the coset space $SL(2,\mathbb{R})/SO(2)$, where the $SO(2)$ appearing in the denominator is the maximal compact subgroup of $SL(2,\mathbb{R})$. The most straightforward way to see this is by computing the metric of H

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \tag{4.8}$$

using the following Kähler potential

$$K(\tau) = -4 \ln \frac{\tau - \bar{\tau}}{2i}. \tag{4.9}$$

This leads to

$$ds^2 = g_{\tau\bar{\tau}} d\tau d\bar{\tau} = d\phi^2 + e^{2\phi} d\chi^2, \tag{4.10}$$

which can be written in terms of the $SL(2,\mathbb{R})/SO(2)$ coset representative

$$\mathcal{V} = e^{\frac{\phi}{2}H} e^{\chi E} \doteq \begin{pmatrix} e^{\frac{\phi}{2}} & e^{\frac{\phi}{2}}\chi \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix} \tag{4.11}$$

following the construction described in Subsec. 2.1.3.

Having the metric we can also compute the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{i\bar{j}} \partial_{\bar{j}}) = \Im(\tau)^2 \partial_\tau \partial_{\bar{\tau}}. \tag{4.12}$$

The kernel of the Laplacian can be easily found to be

$$\mathcal{F}(\tau) = \Im(\tau), \tag{4.13}$$

which will be referred to as the *height function* in the following. The fact that restricting this function to be strictly positive coincides with the defining equation of the half-plane H is not just a coincidence, in other words, the Kähler potential given in Eq. (4.9) is designed to yield the correct height function. We can now write $K(\tau) = -4 \ln \mathcal{F}(\tau)$. Under the $SL(2,\mathbb{R})$ group action the height function transforms as

$$\mathcal{F}(\gamma \cdot \tau) = \frac{\mathcal{F}(\tau)}{|c\tau + d|^2}, \tag{4.14}$$

showing that H is stable under the $SL(2,\mathbb{R})$ action.

4.1.2 Definition of Modular Forms

On the complex half-plane H we have just described, any meromorphic function satisfying the relation

$$f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{4.15}$$

is called a *weakly modular form* of weight k. The factor of automorphy

$$\mu(\gamma, \tau) = c\tau + d \tag{4.16}$$

can be obtained by differentiation

$$\frac{d(\gamma \cdot \tau)}{d\tau} = \frac{1}{\mu(\gamma, \tau)^2}, \tag{4.17}$$

and satisfies the cocycle identity

$$\mu(\gamma\gamma', \tau) = \mu(\gamma, \gamma' \cdot \tau)\mu(\gamma', \tau). \tag{4.18}$$

The relation (4.15) can then be written as

$$f(\gamma \cdot \tau)(d(\gamma \cdot \tau))^{\frac{k}{2}} = f(\tau)(d\tau)^{\frac{k}{2}}, \tag{4.19}$$

i.e., the "differential form" $f(\tau)(d\tau)^{\frac{k}{2}}$ is invariant under the modular group.

The fact that a weakly modular form is invariant under the element T

$$f(T \cdot \tau) = f(\tau + 1) = f(\tau) \tag{4.20}$$

indicates that it can be expanded as a Laurent expansion

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i \tau n}. \tag{4.21}$$

If $f(\tau)$ extends to a meromorphic function at $\tau = i\infty$, then for $n < m$ the coefficients a_n are all vanishing. The number m is the order of the pole of $f(\tau)$ at infinity. If $m \geq 0$ the function $f(\tau)$ is holomorphic. A weakly modular form is called modular if it is meromorphic at infinity.

Just to summarize, a *modular form* is an analytic function on the complex upper half-plane, which transforms according to Eq. (4.15) under the modular group and is meromorphic everywhere (including infinity). When the weight k is zero, it is then called a *modular function*. Moreover, if a modular form is holomorphic everywhere and vanishes at infinity, it is called a *cusp form*. Another way to understand the modular forms is to view them as the invariants of elliptic curves. To each $\tau \in H$ it is possible to assign an elliptic curve $C(\tau)$, in other words the quotient of $\mathbb{C}$ by the lattice $\mathbb{Z} + \mathbb{Z}\tau$, and the elliptic curves corresponding to τ and $\gamma \cdot \tau$ are defined to be isomorphic.

So far we have only described the modular forms based on the modular group $SL(2, \mathbb{Z})$, in general modular forms can be constructed for any discrete subgroup Γ of $SL(2, \mathbb{R})$, as long as $\Gamma \backslash H$ is compact or the complement of finitely many points on a compact Riemann surface. For illustrative purpose we will continue to use the modular group as the symmetry group for the rest of this section.

Examples

It is now time to give some examples of modular forms.

i. The most simple example that can be constructed is the *holomorphic Eisenstein series* of weight $2k$:

$$G_{2k}(\tau) = \sum_{m,n}{}' \frac{1}{(m+n\tau)^{2k}}, \tag{4.22}$$

which satisfies $\partial_{\bar{\tau}} G_{2k}(\tau) = 0$. This series is convergent only for integers $k \geq 2$, and it is holomorphic everywhere. The Fourier expansion of the holomorphic Eisenstein series can be written as

$$G_{2k}(\tau) = 2\zeta(2k)\left(1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1}e^{2\pi i\tau n}}{1-e^{2\pi i\tau n}}\right), \tag{4.23}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

ii. The second example is the *Jacobi theta function*

$$\theta(\tau) = \sum_{m\in\mathbb{Z}} e^{\pi i m^2\tau}. \tag{4.24}$$

This function is a modular form of weight 1/2 for the congruence subgroup

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| c \equiv 0 \mod 2 \right\} \tag{4.25}$$

of the modular group.

iii. The Jacobi theta function can be related to the *Dedekind eta function*

$$\eta(\tau) = e^{\frac{2\pi i\tau}{24}} \prod_{n=1}^{\infty}(1-e^{2\pi i\tau n}) \tag{4.26}$$

via

$$\theta(\tau) = \frac{\eta(\frac{\tau+1}{2})^2}{\eta(\tau+1)}. \tag{4.27}$$

The combination[1]

$$\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24} \tag{4.28}$$

turns out to be a cusp form of weight 12.

[1] This function is called the modular discriminant of the Weierstrass elliptic function, and the number 24 appearing in the definition can be related to the Leech lattice.

iv. The last example given here is the *non-holomorphic Eisenstein series*

$$\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2}{}' \left(\frac{\Im(\tau)}{|m+n\tau|^2}\right)^s \tag{4.29}$$

of *order* s, which is invariant under modular transformations. For convergence the complex order must satisfy $\Re(s) > 1$, but it can be analytically continued to the whole complex plane. The Eisenstein series will then be meromorphic with a unique pole at $s = 1$ (for all τ in the upper half-plane). The non-holomorphic Eisenstein series is itself an eigenfunction of the Laplace-Beltrami operator defined in Eq. (4.12) with eigenvalues $s(s-1)$

$$\Delta\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau) = s(s-1)\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau). \tag{4.30}$$

A function which is invariant under modular transformations and at the same time is an eigenfunction of the Laplacian is called a *Maass wave form*.

The celebrated complete, non-perturbative, four-graviton scattering amplitude at low energies is encoded in a non-holomorphic Eisenstein series with order $s = 3/2$, see Ref. [60], and understanding this kind of modular forms is thus of great interest for theoretical physicists. For the rest of this chapter we will use this series as the main example.

4.2 Towards Automorphic Forms

Having worked through the modular forms, we are now ready to introduce the automorphic forms. For a deeper treatment of the subject see for instance Ref. [163].

4.2.1 Definition of Automorphic Forms

Generalizing the modular group to a discrete subgroup Γ of an arbitrary Lie group G, automorphic forms are the generalizations of the modular forms, living on the coset space $G/\mathcal{K}(G)$. As usual, the notation $\mathcal{K}(G)$ refers to the maximal compact subgroup of G. Thus, a function f on $\mathcal{Z} \equiv G/\mathcal{K}(G)$ is defined as an *automorphic form* if it

i. transforms under the discrete group $G(\mathbb{Z}) \subset G$ according to a given factor of automorphy μ

$$f(\gamma \cdot x) = \mu_\gamma(x) f(x), \quad \gamma \in G(\mathbb{Z}) \text{ and } x \in \mathcal{Z}; \tag{4.31}$$

ii. satisfies certain meromorphic conditions at infinity; and

iii. is an eigenfunction of all the Casimir operators of G.

The *moduli space*, i.e., the space of parameters of the automorphic form is given by the double coset

$$\mathcal{M}_G = G(\mathbb{Z})\backslash G/\mathcal{K}(G). \tag{4.32}$$

When an automorphic form is invariant under $G(\mathbb{Z})$ transformations we call it an *automorphic function*. The modular form is the special case when $G = SL(2,\mathbb{R})$.

4.2.2 Constructing Automorphic Forms

Having defined what an automorphic form is, it is time to explicitly construct one. Since the non-holomorphic Eisenstein series $\mathcal{E}_s^{G(\mathbb{Z})}(\mathcal{Z})$, based on the *principal series representations* of G, are of our main interest, the different constructions described here will be specialized for them. However, the construction based on spherical vectors can be used generally for automorphic forms.

Lattice Construction

The fact that the parameter space of an automorphic form is a *symmetric space*, makes it possible to construct the non-holomorphic Eisenstein series using a representative of the coset. Let $\vec{\omega}$ be a vector of the lattice which is left invariant by $G(\mathbb{Z})$, and let $\mathcal{V}$ be a representative of the coset $G/\mathcal{K}(G)$ transforming according to

$$\mathcal{V} \longmapsto k\mathcal{V}g, \quad g \in G \;\text{ and }\; k \in \mathcal{K}(G). \tag{4.33}$$

This leads to

$$\mathcal{V}\vec{\omega} \longmapsto k\mathcal{V}\vec{\omega}' \tag{4.34}$$

under a discrete $G(\mathbb{Z})$ transformation. We can thus form the non-holomorphic Eisenstein series as

$$\mathcal{E}_s^{G(\mathbb{Z})}(\mathcal{Z}) = \sum_{\vec{\omega}} \delta(\vec{\omega}^\dagger \wedge \vec{\omega}) \left[(\mathcal{V}\vec{\omega})^\dagger \cdot \mathcal{V}\vec{\omega}\right]^{-s}, \tag{4.35}$$

which is by construction invariant under $G(\mathbb{Z})$. Notice that in order to ensure invariance $k^\dagger k = 1$. This function is non-holomorphic due to the Hermitian conjugate appearing. The factor $\delta(\vec{\omega}^\dagger \wedge \vec{\omega})$ is a constraint to ensure that the sum runs over the actual lattice which is invariant under $G(\mathbb{Z})$, in case $\vec{\omega}$ are defined on a larger lattice with simpler structure. When there is no issue of embedding the lattice into a larger but simpler one, then the constraint is simply empty.

For the modular group a representative of $SL(2,\mathbb{R})/SO(2)$ was given in Eq. (4.11). Setting $\vec{\omega} = (m,n) \in \mathbb{Z}^2$ one finds

$$\mathcal{E}_s^{SL(2,\mathbb{Z})}(\phi,\chi) = \sum_{(m,n)\in\mathbb{Z}^2}{}' \left[e^{\phi}\{(m+n\chi)^2 + n^2 e^{-2\phi}\}\right]^{-s}, \tag{4.36}$$

which is precisely the Eisenstein series introduced in Eq. (4.29). In this case $\vec{\omega}$ can be viewed as the vector representation of $SL(2,\mathbb{R})$ restricted to integer values, thus no constraint needs to be posed on the lattice.

In Paper V another example is given, which deals with the group $SU(2,1)$. There, the lattice $SU(2,1;\mathbb{Z}[i])$ was embedded inside $SL(3;\mathbb{Z}[i])$ to make it more manageable, a constraint on the lattice vector is the price one has to pay.

Poincaré Series

The lattice construction described above is entirely based on the group theoretical properties of the moduli space, however, we began this chapter by introducing modular forms based on their complex analytic properties. The fractional transformation in Eq. (4.1) is the bridge between these two points of view. More generally, all the Riemannian symmetric spaces $G/\mathcal{K}(G)$ have been classified by E. Cartan almost a century ago, the complete list can be found for instance in Ref. [164]. In those cases where a geometric interpretation as a complex manifold exists for the symmetric space, there is a nice way to construct non-holomorphic Eisenstein series in terms of *Poincaré series.*

The key here is the kernel of the Laplace-Beltrami operator, which we named the height function $\mathcal{F}(\mathcal{Z})$. This very function is also used to define our complex manifold. Moreover, it is invariant under the nilpotent upper triangular subgroup $N(\mathbb{Z})$, which is generated by the positive step operators in the Lie algebra of G. A Poincaré series for the group $G(\mathbb{Z})$ can then be constructed as

$$\mathcal{P}_s^G(\mathcal{Z}) = \sum_{\gamma\in N(\mathbb{Z})\backslash G(\mathbb{Z})} \mathcal{F}(\gamma\cdot\mathcal{Z})^s, \tag{4.37}$$

where the sum goes over the orbit $\gamma \in N(\mathbb{Z})\backslash G(\mathbb{Z})$. Once the transformation rules of the moduli space variables under the group G have been established, the generators in the Lie algebra of G can then be identified with derivative operations on the corresponding function space. The quadratic Casimir operator will simply coincide with the Laplacian.

Turning to our favorite example $SL(2,\mathbb{R})$, the transformation of the height function there was according to Eq. (4.14)

$$\mathcal{F}(\gamma \cdot \tau) = \frac{\mathcal{F}(\tau)}{|c\tau + d|^2}, \tag{4.38}$$

where $\mathcal{F}(\tau) = \Im(\tau)$. The nilpotent upper triangular subgroup in this case is generated by the element T, which correctly leaves $\mathcal{F}(\tau)$ invariant. Setting $c = n'$ and $d = m'$, the Poincaré series then takes the form

$$\mathcal{P}_s^{SL(2,\mathbb{Z})}(\tau) = \sum_{\substack{(m',n')\in\mathbb{Z}^2 \\ (m',n')=1}}{}' \left(\frac{\Im(\tau)}{|m' + n'\tau|^2} \right)^s. \tag{4.39}$$

Defining $(m,n) = \beta(m',n')$ with $\beta = \gcd(m,n) \in \mathbb{Z}$, we obtain at last

$$\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau) = 2\zeta(2s)\mathcal{P}_s^{SL(2,\mathbb{Z})}(\tau), \tag{4.40}$$

where $\zeta(2s)$ is the Riemann zeta function

$$\zeta(2s) = \sum_{\beta=1}^{\infty} \frac{1}{\beta^{2s}}. \tag{4.41}$$

The fractional transformation in Eq. (4.1) together with Eq. (4.3) lead to the following identifications

$$E \doteq \partial_\chi; \quad F \doteq (y^2 - \chi^2)\partial_\chi - 2\chi y \partial_y; \quad H \doteq 2(\chi\partial_\chi + y\partial_y) \tag{4.42}$$

for the Lie algebra generators of $SL(2,\mathbb{R})$, with $y \equiv e^{-\phi}$. It is then apparent that the quadratic Casimir operator

$$C = \frac{1}{2}H^2 + EF + FE \doteq 2y^2(\partial_\chi^2 + \partial_y^2) \tag{4.43}$$

indeed is the same as the Laplacian

$$C \doteq 2\Delta \tag{4.44}$$

given in Eq. (4.12). Moreover, the non-holomorphic Eisenstein series can be verified explicitly to be an eigenfunction of the Casimir operator.

The Eisenstein series are constructed using the principal continuous series of G. Depending on properties of the representation sometimes more than one Casimir operator have to be taken into consideration. In the case of $SL(2,\mathbb{R})$ only the quadratic Casimir operator needs to be computed. However, already for $SL(3,\mathbb{R})$ the eigenvalue equations imposed by the quadratic and the cubic Casimirs have to be satisfied independently, thus leading to two order numbers, see Ref. [165].

Spherical Vector

The last method to be described is based on the mathematical objects called *spherical vectors*. This way of constructing automorphic forms is very general and has been brought to attention in the physics community in a series of papers [166, 167, 157, 168, 169, 170, 165].

The basic idea is that on a symmetric space $\mathcal{Z} = G/\mathcal{K}(G)$, any automorphic function can be cast into the following form

$$\Psi(g) = < f_{\mathbb{Z}}, \rho(g^{-1}) \cdot f_{\mathcal{K}} >, \quad g \in G. \tag{4.45}$$

The three constituents of the above formula are:

i. a linear representation ρ of G acting on a Hilbert space $\mathcal{H}$ of square integrable functions;

ii. a $\mathcal{K}(G)$-invariant function $f_{\mathcal{K}} \in \mathcal{H}$ called the spherical vector; and

iii. a $G(\mathbb{Z})$-invariant distribution $f_{\mathbb{Z}} \in \mathcal{H}^{\star}$ living in the dual space of $\mathcal{H}$.

The inner product $<,>$ is the natural pairing between $\mathcal{H}$ and $\mathcal{H}^{\star}$.

The Iwasawa decomposition

$$G = KAN \tag{4.46}$$

of G makes it possible to write any group element as $g = k\mathcal{V}$, with $\mathcal{V}$ being a representative of $G/\mathcal{K}(G)$. Since the spherical vector is invariant under $\mathcal{K}(G)$, Eq. (4.45) becomes

$$\Psi(\mathcal{V}) = < f_{\mathbb{Z}}, \rho(\mathcal{V}^{-1}) \cdot f_{\mathcal{K}} > . \tag{4.47}$$

Using the transformation rule in Eq. (4.33) together with the defining properties of $f_{\mathcal{K}}$ and $f_{\mathbb{Z}}$, it is now manifest that $\Psi(\mathcal{V})$ is invariant under $G(\mathbb{Z})$. However, the real challenge boils down to explicitly constructing $f_{\mathcal{K}}$ and $f_{\mathbb{Z}}$. Once the real spherical vector $f_{\mathcal{K}}$ has been found, using *p-adic number theory* the distribution $f_{\mathbb{Z}}$ can be written in terms of a p-adic counterpart of $f_{\mathcal{K}}$. The automorphic function then takes the form

$$\Psi(\mathcal{V}) = \sum_{x \in \mathbb{Q}^n} \Big[\prod_{p < \infty} f_p(x) \Big] \rho(\mathcal{V}^{-1}) \cdot f_{\mathcal{K}}(x), \tag{4.48}$$

where x is a vector of rational numbers in $\mathbb{Q}^n$, and p denotes all the prime numbers. A very brief summery of the p-adic numbers is given in Appendix A, containing only the necessary properties needed for our purpose. For a proper introduction to the field $\mathbb{Q}_p$ of p-adic numbers, Refs. [171] and [172] are nice starting points.

So far this construction has been quite formal, it will become more clear once we apply it to the $SL(2,\mathbb{R})$ example. For the *principal continuous series* of a group G there is a standard way to construct the real spherical vector $f_{\mathcal{K}}$, by viewing it as a square-integrable function on the coset space $P\backslash G$. P here denotes the parabolic subgroup of G, and is generated by the negative step operators together with the Cartan generators. The isomorphism between $P\backslash G$ and the nilpotent subgroup N makes it possible to choose

$$n = e^{xE} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \in N \tag{4.49}$$

as a representative also for $P\backslash G$. Under the group action of G a compensating P action from left is needed to ensure the upper triangular form of n

$$n \longmapsto png, \qquad g \in G \quad \text{and} \quad p \in P. \tag{4.50}$$

This transformation law naturally induces a linear representation $\rho(g)$ acting on any function of the parameter x. The spherical vector will be a function satisfying

$$\rho(g) \cdot f_{\mathcal{K}}(x) = \chi_s(p) f_{\mathcal{K}}(x'), \qquad g \in G \quad \text{and} \quad p \in P, \tag{4.51}$$

where $\chi_s(p)$ is an infinitesimal character chosen so that

$$\rho(k) \cdot f_{\mathcal{K}}(x) = f_{\mathcal{K}}(x), \qquad k \in \mathcal{K}(G). \tag{4.52}$$

Explicitly computing the right hand side of Eq. (4.50) it is easy to see that the spherical vector we seek is uniquely given by the Euclidean norm of the first row in n

$$f_{\mathcal{K}}(x) = (1 + x^2)^{-s}. \tag{4.53}$$

Remembering $\mathcal{V}$ from Eq. (4.11) we then obtain

$$\rho(\mathcal{V}^{-1}) \cdot f_{\mathcal{K}}(x) = \left(\frac{\Im(\tau)}{|-x+\tau|^2} \right)^s . \tag{4.54}$$

The next step is to find the $SL(2,\mathbb{Z})$-invariant distribution. The general recipe is to first replace the Euclidean norm in $f_{\mathcal{K}}$ with the p-adic one

$$f_p(x) = |1, x|_p^{-2s} = \max(1, |x|_p)^{-2s}. \tag{4.55}$$

Letting $x = \frac{m'}{n'}$ with m' and n' being co-prime integers, one can then show that

$$f_{\mathbb{Z}} = \prod_{p<\infty} f_p(x) = \prod_{p<\infty} \max\left(1, \left|\frac{m'}{n'}\right|_p\right)^{-2s} = n'^{-2s}. \tag{4.56}$$

Details concerning the p-adic norm can be found in Appendix A. Putting all the puzzles together and renaming $m' \to -m'$, we find at last

$$\Psi_s^{SL(2,\mathbb{Z})}(\tau) = \sum_{\substack{(m',n')=1 \\ n'\neq 0}} \left(\frac{\Im(\tau)}{|m' + n'\tau|^2} \right)^s . \tag{4.57}$$

This is almost the same function as we found using Poincaré series, but with one caveat. In order to make the redefinition $x = \frac{m'}{n'}$ well-defined, the number n' is not allowed to have zero value. The result is one missing constant term in the Fourier expansion, which corresponds to the value of the automorphic function at the cusp $\tau \to i\infty$. The problem may be cured by extending the parameter space to a reducible one with

$$f_{\mathcal{K}}(x,y) = (x^2 + y^2)^{-s} \tag{4.58}$$

and

$$f_p(x,y) = \max(|x|_p, |y|_p)^{-2s}. \tag{4.59}$$

The group action works then according to

$$\rho(g) \cdot f(x) = f(ax + by, cx + dy), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{4.60}$$

We have used the Eisenstein series to illustrate the spherical vector construction, but it is a much more general and powerful technique. By choosing different unitary representations of G, different automorphic forms for the same group can be constructed. The *minimal* representations, associated to the nilpotent orbits of smallest dimension, of the exceptional Lie groups are of special interest in M-theory, as their corresponding automorphic forms are believed to govern the partition function of supermembrane zero-modes [156].

4.2.3 Fourier Expansion

Having constructed the automorphic forms, they are however not cast in the form that naturally appears in physics. As explained in Section 3.2 the automorphic forms found in string theory are expressed as a *Fourier expansion.* The reason behind the existence of a Fourier expansion for the modular forms is its invariance under the nilpotent upper triangular subgroup, in fact one example has already been given when we introduced the holomorphic Eisenstein series. In the general case, axionic type of symmetries in G will guarantee the Fourier expansion. In this section we will describe the standard procedure in obtaining this expansion for the non-holomorphic Eisenstein series.

The Modular Function Revisited

Before jumping onto the most general Eisenstein series, we will again illustrate the main concepts and ideas using the modular function

$$\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2}{}^{'} \frac{\tau_2^s}{|m+n\tau|^{2s}}, \tag{4.61}$$

with $\tau = \chi + ie^{-\phi} = \tau_1 + i\tau_2$. The convention used here follows Ref. [173], where more details in the computation can be found.

The first *constant term* is obtained by computing the value of $\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau)$ at the cusp $\tau = i\infty$, or equivalently by setting $n = 0$

$$\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau) = 2\zeta(2s)\tau_2^s + \sum_{m\in\mathbb{Z}}\sum_{n\neq 0} \frac{\tau_2^s}{|m+n\tau|^{2s}}. \tag{4.62}$$

Recall that Riemann zeta function is defined as $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$. The next step is to make use of the *integral representation*

$$\frac{1}{|m+n\tau|^{2s}} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{s+1}} e^{-\frac{\pi}{t}|m+n\tau|^2} \tag{4.63}$$

for the summand. Since the summation parameter m is unrestricted we can *Poisson resum* as follows

$$\sum_{m\in\mathbb{Z}} e^{-\pi x(m+a)^2 + 2\pi i m b} = \frac{1}{\sqrt{x}} \sum_{\tilde{m}\in\mathbb{Z}} e^{-\frac{\pi}{x}(\tilde{m}+b)^2 - 2\pi i(\tilde{m}+b)a}, \tag{4.64}$$

with $x = t^{-1}$, $a = n\tau_1$ and $b = 0$. The second constant term can now be extracted by setting $\tilde{m} = 0$

$$\begin{aligned} \mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau) &= 2\zeta(2s)\tau_2^s + 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)\tau_2^{1-s} \\ &\quad + \frac{\pi^s\tau_2^s}{\Gamma(s)} \sum_{\tilde{m}\neq 0}\sum_{n\neq 0} e^{-2\pi i\tilde{m}n\tau_1} \int_0^\infty \frac{dt}{t^{s+\frac{1}{2}}} e^{-\pi t\tilde{m}^2 - \frac{\pi}{t}n^2\tau_2^2}. \end{aligned} \tag{4.65}$$

What remains is to perform the integration

$$\int_0^\infty \frac{dt}{t^{s+\frac{1}{2}}} e^{-\pi t\tilde{m}^2 - \frac{\pi}{t}n^2\tau_2^2} = 2\left|\frac{\tilde{m}}{n}\right|^{s-\frac{1}{2}} \tau_2^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|\tilde{m}n|\tau_2) \tag{4.66}$$

of Bessel type and then rename one of the variables as $N \equiv -\tilde{m}n$. When the dust has settled we find at last

$$\begin{aligned} \mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau) &= 2\zeta(2s)\tau_2^s + 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)\tau_2^{1-s} \\ &\quad + \frac{2\pi^s\sqrt{\tau_2}}{\Gamma(s)} \sum_{N\neq 0} \mu_{1-2s}(N) N^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|N|\tau_2) e^{2\pi i N\tau_1}, \end{aligned} \tag{4.67}$$

with

$$\mu_{1-2s}(N) := \sum_{n|N} n^{1-2s} \tag{4.68}$$

being the so called *instanton measure*. The fact that the sum runs over divisors of N in the instanton measure is due to the change of variable in the last step. Recalling Section 3.2 this is precisely the kind of function appearing in the four graviton scattering amplitude.

A few words concerning the structure of Eq. (4.67) are now in order. One of the main properties of the non-holomorphic Eisenstein series is that it is an eigenfunction of the Laplacian, see Eq. (4.30). Acting Eq. (4.67) with the Laplace-Beltrami operator shows not only the entire sum satisfies the eigenvalue equation, the three constituents are in fact eigenfunctions separately. We can now start from a general Ansatz

$$\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau) = \mathfrak{C}_0(\tau_2) + \sum_{n\neq 0} \mathfrak{C}_n(\tau_2) e^{2\pi i n \tau_1}, \tag{4.69}$$

where the constant terms $\mathfrak{C}_0(\tau_2)$ by definition only depend on the dilaton. By solving the Laplace equation

$$\Delta f_s(\tau) = s(s-1) f_s(\tau) \tag{4.70}$$

separately for $\mathfrak{C}_0(\tau_2)$ and the infinite sum we will already arrive to the result

$$\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau) = A(s)\tau_2^s + B(s)\tau_2^{1-s} + \sqrt{\tau_2}\sum_{n\neq 0} C(n) K_{s-\frac{1}{2}}(2\pi|n|\tau_2) e^{2\pi i n \tau_1}. \tag{4.71}$$

However, to find the coefficients $A(s)$, $B(s)$ and $C(n)$ the Poisson resummation has to be carried out explicitly, after all, we have not used the $SL(2,\mathbb{Z})$-invariance of $\mathcal{E}_s^{SL(2,\mathbb{Z})}(\tau)$ yet.

As can be seen in Eq. (4.67), there are two constant terms

$$\mathfrak{C}_0(\tau_2) = 2\zeta(2s)\left[\tau_2^s + \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\tau_2^{1-s}\right]. \tag{4.72}$$

Physically they correspond to the tree level and one-loop amplitudes, respectively, in the four graviton scattering. In Ref. [174] it is nicely explained that a more natural way to write the above expression is in terms of the so called *complete zeta function*

$$\xi(k) := \pi^{-k/2}\Gamma(k/2)\zeta(k), \tag{4.73}$$

which obeys the functional identity

$$\xi(k) = \xi(1-k). \tag{4.74}$$

It then becomes apparent that the combination

$$\mathfrak{C}_0(\tau_2)\frac{\xi(2s)}{2\zeta(2s)} = \xi(2s)\tau_2^s + \xi(2s-1)\tau_2^{1-s} \tag{4.75}$$

is invariant under the action

$$s \to 1-s. \tag{4.76}$$

Looking closely it turns out to be nothing but the Weyl action of $SL(2,\mathbb{R})$.

The last comment we make is that the Eisenstein series in its Fourier expanded form can readily be obtained by using the spherical vector construction [157], thus demonstrating its generality. In this case a linear representation of $SL(2,\mathbb{R})$

$$E \doteq iz; \quad F \doteq i(z\partial_z + 2 - 2s)\partial_z; \quad H \doteq 2z\partial_z + 2 - 2s \tag{4.77}$$

acts on the real spherical vector

$$f_{\mathcal{K}}(z) = z^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(z), \tag{4.78}$$

whose p-adic counterpart is

$$f_p(z) = \frac{1 - p^{-2s+1}|z|_p^{2s-1}}{1-p^{-2s+1}}\gamma_p(z). \tag{4.79}$$

The function $\gamma_p(z)$ is defined such that it takes value one when z is a p-adic integer and zero otherwise. This real spherical vector is related to the one from Eq. (4.53) precisely by a Fourier transform in the variable z.

General Groups

Most of the properties we have mentioned above can be generalized to Eisenstein series of larger Lie groups. The constant terms and the infinite expansions in the axionic fields are still satisfying the *Laplace equation* separately. Formally the *constant terms* can be obtained by integrating out all the axions

$$\mathfrak{C}_0(\phi) = \int_{\chi_\alpha} d\chi_\alpha \mathcal{E}_s^{G(\mathbb{Z})}(\phi, \chi_\alpha). \tag{4.80}$$

Notice that s denotes the set of all order numbers of the particular Eisenstein series. Moreover, the number of constant terms as well as their relative structures are dictated by the *Weyl group*.

What differs most from the simple modular example is perhaps the structure of the part containing the axions. As we already mentioned, the linear displacement of the axion field under the *nilpotent upper triangular subgroup* in Eq. (4.6) is the motivation behind the Fourier expansion. However, for an arbitrary Lie group the nilpotent subgroup, which is generated by all the positive step operators, is in general non-abelian. This simply means that under these translations the axion fields will transform into each other, and thus destroy the chances for any periodicity in the Eisenstein series. The way to resolve this problem is to factorize the nilpotent upper triangular subgroup into a product of its center and a nilpotent abelian group

$$N = Z \times \tilde{N} \qquad \Rightarrow \qquad \chi_\alpha = \{\tilde{\chi}_a, \psi_i\}. \tag{4.81}$$

The parameters of the abelian group $\tilde{N}$ are then only transforming under $\tilde{N}$

$$\tilde{\chi}_a \longmapsto \tilde{\chi}_a + k_a, \tag{4.82}$$

while the parameters of the center Z are affected by the full N

$$\psi_i \longmapsto \psi_i + l_i + \sum_{a,b} c_{ab} k_a \tilde{\chi}_b, \tag{4.83}$$

with some numerical coefficients c_{ab}. The Eisenstein series is now characterized by three parts: the constant, the abelian and the non-abelian ones

$$\begin{aligned}\mathcal{E}_s^{G(\mathbb{Z})}(\phi, \chi_\alpha) &= \mathcal{I}_0 + \mathcal{I}_{\mathrm{A}} + \mathcal{I}_{\mathrm{NA}} \\ &= \mathfrak{C}_0(\phi) + \sum_{k_a \in \mathbb{Z}} \mathfrak{C}_k(\phi) \exp\left[2\pi i \sum_a k_a \tilde{\chi}_a\right] \\ &\quad + \sum_{l_i \in \mathbb{Z}} \mathfrak{C}_l(\phi, \tilde{\chi}_a) \exp\left[2\pi i \sum_i l_i \psi_i\right].\end{aligned} \tag{4.84}$$

The principles of the Poisson resummation remain the same even for general Lie groups, though sometimes tricks might be needed to actually be able to carry out the computations. In particular lattice constraints described below Eq. (4.91) tend to require special treatment. Also, often one needs to carry out a series of Poisson resummation before arriving at the final result.

In Paper V the Fourier expansion of the Eisenstein series invariant under $SU(2,1;\mathbb{Z}[i])$ is computed. Many properties which are described in general terms here can be illustrated by that example.

4.3 Transforming Automorphic Forms

So far in this chapter we have dealt extensively with automorphic functions, i.e., functions that are invariant under some discrete group $G(\mathbb{Z})$. The generic case relevant for string theory, however, is automorphic forms which transform as representations of $\mathcal{K}(G)$. Most often functions that have been conjectured as coefficients of the higher derivative corrections in supergravity theories are of this more general type.

The *transforming modular form* that generalizes the non-holomorphic Eisenstein series has been well-studied in the physics literature [150, 175, 151, 160]. It has the form

$$f_{s,k}(\tau,\bar{\tau}) = {\sum_{(m,n)}}' \frac{\tau_2^s}{(m+n\tau)^{s+k}(m+n\bar{\tau})^{s-k}}, \tag{4.85}$$

and obeys $\bar{f}_{s,k} = f_{s,-k}$. The bar here denotes the complex conjugation. Under modular transformations this series gains an $SO(2)$ phase

$$f_{s,k}(\tau,\bar{\tau}) \longmapsto \left(\frac{c\tau+d}{c\bar{\tau}+d}\right)^k f_{s,k}(\tau,\bar{\tau}), \tag{4.86}$$

where the $SL(2,\mathbb{Z})$ group elements are parameterized as in Eq. (4.1). The ordinary non-holomorphic Eisenstein series is the special case when $k=0$. A covariant derivative

$$\mathcal{D}_k \equiv 2i\tau_2\partial_\tau + k \tag{4.87}$$

can be defined to step between different orders of k

$$\mathcal{D}_k f_{s,k}(\tau,\bar{\tau}) = (s+k) f_{s,k+1}(\tau,\bar{\tau}). \tag{4.88}$$

Making use of $\bar{f}_{s,k} = f_{s,-k}$ one can show that

$$\bar{\mathcal{D}}_{-(k+1)}\mathcal{D}_k f_{s,k}(\tau,\bar{\tau}) = (s-k-1)(s+k) f_{s,k}(\tau,\bar{\tau}). \tag{4.89}$$

This is the generalization of the Laplace equation.

For an arbitrary Lie group the transforming automorphic forms can readily be obtained by generalizing the spherical vector method. Instead of requiring invariance under the maximal compact subgroup, one simply demands the generalized spherical vector $f_{\mathcal{K}}$ to transform appropriately under $\mathcal{K}(G)$. This will assure the correct transformation properties of the resulting automorphic form. Though the logic is clear, computationally it might be very complicated. In particular finding the p-adic counterpart of a generalized spherical vector can pose a mathematical challenge.

An alternative construction is presented in Ref. [87], where the authors made an intuitive generalization of the lattice construction. The key observation for this construction is Eq. (4.34), namely

$$\mathcal{V}\vec{\omega} \longmapsto k\mathcal{V}\vec{\omega}', \quad k \in \mathcal{K}(G) \tag{4.90}$$

under $G(\mathbb{Z})$ transformations. Recall that $\mathcal{V} \in G/\mathcal{K}(G)$ and $\vec{\omega}$ is a vector in the lattice left invariant by $G(\mathbb{Z})$. If $\mathcal{V}$ was constructed using the fundamental representation of G, then $\mathcal{V}\vec{\omega}$ will transform as the fundamental representation of $\mathcal{K}(G)$. We can now take appropriate tensor product of a number of $\mathcal{V}\vec{\omega}$ to obtain objects transforming as arbitrary finite-dimensional representation of $\mathcal{K}(G)$. Schematically a transforming automorphic form can then be written as

$$f^{G(\mathbb{Z})}_{s,a_1\dots a_n}(\mathcal{Z}) = \sum_{\vec{\omega}} \delta(\vec{\omega}^\dagger \wedge \vec{\omega}) \frac{(\mathcal{V}\vec{\omega})_{a_1} \dots (\mathcal{V}\vec{\omega})_{a_n}}{[(\mathcal{V}\vec{\omega})^\dagger \cdot \mathcal{V}\vec{\omega}]^s}, \tag{4.91}$$

with some (anti)symmetrization of the indices $a_1, \dots, a_n$. It was shown in Ref. [87] that the generalized modular form in Eq. (4.85) indeed can be obtained by this method.

In this thesis we have been concentrating on the non-holomorphic Eisenstein series based on the principal continuous representation, which appear as coefficients of higher order derivative corrections in string theory. At each level in α', these are the functions dictating the expansion in the string coupling constant g_s. However, theta theories based on the minimal representation is another example of automorphic form which play important role in physics. It is believed that the partition function of the supermembrane is related to the theta series, though so far only the zero modes have been studied [156]. Also, there are strong indications that minimal representations and theta series are associated to black hole degeneracies [176, 168, 169].

A

p-Adic Numbers

This appendix provides a list of basic relations of p-adic numbers we need for the construction of Eisenstein series. For a proper introduction to the world of p-adic numbers the reader is referred to for instance Refs. [171, 172], or [161] for a different take of the subject.

For a given prime number $p \in \mathbb{P}$, the field $\mathbb{Q}_p$ of *p-adic numbers* is a completion, in the Cauchy sequence sense, of the rational numbers. The defining property of a p-adic number is that it can be expanded similar to a power series with base p

$$\sum_{i=k}^{\infty} = a_i p^i, \quad k \in \mathbb{Z}, \tag{A.1}$$

where a_i are integers in the interval $\{0, \ldots, p-1\}$. A subring of $\mathbb{Q}_p$ is the p-adic integers $\mathbb{Z}_p$, which consists of those p-adic numbers with $a_i = 0$ for all $i < 0$. The ring of p-adic integers is not only complete but also compact.

For any non-zero rational number there is a unique decomposition

$$x = \frac{a}{b} p^n, \quad x \in \mathbb{Q}, \quad p \in \mathbb{P} \text{ and } n \in \mathbb{Z}, \tag{A.2}$$

where neither a nor b is divisible by p. The norm used for the Cauchy sequences is then defined as

$$|x|_p := p^{-n}, \tag{A.3}$$

satisfying the following properties:

$$\begin{aligned} i.\ & |x|_p \geq 0, \quad \forall x; \\ ii.\ & |x|_p = 0 \iff x = 0; \\ iii.\ & |xy|_p = |x|_p |y|_p, \quad \forall x, y; \\ iv.\ & |x+y|_p \leq |x|_p + |y|_p, \quad \forall x, y; \\ v.\ & |x+y|_p \leq \max(|x|_p, |y|_p), \quad \forall x, y. \end{aligned} \tag{A.4}$$

Generalizing to a vector of p-adic numbers $\vec{x} = (x_1, \dots, x_k)$, the norm is defined as

$$|\vec{x}|_p := \max(|x_1|_p, \dots, |x_k|_p). \tag{A.5}$$

The p-adic spherical vector appearing in the construction of $SL(2,\mathbb{R})$ Eisenstein series is

$$f_p(x) = (|1, x|_p)^{-2s} = [\max(1, |x|_p)]^{-2s}. \tag{A.6}$$

Since $|-x|_p = |x|_p$ we can assume x to be positive. Set $x = \frac{m}{n}$ with m and n being integer and co-prime. By computing the prime factorizing $m = p_1^{k_1} \dots p_m^{k_m}$ and $n = q_1^{l_1} \dots q_n^{l_n}$, where $\{p_i, q_j\}$ are all distinct prime numbers, it then follows that

$$|x|_{p_i} = p_i^{-k_i}, \quad |x|_{q_j} = q_j^{l_j}, \quad |x|_p = 1 \;\; \forall p \notin \{p_i, q_j\}. \tag{A.7}$$

The $G(\mathbb{Z})$-invariant distribution now simplifies to

$$f_{G(\mathbb{Z})} = \prod_{p<\infty} f_p(x) = (q_1^{l_1} \dots q_n^{l_n})^{-2s} = n^{-2s}. \tag{A.8}$$

Following a similar argument

$$\prod_{p<\infty} \max\left(\left|\frac{m_1}{n_1}\right|_p, \left|\frac{m_2}{n_2}\right|_p \right) = n_1 n_2. \tag{A.9}$$

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Paper I

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Membranes for topological M-theory

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Abstract: We formulate a theory of topological membranes on manifolds with G_2 holonomy. The BRST charges of the theories are the superspace Killing vectors (the generators of global supersymmetry) on the background with reduced holonomy $G_2 \subset \mathrm{Spin}(7)$. In the absence of spinning formulations of supermembranes, the starting point is an $N = 2$ target space supersymmetric membrane in seven euclidean dimensions. The reduction of the holonomy group implies a twisting of the rotations in the tangent bundle of the branes with "R-symmetry" rotations in the normal bundle, in contrast to the ordinary spinning formulation of topological strings, where twisting is performed with internal U(1) currents of the $N = (2, 2)$ superconformal algebra. The double dimensional reduction on a circle of the topological membrane gives the strings of the topological A-model (a by-product of this reduction is a Green-Schwarz formulation of topological strings). We conclude that the action is BRST-exact modulo topological terms and fermionic equations of motion. We discuss the rôle of topological membranes in topological M-theory and the relation of our work to recent work by Hitchin and by Dijkgraaf et al.

Keywords: M-Theory, Topological Strings.

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1. Introduction

The notion of topological M-theory was introduced recently by Dijkgraaf et al. [1] (see also [2]). In analogy with topological string theory [8, 9] (for a recent review, see ref. [10]), one expects here a topological membrane world-volume theory to give rise to a field theory in a seven-dimensional target space. In the string case both the world-sheet and the six-dimensional target space theories are fairly well understood, the latter being in fact string field theories constructed from the world-sheet BRST charge. Although Calabi-Yau three-folds have special properties in this context [10], topological strings exist also on special holonomy manifolds of other dimensionalities, see e.g. ref. [3]. The features found in the topological string case would for many reasons be very valuable to understand also in the membrane/M-theory case. One important reason is connected to the rôle topological string amplitudes play in compactification of physical string theories. One may also wonder if a better understanding of topological M-theory may indicate how to approach the problem of finding a microscopic formulation of M-theory, possibly including a quantisation of the membrane.

In ref. [1], the authors took a first step towards this goal by suggesting the form of the effective target space field theory of topological M-theory. Such an effective theory may be obtained by arguing that the theory and its topological properties should be connected to those of the A and/or B topological string models by dimensional reduction in much the same way as the physical field theories in ten and eleven dimensions are related. Similarly, one should be able to connect the topological string world-sheet theories to the topological membrane one by subjecting the latter to a double dimensional reduction.

The target space aspects were discussed in some detail in ref. [1], where the crucial rôle of Hitchin functionals [5, 6] was elaborated upon. These are special functionals of p-forms which can be connected to metric fields by some rather complicated non-linear relations. The resulting theory was given the appropriate name *form-gravity* in ref. [1]. By starting from a Hitchin 3-form on a seven-dimensional G_2 holonomy manifold, the authors of ref. [1] show that by dimensional reduction various well-known topological form-gravity theories in lower dimensions are obtained. In particular, one finds the Kodaira-Spencer theory [11] for the complex structure deformations in the B-model and the Kähler gravity theory [12] of the Kähler deformations in the A-model, albeit produced in a particular interacting form.

At the classical level the connection between form-gravity based on a six-dimensional Hitchin functional and the topological B-model was made explicit by relating the corresponding tree-level partition functions to each other. However at one loop level, where the B-model is known to compute a special combination of Ray-Singer torsion invariants [11], it was recently demonstrated by Pestun and Witten [15] that one needs to use the extended Hitchin functional introduced in ref. [4] to obtain the same one-loop partition function. This connection to the extended Hitchin functionals is intriguing since they play a rôle also in flux compactifications [20] on the generalised Calabi-Yau manifolds discussed by Hitchin in his paper.

The natural next step seems to be to construct a topological membrane theory that may be related to the topological M-theory mentioned above. That is, we want to construct a membrane embedded in a seven-dimensional space with G_2 holonomy whose effective action is the Hitchin functional 3-form gravity theory discussed in ref. [1]. The usual approach to derive topological strings by means of twisting does not seem to work here since it is based on the spinning string, or NSR, formulation which is lacking in the membrane case. Here we will instead approach this problem by starting from the Green-Schwarz (GS) formulation of the membrane [21]. Of course, since the superstring, and perhaps also the supermembrane, are quantized most easily using Berkovits' pure spinor formulation [22], this is probably an even more suitable starting point. This point was discussed recently also in [23]. We note here that although the GS formulation of string theory is as standard as the NSR one, it does not seem to have been used yet in the construction of topological strings. As will be clear below such a GS formulation will come out of the results presented here for the topological membrane.

One important aspect of twisting in the construction of the topological string from a two-dimensional supersymmetric sigma model is that it turns a spin-$\frac{3}{2}$ supersymmetry current into a spin-1 object that can be interpreted as a BRST current. This kind of twisting is accomplished by enforcing the identification of the world-sheet Lorentz symmetry with an $so(2)$ R-symmetry giving fermionic quantities unphysical integer spin values. In the GS formulation of the membrane, which is the starting point in our approach to the topological membrane, such an unphysical spin-statistics relation on the world-volume is already in effect since the supercoordinates in the target space $(x^m, \psi^{\hat{\mu}I})$ contain the anticommuting world-volume scalars $\psi^{\hat{\mu}I}$ (the ranges of the various indices will be specified later). For trivial target spaces like eleven-dimensional flat space, the gauge-fixing of the κ-symmetry generates an ordinary supermultiplet in three dimensions with physical spin fermions [21]. However, in the context discussed here no twisting needs to be done by hand, a fact that has been noticed before in ref. [7]. As discussed in detail in section 3, a similar phenomenon to twisting does occur but now as an automatic consequence of combining G_2 holonomy and the tangent space symmetry remaining after the introduction of the membrane into target space. This twisting leaves the bosonic and fermionic fields in the same representation of the surviving symmetry. We will however not fix the gauge, and for the most part work with a fully κ-symmetric theory with a (1+7)-dimensional parameter.

This paper is organised as follows. In section 2 we start by discussing the G_2 3-form gravity theory that the topological membrane is supposed to generate in the seven-dimensional target space. Different action functionals are presented for this theory, one of which we believe is new. This section also describes the supergeometry into which the bosonic seven-dimensional G_2 holonomy manifolds can be embedded. The supercoordinates are $Z^M = (x^m, \theta^{\hat{\mu}I})$ where m runs over seven values and $\hat{\mu}I$ enumerates two $(I = 1, 2)$ eight-dimensional spinors $(\hat{\mu} = 1, .., 8)$. The supergeometry in encoded by a standard vielbein (supersiebenbein) and a superspace 3-form $C_{MNP}(Z)$. The Bianchi identities are discussed and an explicit 3-form superfield is derived, but only in the flat space limit. As also explained, the full expansions in fermionic coordinates of the curvature dependent 3-form and vielbein superfields can be obtained by a lengthy iterative procedure which we hope to come back to in a future publication (for a similar discussion, see refs. [13, 14]).

In section 3 we discuss the κ-symmetric membrane theory that we propose as the starting point for deriving a topological membrane. The rôle of G_2 in obtaining the BRST charge from a partially gauge fixed world-volume action is explained and arguments indicating the topological nature of the action, namely the fact that it is BRST exact, are presented. This discussion is carried out in the full theory but the calculation of the action is performed only to lowest order in the curvature and a full proof will require more work.

In the concluding section 4 we make a few additional remarks and comments. Properties of the octonions are used heavily in this paper and some aspects can be found in the appendices. In appendix A we discuss G_2 tensors, projection operators and the relation to quaternions, while in appendix B we give the explicit form of the flat superspace 3-form based on the octonionic structure constants.

2. G_2 holonomy

Seven-dimensional manifolds with G_2 holonomy have special properties, among which are Ricci-flatness and a single covariantly constant (Killing) spinor.

When holonomy is restricted to lie in a G_2 subgroup, a (partial) gauge choice can be made for the spin connection to make it lie entirely in the Lie algebra G_2. Then, G_2 singlets can be defined as constant over the manifold, and this thus applies to special elements of any Spin(7) representation containing a G_2 singlet. So, there is a constant spinor, since

$8 \to 1 \oplus 7$, and a constant 3-form Ω, since $35 \to 1 \oplus 7 \oplus 27$. In a flat frame the 3-form may be chosen as $\Omega_{abc} = \sigma_{abc}$, the octonionic structure constants, invariant under the action of G_2, the automorphism group of $\mathbb{O}$ (see appendix A for details).

2.1 The 3-form

Hitchin [5, 6] has constructed a model containing a 3-form field, whose solutions are G_2 manifolds. This is certainly a part of topological M-theory. The metric is constructed from the 3-form as

$$K_{mn} = \sqrt{g}\, g_{mn} = -\frac{1}{144}\varepsilon^{m_1 \dots m_7}\Omega_{mm_1m_2}\Omega_{nm_3m_4}\Omega_{m_5m_6m_7}\,, \tag{2.1}$$

Hitchin gives the action $S = \int d^7x K^{1/9}$. A Polyakov type action, due to Nekrasov [26], giving both the relation (2.1) and the covariant constancy of the 3-form, is

$$S' = \frac{2}{9}\int d^7x \left(\sqrt{g} - \frac{1}{288}g^{mn}\varepsilon^{m_1\dots m_7}\Omega_{mm_1m_2}\Omega_{nm_3m_4}\Omega_{m_5m_6m_7}\right)\,, \tag{2.2}$$

The metric is auxiliary and determined by its equation of motion. The constant in front is chosen so that the action is normalised to the volume. In a frame where (locally) $\Omega_{abc} = \sigma_{abc}$, one thus has $g_{ab} = \delta_{ab}$, which is checked by $\sigma_{acd}\sigma_{bef}\star\sigma_{cdef} = -24\delta_{ab}$ (see appendix A).

Varying the action (2.2) w.r.t. Ω gives

$$\begin{aligned} -\tfrac{1}{9\times 144}\textstyle\int d^7x(\; & 2g^{rn}\varepsilon^{stm_3\dots m_7}\Omega_{nm_3m_4}\Omega_{m_5m_6m_7} \\ & +g^{mn}\varepsilon^{m_1\dots m_4rst}\Omega_{mm_1m_2}\Omega_{nm_3m_4})\delta\Omega_{rst}\,. \end{aligned} \tag{2.3}$$

Using the relations of appendix A to calculate the two terms, one finds using the expression for g_{mn} that they both are proportional to the same expression, and that the variation (2.3) becomes

$$\frac{1}{3}\int \star\Omega\wedge\delta\Omega = \frac{1}{18}\int d^7x\sqrt{g}\,\Omega^{mnp}\delta\Omega_{mnp}\,. \tag{2.4}$$

The relation (2.1) for the metric may equivalently be written in the implicit form

$$g_{mn} = \frac{1}{6}g^{p_1q_1}g^{p_2q_2}\Omega_{mp_1p_2}\Omega_{nq_1q_2}\,, \tag{2.5}$$

which is used by Hitchin in expressing the variation of his action in the "linear form" (2.4). This latter relation could as well be obtained from an action, which now takes a much more conventional form:

$$S'' = -\frac{1}{6}\int d^7x\sqrt{g}\left(1 - \frac{1}{6}g^{m_1n_1}g^{m_2n_2}g^{m_3n_3}\Omega_{m_1m_2m_3}\Omega_{n_1n_2n_3}\right) \tag{2.6}$$

(what varying this action w.r.t. g^{mn} really gives is $\Omega_{mpq}\Omega_n{}^{pq} = -g_{mn}(1 - \frac{1}{6}\Omega_{pqr}\Omega^{pqr})$, which after contracting the free indices with g^{mn} gives $\Omega_{pqr}\Omega^{pqr} = 42$, and thus $g_{mn} = \frac{1}{6}\Omega_{mpq}\Omega_n{}^{pq}$). Variation of the action (2.6) w.r.t. Ω gives an expression proportional to (2.4) directly, without any use of the algebraic identities of appendix A.

The 3-form is part of the geometric background for propagation of membranes. The expression "$\Omega = \sigma$" is purely bosonic. In a superspace, Ω will contain more components when expressed in flat basis, due to torsion (see appendix B).

2.2 Superspace and supersymmetry

The superspace we want to consider has bosonic coordinates which are the coordinates of a euclidean manifold with G_2 holonomy. In addition there will be fermionic coordinates. These are *a priori* a set of real spinors in the 8-dimensional representation of Spin(7), but when Spin(7) $\to G_2$ each spinor decomposes as $8 \to 1 \oplus 7$. The γ-matrices of Spin(7) are real and antisymmetric, so it is clear that an even number of spinors are needed, together with an internal Sp($2n$) in order to have a non-vanishing torsion. We will choose the simplest possibility, $n = 1$, giving a doublet of spinors, for reasons that become obvious in the following subsection. This superspace is obtained from $D = 11$ superspace, with twice as many fermionic coordinates, as a truncation of the Spin(7) $\times$ SL$(2, \mathbb{C})$ subgroup of Spin$(1, 10)$ to Spin(7) $\times$ SL$(2, \mathbb{R})$, where the spinors in the representation $32 \to (8, 2_{\mathbb{C}})$ are demanded to be real.

A convenient realisation is to consider a vector as an imaginary octonion, $v \in \mathbb{O}'$, and a spinor as an arbitrary octonion, $s \in \mathbb{O}$. Letting the orthonormal basis of $\mathbb{O}'$ be $\{e_a\}_{a=1}^{7}$, multiplication by γ^a is identified with left multiplication of a spinor with e_a, i.e., vs is again a spinor. The octonionic multiplication table, $e_a e_b = -\delta_{ab} + \sigma_{abc} e_c$, tells us that the real γ-matrices square to -1 (a property which will be crucial for supermembranes).

Before moving on let us fix some notation. Superspace coordinates are written

$$Z^M = (x^m, \psi^{\hat{\mu} I}), \quad m = 1, \ldots 7, \quad \hat{\mu} = 0, \ldots, 7, \quad I = 1, 2. \tag{2.7}$$

Flat indices are written $(a, \hat{\alpha} I)$. The spinor index will often be divided into $(0, \alpha = 1, \ldots, 7)$, reflecting the decomposition $8 \to 1 \oplus 7$. This division applies also to curved indices, as long as one only considers super-diffeomorphisms that leave the singlet inert, and we use the notation

$$\psi^{\hat{\mu} I} = (\theta^I, \psi^{\mu I}). \tag{2.8}$$

Bosonic and fermionic vielbeins are written,

$$E^a, \quad \mathcal{E}^{\hat{\alpha} I} = (\mathcal{E}^I, \mathcal{E}^{\alpha I}), \tag{2.9}$$

and the purely bosonic vielbein, $e_m{}^a$.

The γ matrices encoded in the left multiplication of a spinor $\lambda = \lambda^{\hat{\alpha}} e_{\hat{\alpha}}$ by an imaginary unit e_a are

$$\begin{aligned} (\gamma^a)_{\alpha\beta} &= \sigma^a{}_{\alpha\beta}, \\ (\gamma^a)_{0\alpha} &= \delta^a_\alpha. \end{aligned} \tag{2.10}$$

They satisfy $\{\gamma^a, \gamma^b\} = -2\delta^{ab}$, where the minus sign is necessary for real γ-matrices.

The Clifford algebra is spanned by the $so(7)$-invariant tensors $\delta^{\hat{\alpha}}{}_{\hat{\beta}}$, $(\gamma^a)^{\hat{\alpha}}{}_{\hat{\beta}}$, $(\gamma^{ab})^{\hat{\alpha}}{}_{\hat{\beta}}$ and $(\gamma^{abc})^{\hat{\alpha}}{}_{\hat{\beta}}$, of which the first and last are symmetric and the second and third antisymmetric matrices. The decomposition in terms of G_2-invariant tensors is

$$
\begin{aligned}
\delta^{\hat\alpha}{}_{\hat\beta} &= \begin{bmatrix} 1 & 0 \\ 0 & \delta^\alpha{}_\beta \end{bmatrix} \\
(\gamma^a)^{\hat\alpha}{}_{\hat\beta} &= \begin{bmatrix} 0 & \delta^a{}_\beta \\ -\delta^{a\alpha} & \sigma^{a\alpha}{}_\beta \end{bmatrix} \\
(\gamma^{ab})^{\hat\alpha}{}_{\hat\beta} &= \begin{bmatrix} 0 & -\sigma^{ab}{}_\beta \\ \sigma^{ab\alpha} & -\star\,\sigma^{ab\alpha}{}_\beta - 2\delta^{ab}_{\alpha\beta} \end{bmatrix} \\
(\gamma^{abc})^{\hat\alpha}{}_{\hat\beta} &= \begin{bmatrix} \sigma^{abc} & -\star\,\sigma^{abc}{}_\beta \\ -\star\,\sigma^{abc\alpha} & 6\delta^{[a}_{(\alpha}\sigma_{\beta)}{}^{bc]} - \delta^\alpha{}_\beta\sigma^{abc} \end{bmatrix} .
\end{aligned}
\tag{2.11}
$$

Solving the dimension-0 part of the torsion Bianchi identities reveals a possible solution in terms of SO(7) γ-matrices. We choose $T^a_{\alpha I,\beta J} = 2\varepsilon_{IJ}(\gamma^a)_{\alpha\beta}$, implying

$$
\begin{aligned}
T_{\alpha I,\beta J}{}^a &= 2\varepsilon_{IJ}\sigma^a{}_{\alpha\beta} , \\
T_{0I,\alpha J}{}^a &= 2\varepsilon_{IJ}\delta^a_\alpha , \\
T_{0I,0J}{}^a &= 0 .
\end{aligned}
\tag{2.12}
$$

(A wider class of G_2-invariant solutions exists but κ-symmetry restricts the choice to eq. (2.12).) The background will contain a 3-form potential C (descending from the one in $D = 11$) with 4-form field strength, G, whose dimension-0 part is taken to be $G_{ab,\hat\alpha I,\hat\beta J} = -2\varepsilon_{IJ}(\gamma_{ab})_{\hat\alpha\hat\beta}$:

$$
\begin{aligned}
G_{ab,\alpha I,\beta J} &= 2\varepsilon_{IJ}(2\delta^{ab}_{\alpha\beta} + \star\sigma_{ab\alpha\beta}) , \\
G_{ab,0I,\beta J} &= 2\varepsilon_{IJ}\sigma_{ab\beta} , \\
G_{ab,0I,0J} &= 0 .
\end{aligned}
\tag{2.13}
$$

The Fierz identity in $D = 7$ ensuring the Bianchi identity for G is

$$
(\gamma^b)_{\hat\alpha\hat\beta}(\gamma_{ab})_{\hat\gamma\hat\delta}|_{\boxplus} = 0 ,
\tag{2.14}
$$

where the Young tableau indicates the symmetry structure of the spinor indices. The expression $(\gamma^b)_{\hat\alpha\hat\beta}(\gamma_{ab})_{\hat\gamma\hat\delta}$ contains only terms that are antisymmetric in at least three spinor indices, implying that $\varepsilon_{IJ}\varepsilon_{KL}(\gamma^b)_{\hat\alpha\hat\beta}(\gamma_{ab})_{\hat\gamma\hat\delta}$ completely symmetrised in the four composite indices $(\hat\alpha I, \hat\beta J, \hat\gamma K, \hat\delta L)$ vanishes.

The potential C, which will be the field that the supermembranes couples minimally to, is *a priori* thought of as a 3-form with vanishing cohomology class, so that, modulo gauge transformations, $C_{abc} = 0$. Of course, changing C to $C^{(k)} = C + k\Omega$ leaves the field strength invariant.

The constraints for torsion and field strength used are standard, and the ones obtained by reduction from $D = 11$ and truncation to real fermions. In order to use them to extract an explicit form for the dynamics of the supermembrane introduced in the following section, one would need to solve these constraints explicitly for the vielbeins and components of C in terms of the bosonic and fermionic coordinates. This has not been done, except for in the case of flat manifolds (orbifolds of tori). In principle, this can be done order by order in the fermions, and we will indicate how this expansion starts.

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The target space coordinates[1] are x^m, ψ^{mI} and θ^I . Under (bosonic) diffeomorphisms, $\delta_\chi x^m = \chi^m$, $\delta_\chi \psi^{mI} = \psi^{nI}\partial_n\chi^m$, $\delta_\chi\theta^I = 0$. This means that the derivatives and dual differentials that transform covariantly are

$$\begin{array}{ll} dx^m & \mathcal{D}_m = \partial_m - \Gamma^p_{mn}\psi^{nI}\frac{\partial}{\partial\psi^{pI}} \\ D\psi^{mI} = d\psi^{mI} + dx^n\psi^{pI}\Gamma^m_{np} & \frac{\partial}{\partial\psi^{mI}} \\ d\theta^I & \frac{\partial}{\partial\theta^I} \end{array} \tag{2.15}$$

(if we have differentials that transform covariantly, we can just contract them with ${e_m}^a$ to get something that is invariant). In order to reproduce the dimension-0 torsion, the vielbeins are constructed from the covariant differentials as

$$\begin{aligned} E^a &= (dx^m + \varepsilon_{IJ}{\Omega^m}_{np}D\psi^{nI}\psi^{pJ} + 2\varepsilon_{IJ}d\theta^I\psi^{mJ}){e_m}^a + \ldots\,, \\ \mathcal{E}^{aI} &= D\psi^{mI}{e_m}^a + \ldots\,, \\ \mathcal{E}^I &= d\theta^I\,. \end{aligned} \tag{2.16}$$

We also let $\omega = dx^m\omega_m(x)$ and $D = d + \omega$. These terms generate torsion, however, which contains the Riemann tensor ($\mathcal{T}^{aI} \equiv T^{\alpha I}\delta^a_\alpha$):

$$\begin{aligned} T^a &= \varepsilon_{IJ}({\Omega^m}_{np}D\psi^{nI}\wedge D\psi^{pJ} + 2d\theta^I\wedge D\psi^{mJ}){e_m}^a + {\Omega^m}_{np}{R_q}^n\psi^{qI}\psi^{pJ}{e_m}^a\,, \\ \mathcal{T}^{aI} &= \psi^{nI}{R_n}^m{e_m}^a\,, \end{aligned} \tag{2.17}$$

where $D\Omega = 0$ has been used. The curvature enters with ${R_m}^n \equiv \frac{1}{2}dx^p\wedge dx^q {R_{qpm}}^n$. So, while the correct torsion terms are generated, the curvature-dependent ones have to be compensated for by adding terms of higher order in fermions in the vielbeins. Note, however, that this does not apply to the coefficients of $d\theta^I$, which is a G_2 singlet, hence not affected by the spin connection, and furthermore exact.

2.3 G_2 manifolds and supersymmetry

The existence of a constant spinor allows for a Killing spinor, a fermionic "isometry" of the superspace, i.e., a global supersymmetry. In our superspace with an internal SL(2) index, there will be a doublet of supersymmetries. We choose a parametrisation where the superspace Killing vectors, i.e., the supersymmetry generators are

$$Q_I = \frac{\partial}{\partial\theta^I}\,, \tag{2.18}$$

which obviously fulfill

$$\{Q_I, Q_J\} = 0\,. \tag{2.19}$$

All vielbeins in eq. (2.16) are invariant under Q_I. We may remark that the simple form (2.18) of the supersymmetry generators depends on the the form of the bosonic vielbeins. If the fermion bilinears in the bosonic vielbein had been chosen to contain $\varepsilon_{IJ}(d\theta^I\psi^{mJ} - \theta^I d\psi^{mJ})$ instead of $2\varepsilon_{IJ}d\theta^I\psi^{mJ}$ (which to lowest order corresponds to a

[1] The identification of part of the spinor as vectors involves gauge-fixing all except the bosonic diffeomorphisms.

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change of bosonic coordinate), one would also have had a term $-\varepsilon_{IJ}\psi^{mI}\frac{\partial}{\partial x^m}$ in Q_I. Diffeomorphism covariance would then demand that $\frac{\partial}{\partial x^m}$ is replaced by $\mathcal{D}_m$, so also ψ is transformed. It turns out (by trial and error) that it is impossible to construct a supersymmetry doublet that starts out this way and fulfills the nilpotency relation (2.19), due to curvature terms, so we are left with the choice of eq. (2.18).

In the discussion on a topological theory of membranes below, the Q_I's are the nilpotent operators that will be promoted to BRST operators.

3. Topological membranes

In this section, we will describe in detail how we obtain a topological membrane by imposing a supersymmetry constraint on supermembranes embedded in a superspace extending a manifold of G_2 holonomy. First we will introduce supermembranes, and investigate the structure of κ-symmetry in the background at hand. We proceed to promote the global supersymmetry generators to BRST operators, thereby turning the theory into a topological theory. We show that the action, modulo topological terms and fermionic equations of motion, is not only BRST-invariant, but also BRST-exact.

3.1 Supermembranes on G_2 manifolds

A supermembrane in seven dimensions should have $N = 2$ supersymmetry, i.e., propagate in a background superspace with two fermionic spinorial coordinates $\psi^{\hat{\mu}I}$. Then the four transverse bosons match the fermions in number, with κ-symmetry and equations of motion taken into account. Both bosons and fermions are *a priori* scalars on the world-volume. This can of course change after some gauge-fixing, e.g. choosing a static gauge. The superspace we choose for the propagation of the membrane is thus taken to be the one described in the previous section.

When formulating a theory of topological *strings* it is convenient to start from the action of a spinning (world-sheet supersymmetric) string. For a membrane, no such formulation exists that is equivalent to the space-time supersymmetric one. Since we want our membrane to describe part of M-theory, we seem to be forced to use the ordinary supermembrane action. The generic action for a supersymmetric membrane is

$$S = \int d^3\xi\sqrt{g} + \int C\,. \tag{3.1}$$

where g and C are pullbacks from target superspace to the world-volume.

The 7-dimensional R-symmetry is SL(2). R-symmetry is typically something one wants to use in a topological twist, but the real forms of R-symmetry and local world-volume rotations $su(2)$ do not match. On the other hand, once one decomposes rotations into longitudinal and transverse, there are lots of $su(2)$'s. When $so(7) \to so(3) \oplus so(4) \approx su(2) \oplus su(2) \oplus su(2)$, $7 \to (1,2,2) \oplus (3,1,1)$ and $8 \to (2,1,2) \oplus (2,2,1)$. But if we also have the breaking $so(7) \to G_2$, $7 \to 7$, $8 \to 1 \oplus 7$, we have to consider the maximal unbroken subalgebra contained in both G_2 and $su(2)^3$. In the case that the embedding of the membrane world-volume is associative, i.e., if $\star\sigma_{ijka} = 0$, or equivalently $\sigma_{ijk} = \pm\varepsilon_{ijk}$, this is $su(2) \oplus su(2)$, which is a maximal subalgebra of G_2, and where the second $su(2)$ is the last of the three in $so(7) \to su(2) \oplus su(2) \oplus su(2)$ and the first is the diagonal subalgebra of the first two (this is shown in detail in appendix A, using the splitting of an octonion into a pair of quaternions). For a more general embedding, the same representations are obtained in a static gauge based on coordinate directions spanning a quaternion.

From a 3-dimensional perspective, we have (before G_2 is imposed) scalars transforming as vectors under R-symmetry $so(4)$, $\phi \in (1,2,2)$, and spinors transforming as either of the chiralities of $so(4)$, $\psi \in (2,2,1)$ and/or $\psi' \in (2,1,2)$. *Introduction of G_2 implies a twist of one of the spinor representations*, since it identifies one of the two R-symmetry $su(2)$'s with the $su(2)$ of space rotations. This twisting has been observed earlier in ref. [7].

The lesson from the behaviour of the representations and the effective twisting is that when one wants to formulate a topological membrane theory, no twisting "by hand" is needed — it is automatically provided in a space-time supersymmetric formulation.

3.2 Fermionic symmetries

The supermembrane action is invariant under global supersymmetry as well as κ-symmetry. Let us discuss these symmetries in some more detail, beginning with supersymmetry, generated by the vector fields $Q_\varepsilon = \varepsilon^I \frac{\partial}{\partial \theta^I}$, with constant parameters ε^I.

All vielbeins, both the bosonic ones E^a and the fermionic ones $\mathcal{E}^{\hat{\alpha} I} = (d\theta^I, \mathcal{E}^{\alpha I})$, are invariant under supersymmetry — this is just the statement that supersymmetry is an isometry of superspace. This accounts for the invariance of the kinetic volume term in the supermembrane action.

Invariance of the Wess-Zumino term $\int C$ is guaranteed by the invariance of the field strength G of eq. (2.13). The field strength is expressible as constant coefficients times wedge products of vielbeins, and thus invariant. This implies that the supersymmetry transformation of C is a total derivative, $Q_\varepsilon C = \varepsilon^I d\Lambda_I$. It is indeed possible to choose a gauge where a stronger statement, namely local invariance, $Q_\varepsilon C = 0$, holds. We have constructed C explicitly in such gauges (to lowest order in curvature), see appendix B. The fact that C can be chosen to be completely independent of θ^I will later, when Q_I are used as BRST operators, be a crucial property.

To begin our exposé of κ-symmetry for the topological membrane we recount some well known facts concerning its properties. In order to reduce clutter we drop the $sl(2)$-indices temporarily, reinserting them when returning to the topological membrane. We begin by introducing the superspace vector field,

$$\kappa = \kappa^M \partial_M = \kappa^\alpha \mathcal{E}_\alpha^M \partial_M \quad (\kappa^a = 0)\,, \tag{3.2}$$

the action of which transforms the pullback of a superspace form as,

$$\delta_\kappa(f^*\Omega) = f^* \mathcal{L}_\kappa \Omega = f^*(i_\kappa d + d i_\kappa)\Omega\,, \tag{3.3}$$

where f^* is a pullback and $\mathcal{L}$ a Lie derivative. From here on we will not write out pullbacks explicitly. The action of this vector field on the Wess-Zumino term then follows,

$$\delta_\kappa \int C = \int (i_\kappa d + d i_\kappa) C = \int (i_\kappa G + d i_\kappa C) = \int i_\kappa G\,, \tag{3.4}$$

and the variation of the vielbein,

$$\begin{aligned}\delta_\kappa E^A &= i_\kappa(T^A - E^B \wedge \omega_B{}^A) + D i_\kappa E^A - (i_\kappa E^B) \wedge \omega_B{}^A \\ &= i_\kappa T^A - E^B \wedge i_\kappa \omega_B{}^A + D i_\kappa E^A\,.\end{aligned} \tag{3.5}$$

By adding a local Lorentz transformation with parameter $i_\kappa \omega_B{}^A$, we can reduce the expression to $\delta_\kappa E^A = i_\kappa T^A + D i_\kappa E^A$, and furthermore, by considering the relevant part of this expression, to

$$\delta_\kappa E^a = i_\kappa T^a\,. \tag{3.6}$$

The variation of the kinetic term then becomes (with pullbacks written out)

$$\delta_\kappa \sqrt{g} = \frac{1}{2}\sqrt{g} g^{ij} \delta_\kappa g_{ij} = \sqrt{g} g^{ij} E^a_{(i} E^B_{j)} \kappa^\alpha T_{\alpha B}{}^a\,, \tag{3.7}$$

where we have used $\delta_\kappa g_{ij} = \delta_\kappa (E^a_i E^a_j) = 2 E^a_{(i} E^B_{j)} \kappa^\alpha T_{\alpha B}{}^a$. At the level of (length-)dimension 0 this term varies as

$$\delta_\kappa \sqrt{g} = \sqrt{g} g^{ij} E^a_i E^\beta_j \kappa^\alpha T_{\alpha\beta}{}^a = \sqrt{g} E^\beta_j T_{\alpha\beta}{}^j \kappa^\alpha\,, \tag{3.8}$$

whereupon the action consequently transforms as

$$\delta_\kappa \left(\int d^3\xi \sqrt{g} + \int C \right) = \int d^3\xi \sqrt{g} \left(E^\beta_i T_{\alpha\beta}{}^i \kappa^\alpha + \frac{1}{2}\frac{\varepsilon^{ijk}}{\sqrt{g}} E^\beta_k \kappa^\alpha G_{ij\alpha\beta} \right).. \tag{3.9}$$

Turning to the case of the G_2-membrane this transformation, after insertion of the torsion and field strength, looks like,

$$\begin{aligned}\delta_\kappa S &= \int d^3\xi \sqrt{g} \left(E^{\hat\alpha I}_i T_{\hat\alpha I, \hat\beta J}{}^i \kappa^{\hat\beta J} + \frac{1}{2}\frac{\varepsilon^{ijk}}{\sqrt{g}} E^{\hat\alpha I}_k G_{ij\hat\alpha I, \hat\beta J} \kappa^{\hat\beta J} \right) \\ &= \int d^3\xi \sqrt{g} E^{\hat\alpha I}_i \left(2(\gamma^i)_{\hat\alpha\hat\beta} - \frac{\varepsilon^{ijk}}{\sqrt{g}} (\gamma_{jk})_{\hat\alpha\hat\beta} \right) \kappa^{\hat\beta J} \varepsilon_{IJ}\,,\end{aligned} \tag{3.10}$$

which can be rewritten as

$$\delta_\kappa S = 4 \int d^3\xi \sqrt{g} E^{\hat\alpha I}_i (\gamma^i \Pi_+)_{\hat\alpha\hat\beta} \kappa^{\hat\beta J} \varepsilon_{IJ} \tag{3.11}$$

The κ-symmetry condition is thus $(\Pi_+)^{\hat\alpha}{}_{\hat\beta} \kappa^{\hat\beta I} = 0$, where

$$(\Pi_+)^{\hat\alpha}{}_{\hat\beta} \equiv \frac{1}{2}\left\{ \delta^{\hat\alpha}{}_{\hat\beta} + \frac{1}{6\sqrt{g}} \varepsilon^{ijk} (\gamma_{ijk})^{\hat\alpha}{}_{\hat\beta} \right\} \tag{3.12}$$

is the operator which annihilates an infinitesimal κ-parameter. The fact that

$$\Gamma^{\hat\alpha}{}_{\hat\beta} \equiv \frac{1}{6\sqrt{g}} \varepsilon^{ijk} (\gamma_{ijk})^{\hat\alpha}{}_{\hat\beta} = \frac{1}{6\sqrt{g}} \varepsilon^{ijk} \begin{bmatrix} \sigma_{ijk} & -\star\sigma_{ijk\beta} \\ -\star\sigma_{ijk}{}^\alpha & 6\left(\delta^{(\alpha}_{[i}\sigma^{\beta)}{}_{jk]} - \frac{1}{6}\delta^{\alpha\beta}\sigma_{ijk}\right) \end{bmatrix} \tag{3.13}$$

fulfills the conditions $\mathrm{Tr}(\Gamma) = 0$ and $\Gamma^2 = 1$ implies that Π_+ is a projection operator[2]. It is then obvious that the κ-symmetry condition can be solved by $\kappa = \Pi_- \xi$, where Π_- is defined

[2] An essential observation for the working of κ-symmetry is that the euclidean signature of the world-volume is compensated by the fact that the gamma matrices square to minus one. Compared to 11-dimensional Minkowski space there are two changes of sign. Had only one of these changes occurred, idempotent projection matrices could not have been constructed.

as $(\Pi_-)^{\hat\alpha}{}_{\hat\beta} \equiv \frac{1}{2}(\delta^{\hat\alpha}{}_{\hat\beta} - \Gamma^{\hat\alpha}{}_{\hat\beta})$ and ξ is an arbitrary spinor. Since Π_- projects out half of the degrees of freedom of ξ, κ is parametrised by two scalars and two world-volume vectors $\{\lambda^{0I}, \lambda^{iI}\}$. It can be shown that $\Pi_\pm$ are the only projection operators, which project out precisely half of the spinors, that can be formed using the G_2 invariant tensors only, and hence we have found the most general κ-variation.

A byproduct of the above calculation is that the fermionic equations of motion are $\Pi_+\gamma^i\mathcal{E}_i^{\hat\alpha I} = 0$.

A general background will of course contain fermionic excitations, demanding that κ-symmetry is checked also at dimension $\frac{1}{2}$. In the present context, however, we are only interested in superspaces extending any bosonic manifold of G_2 holonomy. We do not consider deformations of the geometry. In topological M-theory, such deformations should be parametrised by solutions of the Hitchin model, and purely bosonic.

The algebra of κ-symmetry is obtained by commuting κ-variations of a fermionic variable, which after some calculation, mainly involving transformation of the projection matrix, yields

$$\begin{aligned}[\delta_{\tilde\kappa}, \delta_\kappa]\psi^{\hat\alpha I} = {} & \varepsilon_{LK}(\Pi_+\gamma^i\mathcal{E}_i)^{\hat\alpha K}\tilde\kappa^{(L}\kappa^{I)} - \varepsilon_{LK}(\gamma^j)^{\hat\alpha}{}_{\hat\beta}(\Pi_+\gamma^i\mathcal{E}_i)^{\hat\beta K}\tilde\kappa^{[L}\gamma_j\kappa^{I]} \\ & +(\Pi_-)^{\hat\alpha}{}_{\hat\beta}\left\{\frac{1}{2}[(\delta_{\tilde\kappa}\Pi_-)\xi - (\delta_\kappa\Pi_-)\tilde\xi]^{\hat\beta I} + (\mathcal{E}_i)^{\hat\beta I}\varepsilon_{KL}\tilde\kappa^K\gamma^i\kappa^L\right\} \\ & -(\mathcal{E}_i)^{\hat\alpha I}\varepsilon_{KL}\tilde\kappa^K\gamma^i\kappa^L\,. \end{aligned} \tag{3.14}$$

It is straight-forward to see that the three rows represent fermionic equations of motion, κ-transformations and world-volume diffeomorphisms, respectively. This is the point where it becomes clear that the formulation, due to the mismatch between fermions and bosons off-shell, is an on-shell formulation — part of the gauge symmetry only works modulo equations of motion.

Although we will not elaborate on this in the present paper, it is worth mentioning that κ-symmetry can in fact be treated in a completely covariant manner on a G_2 manifold. The projection $\kappa = \Pi_+\kappa$ may be solved by parametrising κ in terms of a scalar and a world-volume vector as

$$\begin{aligned}\kappa^0 &= (1-y)\xi\,, \\ \kappa^\alpha &= z^\alpha\xi + (E_i^\alpha - \tfrac{1}{2\sqrt{g}}\varepsilon_i{}^{jk}\sigma^\alpha{}_{jk})\zeta^i\,, \end{aligned} \tag{3.15}$$

where $y = \frac{1}{6\sqrt{g}}\varepsilon^{ijk}\sigma_{ijk}$, $z_\alpha = \frac{1}{6\sqrt{g}}\varepsilon^{ijk}\star\sigma_{ijk\alpha}$. In a situation where the scalar part has been fixed, the remaining gauge symmetry (closing on-shell) will be a super-diffeomorphism algebra with an SL(2) doublet of world-volume vectors as fermionic generators.

There is an interplay between the global supersymmetry and the local κ-symmetry, in the sense that both transform the singlet fermions θ^I. Even if the supersymmetry generators obey eq. (2.19) exactly and without reference to the embedding of the membrane world-volume, this ceased to be true once κ-symmetry is gauge fixed. When some gauge is chosen that involves θ^I (which any gauge has to), compensating gauge transformations have to be introduced in order that the redefined supersymmetry generator transforms within the constraint surface defined by the gauge choice. Then, due to the commutation relation (3.14), the nilpotency relation (2.19) only holds on-shell, i.e., modulo fermionic equations of motion.

3.3 Topological membranes

In order to restrict the supermembrane theory to a topological theory, we want to promote the two supercharges Q_I to BRST operators, and let the theory be defined by cohomology of these. Unlike the theory of topological strings, where one in a conformal gauge has a split in left- and right-movers (or holomorphic and anti-holomorphic dependence of the world-sheet coordinate), there is no such natural split, and one has to treat the two supersymmetry generators simultaneously.

We have already shown how the invariance of the supermembrane action works. If the theory is to become a cohomological field theory, it is important that the action not only is invariant, but also trivial in cohomology, i.e., BRST-exact. This means that there should exist a functional Σ^I with

$$Q_I\Sigma^J = \delta_I^J S\,. \tag{3.16}$$

The simple form of the supersymmetry generators assures that this is achieved by $\Sigma^I = \int \mathcal{L}\theta^I$, if $Q_I\mathcal{L} = 0$ locally on the world-volume, and the action is then invariant without resort to partial integration. We have demonstrated earlier that this is actually the case, due to the fact that a gauge can be chosen where the 3-form C is independent of θ. The proof of this statement involved the explicit construction of the superspace 3-form, which used flat space expressions, but should be possible to generalise.

The "pre-action" $\Sigma^I$is defined modulo Q-exact terms, encoded in $Q_I\Xi^{JK} = \delta_I^K\Delta\Sigma^J$, which can be seen as "gauge transformations" in the complex. It is important that other gauge symmetries in the model are consistent with this one, in the sense that Σ^I must be invariant modulo terms of this trivial type. This applies especially to κ-symmetry, which is not manifest. Indeed, the fact that the κ-variation of $\mathcal{L}$ is a total derivative ensures that, with the above form of Σ^I, $\delta_\kappa\Sigma^I$ is trivial. This property becomes essential e.g. when one wants to perform a gauge-fixing of a part of κ-symmetry that transforms θ^I. Then, $Q_I$has to be supplemented with a compensating gauge transformation, which can not be allowed to interfere with cohomology. Consider an infinitesimal "deformation" of the supersymmetry generator by a κ-transformation, $\tilde{Q}_I = Q_I + M_I^{\mathcal{A}}t_{\mathcal{A}}$, where $t_{\mathcal{A}}$ are generators of some gauge transformations labelled by the index $\mathcal{A}$, and $M_I^{\mathcal{A}}$ are infinitesimal parameters. If $Q_I\Xi_{\mathcal{A}}^{JK} = \frac{1}{2}\delta_I^K Q_L\Xi_{\mathcal{A}}^{JL} = \delta_I^K t_{\mathcal{A}}\Sigma^J$ as above, one can define $\tilde{\Sigma}^I = \Sigma^I - M_J^{\mathcal{A}}\Xi_{\mathcal{A}}^{IJ}$, and still have $\tilde{Q}_I\tilde{\Sigma}^J = \delta_I^J S$. A finite deformation, as when gauge-fixing is performed, will require the discussion to be extended to an infinite sequence of descent equations.

An interesting parallel to topological string theory can be observed when one tries to construct a Σ^I that is "as κ-invariant as possible", order by order in fermions. An Ansatz would, apart from the expression above, include terms that are independent of θ,

$$\Sigma^I = \int d^3\xi\sqrt{g}\theta^I + \int((C + k\Omega)\theta^I + R^I)\,. \tag{3.17}$$

Here, R is a 3-form with $Q_I R = 0$, and the the term containing Ω modifies eq. (3.16) with a purely topological term,

$$Q_I \Sigma^J = \delta_I^J (S + k\Omega)\,. \tag{3.18}$$

Using elements of the calculation yielding κ-symmetry of the action, one finds

$$\delta_\kappa \Sigma^I = \int i_\kappa((\star 1 + C + k\Omega)\wedge d\theta^I + dR^I)\,. \tag{3.19}$$

Invariance at lowest order can be achieved if $k = 1$ and $R^I_{abc} = -\star\sigma_{abc\alpha}\psi^{\alpha I}$, in which case the lowest order variation becomes $\int 2(\Pi_+\kappa^I)^0 = 0$, which is seen from the decomposition (3.13) of the projection matrix in G_2 tensors. However, exact cancellation to all orders is not possible by addition of further terms in R^I. Again, of course, the non-zero terms in the variation are trivial. The relation (3.18), with $k = 1$, is the exact correspondence to the fact that in topological string theory, the BRST-trivial object is the action plus the integral of the Kähler form, which is obtained from Ω on dimensional reduction.

4. Topological membranes in topological M-theory

We have shown how a supermembrane in seven dimensions with euclidean signature can be turned into a topological theory. It would be interesting to study the quantum mechanical properties of the topological membrane theory, and investigate to what extent the quantum theory reproduces topological M-theory. The best framework for doing this would be one including a proper set of auxiliary fields that makes the symmetries of the theory valid of-shell. It seems much harder to reach such a formulation in the present situation than for the usual world-sheet supersymmetric sigma model on which topological string theory is based.

It is clear that associative cycles [17] are solutions of the theory. These are calibrating cycles for the 3-form Ω. An easy way to see that associative cycles are supersymmetric is to partially fix gauge for κ-symmetry by demanding $\theta^I = 0$. The supersymmetry, including a compensating κ-transformation, on the remaining fermions becomes

$$\delta_\varepsilon \psi^{\alpha I} = -\frac{z^\alpha}{1-y}\varepsilon^I\,, \tag{4.1}$$

where $y = \frac{1}{6\sqrt{g}}\varepsilon^{ijk}\sigma_{ijk}$, $z_\alpha = \frac{1}{6\sqrt{g}}\varepsilon^{ijk}\star\sigma_{ijk\alpha}$. A configuration is supersymmetric if $z^\alpha = 0$, giving the possibilities $y = \pm 1$, and if eq. (4.1) is to be well behaved only $y = -1$ is possible. With a non-zero Wess-Zumino term in the membrane action we are actually dealing with a generalised calibration, see e.g. refs. [18, 19]. It is however of a trivial type since the bosonic 3-form is closed and hence the Wess-Zumino term contributes equally to all cycles minimal or not.

Looking for local observables seems more problematic. In the A-model, considering collapsed, point-like, world-sheets is straightforward, and cohomology of the BRST-operator is directly translated into cohomology for a de Rahm-complex for the Calabi-Yau manifold. In the present situation, we have to take κ-symmetry into account, with its projection that depends on the orientation of the embedded world-volume. We have not yet been able to

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address this question in a constructive way, and thus can not present a direct connection between observables for the topological membrane and Hitchin's theory.

It is clear that a double dimensional reduction of the topological membrane produces the strings of the topological A-model, although formulated in a space supersymmetric rather than world-sheet supersymmetric way. Associative cycles will map to holomorphic cycles. For the same reasons as above, we are not able to make a corresponding statement concerning local observables (although investigating this question for the A-model starting from a Green-Schwarz formulation might give some insight)[3]. A direct reduction will of course give A-model 2-branes. These are not D-branes. An A-model D2-brane must be represented by a 3-brane in $D = 7$, since the boundary of an open membrane winding the compactified circle also winds. This, along with the existence of the dual form $\star\Omega$, makes it clear that 3-branes, on which the membranes may end, are needed in topological M-theory. The 3-branes, living on the same superspace, should support a world-volume 2-form potential, with a 3-form field strength. This field, that can be dualised to a scalar, accounts for the correct matching of bosonic and fermionic degrees of freedom. An interesting observation, on which we would like to elaborate in the future, is that although $(\gamma_{abc})_{\hat{\alpha}\hat{\beta}}$ is symmetric in spinor indices, and thus cannot be used in a dimension-0 component of the 5-form field strength for the 4-form potential coupling to the 3-brane, there exists a closed 5-form constructed from G_2-invariant tensors.

It would be a great step forward to find a good set of auxiliary fields for the membrane theory, that would allow for an off-shell formulation, and hopefully make quantisation more manageable. Although this, in general backgrounds, would probably be to ask too much, it is maybe not unrealistic to hope that the G_2 structure would help. It turns spinors into scalars and vectors, and even κ-symmetry can be parametrised covariantly, as in eq. (3.15). One possible starting point could be the construction of a super-diffeomorphism algebra on the world-volume containing an SL(2) doublet of fermionic vector generators, similar to what one obtains after gauge-fixing the scalar part of κ-symmetry.

Although we do not claim to have a microscopic definition of topological M-theory, we hope that the present work represents a step in that direction. Maybe it can be a point of departure for a refined formulation, where urgent questions, such as the connection to Hitchin's theory of G_2 moduli, can be answered. Such a formulation might also give valuable insight into the question of how membrane functional integrals are performed (see e.g. the discussion in ref. [7]). Earlier experience of instanton counting on compact submanifolds have shown that naive counting of membrane configurations may lead to incorrect results [24, 25], and a proper theory of topological membranes may be a place where such issues can be addressed in a precise manner.

A. Some details on G_2 tensors

We use e.g. the expressions $\sigma_{a,a+1,a+3} = 1$ (where indices are counted modulo 7), giving $\star\sigma_{a,a+1,a+2,a+5} = 1$. $\star\sigma$ is the octonionic associator, $[e_a, e_b, e_c] = (e_a e_b)e_c - e_a(e_b e_c) =$

[3]Such a formulation will be possible directly in six dimensions for both the A- and B-models. One has a priori an SL(2) doublet of complex supersymmetries, of which different real combinations may be chosen.

$-2\star\sigma_{abcd}e_d$. Useful relations between octonionic structure constants:

$$
\begin{aligned}
\sigma_{acd}\sigma_b{}^{cd} &= 6\delta_{ab}\,,\\
\star\sigma_{abde}\sigma_c{}^{de} &= -4\sigma_{abc}\,,\\
\sigma_{abc}\sigma^{abc} &= 42\,,\\
\sigma_{abe}\sigma_{cd}{}^{e} &= 2\delta^{ab}_{cd} - \star\sigma_{abcd}\,,\\
\sigma_{abf}\star\sigma_{cde}{}^{f} &= 6\delta^{[a}_{[c}\sigma^{b]}{}_{de]}\,,\\
\star\sigma_{abcg}\star\sigma_{def}{}^{g} &= 6\delta^{abc}_{def} - 3\delta^{[a}_{[d}\star\sigma^{bc]}{}_{ef]} - 3\sigma_{[ab}{}^{[d}\sigma_{c]}{}^{ef]}\,,\\
\star\sigma_{abef}\star\sigma_{cd}{}^{ef} &= 8\delta^{ab}_{cd} - 2\star\sigma_{abcd}\,.
\end{aligned}
\tag{A.1}
$$

The last of these relations can be used to find projections on the 7- and 14-dimensional vector spaces in $21 \to 14 \oplus 7$ under $\mathrm{Spin}(7) \to G_2$ as

$$
\begin{aligned}
\Pi^{(14)}{}_{ab}{}^{cd} &= \frac{2}{3}(\delta^{cd}_{ab} + \frac{1}{4}\star\sigma_{ab}{}^{cd})\,,\\
\Pi^{(7)}{}_{ab}{}^{cd} &= \frac{1}{3}(\delta^{cd}_{ab} - \frac{1}{2}\star\sigma_{ab}{}^{cd})\,.
\end{aligned}
\tag{A.2}
$$

It can be noted that σ_{abc}, seen as a set of seven matrices $(\sigma_a)_{bc}$, are in the 7-dimensional subspace: $(\Pi^{(14)}\sigma_a)_{bc} = 0$, and actually provide a basis for it.

Consider the split of the octonions $\mathbb{O}$ as $\mathbb{H} \oplus \mathbb{H}$, and write $x = \xi + j\eta$, where ξ and η are quaternions and j is an imaginary unit orthogonal to $\mathbb{H}$. The octonionic multiplication is encoded in terms of the quaternions by the multiplication rules $ja = a^*j$, $(ja)b = j(ba)$ for all $a, b \in \mathbb{H}$. Then

$$
xx' = \xi\xi' - \eta'\eta^* + j(\xi^*\eta' + \xi'\eta)\,. \tag{A.3}
$$

We want to examine which of the rotations in $\mathrm{SO}(3) \times \mathrm{SO}(4)$ acting on imaginary octonions and preserving this split are automorphisms, i.e., belong to G_2. The rotations are parametrised as $\xi \to \sigma^*\xi\sigma$, $\eta \to e^*\eta e'$, where all three parameters are unit quaternions. A direct check with eq. (A.3) yields that the necessary condition for this to be an automorphism is $\sigma = e$, verifying that the common subgroup of this $\mathrm{SO}(3) \times \mathrm{SO}(4)$ and G_2 is $\mathrm{SU}(2) \times \mathrm{SU}(2)$, and that the twisting — the identification of world-volume SO(3) rotations with a transverse SU(2)—takes place.

The remaining part of the G_2 algebra transforms as $(4, 2)$, and is realised infinitesimally with a "vector-spinor" h_i, $i = 1, 2, 3$, in $\mathbb{H}^{\otimes 3}$ with $e_ih_i = 0$. The transformations are $\delta\xi = e_i\eta h_i^*$, $\delta\eta = e_i\xi h_i$ $(= -2\xi_ih_i)$, and the derivation property may be checked explicitly.

The split into two quaternions can also be seen as a split in four complex numbers with imaginary unit j. With $x = z_0 + z^ie_i$, the multiplication table is $xx' = z_0z_0' - z^i\bar{z}_i' + (z_0z'^i + z^i\bar{z}_0' + \varepsilon^{ijk}\bar{z}_j\bar{z}_k')e_i$, in which $\mathrm{SU}(3) \subset G_2$ is a manifest automorphism. The rest of the automorphisms are parametrised by λ^i, $\bar{\lambda}_i$ in $3 \oplus \bar{3}$, acting as $\delta z_0 = \lambda^i\bar{z}_i - \bar{\lambda}_iz^i$, $\delta z^i = \lambda^i(z_0 - \bar{z}_0) + \varepsilon^{ijk}\bar{\lambda}_j\bar{z}_k$.

B. 3-forms in superspace

The field strength G is related to the potential C in the conventional way

$$
G = dC \qquad \Rightarrow \qquad G_{ABCD} = 4\delta_{[A}C_{BCD)} + 6T_{[AB}{}^{F}C_{|F|CD)}\,, \tag{B.1}
$$

where the indices in capital letters are the entire superspace indices. The bracket $[*)$ denotes a graded symmetrisation. Using the fact that in a flat background, the only non-vanishing components of G_{ABCD} and $T_{AB}{}^{C}$ are $G_{ab,\hat{\gamma}I,\hat{\delta}J} = -2\varepsilon_{IJ}(\gamma_{ab})_{\hat{\gamma}\hat{\delta}}$ and $T_{\hat{\alpha}I,\hat{\beta}J}{}^{c} = 2\varepsilon_{IJ}(\gamma^{c})_{\hat{\alpha}\hat{\beta}}$, respectively, the equations (B.1) can be solved for C_{ABC}. The solution we are interested in has the property that the only coordinate dependence is through the seven-dimensional fermionic coordinates $\psi^{\alpha I}$. By looking at the group representation structures of the different components of C, we make an Ansatz for the potential using G_2 invariants. Due to invariance under the gauge transformations $\delta C = d\Lambda$, some of the parameters in the Ansatz are free, and set to zero for simplicity. We use flat space or work to lowest order in curvatures. The potential we have found can be written as $C^{(k)} = C + k\Omega$, where

$$\begin{aligned}
C_{abc} &= 0\\
C_{ab,0I} &= \varepsilon_{IL}\psi^{\delta L}2\sigma_{ab\delta}\\
C_{ab,\alpha I} &= \varepsilon_{IL}\psi^{\delta L}(2\delta^{ab}_{\alpha\delta} + \star\sigma_{ab\alpha\delta})\\
C_{a,0I,0J} &= \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}(-4)\sigma_{a\delta\epsilon}\\
C_{a,\alpha I,0J} &= \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}\left(-4\delta^{a\alpha}_{\delta\epsilon} - \frac{2}{3}\star\sigma_{a\alpha\delta\epsilon}\right)\\
C_{a,\alpha I,\beta J} &= \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}\left(-\frac{4}{3}\sigma_{a\delta\epsilon}\delta_{\alpha\beta} - \frac{8}{3}\sigma_{a\alpha\delta}\delta_{\beta\epsilon}\right)\\
C_{0I,0J,0K} &= \varepsilon_{I(L}\varepsilon_{|J|M}\varepsilon_{|K|N)}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}8\sigma_{\delta\epsilon\phi}\\
C_{\alpha I,0J,0K} &= \varepsilon_{IL}\varepsilon_{J(M}\varepsilon_{|K|N)}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}\left(-\frac{8}{3}\right)\star\sigma_{\alpha\delta\epsilon\phi}\\
C_{\alpha I,\beta J,0K} &= \varepsilon_{IL}\varepsilon_{JM}\varepsilon_{KN}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}\left(\frac{8}{3}\delta_{\alpha\beta}\sigma_{\delta\epsilon\phi} + 8\sigma_{\alpha\epsilon\phi}\delta_{\beta\delta}\right)\\
C_{\alpha I,\beta J,\gamma K} &= \varepsilon_{IL}\varepsilon_{JM}\varepsilon_{KN}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}8\delta_{\alpha\phi}\star\sigma_{\beta\gamma\delta\epsilon}
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
\Omega_{abc} &= \sigma_{abc}\\
\Omega_{ab,0I} &= \varepsilon_{IL}\psi^{\delta L}(-2)\sigma_{ab\delta}\\
\Omega_{ab,\alpha I} &= \varepsilon_{IL}\psi^{\delta L}(-2\delta^{ab}_{\alpha\delta} + \star\sigma_{ab\alpha\delta})\\
\Omega_{a,0I,0J} &= \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}4\sigma_{a\delta\epsilon}\\
\Omega_{a,\alpha I,0J} &= \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}(4\delta^{a\alpha}_{\delta\epsilon} + 2\star\sigma_{a\alpha\delta\epsilon})\\
\Omega_{a,\alpha I,\beta J} &= \varepsilon_{I(L}\varepsilon_{|J|M)}\psi^{\delta L}\psi^{\epsilon M}\left(\frac{8}{3}\delta_{a\alpha}\sigma_{\beta\delta\epsilon} - \frac{4}{3}\sigma_{a\delta\epsilon}\delta_{\alpha\beta} - \frac{8}{3}\sigma_{a\alpha\delta}\delta_{\beta\epsilon}\right)\\
\Omega_{0I,0J,0K} &= \varepsilon_{I(L}\varepsilon_{|J|M}\varepsilon_{|K|N)}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}(-8)\sigma_{\delta\epsilon\phi}\\
\Omega_{\alpha I,0J,0K} &= \varepsilon_{IL}\varepsilon_{J(M}\varepsilon_{|K|N)}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}4\star\sigma_{\alpha\delta\epsilon\phi}\\
\Omega_{\alpha I,\beta J,0K} &= \varepsilon_{IL}\varepsilon_{JM}\varepsilon_{KN}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}\left(-\frac{8}{3}\delta_{\alpha\beta}\sigma_{\delta\epsilon\phi} - \frac{56}{3}\sigma_{\alpha\epsilon\phi}\delta_{\beta\delta}\right)\\
\Omega_{\alpha I,\beta J,\gamma K} &= \varepsilon_{IL}\varepsilon_{JM}\varepsilon_{KN}\psi^{\delta L}\psi^{\epsilon M}\psi^{\phi N}8\delta_{\alpha\phi}\star\sigma_{\beta\gamma\delta\epsilon}
\end{aligned} \tag{B.3}$$

and k is a free parameter. Symmetrisation in composite fermionic indices is implicitly understood in eqs. (B.2) and (B.3). Eq. (B.3) can of course be obtained directly (modulo an exact form) by expanding the bosonic differentials in $\Omega = \frac{1}{6}dx^p \wedge dx^n \wedge dx^m e_p{}^c e_n{}^b e_m{}^a \sigma_{abc}$ using the vielbeins of eq. (2.16).

The fact that θ^I are G_2-invariant should make it clear that the proof of local θ-independence of the lagrangian, on which the BRST-exactness relies, may be generalised to curved backgrounds, involving modifications of the explicit forms of the supervielbeins of eq. (2.16) and the super-3-forms of eqs. (B.2) and (B.3).

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Paper II

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A note on topological M5-branes and string-fivebrane duality

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Abstract: We derive the stability conditions for the M5-brane in topological M-theory using κ-symmetry. The non-linearly self-dual 3-form on the world-volume is necessarily non-vanishing, as is the case also for the 2-form field strengths on coisotropic branes in topological string theory. It is demonstrated that the self-duality is consistent with the stability conditions, which are solved locally in terms of a tensor in the representation **6** of SU(3) $\subset G_2$. The double dimensional reduction of the M5-brane is the D4-brane, and its direct reduction is an NS5-brane. We show that the equation of motion for the 3-form on the NS5-brane wrapping a Calabi-Yau space is exactly the Kodaira-Spencer equation, providing support for a string-fivebrane duality in topological string theory.

Keywords: p-branes, Topological Strings, M-Theory.

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Contents

1. Introduction and conclusions

The purpose of this paper is to formulate and examine the stability conditions (generalised calibration relations) for M5-branes in the topological M-theory formulated in ref. [17] (see also refs. [9, 39]). The stability conditions, which have been discussed from a world sheet point of view for D-branes in string theory in ref. [25] and for topological string theory in refs. [26, 8, 27], can also be seen as a direct consequence of calibration [28] or demanding supersymmetry [29, 7, 30, 31]. As is the case e.g. for the D4-brane in the A-model, the stability conditions demand non-vanishing world-volume field strength. Here we derive the corresponding stability conditions for the M5-brane in topological M-theory and its close relative the NS5-brane in the topological A-model. This is achieved using the κ-symmetric top-form formulation applied to the physical M5-brane in ref. [1]. In this approach there is in the 7-dimensional G_2 superspace, apart from the super-4-form field strength, also a super-7-form field strength obeying the appropriate Bianchi identities, but without a bosonic component.

The M5-brane is, apart from the topological membrane constructed in ref. [2] (for a different approach see ref. [23, 32]), the only brane present in topological M-theory.[1] Their direct and double dimensional reductions on a circle to a Calabi-Yau space give all NS-branes and D-branes in the A-model save for the isotropic D-branes with one-dimensional world sheets introduced in ref. [27] which should probably be viewed as Kaluza-Klein modes.

We proceed to demonstrate how the direct reduction of the M5-brane on CY$\times S^1$ gives the NS5-brane in the topological A-model introduced in ref. [18, 19] (see refs. [11, 33] for a review of topological string theory), whose world-volume inherits the dynamical Kodaira-Spencer deformation theory [34] from the M5-brane. The related connection between the M5-brane instantons in the physical M-theory and Kodaira-Spencer theory was first pointed out in ref. [7]. The double reduction will give the D4-brane, with the stability

[1] G_2 target spaces occur also in the topological string constructed in ref. [3]; its relation to topological M-theory is, however, unclear to us.

conditions formulated by Kapustin and Li [8] (although that correspondence is not shown in the present paper). The NS5-brane provides a precise description of how duality between Kähler gravity [22] and Kodaira-Spencer theory [21], describing deformations of the Kähler and complex structures, respectively, is realised in the A-model as a "string-fivebrane duality" [35]. A forthcoming paper [24] will extend the discussion to the full sets of D-branes and RR fields in the A- and B-models.

Related conjectures have been made earlier. In ref. [10] Dijkgraaf, Verlinde and Vonk used T-duality to relate the partition function on coinciding NS5-branes (with linear self-duality) in the A-model to a B-model calculation. S-duality, relating the A- and B-models on the same manifold, for topological strings, was conjectured on a twistorial CY by Neitzke and Vafa [11], and clarified, mainly using D-instantons, by Nekrasov, Ooguri and Vafa in ref. [12], where the existence of the topological NS5-brane was also pointed out. The relevance of the calculation of ref. [10] in this context was observed in ref. [36]. Gerasimov and Shatashvili, in their paper pointing towards a topological M-theory [9], relate Kodaira-Spencer theory to a 7-dimensional theory. Mariño et al. [7] derive conditions for $D = 11$ M5-branes wrapping a Calabi-Yau space to preserve supersymmetry, and derive the Kodaira-Spencer equation. We comment to the relation of the present paper to the latter work in section 3.

2. Topological M5-branes

The reduction of topological M-theory on a circle contains the A-model [17]. The presence in the A-model of a D4-brane and an NS5-brane implies that there has to be a 5-brane in topological M-theory. The purpose of this section is to derive, using superspace techniques and κ-symmetry, the stability conditions for this topological M5-brane, and to demonstrate the consistency between these conditions and the non-linear self-duality for the 3-form field strength on the brane. Open topological membranes have boundaries on the 5-brane, just as fundamental strings end on D-branes and D-branes on NS5-branes in the A-model.

As in ref. [2], where topoloical membranes were considered, the background for the branes is described by superspace geometry. This approach was motivated by the absence of "spinning" supermembranes. In such a formulation there is no need for explicitly performing a twist to obtain the representations of the fields in the topological model; the "twisting" is automatically implied by the G_2 holonomy. The relevant superspace is that of minimal (euclidean) 7-dimensional supergravity, with 16 real fermionic directions and R-symmetry group SL(2). The background is not treated as dynamical. The formalism makes the connection to M-theory and its instantons direct. It is possible to show that the 7-dimensional Hitchin model corresponding to deformations of the G_2 structure is obtained by considering the cohomology of the surviving supercharge in this half-maximal supergravity theory, and we will do this in detail in a forthcoming publication [24].

Let us begin with some details and conventions concerning the superspace background. The number of fermionic coordinates, 16, is half of that superstring theory or M-theory, and appropriate for the formulation of a topological 7-or 6-dimensional theory. The dimension-0 components of the torsion and the 4-form field strength are

$$T^a_{\alpha I,\beta J} = 2\varepsilon_{IJ}\gamma^a_{\alpha\beta}\,, \tag{2.1}$$
$$H_{ab,\alpha I,\beta J} = 2\varepsilon_{IJ}(\gamma_{ab})_{\alpha\beta}\,. \tag{2.2}$$

The real γ-matrices, which can be viewed as imaginary unit octonions multiplying octonionic spinors of Spin(7), square to -1. For details on γ-matrices etc., we refer to the appendix and to ref. [2].

Even though there is no bosonic 7-form field strength in the supergravity multiplet, there is a 7-form field strength on superspace, namely

$$\mathcal{H}_{abcde,\alpha I,\beta J} = 2\varepsilon_{IJ}(\gamma_{abcde})_{\alpha\beta}\,,$$

with the Bianchi identity $d\mathcal{H}+\frac{1}{2}H\wedge H = 0$, following from the 7-dimensional Fierz identities. This presence of a superspace field strength that does not contain a purely bosonic part, or, more precisely, the absence of an invariant cohomologically non-trivial 6-form to calibrate the 6-cycle of the brane world-volume, is symptomatic for the cases of high-dimensional branes where non-vanishing world-volume field strength is demanded by the generalised calibration (stability) conditions.

We write an action for the 5-brane in complete analogy with ref. [1], the only difference being that the signature of the world-volume is euclidean,

$$S = \int d^6\xi\sqrt{g}\lambda\left[1+\Phi(F)+(\star\mathcal{F})^2\right]\,,$$

where the field λ is a Lagrange multiplier and Φ a functional to be determined. $\mathcal{F}$ is the modified 6-form field strength of a 5-form potential $\mathcal{A}$ and the 3-form F is the field strength of the 2-form A:

$$F = dA - C\,, \tag{2.3}$$
$$\mathcal{F} = d\mathcal{A} - \mathcal{C} - \frac{1}{2}A\wedge H \tag{2.4}$$

where the pullbacked superfield potentials C and $\mathcal{C}$ provide the coupling to the background. These field strengths are constructed with background gauge invariance as guideline. The Bianchi identities are $dF = -H$, $d\mathcal{F} = -\mathcal{H}+\frac{1}{2}F\wedge H$. The action has of course to be supplemented by some self-duality condition. The advantage of actions of this type [13–15, 1, 16], with world-volume fields corresponding to all background fields the brane couples to, is (apart from complete control over background couplings and possible boundary conditions for lower-dimensional branes) that consistency of the non-linear self-duality relation is restrictive enough that demanding κ-symmetry gives its explicit form, which can be obtained without *a priori* specifying the function Φ. At the same time, the corresponding projector on κ is derived, and Φ can be constructed.

We define $K^{ijk}\equiv\frac{\partial\Phi}{\partial F_{ijk}}$. The equations of motion for A, $\mathcal{A}$ and λ are

$$d(\lambda\star K) - \lambda(\star\mathcal{F})H = 0\,, \tag{2.5}$$
$$d(\lambda\star\mathcal{F}) = 0\,, \tag{2.6}$$
$$1+\Phi+(\star\mathcal{F})^2 = 0\,, \tag{2.7}$$

respectively. These must be consistent with the Bianchi identities, thus, combining the first two equations of motion with the Bianchi identity $dF = -H$ we find $K = (\star\mathcal{F}) \star F$. By varying the action using $\delta_\kappa F = -i_\kappa H$ and $\delta_\kappa \mathcal{F} = -i_\kappa \mathcal{H} + \frac{1}{2} F \wedge i_\kappa H$ and inserting the relation between K and the field strengths, the projection matrix on κ and the non-linear self-duality of the field strengths are obtained. We leave out the details, since they are in close parallel to ref. [1], and state the result. For the action to be invariant under the κ-symmetry the parameter κ must satisfy $(1-\Gamma)\kappa = 0$, with

$$\Gamma = \frac{i}{N\sqrt{g}} \varepsilon^{ijklmn} \left[\frac{1}{6!} \gamma_{ijklmn} + \frac{1}{2(3!)^2} F_{ijk} \gamma_{lmn} \right] \tag{2.8}$$

and $N \equiv \sqrt{1+\Phi}$. The self-duality relation is

$$iN\star F_{ijk} = N^2 F_{ijk} + \frac{1}{2} q_{[i}{}^l F_{jk]l} \,, \tag{2.9}$$

where the sign choice $\star\mathcal{F} = -i\sqrt{1+\Phi} = -iN$ has been used. Here we have introduced the symmetric matrix $k_{ij} = \frac{1}{2} F_i{}^{kl} F_{jkl}$ and the traceless $q = k - \frac{1}{6} \mathrm{tr}\, k$. Inserting eq. (2.9), together with the Bianchi identities, into the equations of motion we find $\Phi = -\frac{1}{6} \mathrm{tr}\, k - \frac{1}{24} \mathrm{tr}\, q^2 + \frac{1}{144} (\mathrm{tr}\, k)^2$. On the other hand, contracting the self-duality relation (2.9) with F^{ijk} gives $\mathrm{tr}\, q^2 = -24 N^2 (1-N^2)$, which by representation theory turns out to be the stronger relation $q^2 = -4N^2(1-N^2) \mathbb{1}$. The equation of motion for the Lagrange multiplier λ now becomes

$$N^2 = 1 - \frac{1}{12} \mathrm{tr}\, k \,. \tag{2.10}$$

This relation follows in fact also from $\Gamma^2 = \mathbb{1}$. Dualising the self-duality relation (and using all the known relations between N, k and q as well as "$\star(qF) = -q\star F$") gives consistency.

After elimination of the top-form $\mathcal{F}$, we may write an action of a more standard type giving the same equations of motion,

$$S = \int d^6 \xi \sqrt{g(1+\Phi)} + i \int \left(\mathcal{C} - \frac{1}{2} F \wedge C \right) . \tag{2.11}$$

Although this type of action (supplemented with some self-duality[2]) is less convenient as a starting point, the calibration relations we derive below has a clearer interpretation as relating kinetic and Wess-Zumino terms, as usual.

We are now ready to consider this M5-brane in a manifold with G_2 holonomy, and look for 6-cycles that, together with the appropriate values of F, preserve supersymmetry. There is a covariantly constant spinor η^I (for each value of the SL(2) index I), which we take to be the real part of the octonion. We expect the global supersymmetry to play the rôle of BRST charges, in analogy with the situation for the topological membrane of ref. [2]. In that reference it was shown that the 3-form field strength, to which the topological membrane was coupled, can be made invariant under local supersymmetric transformations by an appropriate gauge choice. This observation is then used to show that the action is

[2]Note that the implementation of the self-duality condition [1] can only be done on the level of the partition function, see [6, 4, 5].

BRST-exact. We expect that the same can be shown for the 6-form field strength of the M5-brane, so that the action (2) is not only BRST-invariant (supersymmetric) but also BRST-exact, however the calculations involved would be quite extensive.

Using the explicit expressions for γ-matrices in terms of G_2-invariant tensors we have the action of Γ on the covariantly constant spinor:

$$\Gamma \begin{bmatrix}1\\0\end{bmatrix} = \frac{i}{N\sqrt{g}}\varepsilon^{ijklmn}\begin{bmatrix} \frac{1}{2(3!)^2}F_{ijk}\sigma_{lmn} \\ -\frac{1}{6!\sqrt{g}}\varepsilon_{ijklmn}\delta^{\alpha}_{7} + \frac{1}{2(3!)^2}F_{ijk}\star\sigma_{lmn\alpha}\end{bmatrix}, \tag{2.12}$$

where we for convenience have used a local basis where the direction dx^7 is normal to the world-volume. The tensor σ is the covariantly constant G_2-invariant 3-form. The criterion for supersymmetry is that $(1-\Gamma)\eta = 0$, which yields the stability conditions for the brane:

$$\frac{i}{2}F\wedge f^*\sigma = N\mathrm{Vol}_6\,, \tag{2.13}$$

$$\frac{1}{2}F\wedge\star\sigma = -\mathrm{Vol}_7\,, \tag{2.14}$$

$$F\wedge i_v f^*\star\sigma = 0\,, \tag{2.15}$$

where Vol_6 and Vol_7 are the world-volume and space volume forms, respectively, and v is any world-volume vector. In order to solve these relations locally, and check their consistency, we parametrise the tensors using the local breaking of G_2 to SU(3), and use the standard relations $\sigma = \mathrm{Re}\,\Omega + \omega\wedge dx^7$, $\star\sigma = -\mathrm{Im}\,\Omega\wedge dx^7 - \frac{1}{2}\omega\wedge\omega$ (see appendix for conventions). At the moment this is not necessarily to be seen as the direct reduction to an A-model NS5-brane, although the local parametrisation suits this case. The SU(3)-covariant version of the stability conditions is

$$\frac{i}{2}F\wedge f^*\mathrm{Re}\,\Omega = N\mathrm{Vol}_6\,, \tag{2.16}$$

$$\frac{1}{2}F\wedge f^*\mathrm{Im}\,\Omega = \mathrm{Vol}_6\,, \tag{2.17}$$

$$F\wedge f^*\omega = 0\,. \tag{2.18}$$

From the conditions (2.18) it follows immediately that $F_{abc} = -\frac{1+N}{4}\Omega_{abc}$, $F_{\bar{a}\bar{b}\bar{c}} = -\frac{1-N}{4}\bar{\Omega}_{\bar{a}\bar{b}\bar{c}}$ (we suppress explicit pullbacks from now on), and that $g^{b\bar{c}}F_{ab\bar{c}} = 0$ and $g^{a\bar{b}}F_{a\bar{b}\bar{c}} = 0$ (the last two equations leave only the representations $\bar{\mathbf{6}}$ out of $\bar{\mathbf{6}}\oplus\mathbf{3}$ in $F_{(2,1)}$ and $\mathbf{6}$ out of $\mathbf{6}\oplus\bar{\mathbf{3}}$ in $F_{(1,2)}$). It is not *a priori* clear that the stability conditions, derived from the G_2 structure, are consistent with the self-duality relations. We will however show that this is indeed the case, and that, given the value of $F_{(3,0)}$ from the stability condition, the self-duality relation dictates exactly the value of $F_{(0,3)}$ given after eq. (2.18).

It is convenient to parametrise the non-linearly self-dual 3-form F in terms of a linearly self-dual one, h. It is straightforward to show that $h_{ijk} = F_{ijk} + \frac{1}{2N(1+N)}q_i{}^lF_{jkl}$ satisfies $i\star h = h$. Forming the matrix $r_{ij} = \frac{1}{2}h_i{}^{kl}h_{jkl}$, the relations above give $r = \frac{2}{N(1+N)}q$, so the relation between h and F becomes $h_{ijk} = m_i{}^lF_{jkl}$, where $m = 1\!\!1 + \frac{1}{4}r$. Inverting the matrix m, $m^{-1} = \frac{(1+N)}{2}(1\!\!1 - \frac{1}{4}r)$, finally gives the explicit parametrisation of F in terms of h,

$$F_{ijk} = \frac{1+N}{2}\left(h_{ijk} - \frac{1}{4}r_i{}^lh_{jkl}\right)\,, \tag{2.19}$$

where the scalar N now is defined by $r^2 = -16\frac{1-N}{1+N}1\!\!1$.

The general Ansatz for h in terms of SU(3) tensors contains a singlet in $h_{(3,0)}$ (ξ), a triplet **3** in $h_{(2,1)}$ and the representation **6** in $h_{(1,2)}$ (u). It is clear that the triplet generates triplets in F violating the last equation in (2.18), so we set it to zero. The Ansatz becomes

$$h_{abc} = \frac{1}{2}\xi\Omega_{abc}\,, \tag{2.20}$$
$$h_{ab\bar{c}} = 0\,, \tag{2.21}$$
$$h_{a\bar{b}\bar{c}} = \frac{1}{2}u_a{}^{\bar{d}}\bar{\Omega}_{\bar{b}\bar{c}\bar{d}}\,, \tag{2.22}$$
$$h_{\bar{a}\bar{b}\bar{c}} = 0\,. \tag{2.23}$$

The matrix r has the non-vanishing components $r_{ab} = 4\xi u_a{}^{\bar{c}}g_{b\bar{c}}$, $r_{\bar{a}\bar{b}} = \frac{1}{4}\bar{\Omega}_{\bar{a}}{}^{cd}\bar{\Omega}_{\bar{b}\bar{e}\bar{f}}u_c{}^{\bar{e}}u_d{}^{\bar{f}}$. Calculating F from this Ansatz gives immediately $F_{abc} = \frac{1+N}{4}\xi\Omega_{abc}$, so $\xi = 1$ by the stability conditions. We have $\operatorname{tr} r^2 = 96\det u$ (note that $\det u = \frac{1}{8\cdot 3!}\bar{\Omega}^{abc}\bar{\Omega}_{\bar{a}\bar{b}\bar{c}}u_a{}^{\bar{a}}u_b{}^{\bar{b}}u_c{}^{\bar{c}}$), and thus $\det u = \frac{1-N}{1+N}$. The complete non-linearly self-dual tensor is

$$F_{abc} = -\frac{1+N}{4}\Omega_{abc}\,, \tag{2.24}$$
$$\begin{aligned} F_{ab\bar{c}} &= \frac{1+N}{4}\bar{\Omega}_{\bar{c}\bar{d}\bar{e}}u_a{}^{\bar{d}}u_b{}^{\bar{e}} \\ &= \frac{1-N}{4}\Omega_{abd}(u^{-1})_{\bar{c}}{}^{d}\,, \end{aligned} \tag{2.25}$$
$$F_{a\bar{b}\bar{c}} = \frac{1+N}{4}u_a{}^{\bar{d}}\bar{\Omega}_{\bar{b}\bar{c}\bar{d}}\,, \tag{2.26}$$
$$\begin{aligned} F_{\bar{a}\bar{b}\bar{c}} &= -\frac{1+N}{4}\bar{\Omega}_{\bar{a}\bar{b}\bar{c}}\det u \\ &= -\frac{1-N}{4}\bar{\Omega}_{\bar{a}\bar{b}\bar{c}}\,. \end{aligned} \tag{2.27}$$

We notice that the value of $F_{(0,3)}$ consistent with the stability conditions is exactly the one that follows from non-linear self-duality. This concludes the check of algebraic consistency of the stability conditions (2.15) with the self-duality relation (2.9), and provides an explicit parametrisation for the following section.

3. NS5-branes in the A-model and Kodaira-Spencer theory

So far, the analysis is completely local and algebraic. We will show that the equation of motion (or equivalently, the Bianchi identity) for the 3-form is the Kodaira-Spencer equation. We will now suppose that the M5-brane actually winds a Calabi-Yau space, so that it becomes an NS5-brane in the A-model. The components of $dF = 0$ are (we assume that the RR field strengths vanish)

$$(dF)_{(1,3)}: \quad \partial_a N - \bar{\partial}_{\bar{b}}[(1+N)u_a{}^{\bar{b}}] = 0\,, \tag{3.1}$$
$$(dF)_{(2,2)}: \quad \bar{\Omega}^{acd}\partial_c[(1+N)u_d{}^{\bar{b}}] + \Omega^{\bar{b}\bar{c}\bar{d}}\bar{\partial}_{\bar{c}}[(1-N)(u^{-1})_{\bar{d}}{}^{a}] = 0\,, \tag{3.2}$$
$$(dF)_{(3,1)}: \quad \bar{\partial}_{\bar{a}}N + \partial_b[(1-N)(u^{-1})_{\bar{a}}{}^{b}] = 0\,. \tag{3.3}$$

It is straightforward to show, using $dN = -\frac{1}{2}(1-N^2)\mathrm{tr}\,(u^{-1}du)$, that the first two equations imply the third. The first equation can be seen as a gauge-fixing condition, while the second one reads

$$0 = \partial_{[a}[(1+N)u_{b]}{}^{\bar{c}}] + \bar{\partial}_{\bar{d}}[(1+N)u_{[a}{}^{\bar{d}}u_{b]}{}^{\bar{c}}] \tag{3.4}$$

$$= (\partial_{[a}N - \bar{\partial}_{\bar{d}}[(1+N)u_{[a}{}^{\bar{d}}])u_{b]}{}^{\bar{c}} + (1+N)(\partial_{[a}u_{b]}{}^{\bar{c}} - u_{[a}{}^{\bar{d}}\bar{\partial}_{\bar{d}}u_{b]}{}^{\bar{c}})\,, \tag{3.5}$$

which, using the gauge-fixing condition, implies that u fulfills the Kodaira-Spencer equation

$$\partial_{[a}u_{b]}{}^{\bar{c}} - u_{[a}{}^{\bar{d}}\bar{\partial}_{\bar{d}}u_{b]}{}^{\bar{c}} = 0\,, \tag{3.6}$$

corresponding to the deformation of the complex structure encoded in the differential $\partial' = dz^a(\partial_a - u_a{}^{\bar{b}}\bar{\partial}_{\bar{b}})$.

The non-linearly self-dual closed 3-form F is exactly the deformation of the form $\frac{1}{2}\Omega$ defining the complex structure. It will be linearly self-dual under the deformed metric. It is possible to be quite explicit about the deformed metric G, such that $i\star_G F = F$. From the form of the non-linear self-duality relation, it is clear that the metric G satisfies (using that the antisymmetry of $G_{[i}{}^l F_{jk]l}$ is automatic provided G is expressible in terms of F)

$$\frac{G^3}{\sqrt{\det G}} = N1\!\!1 - \frac{1}{2N}q\,, \tag{3.7}$$

where contractions are made with the undeformed metric (which we for calculations have taken to be locally $1\!\!1$). The right hand side has unit determinant. The expressions become more transparent if we use the normalised matrix $s = \frac{1}{2N\sqrt{1-N^2}}q$ with $s^2 = -1\!\!1$. We then have $(\det G)^{-1/2}G^3 = N1\!\!1 - \sqrt{1-N^2}s = e^{-s\theta}$, where θ is defined by $\cos\theta = N$. The deformed metric is thus defined, up to a scale factor, by

$$(\det G)^{-1/6}G = e^{-\frac{1}{3}s\theta}\,. \tag{3.8}$$

It will of course be hermitean only with respect to the deformed complex structure.

We would like to comment on the relation to the treatment of the 11-dimensional M5-brane instantons winding on CY spaces of ref. [7]. The projection matrix on the κ parameter stated there does not contain the actual Γ of eq. (2.8), but only its linearisation in h, which is the projection arising from a superembedding treatment [37]. It was shown in ref. [38] how the two apparently different projections "$\frac{1}{2}(1-\Gamma)$" are related, and that they both project on the fermionic gauge degrees of freedom. Here we start from a topological M5-brane, in a superspace with 7 bosonic coordinates and half the number of fermions compared to M-theory, whose presence in topological M-theory is necessitated by the existence of D4- and NS5-branes in the A-model.

A. Conventions

In 7 euclidean dimensions, we use γ matrices that satisfy

$$\{\gamma^a, \gamma^b\} = -2\delta^{ab}\,, \tag{A.1}$$

where the minus sign is necessary for real γ-matrices. The spinors are real ψ^I_α, where $\alpha = 1, \ldots, 8$ and the $I = 1, 2$ is an $\mathrm{SL}(2, R)$ R-symmetry index [2].

For the 3-form σ, we use $\sigma_{124} = 1$ and cyclic. On the CY space, with 3 complex dimensions, we use locally $\Omega_{abc} = \varepsilon_{abc}$, so that $\Omega \wedge \bar{\Omega} = 8i\mathrm{Vol}_6$. We have $g_{a\bar{b}} = \frac{1}{2}\delta_{a\bar{b}}$ and $\omega_{a\bar{b}} = \frac{i}{4}\delta_{a\bar{b}}$, so that $\omega \wedge \omega \wedge \omega = -6\mathrm{Vol}_6$. The relations between 7-dimensional and 6-dimensional forms are

$$\sigma = \mathrm{Re}\,\Omega + \omega \wedge dx^7\,, \tag{A.2}$$

$$\star\sigma = -\mathrm{Im}\,\Omega \wedge dx^7 - \frac{1}{2}\omega \wedge \omega\,. \tag{A.3}$$

The real 7-dimensional γ matrices encoded in the left multiplication of a spinor $\lambda = \lambda^{\hat{\alpha}} e_{\hat{\alpha}}$ by an imaginary unit e_a are

$$(\gamma^a)_{\alpha\beta} = \sigma^a{}_{\alpha\beta}\,, \tag{A.4}$$

$$(\gamma^a)_{0\alpha} = \delta^a_\alpha\,. \tag{A.5}$$

The Clifford algebra is spanned by the $so(7)$-invariant tensors $\delta^{\hat{\alpha}}{}_{\hat{\beta}}$, $(\gamma^a)^{\hat{\alpha}}{}_{\hat{\beta}}$, $(\gamma^{ab})^{\hat{\alpha}}{}_{\hat{\beta}}$ and $(\gamma^{abc})^{\hat{\alpha}}{}_{\hat{\beta}}$, of which the first and last are symmetric and the second and third antisymmetric matrices. The decomposition in terms of G_2-invariant tensors is

$$\delta^{\hat{\alpha}}{}_{\hat{\beta}} = \begin{bmatrix} 1 & 0 \\ 0 & \delta^\alpha{}_\beta \end{bmatrix}, \tag{A.6}$$

$$(\gamma^a)^{\hat{\alpha}}{}_{\hat{\beta}} = \begin{bmatrix} 0 & \delta^a{}_\beta \\ -\delta^{a\alpha} & \sigma^{a\alpha}{}_\beta \end{bmatrix}, \tag{A.7}$$

$$(\gamma^{ab})^{\hat{\alpha}}{}_{\hat{\beta}} = \begin{bmatrix} 0 & -\sigma^{ab}{}_\beta \\ \sigma^{ab\alpha} & -\star\sigma^{ab\alpha}{}_\beta - 2\delta^{ab}_{\alpha\beta} \end{bmatrix}, \tag{A.8}$$

$$(\gamma^{abc})^{\hat{\alpha}}{}_{\hat{\beta}} = \begin{bmatrix} \sigma^{abc} & -\star\sigma^{abc}{}_\beta \\ -\star\sigma^{abc\alpha} & 6\delta^{[a}_{(\alpha}\sigma_{\beta)}{}^{bc]} - \delta^\alpha{}_\beta\sigma^{abc} \end{bmatrix}. \tag{A.9}$$

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Paper III

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Aspects of higher curvature terms and U-duality

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Abstract
We discuss various aspects of the dimensional reduction of gravity with the Einstein–Hilbert action supplemented by a lowest-order deformation formed as the Riemann tensor raised to powers two, three or four. In the case of R^2 we give an explicit expression, and discuss the possibility of extended coset symmetries, especially $SL(n+1, \mathbb{Z})$ for reduction on an n-torus to three dimensions. Then we start an investigation of the dimensional reduction of R^3 and R^4 by calculating some terms relevant for the coset formulation, aiming in particular towards $E_{8(8)}/(\mathrm{Spin}(16)/\mathbb{Z}_2)$ in three dimensions and an investigation of the derivative structure. We emphasize some issues concerning the need for the introduction of non-scalar automorphic forms in order to realize certain expected enhanced symmetries.

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1. Introduction and summary

M-theory, when compactified on an n-torus, is conjectured to have a global U-duality symmetry $E_{n(n)}$ in the low-energy limit described by maximal supergravity in $d = 11 - n$ dimensions. It is known from string theory that this continuous symmetry is broken in the quantum theory to a discrete version $E_{n(n)}(\mathbb{Z})$. The massless scalars in the compactified theory belong to the coset $E_{n(n)}(\mathbb{Z})\backslash E_{n(n)}/K(E_{n(n)})$, where $K(E_{n(n)})$ is the (locally implemented) maximal compact subgroup of the split form $E_{n(n)}$. When $d \leqslant 3$, no local massless bosonic degrees of freedom remain except the scalar ones. It has been proposed that it may even be possible to define M-theory itself as a theory on the coset obtained when going to $d = 1$ (E_{10}) or $d = 0$ (E_{11}), although it is unclear whether or not such a formulation incorporates degrees of freedom beyond supergravity.

Some aspects of these discrete symmetries are well investigated. This primarily concerns calculations in the cases with low dimension of the torus. For $n < 3$, non-perturbative string theory results are obtained from loop calculations in $D = 11$ supergravity. For $n \geqslant 3$ one expects that there will also be contributions from membrane instantons and for $n \geqslant 6$ from five-brane instantons. This makes results for higher-dimensional tori difficult to obtain. On the other hand, one may turn the argument around and ask what kind of restrictions U-duality puts on the possible quantum corrections of the theory. It is convenient to work in the massless sector, obtained by dimensional reduction, and let quantum effects manifest themselves in an effective action, which will then contain higher orders of curvatures (and other fields), i.e., higher derivative terms.

Some partial results have been obtained by investigating the general structures of higher derivative terms to determine if they can be made to fit into something U-duality invariant. For example, it has been shown that the Riemann tensor in $D = 11$ comes only in powers $3k + 1$, where k is an integer. The purpose of the paper is to initiate a more detailed analysis aiming at actually checking the invariance. The scope of the paper is modest; we restrict our attention to the D-dimensional gravitational sector alone. Then we set out to form higher-derivative corrections to the Einstein–Hilbert action in the form of second, third and fourth powers of the Riemann tensor. The full U-duality group is not accessible with gravity only, but on compactification to $d = 3$ there still has to be an enhancement from $SL(8)$ to $SL(9)$, which is the subgroup of $E_{8(8)}$ of which the gravitational scalars form a coset (more generally, on reduction from $n+3$ to 3 dimensions, we expect an enhancement from $SL(n)$ to $SL(n + 1)$). Some aspects about the general structures of the higher curvature terms at hand are investigated, before we turn to examining chosen subsets of terms and thereby extracting concrete information concerning the possibility of implementing $SL(n + 1)$. We draw some definite conclusions about the necessity of introducing transforming automorphic forms, and show that they can always be chosen to reproduce the results in the dimensionally reduced theory. The interpretation of the dimensionally reduced actions is not as U-duality invariant object *per se*, but as properly taken large volume limits of U-duality invariant actions involving transforming automorphic forms. The investigation is very much a partial one, and we point out some further directions, such as a more complete expansion in fields, and a concrete examination of cosets and discrete groups based on exceptional groups.

We refer to [1, 2] for an overview of U-duality. Topics on E_{10} and E_{11} as fundamental symmetries are dealt with in [3–6] and references therein. Recent developments concerning the connections between U-duality and higher curvature terms are found in [7–10]. For different approaches to higher curvature terms in supergravity and string theory, see [11–17]. The $3k + 1$ restriction on powers of the Riemann tensor in 11-dimensional supergravity is discussed in [18].

2. The torus dimensional reduction procedure

Our ansatz for dimensional reduction on an n-torus to three dimensions is given by

$$\hat{E}^a = e^{-\phi} e^a, \qquad \hat{E}^i = (dy^\mu - A^\mu) e_\mu{}^i. \tag{2.1}$$

Here, e^ϕ is not an independent field, but the determinant of the internal vielbein $e_\mu{}^i$. The prefactor $e^{-\phi}$ is chosen so that a canonically normalized Einstein–Hilbert term results in three dimensions from the reduction of such a term in the higher-dimensional theory. Our conventions are such that $D = d + n$, where D is the spacetime dimension before the dimensional reduction and d the one after and n is the dimension of the internal torus on which we are performing the dimensional reduction. Flat indices are denoted $a, b, \ldots$ in spacetime and $i, j, \ldots$ on the internal manifold which is parametrized by coordinates y^μ. The one-forms A^μ in the above ansatz are the n graviphoton potentials, while $e_\mu{}^i$ is the internal vielbein and hence an element of $GL(n)$. One of our goals will be to see if this global symmetry (or, strictly speaking, $SL(n, \mathbb{Z})$, the mapping class group of the internal torus) is extended to larger groups when considering Lagrangians which consist of the Einstein–Hilbert term plus the terms containing the Riemann tensor raised to powers 2, 3 and 4. This issue has previously been investigated by the authors of [9] where the root and weight structure of the scalar prefactors arising in the reduction is studied. These prefactors are in [9] extracted by applying some general arguments about the properties of higher derivative terms. In a continued work [10] they conclude that when weights instead of roots occur in the scalar exponent prefactors this should be compensated for by tensorial automorphic forms. The results obtained here by explicitly computing some of the relevant terms in the dimensional reduction lend further support to such a construction. Automorphic forms of $SL(2,\mathbb{Z})$ with similar non-trivial properties have already been seen to arise in the type II B superstring multiplying a term containing the product of 16 dilatinos [19].

From the above ansatz one easily obtains, using the zero-torsion condition, the dimensionally reduced form of the spin connection one-form and from it the Riemann tensor two-form. By reading off the components of these tensors using the basis indicated by the ansatz above, i.e., $\hat{e}^a = e^a, \hat{e}^i = (dy^\mu - A^\mu)e_\mu{}^i$, we get an answer without explicit graviphoton potentials since this basis is manifestly translation invariant on the torus [20]. In order to examine the possibility of symmetry enhancement in reduction to $d = 3$, we need the following expressions for the components of the Riemann tensor:

$$
\begin{aligned}
\hat{R}_{ab}{}^{cd} &= e^{2\phi}\big[R_{ab}{}^{cd} + 4\delta^{[c}_{[a}D_{b]}\phi D^{d]}\phi + 4\delta^{[c}_{[a}D_{b]}\phi D^{d]}\phi - 2\delta^{cd}_{ab}D_e\phi D^e\phi\big]\\
&\quad - e^{4\phi}\big[\tfrac{1}{2}(F_{ab}F^{cd}) + \tfrac{1}{2}\big(F_{[a}{}^cF_{b]}{}^d\big)\big],\\
\hat{R}_{ab}{}^{cl} &= e^{3\phi}\big[\tfrac{1}{2}D^cF_{ab}{}^l + D^c\phi F_{ab}{}^l - D_{[a}\phi F_{b]}{}^{cl} + \delta^c_{[a}D^d\phi F_{b]d}{}^l + \tfrac{1}{2}(F_{ab}P^c)^l + \big(F_{[a}{}^cP_{b]}\big)^l\big],\\
\hat{R}_{ab}{}^{kl} &= -2e^{2\phi}(P_{[a}P_{b]})^{kl} - \tfrac{1}{2}e^{4\phi}F_a{}^{c[k}F_{bc}{}^{l]},\\
\hat{R}_{aj}{}^{cl} &= +\tfrac{1}{4}e^{4\phi}F^{ce}{}_jF_{ae}{}^l - e^{2\phi}\big[D_aP^c + D_a\phi P^c + D^c\phi P_a - \delta^c_aD_e\phi P^e + P_aP^c\big]_j{}^l,\\
\hat{R}_{aj}{}^{kl} &= -e^{3\phi}F_a{}^{e[k}P_{ej}{}^{l]},\\
\hat{R}_{ij}{}^{kl} &= -2e^{2\phi}(P_e)_{[i}{}^k(P^e)_{j]}{}^l,
\end{aligned}
\tag{2.2}
$$

where $F^i_{ab} := F^\mu_{ab}e_\mu{}^i$, with $F^\mu_{mn} = 2\partial_{[m}A_{n]}{}^\mu$, are the graviphoton field strengths. We use the notation $(AB) = A^iB^i$ for the scalar product of $SO(n)$ vectors. The covariant derivative is $D_m = \partial_m + \omega_m + Q_m$. We have also defined P and Q as the symmetric and antisymmetric parts of the Maurer–Cartan one-form constructed from the internal vielbein $e_\mu{}^i$ (remember that they form the Maurer–Cartan form of $GL(n)$, so that $\mathrm{tr}P = \mathrm{d}\phi$). Q belongs to the $so(n)$ subalgebra and P spans the tangent directions of the corresponding coset $GL(n)/SO(n)$. As a direct consequence of their definition, P and Q satisfy

$$
DP := dP + PQ + QP = 0, \qquad F_Q := dQ + Q^2 = -P^2. \tag{2.3}
$$

We also have that the graviphotons satisfy the Bianchi identity $DF - F\wedge P = 0$.

Reduction of the D-dimensional Einstein–Hilbert term using these expressions leads directly to the following Lagrangian in $d = 3$:

$$
\hat{E}\hat{R} = e\big[R - \mathrm{tr}(P_aP^a) - \tfrac{1}{4}e^{2\phi}(F_{ab}F^{ab}) - D_a\phi D^a\phi\big], \tag{2.4}
$$

where one should keep in mind that there is a hidden contribution to the kinetic term of the dilaton ϕ in the $GL(n)$ coset term. Note, however, that even after putting the two singlet terms together the kinetic term is not conventionally normalized in our conventions; see below for further details. The equations of motion that we will need in the following are (in $d = 3$)

$$
\begin{aligned}
R_{ab} &= \mathrm{tr}(P_aP_b) + D_a\phi D_b\phi + \tfrac{1}{2}e^{2\phi}\big[\big(F_a{}^cF_{bc}\big) - \tfrac{1}{2}\eta_{ab}(F^{cd}F_{cd})\big],\\
(D^aF_{ab})^i &= -(P^aF_{ab})^i - 2D^a\phi F^i_{ab},\\
(D^aP_a)^{ij} &= \tfrac{1}{4}e^{2\phi}F^i_{ab}F^{jab}.
\end{aligned}
\tag{2.5}
$$

Note that the equation of motion for ϕ, $D^a D_a\phi = \frac{1}{4}e^{2\phi}(F^{ab}F_{ab})$, follows directly from the last equation above since $\mathrm{tr}P_a = D_a\phi$. In the following section we will apply this ansatz to derive the compactification of the $\hat{R}^2$ term.

Before leaving this review of the dimensional reduction, we would like to make more explicit the relation of our conventions to the ones in, e.g., [9]. In that paper, the ansatz is written as

$$\hat{E}^a = e^{\alpha\phi}e^a, \qquad \hat{E}^i = e^{\beta\phi}(dy^\mu + A^\mu)\tilde{e}_\mu{}^i, \tag{2.6}$$

where the internal vielbein $\tilde{e}_\mu{}^i$ is an element of $SL(n)$. Furthermore, the parameters α and β are determined to satisfy $\alpha^2 = \frac{n}{2(d-2)(n+d-2)}$ and $\beta = -\frac{d-2}{n}\alpha = -\sqrt{\frac{d-2}{2n(D-2)}}$ in order for the reduction to produce a canonical Einstein–Hilbert term and a properly normalized kinetic term for the scalar ϕ. In fact, using the above ansatz, the coefficient in front of the scalar kinetic terms reads

$$(d-1)(d-2)\alpha^2 + 2n(d-2)\,\alpha\beta + n(n+1)\beta^2. \tag{2.7}$$

Since our ansatz corresponds to $d = 3, \alpha = -1$ and $\beta = \frac{1}{n}$, we find the coefficient to be $1 + \frac{1}{n}$. This is consistent with our action in equation (2.4) if one extracts the contribution to the scalar kinetic term from the coset term. The choice $\beta = \frac{1}{n}$ is natural, since it keeps intact the $GL(n)$ element that will be a building block of $SL(n+1)$ in the following section. Finally, note that the field strength F^i appearing in equation (2.4) has an extra ϕ dependence hidden in the internal vielbein.

3. Toroidal dimensional reduction of R^2

We now consider adding to the Einstein–Hilbert action terms of higher order in the Riemann tensor. In this paper, we only treat one such deformation at the time, and think of it as the next-to-leading term in an infinite expansion in a dimensionful parameter formed from α' or Newton's constant.

At the level of R^2 there is only one possible term, modulo field redefinitions, namely $\hat{R}_{ABCD}\hat{R}^{ABCD}$. At the next-to-leading order, field redefinitions give changes in the action containing the lowest-order field equations, so any term containing the Ricci tensor can be thrown away without loss of generality. The dimensional reduction (setting $A = (a, i)$, etc) will result in an expression that contains the following kind of terms: the square of R_{abcd}, two F^i_{ab} field strengths contracted to one R_{abcd}, plus F^i_{ab}, P^{ij}_a and $D_a\phi$ combined into terms with four such fields, or to terms with three or two fields together with one or two covariant derivatives D_a, respectively.

We note at this point that modulo field equation $\hat{R}_{ABCD}\hat{R}^{ABCD}$ is equivalent to the Gauss–Bonnet term $\mathcal{L}_{GB} = \hat{E}(\hat{R}_{ABCD}\hat{R}^{ABCD} - 4\hat{R}_{AB}\hat{R}^{AB} + \hat{R}^2)$. The fact that the integral of this expression, $\int \mathrm{d}^D x\, \mathcal{L}_{GB} \sim \int \varepsilon_{A_1\ldots A_D}\hat{R}^{A_1A_2}\wedge\hat{R}^{A_3A_4}\wedge\hat{E}^{A_5}\wedge\cdots\wedge\hat{E}^{A_D}$, is a topological invariant in some dimension ($D = 4$) implies that it has no two-point function (the terms quadratic in fields are total derivatives). Perhaps less well known is that this feature repeats itself at the level of three fields in the scalar sector. This is an effect of the dimensional reduction. It is quite trivial to convince oneself that any three-point coupling P^2DP, modulo the lowest-order field equation (representing the freedom of field redefinitions) is a total derivative. However, as we will discuss more later, for R^3 and R^4 terms related to topological invariants in six and eight dimensions, this property holds only for terms containing three and four fields, respectively.

To present the result of the dimensional reduction of $\hat{R}_{ABCD}\hat{R}^{ABCD}$, it is convenient to first note that the splitting of the indices $A = (a, i)$, etc, gives

$$\hat{R}_{ABCD}\hat{R}^{ABCD} = \hat{R}_{abcd}\hat{R}^{abcd} + 4\hat{R}_{ibcd}\hat{R}^{ibcd} + 2\hat{R}_{ijcd}\hat{R}^{ijcd} + 4\hat{R}_{ibkd}\hat{R}^{ibkd} + 4\hat{R}_{ijkd}\hat{R}^{ijkd} + \hat{R}_{ijkl}\hat{R}^{ijkl}. \tag{3.1}$$

At this point, we suppress the dilaton dependence in the higher curvature terms. It should of course be kept for a complete treatment, but will be irrelevant for the considerations in this and the following sections. Formally, this amounts to setting $\phi = 0$, which implies $\mathrm{tr}P = 0$. We then get

$$\begin{aligned}
\hat{R}_{abcd}\hat{R}^{abcd} &= R_{abcd}R^{abcd} - \tfrac{3}{2}R_{abcd}(F^{ab}F^{cd}) \\
&\quad + \tfrac{3}{8}[(F^{ab}F^{cd})(F_{ab}F_{cd}) + (F^{ab}F^{cd})(F_{ac}F_{bd})], \\
\hat{R}_{ibcd}\hat{R}^{ibcd} &= (D_{[c}F_{d]b}D^cF^{db}) - 2(F_{cd}P_bD^cF^{db}) + (F_{ab}P_cP^cF^{ab}), \\
\hat{R}_{ijcd}\hat{R}^{ijcd} &= \tfrac{1}{8}\big[\big(F_a{}^cF_{bc}\big)\big(F^a{}_dF^{bd}\big) - \big(F^{ab}F^{cd}\big)(F_{ac}F_{bd})\big] + 2\big(F^{ae}P_{[a}P_{b]}F^b{}_e\big) \\
&\quad - 2\mathrm{tr}(P_aP_bP_aP_b) + 2\mathrm{tr}(P^aP_aP^bP_b), \\
\hat{R}_{ibkd}\hat{R}^{ibkd} &= \tfrac{1}{16}\big(F_a{}^cF_{bc}\big)\big(F^a{}_dF^{bd}\big) + \mathrm{tr}\big(P^aP_aP^bP_b\big) - \tfrac{1}{2}\big(F^a{}_eP_bP_aF^{be}\big) \\
&\quad + 2\mathrm{tr}(D_aP_bP^aP^b) - \tfrac{1}{2}\big(F^a{}_eD_aP_bF^{be}\big) + \mathrm{tr}(D_aP_bD^aP^b), \\
\hat{R}_{ijkd}\hat{R}^{ijkd} &= \tfrac{1}{2}\big(F^a{}_cF^{bc}\big)\,\mathrm{tr}(P_aP_b) - \tfrac{1}{2}\big(F^a{}_eP_bP_aF^{be}\big), \\
\hat{R}_{ijkl}\hat{R}^{ijkl} &= 2\mathrm{tr}(P_aP_b)\,\mathrm{tr}(P^aP^b) - 2\mathrm{tr}(P_aP_bP^aP^b).
\end{aligned} \tag{3.2}$$

All traces and scalar products are over internal indices, all spacetime indices are explicit. Two of the above Riemann tensor components depend explicitly, as well as implicitly after integration by parts, on the field equations. After using the lowest-order field equations obtained from the reduction of the Einstein–Hilbert term, we find that the expressions for these components become (modulo total derivative terms and including the combinatorial factors above)

$$\begin{aligned}
4\hat{R}_{ibcd}\hat{R}^{ibcd} &= R^{abcd}(F_{ab}F_{cd}) - 2R^{ab}\big(F_a{}^cF_{bc}\big) - \tfrac{1}{2}(F^{ab}F^{cd})(F_{ab}F_{cd}) \\
&\quad + 6\big(F_a{}^cP^bP^aF_{bc}\big) + 2(F^{ab}P^cP_cF_{ab}), \\
4\hat{R}_{ibkd}\hat{R}^{ibkd} &= -4R_{ab}\,\mathrm{tr}(P^aP^b) + \tfrac{1}{4}\big(F_a{}^cF_{bc}\big)\big(F^a{}_dF^{bd}\big) \\
&\quad + \tfrac{1}{8}(F^{ab}F^{cd})(F_{ab}F_{cd}) - 4\mathrm{tr}(P^aP_aP^bP_b) + 8\mathrm{tr}(P_aP_bP^aP^b) \\
&\quad - 2\big(F^a{}_eP_aP_bF^{be}\big) - 2(F^{ab}P^cP_cF_{ab}).
\end{aligned} \tag{3.3}$$

Note that we have not yet implemented the Einstein equation since it will only produce terms with short traces, that is over two $P's$, and these will not enter the discussion below. It is for the same reason that we can neglect the dependence on the scalar ϕ in the above formulae. Here we have also made use of the Maurer–Cartan equations and Bianchi identities which in the particular case of R^2 terms imply that no derivatives appear anywhere (it is straightforward to show that this is true also for the non-constant ϕ). As we will see in later sections, this nice feature will not occur for R^n with $n > 2$.

In $d = 3$ the two-forms F can be dualized to one-forms f, turning the graviphoton degrees of freedom into scalars. Dualization is performed by adding a term $\int u_\mu\,\mathrm{d}F^\mu$ to the action, thus enforcing the Bianchi identity of F with a Lagrange multiplier, and treating F as an independent field. Solving the algebraic field equations for F in terms of $\mathrm{d}u$ and reinserting the solution into the action gives the action in terms of the scalar dual graviphotons u_μ. At the level of the Einstein–Hilbert action, reintroducing the scalar, this procedure gives the Lagrangian

$$\mathcal{L}_{\text{dual}} = e\big[R - \mathrm{tr}(P_aP^a) - \tfrac{1}{2}(f_af^a) - D_a\phi D^a\phi\big], \tag{3.4}$$

where the dualized field strength is given by $F^i = e^{-\phi}\star f^i$. It has the Bianchi identity $Df + f \wedge P + f \wedge \mathrm{d}\phi = 0$ and the equation of motion $D^a f_a - P^a f_a - D^a \phi f_a = 0$, and is obtained from the scalar as $f = e^{-\phi} e^{-1}\,\mathrm{d}u$. The dualized scalars fit together with the $GL(n)$ ones parametrizing the internal torus into an element of $SL(n+1)$ as

$$G = \begin{bmatrix} e^{-\phi} & 0 \\ e^{-\phi}u & e \end{bmatrix}, \tag{3.5}$$

which gives the $SL(n+1)$ Maurer–Cartan form

$$\mathcal{P} + \mathcal{Q} = G^{-1}\,\mathrm{d}G = \begin{bmatrix} -\mathrm{d}\phi & 0 \\ f = e^{-\phi}e^{-1}\,\mathrm{d}u & e^{-1}\,\mathrm{d}e \end{bmatrix}. \tag{3.6}$$

The $SL(n+1)$ symmetry of the dimensionally reduced Einstein–Hilbert action is manifested as

$$\mathcal{L}_{\text{dual}} = e[R - \mathrm{tr}(\mathcal{P}_a\mathcal{P}^a)]. \tag{3.7}$$

We note that, at lowest order, the Lagrange multiplier term contributes to the action (in fact, so that the kinetic term keeps its correct sign after dualization). When the action contains higher-order interaction terms, the equations of motion for F become nonlinear, and one will get a nonlinear duality relation between F and f. In general, one has to be careful about this, but it is straightforward to check that for any next-to-leading term, the nonlinearities cancel between the F^2 term and the Lagrange multiplier term. To the next-to-leading order, which is all we treat in this paper, the correct dualized version of the higher-curvature term is obtained by direct insertion of the linearly dualized graviphotons.

In view of this it is of course interesting to check if the pure P terms, respecting the manifest $\mathfrak{so}(n)$ symmetry, can combine with the graviphotonic scalars to form the enlarged symmetry $\mathfrak{sl}(n+1)$ also when the R^2 terms are included. To this end, we collect the terms of the form $\mathrm{tr}(P_aP_bP^aP^b)$ and $\mathrm{tr}(P_aP^aP_bP^b)$ together with the terms containing F's that would mix with them under $\mathfrak{sl}(n+1)$.

The result is

$$2\mathrm{tr}(P_aP_bP^aP^b) + 2\big(F_{ac}P^bP^aF_b{}^c\big) \tag{3.8}$$

(i.e., the terms $\mathrm{tr}(P_aP^aP_bP^b)$ cancel out), which becomes, after dualization of the two-forms F^i to one-forms f^i as discussed above:

$$2\mathrm{tr}(P_aP_bP^aP^b) + 2(f^aP_aP^bf_b) - 2(f^aP_bP^bf_a). \tag{3.9}$$

This should then be compared to the $SL(n+1)$-covariant expression $\mathrm{tr}\mathcal{P}^4$. The terms contributing uniquely to this 'long trace', and not to $(\mathrm{tr}\mathcal{P}^2)^2$, are of the types $\mathrm{tr}P^4$ and $(fPPf)$ as above, together with $\partial\phi(fPf)$, with tangent indices placed in all possible ways. With the parametrization of the $SL(n+1)/SO(n+1)$ coset as above, we get

$$\begin{aligned} \mathrm{tr}(\mathcal{P}^a\mathcal{P}_a\mathcal{P}^b\mathcal{P}_b) &= \mathrm{tr}(P^aP_aP^bP_b) + \tfrac{1}{2}[(f^aP_bP^af_b) + (f^aP^bP_bf_a)] \\ &\quad + \tfrac{1}{16}[(f^af_a)(f^bf_b) + (f^af^b)(f_af_b)] + \pi^a(f_aP^bf_b) \\ &\quad + \tfrac{1}{2}[\pi^a\pi_a(f^bf_b) + \pi^a\pi^b(f_af_b)] + \pi^a\pi_a\pi^b\pi_b, \\ \mathrm{tr}(\mathcal{P}^a\mathcal{P}^b\mathcal{P}_a\mathcal{P}_b) &= \mathrm{tr}(P^aP^bP_aP_b) + (f^aP^bP_af_b) \\ &\quad + \tfrac{1}{8}(f^af^b)(f_af_b) + \pi^a(f^bP_af_b) + \pi^a\pi^b(f_af_b) + \pi^a\pi_a\pi^b\pi_b, \end{aligned} \tag{3.10}$$

where $\pi = -\mathrm{d}\phi$ is the upper left corner component of $\mathcal{P}$. It seems hard to reconcile equation (3.9) with a possible $\mathfrak{sl}(n+1)$. In fact, the coefficients of the two terms are dictated by the $\mathrm{tr}P^4$ terms. Of the three structures $(fPPf)$ consistent with $SL(n)$, only two linear combinations are allowed by $SL(n+1)$. The terms from dimensional reduction in equation (3.9) are not the ones required by equation (3.10).

In the above calculation, the volume factor e^{ϕ} of the internal torus has been omitted (set to 1). After dualization, any term from $\hat{R}^p$ carries an overall factor $e^{2(p-1)\phi}$. This factor tells us that the terms obtained by dimensional reduction cannot be $SL(n+1)$-invariant, since ϕ is one of the scalars parametrizing the coset $SL(n+1)/SO(n+1)$. Neither is this expected from string theory or M-theory, since quantum corrections break the global symmetry group to a discrete version. The terms obtained from the reduction will not be the whole answer, but its large volume limit. The torus volume factor may be obtained as the large volume limit of an automorphic form. As we will see later, the observation that the tensor structure does not match with $SL(n+1)$ covariance means that scalar ($SO(n+1)$-invariant) automorphic forms (i.e., functions) do not suffice, and calls for the introduction of automorphic forms transforming under $SO(n+1)$. Similar conclusions are reached in [10] based on an investigation of the root and weight structure of the scalar prefactors.

At this point, we could of course extend the investigation to other terms by including $\mathrm{d}\phi$ and considering also 'short' traces. However, as we have already demonstrated the need for transforming automorphic forms, we will now show how any term obtained in the reduction can be matched to such constructions.

4. Transforming automorphic forms

Previous work by Green *et al* [19] (see also [11]) indicates how the apparent contradiction found in the previous section should be resolved. In fact, as we will see in later sections, there are also terms arising in the compactification of R^4 from $D = 11$ to $d = 3$ that are not immediately compatible with the $SL(9)$ subgroup of $E_{8(8)}$. We suggest that the proper interpretation of these results is that they should be viewed as the large volume limit of an $SL(9,\mathbb{Z})$-invariant constructed from transforming automorphic forms and non-scalar products of the fields in question. This turns out to hold for the R^2 terms of the previous section on reduction from any D to $d = 3$. Of course, consistency with decompactification requires that the automorphic form, in the large volume limit, does not diverge and has as its only remnant after decompactification the very term that was used as the starting point for the compactification.

Appendix A describes the construction of automorphic forms, scalar as well as transforming ones. (For a partly overlapping discussion, see the appendix of [10].) We now apply this construction to the quartic terms of the previous section, although it will be obvious that the treatment is general. For any irreducible $SO(n+1)$ representation r contained in the symmetric product of four symmetric traceless tensors, we can form the combination $\psi^{(r)}_{IJ,KL,MN,PQ}\mathcal{P}^{aIJ}\mathcal{P}_a^{KL}\mathcal{P}^{bMN}\mathcal{P}_b^{PQ}$, where ψ is an automorphic form transforming in the representation r. The symmetric product of four symmetric traceless $SO(n+1)$ tensors contains 23 irreducible representations for any $n \geqslant 8$, and this is then the number of $SL(n+1,\mathbb{Z})$-invariant terms we can write down starting from the symmetric traceless representation[1]. This is however true only when all the indices on $\mathcal{P}$'s are contracted with indices on an automorphic form constructed as in appendix A. The actual number is larger, since nothing prevents us from taking products of such automorphic forms and invariant tensors without symmetrizing all indices—there is no *a priori* reason to symmetrize in $\mathcal{P}$'s with different spacetime indices.

[1] The 'weight' of each automorphic form, as defined in appendix A, is fixed by the overall volume factor. We ignore ambiguities from products of automorphic forms, where only the sum of weights will be determined, as well as from the use of different Casimirs in the sum defining the automorphic form. Terms differing in these respects are indistinguishable in the large volume limit.

In the case of an SL group, it is preferable to build automorphic forms from the fundamental representation (although this option does not exist if we want, e.g., $SL(9)$ as a subgroup of $E_{8(8)}$). By using the automorphic forms built from the fundamental representation, we have seen in appendix A that the only surviving part in the large volume limit is the one with all indices equal to 0 (the first component in our $SL(n+1)$ matrices) [10]. The part of $\hat{R}^2$ containing P^4 comes from an $SO(n+1)$ scalar automorphic form. Since $\mathcal{P}_{0i} = \frac{1}{2} f_i$ and $\mathcal{P}_{00} = \pi$, we can always choose to insert an even number of zeros in the positions we like, and thereby arrange for products of transforming automorphic forms and $SO(n+1)$-invariant tensors to have a large volume limit reproducing any of the $SO(n)$-invariant terms occurring in the reduction. The matching can be made recursively, in an increasing number of 0 indices. We take the long trace as an example. The terms with scalar automorphic forms are determined from the $\mathrm{tr} P^4$ terms to be proportional to $\psi^{(1)}\,\mathrm{tr}(\mathcal{P}^a\mathcal{P}^b\mathcal{P}_a\mathcal{P}_b)$, with the notation of appendix A. Subtracting its large volume limit from the actual result of the reduction, given in part by equation (3.9), there is a remainder proportional to $(f^a P_a P^b f_b) - (f^a P^b P_b f_a) - (f^a P^b P_a f_b)$. This implies a term proportional to $\psi^{(2,1)}_{IJ}(\mathcal{P}^a\mathcal{P}_a\mathcal{P}^b\mathcal{P}_b - \mathcal{P}^a\mathcal{P}^b\mathcal{P}_b\mathcal{P}_a - \mathcal{P}^a\mathcal{P}^b\mathcal{P}_a\mathcal{P}_b)^{IJ}$. With a more complete expansion of the reduced curvature term, it can always be matched to the large volume limit of an expression in terms of $\mathcal{P}$'s and automorphic forms.

It should of course be checked that automorphic forms exist that give the correct power of the torus volume factor obtained from the reduction. A term $\hat{R}^p$ gives an overall factor $e^{2(p-1)\phi}$. Suppose we try to obtain some corresponding terms with a product of M automorphic forms, each with $2l_k$ fundamental indices and weight w_k, $k = 1, \ldots, M$. Convergence of the sum defining the automorphic forms demands $2(w_k - l_k) > 1$, and the large volume limit will yield a dilaton dependence $\exp\left(\sum_{k=1}^{M} 2(w_k - l_k)\phi\right)$. Matching with the reduction gives $p - 1 = \sum_{k=1}^{M}(w_k - l_k)$ and thus $M < 2(p-1)$. The $\hat{R}^2$ term gives room for one automorphic form, which is exactly what we need.

It is not clear what to expect for this kind of symmetry enhancement to $SL(n+1)$ in the context of $\hat{R}^2$ terms. There are, for example, no known examples from string/M-theory that make use of such a step, so the $\hat{R}^2$ term should probably be seen as a toy model to set the framework for the higher curvature terms. (The heterotic string has an $\hat{R}^2$ term, but the symmetry enhancement is to $SO(n,n)$; this case is treated in [21].) The situation is different for the $\hat{R}^4$ terms, which are the first higher derivative terms to arise in the maximally supersymmetric string theories in 10 dimensions, as well as in the 11-dimensional M-theory. When compactified to three dimensions all degrees of freedom are collected into a coset based on $E_{8(8)}$ when starting from a two-derivative action. We now proceed to discuss the $\hat{R}^3$ and $\hat{R}^4$ terms in this context with the goal of understanding the role of $SL(n+1)$, and to develop methods that might eventually be useful in dealing with the more complicated case of $E_{8(8)}$.

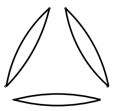
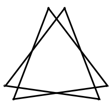
Automorphic forms of $SL(2,\mathbb{Z})$ transforming under $U(1)$ have been encountered in loop calculations with external fermions in string theory compactified on a circle to $d = 9$ [19]. We expect that the appearance of transforming automorphic forms is generic in a situation where the external fields of the diagram transforms under K. In the specific case in [19], the contribution was shown to disappear in the M-theoretic large volume limit. We note that this limit is a quite different one in terms of the SL group involved than in our case. We deal with a larger symmetry $SL(n+1)$ appearing after dualization of the graviphotons, and blow up a certain parameter, that, had our $SL(n+1)$ element been a vielbein on T^{n+1}, would have corresponded to shrinking one direction and consequently blowing up the other n directions. In the $SL(2)$ case, the large volume limit has nothing to do with the $SL(2)$ element parametrizing the shape of T^2, but with the determinant of a $GL(2)$ element blowing up. We have shown how to match combinations of $\mathcal{P}$'s and automorphic forms that are designed to survive in the large volume limit. It would be interesting to compare such a construction (not for terms corresponding to $\hat{R}^2$, but presumably to $\hat{R}^4$) to actual loop calculations.

5. The case of R^3

Our main concern in the rest of the paper is the investigation of the $\hat{R}^4$ terms which are part of the first non-trivial correction in M-theory and type II string theory. Before doing that we would however like to emphasize some aspects of the $\hat{R}^3$ terms. The $\hat{R}^2$ terms of section 3 were a testing ground for the ideas but turned out to have some special non-generic features, such as the effective vanishing of all terms with second derivatives on scalar fields. As we will see below, this feature is not found for $\hat{R}^3$ and higher terms. Here we also take the opportunity to introduce some diagrammatic methods that will be tremendously helpful in keeping track of index structures of increasing complexity as we go to higher powers of the Riemann tensor.

Again, the $\hat{R}^3$ terms are seen as a next-to-leading order correction to the Einstein–Hilbert action (i.e., there are no $\hat{R}^2$ terms). Any term which contains the lowest-order field equation can be removed by a field redefinition, so we leave them out from the start. We thus want to list all possible terms where indices are contracted between different Riemann tensors. We represent each contracted index by a line, and each Riemann tensor by the endpoints of four such lines. The lines whose endpoints meet represent an antisymmetric pair of indices. The sign is fixed by letting the indices, as they sit on $\hat{R}$, run in the clockwise direction in the diagram. The only structure not accounted for is $\hat{R}_{A[BCD]} = 0$, which has a simple diagrammatic expression.

A basis for the two inequivalent R^3 terms can be taken as

(1):  (2): 

thus representing $\hat{R}_{AB}{}^{CD}\hat{R}_{DC}{}^{EF}\hat{R}_{FE}{}^{BA}$ and $\hat{R}_A{}^B{}_D{}^E\hat{R}_B{}^C{}_E{}^F\hat{R}_C{}^A{}_F{}^D$, respectively. One may also consider the contraction , but it is related to the ones in the basis, using the Bianchi identity $\hat{R}_{A[BCD]} = 0$, as  $= \frac{1}{4}(1) + (2)$.

At this level, there is one obvious combination that does not give any three-point couplings. This is the 'Gauss–Bonnet' term,

$$\begin{aligned}\varepsilon\varepsilon\hat{R}^3 &= \varepsilon^{A_1\ldots A_D}\varepsilon_{B_1\ldots B_D}\hat{R}_{A_1A_2}{}^{B_1B_2}\hat{R}_{A_3A_4}{}^{B_3B_4}\hat{R}_{A_5A_6}{}^{B_5B_6}\delta^{B_7\ldots B_D}_{A_7\ldots A_D}\\ &= 32(D-6)!\{(1)+2(2)\},\end{aligned} \tag{5.1}$$

which is topological in $D = 6$ and lacks three-point couplings in any dimension. The general form of the scalar terms will be $(D\mathcal{P})^3 + \mathcal{P}^2(D\mathcal{P})^2 + \mathcal{P}^4D\mathcal{P} + \mathcal{P}^6$, where '$\mathcal{P}$' denotes any of P, f and $\partial\phi$, but we are guaranteed that the first term vanishes for this specific combination.

To see this explicitly, and to derive further properties relying on the dimensional reduction, we concentrate on the pure P terms (note that this truncation is consistent and implies $\mathrm{tr}P = 0$). They are extracted in the Riemann tensor derived in section 2:

$$\begin{aligned}\hat{R}_{ab}{}^{cd} &= -4\delta^{[c}_{[a}\,\mathrm{tr}(P_{b]}P^{d]}) + \delta^c_{[a}\delta^d_{b]}\,\mathrm{tr}(P_eP^e),\\ \hat{R}_{ab}{}^{cl} &= 0,\\ \hat{R}_{ab}{}^{kl} &= -2(P_{[a}P_{b]})^{kl},\\ \hat{R}_{aj}{}^{cl} &= -(D_aP^c)_j{}^l - (P_aP^c)_j{}^l,\\ \hat{R}_{aj}{}^{kl} &= 0,\\ \hat{R}_{ij}{}^{kl} &= -2(P_e)_{[i}{}^k(P^e)_{j]}{}^l.\end{aligned} \tag{5.2}$$

Note that Einstein's equations in three dimensions have been used to obtain the specific form of $\hat{R}_{ab}{}^{cd}$.

The two independent cubic contractions of the Riemann tensor components above become after compactification, and keeping only terms which give pure P contributions[2],

$$\begin{aligned}\hat{R}_{AB}{}^{CD}\hat{R}_{DC}{}^{EF}\hat{R}_{FE}{}^{BA} = {}& R_{ij}{}^{kl}R_{lk}{}^{mn}R_{nm}{}^{ji} + 3R_{ab}{}^{kl}R_{lk}{}^{mn}R_{nm}{}^{ba} + 8R_{aj}{}^{cl}R_{lc}{}^{en}R_{ne}{}^{ja}\\ &+ R_{ab}{}^{cd}R_{dc}{}^{ef}R_{fe}{}^{ba} + 3R_{ab}{}^{cd}R_{dc}{}^{ij}R_{ji}{}^{ba}\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\hat{R}_A{}^B{}_D{}^E\hat{R}_B{}^C{}_E{}^F\hat{R}_C{}^A{}_F{}^D = {}& R_i{}^j{}_l{}^m R_j{}^k{}_m{}^n R_k{}^i{}_n{}^l + 3R_a{}^j{}_d{}^m R_j{}^k{}_m{}^n R_k{}^a{}_n{}^d + 2R_i{}^j{}_d{}^e R_j{}^k{}_e{}^f R_k{}^i{}_f{}^d\\ &+ 6R_i{}^b{}_d{}^m R_b{}^k{}_m{}^f R_k{}^i{}_f{}^d + R_a{}^b{}_e{}^f R_b{}^c{}_f{}^g R_c{}^d{}_g{}^e + 3R_a{}^b{}_c{}^d R_b{}^i{}_d{}^j R_i{}^a{}_j{}^c.\end{aligned} \tag{5.4}$$

We now insert the P-dependent terms from above. For the purposes here it is sufficient to collect only the $(DP)^3$ and $P^2(DP)^2$ terms, while remembering that $\mathrm{tr}P = 0$.

For $(DP)^3$, which gets a contribution entirely from (1)[3], it is straightforward to show that it is a total derivative

$$(DP)^3 = \mathrm{tr}\big(S_a{}^b S_b{}^c S_c{}^a\big) = D_c\big[\mathrm{tr}(P^a S_a{}^b S_b{}^c) - \tfrac{1}{2}\mathrm{tr}(P^c S_a{}^b S_b{}^a)\big] \tag{5.5}$$

(as always, modulo the lowest-order equations of motion), where $S_{ab} = D_a P_b$. The tensor $(S_{ab})_{ij}$ is symmetric in both (ab) and (ij), and $S_a{}^a$ is the kinetic term in the equation of motion for P_a. The fact that, modulo equations of motion, the $(DP)^3$ term is a total derivative is expected for the highest derivative term in a Gauss–Bonnet combination of any order, but we see that after dimensional reduction the scalar three-point couplings vanish for any $\hat{R}^3$ term.

Doing a similar analysis for the $(DP)^2P^2$ terms, there are ten algebraically independent structures:

$$\begin{aligned}(i) &= \mathrm{tr}\big(S_a{}^b S_b{}^a P_c P^c\big),\\ (ii) &= \mathrm{tr}\big(S_a{}^b S_b{}^c P_c P^a\big),\\ (iii) &= \mathrm{tr}\big(S_a{}^b S_b{}^c P^a P_c\big),\\ (iv) &= \mathrm{tr}\big(S_a{}^b P_c S_b{}^a P^c\big),\\ (v) &= \mathrm{tr}\big(S_a{}^b P_b S^a{}_c P^c\big),\\ (vi) &= \mathrm{tr}\big(S_a{}^b S_b{}^a\big)\,\mathrm{tr}(P_c P^c),\\ (vii) &= \mathrm{tr}\big(S_a{}^b S_b{}^c\big)\,\mathrm{tr}\big(P_c P^a\big),\\ (viii) &= \mathrm{tr}\big(S_a{}^b P_c\big)\,\mathrm{tr}\big(S_b{}^a P^c\big),\\ (ix) &= \mathrm{tr}\big(S_a{}^b P_c\big)\,\mathrm{tr}\big(S_c{}^a P_b\big),\\ (x) &= \mathrm{tr}\big(S_a{}^b P_b\big)\,\mathrm{tr}\big(S_c{}^a P^c\big).\end{aligned} \tag{5.6}$$

Since (i)–(v) will not mix with (vi)–(x), we will consider the two groups separately. For the single-trace terms, neglecting the equations of motion, the combination $x_1(i) + 2x_2(ii) - (x_2 - 2x_3)(iii) + x_3(iv) + (2x_1 - x_2)(v)$ is a total derivative for arbitrary values of $\{x_*\}$. Correspondingly, a total derivative consisting of the double-trace terms must be written as $y_1(vi) + (y_2 + y_3)(vii) + y_2(viii) + (y_2 - 2y_3)(ix) + (2y_1 + y_3)(x)$ for arbitrary values of $\{y_*\}$. Extracting the pure $(DP)^2P^2$ terms from (5.3) and (5.4), we find that

$$(1) + z(2) = 6(4 - z)(ii) + 6z(iii) + 3z(iv) + z\left[-\tfrac{3}{2}(vi) + 6(vii) - 3(viii)\right], \tag{5.7}$$

[2] The combinatorial factors are easily read off from the diagrams. Splitting of the indices into two classes, spacetime and internal, corresponds to colouring the lines in the diagrams with two colours. The factors are given by the number of ways this can be done.

[3] This term comes only from $R_{aj}{}^{cl}$, which means that it gets contributions only from coloured diagrams with alternating colour on all cycles. A diagram containing a cycle with an odd number of lines cannot contribute.

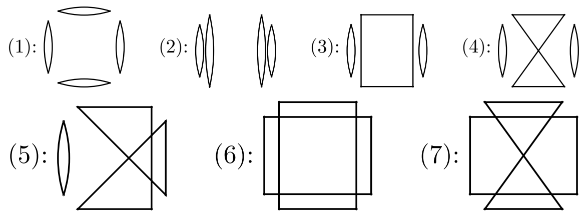
with an arbitrary parameter z. For the single-trace terms in equation (5.7), $z = 4$ is the only choice where they can form a total derivative, this corresponds to the case $x_1 = x_2 = 0$, $x_3 = 12$ (which is not the Gauss–Bonnet combination from equation (5.1)). For the double-trace terms in equation (5.7), however, no choice of z can make them a total derivative. We have thus shown that the $(DP)^2P^2$ cannot vanish by partial integrations. Unlike in the $\hat{R}^2$ terms, derivatives of Maurer–Cartan forms necessarily appear.

A more complete treatment should include also the other fields in $\mathcal{P}$. One should also continue with terms of the types $\mathcal{P}^4 D\mathcal{P}$ and $\mathcal{P}^6$. This would imply quite some work which we do not find motivated for $\hat{R}^3$. In order to access the complete expressions, care has to be taken when using partial integrations, since terms with a certain number of derivatives contribute to terms with fewer derivatives via equations of motion, Bianchi identities and curvatures (R and F_Q).

6. R^4 terms

In this section, we start the analysis of the $\hat{R}^4$ terms by presenting the content of $t_8 t_8 \hat{R}^4$ and $\varepsilon\varepsilon\hat{R}^4$ in terms of an explicitly given basis of seven elements. That this basis is seven-dimensional is well known [22]. We then concentrate on the terms that after the dimensional reduction contain only the coset variable P_a^{ij}. These are of the types $(DP)^4$, $P^2(DP)^3$, $P^4(DP)^2$, $P^6(DP)$ and P^8. A test of the possible role of the octic invariant of $E_{8(8)}$ derived in [23] is spelt out (for details see appendix B). This would involve the P^8 terms and be rather lengthy. For that reason we turn in the section to the much simpler terms $(DP)^4$ from which we are able to draw the conclusions we are looking for.

Using the same diagrammatic notation as in the previous section, a basis for the seven $\hat{R}^4$ terms can be taken as



A contraction that also occurs naturally (e.g. in $\varepsilon\varepsilon R^4$) is , and it can be related to the others by using $\hat{R}_{[ABC]D} = 0$ as follows: cycling on gives $=$ $-\frac{1}{2}$. Cycling on (5) $=$ gives $=$ $+\frac{1}{2}$, and on (7) $=$ gives $=$ $+\frac{1}{2}$. Eliminating the diagrams not present in the basis, and , gives the relation $= \frac{1}{4}(4) - (5) + (7)$.

In $D = 10$ and 11, the structures

$$\varepsilon\varepsilon\hat{R}^4 = \varepsilon^{A_1\ldots A_D}\varepsilon_{B_1\ldots B_D}\,\hat{R}_{A_1A_2}{}^{B_1B_2}\,\hat{R}_{A_3A_4}{}^{B_3B_4}\,\hat{R}_{A_5A_6}{}^{B_5B_6}\,\hat{R}_{A_7A_8}{}^{B_7B_8}\,\delta^{B_9\ldots B_D}_{A_9\ldots A_D} \tag{6.1}$$

(the 'Gauss–Bonnet term') and $t_8t_8\hat{R}^4$ are of special interest, since they, or combinations of these, are dictated by string theory calculations and by supersymmetry; see for instance the explicit evaluation in appendix B2 of [12] of the appropriate superspace term given in [24]. The invariant tensor $t_8^{A_1A_2,A_3A_4,A_5A_6,A_7A_8}$ is defined to be antisymmetric in the indices composing the pairs and symmetric in the four pairs. When contracted with the antisymmetric matrix M, it is defined to give

$$t_8^{A_1A_2,A_3A_4,A_5A_6,A_7A_8}M_{A_1A_2}M_{A_3A_4}M_{A_5A_6}M_{A_7A_8} = 24\,\mathrm{tr}M^4 - 6(\mathrm{tr}M^2)^2. \tag{6.2}$$

In $t_8t_8\hat{R}^4$, the indices are contracted according to

$$t_8t_8\hat{R}^4 = t_8^{A_1A_2,A_3A_4,A_5A_6,A_7A_8}t_{8B_1B_2,B_3B_4,B_5B_6,B_7B_8}$$
$$\times \hat{R}_{A_1A_2}{}^{B_1B_2}\hat{R}_{A_3A_4}{}^{B_3B_4}\hat{R}_{A_5A_6}{}^{B_5B_6}\hat{R}_{A_7A_8}{}^{B_7B_8}. \tag{6.3}$$

A direct evaluation gives, in D dimensions,

$$\frac{1}{12}t_8t_8\hat{R}^4 = 2(1) + (2) - 16(3) - 8(4) + 16(6) + 32(7)$$
$$-\frac{1}{48(D-8)!}\varepsilon\varepsilon\hat{R}^4 = 2(1) + (2) - 16(3) + 32(5) + 16(6) - 32 \tag{6.4}$$

or, with expressed in the basis as above,

$$\frac{1}{12}t_8t_8\hat{R}^4 = 2(1) + (2) - 16(3) - 8(4) + 16(6) + 32(7)$$
$$-\frac{1}{48(D-8)!}\varepsilon\varepsilon\hat{R}^4 = 2(1) + (2) - 16(3) - 8(4) + 64(5) + 16(6) - 32(7). \tag{6.5}$$

These expressions agree with the ones in, e.g., [17], where the basis $\{A_1 = (2), A_2 = (3), A_3 = (1), A_4 = (4), A_5 = - = \frac{1}{2}(4) - (5), A_6 = (6), A_7 = = \frac{1}{4}(4) - (5) + (7)\}$ is used.

The P^8 terms obtained when compactifying from 11 dimensions to 3 will of course form a scalar of $SO(8)$. Assuming that these terms combine to a scalar also of the $\mathrm{Spin}(16)/\mathbb{Z}_2$ that is associated with coset $E_{8(8)}/(\mathrm{Spin}(16)/\mathbb{Z}_2)$ arising in the two-derivative sector of M-theory, the only invariant possible (apart from the fourth power of the quadratic one) would be the octic invariant constructed in [23]. As explained in appendix B, when reducing this to an invariant of $SO(9)$ one finds a certain polynomial in the $SL(9)/SO(9)$ coset element that if valid puts severe restrictions on the structure of the P^8 terms. However, checking this is lengthy and instead we turn to the $(DP)^4$ terms where, as we will see below, some qualitative results we are looking for can be obtained with much less effort.

Thus, we now concentrate on the four-point couplings, which consequently have four derivatives. Assume, for the moment, that $E_{8(8)}(\mathbb{Z})$ invariance were to be achieved with a scalar automorphic form. Since E_8 has no invariant of order four other than the square of the quadratic Casimir (and thus the only $\mathfrak{so}(16)$ invariant quartic in spinors is the square of the quadratic one), we would get the restriction that any trace $\mathrm{tr}(DP)^4$ has to vanish, since this $\mathfrak{so}(8)$ invariant cannot be lifted via $\mathfrak{so}(9)$ to $\mathfrak{so}(16)$.

Using the Riemann tensor with only P terms (see the previous section), it is not very difficult to derive the $(DP)^4$ terms from diagrams (1)–(7). Since we only want contributions with the components R_{aibj}, one gets one contribution from each colouring with two colours (for spacetime and internal indices) of the graphs, where the two colours alternate on every cycle. It follows directly that any diagram with a cycle of odd length does not contribute. There are none in the basis, but in the process of cycling above we encountered the contraction $= (5) - \frac{1}{2}(4)$ that then does not contribute to $(DP)^4$.

There are eight algebraically independent structures containing $(DP)^4$. We enumerate them as

$$
\begin{aligned}
(i) &= \mathrm{tr}(S_{ab}S^{ab}S_{cd}S^{cd}),\\
(ii) &= \mathrm{tr}(S_{ab}S_{cd}S^{ab}S^{cd}),\\
(iii) &= \mathrm{tr}(S_{ab}S^{bc}S_{cd}S^{da}),\\
(iv) &= \mathrm{tr}(S_{ab}S_{cd}S^{ac}S^{bd}),\\
(v) &= \mathrm{tr}(S_{ab}S^{ab})\,\mathrm{tr}(S_{cd}S^{cd}),\\
(vi) &= \mathrm{tr}(S_{ab}S^{cd})\,\mathrm{tr}(S_{ab}S^{cd}),\\
(vii) &= \mathrm{tr}(S_{ac}S^{bc})\,\mathrm{tr}(S^{ad}S_{bd}),\\
(viii) &= \mathrm{tr}(S_{ab}S_{cd})\,\mathrm{tr}(S^{ac}S^{bd}),
\end{aligned}
\tag{6.6}
$$

where $S_{ab} = D_a P_b$. One also has to take total derivatives into account. This can be done by writing out all possible terms $(PS^3)_a$ (there are 12) and take the divergence. As long as we only consider $(DP)^4$, we let $S^a{}_a \to 0$ and $S_{[ab]} \to 0$. It turns out that only two combinations of these do not produce terms $P(DP)^2D^2P$ (the second derivative of P can again be considered as symmetric and traceless), and they lead to the combinations $(i) + \frac{1}{2}(ii) - (iii) - 2(iv)$ and $\frac{1}{2}(v) + (vi) - 2(vii) - (viii)$ being total derivatives. (These in fact arise from $\mathrm{tr}\big(P_{[a}S_b{}^bS_c{}^cS_{d]}{}^d\big)$ and $\mathrm{tr}\big(P_{[a}S_b{}^b)\,\mathrm{tr}(S_c{}^cS_{d]}{}^d\big)$, where the antisymmetry, by the Bianchi identity, prevents $P(DP)^2D^2P$ from arising. The counting also holds for reduction to $d = 3$, but with the combinations being total derivatives in higher dimensions now being identically zero.)

Evaluating the contributions to the 4-point couplings from the terms $(1), \ldots, (7)$ then gives

$$
\begin{aligned}
(1) &\longrightarrow 16(iii),\\
(2) &\longrightarrow 16(v),\\
(3) &\longrightarrow 4(i) + 4(vii),\\
(4) &\longrightarrow 8(iv),\\
(5) &\longrightarrow 4(iv),\\
(6) &\longrightarrow 2(iii) + 2(vi),\\
(7) &\longrightarrow (ii) + 2(iv) + (viii).
\end{aligned}
\tag{6.7}
$$

Demanding that the contribution vanishes, modulo total derivatives, tells that the R^4 term is proportional to $2(1) + (2) - 16(3) + x(4) + (48 - 2x)(5) + 16(6) - 32(7)$ for some number x. $\varepsilon\varepsilon\hat{R}^4$ (of course) passes the test, but $t_8t_8\hat{R}^4$ does not. The combination $(4) - 2(5)$ does, as seen above. In this calculation, $t_8t_8\hat{R}^4$ does not even contribute with $(\mathrm{tr}S^2)^2$ terms only, as would be demanded from E_8 invariance. (The condition that the long contractions vanish can be expressed as conditions on the coefficients in front of (5)–(7), given the ones in front of (1)–(4). The latter are identical in $t_8t_8\hat{R}^4$ and $\varepsilon\varepsilon\hat{R}^4$.) The 'difference' between $t_8t_8\hat{R}^4$ and $\varepsilon\varepsilon\hat{R}^4$ (with the normalizations above) is another very simple expression, proportional to $(7) - (5)$ or to  $- \frac{1}{4}(4)$, whose contribution to the long contractions is $(ii) - 2(iv) \neq 0$. In the conclusion, if the term $t_8t_8\hat{R}^4$ is present, there are four-point couplings not only in the gravity sector but also in the scalar sector. The term $t_8t_8\hat{R}^4$ cannot be obtained without transforming E_8 automorphic forms.

We thus find a contradiction with E_8 unless transforming automorphic forms are introduced. The fact that E_8 does not have primitive fourth-order invariant means that the $SL(8)$-invariant D^4P^4 terms derived here must come from an E_8 term which is a double trace. Since we find nonzero single-trace terms, this means that the enhanced symmetries do not generalize to higher derivative terms obtained through compactification as described here with scalar automorphic forms.

Given that the number of automorphic forms of E_8 is smaller than that of $SL(9)$, for the same number of $\mathfrak{so}(16)$ spinors or $\mathfrak{so}(9)$ symmetric traceless tensors (see appendix A), it seems reasonable to believe that E_8 puts some constraints on the possible terms obtained by reduction of pure gravity. Checking this would require more concrete knowledge of E_8 automorphic forms and their large volume limit, as well as (presumably) a much more complete expansion of the seven R^4 terms. It is not at all clear to what degree E_8 will single out some specific combination of these.

Performing a loop calculation with external scalars analogous to the ones in [19, 25] would give information on what kind automorphic functions actually appear in an M-theory context (although such a calculation leaves out non-perturbative information from winding membranes and five-branes).

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Appendix A. Automorphic forms

Consider an element $g \in G$, where G is a Lie group. In the context of the supergravities (or sigma models) we are considering, g represents the scalar degrees of freedom. These belong to a coset G/K, where K is a subgroup of G. In all cases under consideration, G has the split (maximally non-compact) real form and K is the maximal compact subgroup of G. The coset is realized by gauging the local right action of K, $g \to gk, k \in K$. This still leaves room for a global left action of G on g, $g \to \gamma g, \gamma \in G$. These global G transformations are however symmetries only of the undeformed supergravities or sigma models, and are broken by quantum effect in string theory. Higher-derivative corrections to effective actions in string theory are expected to break G to a discrete duality subgroup $G(\mathbb{Z})$, and the correct moduli space for the scalars is not G/K but $G(\mathbb{Z})\backslash G/K$.

The definition of $G(\mathbb{Z})$ has to be clear, of course. If G is a classical matrix group, it can be defined as the group of elements in G with integer entries in the fundamental representation. For exceptional groups, care has to be taken to choose the relevant discrete subgroup. A definition of $G(\mathbb{Z})$ in terms of generators of the Lie algebra $\mathfrak{g}$ of G in the Chevalley basis is given in [26] (see also [2]). In the following, it will be understood that $G(\mathbb{Z})$ is the discrete duality group relevant to M-theory compactifications, although the construction in principle holds also for other discrete subgroups of G.

The general method for building automorphic forms [10, 11, 25, 27, 28] is to combine g with some element in the discrete group (or a representation of it) so that the resulting entity only transforms under K, in the sense defined below. The invariance under $G(\mathbb{Z})$ is then obtained by summation over $G(\mathbb{Z})$ (or some representation). Let $g \in G$ and $\mu \in G(\mathbb{Z})$, with the transformation rules under $G(\mathbb{Z}) \times K$ with group element $\gamma \otimes k$: $g \to \gamma g k$, $\mu \to \gamma\mu\gamma^{-1}$. If one forms $g^{-1}\mu g$, it transforms as $g^{-1}\mu g \to k^{-1}(g^{-1}\mu g)k$, i.e., only under K. One may then K-covariantly project $g^{-1}\mu g$ on the representation '$\mathfrak{g}/\mathfrak{k}$', i.e., the complement to $\mathfrak{k}$ in the Lie algebra $\mathfrak{g}$, which forms a representation of K.[4] We denote the obtained building block $\Gamma = \Pi_{\mathfrak{g}/\mathfrak{k}}(g^{-1}\mu g)$. When using tensor notation, we write Γ_α, inspired by the $\mathfrak{so}(16)$ spinor index carried by the tangent space to $E_8(8)/(\mathrm{Spin}(16)/\mathbb{Z}_2)$.

[4] This projection may be performed by letting g and μ be represented as matrices in any faithful representation of G, the result of course being independent of the choice of representation.

Let us start with the simplest kind of automorphic forms, the scalar ones. In order for the function not to transform under K, we need to form scalars from a number of Γ's. This is straightforward—the algebraically independent polynomial invariants have the same number and degree of homogeneity as the Casimir operators of $\mathfrak{g}$. In fact, as observed in [23], they are simply the restrictions of the Casimir operators to $\mathfrak{g}/\mathfrak{k}$. Let us denote them $C_i(\Gamma)$, $i = 1, \ldots, r$, r being the rank of $\mathfrak{g}$. Finally, in order to achieve invariance under $G(\mathbb{Z})$, one has to form some function of the C_i's and sum over the discrete group element μ. The function should be conveniently formed so that the sum converges, e.g. a power function. For some 'weight' w, we thus define

$$\phi^{(i,w)}(g) = \sum_{\mu \in G(\mathbb{Z})} [C_i(\Gamma(\mu, g))]^{-w}. \tag{A.1}$$

This automorphic function is clearly a function on the double coset $G(\mathbb{Z})\backslash G/K$.

The construction above is entirely based on Γ, which is obtained as (a projection of) the action of g by conjugation on a discrete group element μ. Alternatively, one may start from some representation. Especially, when G is a classical matrix group, it is simpler to let m lie in the fundamental module (a row vector with integer entries) [28]. Consider the case $G = SL(n)$ with $K = SO(n)$. We form mg, which if m transforms as $m \to m\gamma^{-1}$ transforms under $G(\mathbb{Z}) \times K$ as $mg \to (mg)k$. Then, mg is taken as a building block, and one forms the invariant $|mg|^2 = (mg)(mg)^t$. The automorphic function is

$$\psi^{(w)}(g) = \sum_{\mathbb{Z}^n\backslash 0} |mg|^{-2w}. \tag{A.2}$$

This construction has the advantage that the summation is easier to perform than the one over the discrete group, but it is not available for exceptional groups G. Choosing other modules yields algebraically independent automorphic functions, as long as these modules are formed by anti-symmetrization from the fundamental one. One gets again a number of functions equating the rank.

We expect that the summation in the defining equation (A.1), which is over a single orbit of the discrete group, namely the group itself, can be lifted to the summation over a lattice, quite analogously to how the summation in equation (A.2) can be decomposed into an infinite number of orbits. Such a lattice summation might make even automorphic forms of exceptional groups reasonable to handle. Equation (A.1), with the replacement of the discrete group by a lattice, is well suited for the bosonic degrees of freedom of the sigma model obtained by dimensional reduction, since the object Γ carries the same index structure as P. When it comes to fermions, these transform under another representation which is (an enlargement of) a spinor representation of $\mathfrak{so}(n)$, and it seems natural to consider spinorial automorphic forms.

One attractive feature of invariant automorphic forms, automorphic functions, is that their structure and number closely reflect the properties of the Lie algebra $\mathfrak{g}$. Once one takes the step to transforming automorphic forms, the freedom is much bigger. Remember that the scalar degrees of freedom reside in the coset $G(\mathbb{Z})\backslash G/K$, and that they appear through the 'physical' part P of the Maurer–Cartan form $g^{-1}\,\mathrm{d}g$, $P = \Pi_{\mathfrak{g}/\mathfrak{k}}(g^{-1}\,\mathrm{d}g)$. Any higher-derivative term (considering purely scalar terms) contain a number of P's, perhaps with covariant derivatives, contracted with something that cancels the K transformation of P in the appropriate way. Note that $\Gamma(\mu, g)$ transforms correctly, so that a K-invariant object may be formed by contracting P's either with each other, or with Γ's. Again, summation over $G(\mathbb{Z})$ is of course needed. We arrive at automorphic forms of the generic form

$$\phi^{(i,w,k)}_{\alpha_1\ldots\alpha_k}(g) = \sum_{\mu \in G(\mathbb{Z})} \Gamma_{\alpha_1}\ldots\Gamma_{\alpha_k} C_i(\Gamma)^{-w}, \tag{A.3}$$

where again $\Gamma_\alpha = \Gamma_\alpha(\mu, g) = [\Pi_{\mathfrak{g}/\mathfrak{k}}(g^{-1}\mu g)]_\alpha$. The automorphic form ϕ is symmetric in the α_k indices. The restricted Casimir C_i is inserted for convergence of the sum. We see that, for a given choice of C_i and w, there is one automorphic form for each irreducible K-module in the symmetric tensor product of n elements in $\mathfrak{g}/\mathfrak{k}$, generically a much larger number than the number of invariant automorphic forms.

Also here, the simpler construction for $G = SL(n)$ is obtained with an even number of $(mg)_I$'s (I is the fundamental index) as

$$\psi^{(w,l)}_{I_1\ldots I_{2l}}(g) = \sum_{\mathbb{Z}^n\backslash 0}(mg)_{I_1}\ldots(mg)_{I_{2l}}|mg|^{-2w}. \tag{A.4}$$

We would like to comment on the transformation properties of the transforming automorphic forms. As they are written (as functions of g), they are a collection of functions on $G(\mathbb{Z})\backslash G$, transforming under K transformations as specified by the index structure. If we, on the other hand, view g as a representative of the right coset G/K by fixing a gauge encoded in some parametrization $g = g(\tau)$, the picture changes. The coset coordinates τ transform nonlinearly under $G(\mathbb{Z})$, and a compensating gauge transformation is required to get back on the gauge hypersurface. The transformations under $G(\mathbb{Z})$ are of the form $g(\tau) \to \gamma g(\tau)k(\gamma,\tau)$. In this picture, a $G(\mathbb{Z})$ transformation of the automorphic forms induces a K-transformation with the element $k(\gamma,\tau)$ on the appropriate module given by the index structure. This can of course be mimicked without gauge fixing by replacing the element $\gamma\otimes 1 \in G(\mathbb{Z})\times K$ by the element $\gamma\otimes k(\gamma,\tau)$, which allows us to interpret the automorphic forms as collections of functions on G/K with a specific nonlinear transformation property under $G(\mathbb{Z})$.

We are sometimes interested in certain limiting values of automorphic forms. In the present paper, the terms obtained after dimensional reduction should correspond to leading terms in an asymptotic expansion at large volume of a torus. We consider the possibility of collecting the terms we obtain in sums of automorphic functions of $SL(n+1)$ after reducing on T^n. With the embedding of $GL(n)$ in $SL(n+1)$ discussed earlier in section 3, where e^ϕ is the determinant of the metric on T^n, the $SL(n+1)$ group element may (with a partial gauge choice) be parametrized as

$$G = \begin{bmatrix} e^{-\phi} & 0 \\ e^{-\phi}u & e^{\frac{\phi}{n}}\tilde{e}\end{bmatrix}, \tag{A.5}$$

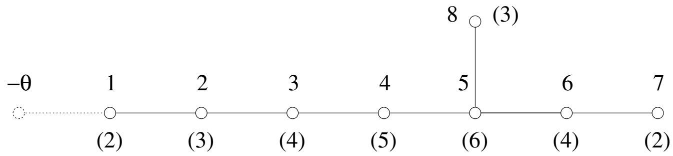
where $\tilde{e}$ is a group element of $SL(n)$ parametrizing the shape of T^n. The large volume limit is $\phi\to\infty$. The shape of T^n should be irrelevant in this limit, as long as it is non-degenerate, and we take $\tilde{e} = \mathbb{1}$. An automorphic form of the type in equation (A.4) with $2w > 2l+1$ (for convergence) is then dominated by terms with $m = (m_0, 0, \ldots, 0)$ and has the limiting value [10]

$$\psi^{(w,l)}_{I_1\ldots I_{2l}}(g) \underset{\phi\to\infty}{\longrightarrow} e^{2(w-l)\phi}\zeta(2(w-l))\delta_{I_1,0}\ldots\delta_{I_{2l},0}. \tag{A.6}$$

Finally, it is interesting to count the number of possible terms one can write down in a concrete situation. Much of the present paper aims at reduction to $d = 3$ and the coset $E_{8(8)}(\mathbb{Z})\backslash E_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$. An R^4 correction contains terms with up to eight P's. Just considering these for a given w (i.e., for the moment omitting the terms with derivatives of P), and assuming that we use the quadratic Casimir, the number of possible terms obtainable are labelled by irreducible $\mathfrak{so}(16)$ representations in the symmetric product of eight chiral spinors. The number of representations, i.e., of automorphic forms, is 222. This can be compared to the number of scalars, two, which is obtained directly from the E_8 Casimir operators. The corresponding number relevant for gravity, i.e., the number of irreducible $\mathfrak{so}(9)$ representations in the symmetric product of eight symmetric traceless tensors, is 609. It seems that demanding $E_{8(8)}$ invariance gives some restriction even on the possible $SL(9,\mathbb{Z})$-invariant terms involving the gravitational scalars only, but it will take some ingenuity to extract the information. It is tempting to believe that the octic E_8 invariant [23] has some special role in the R^4 terms, but this remains unclear in the light of the large number of transforming automorphic functions.

Appendix B. Reduction of the octic invariant to matrices

By assuming that E_8 organizes the scalars after compactification to three dimensions also after the inclusion of R^4 terms, we can obtain constraints related to $SL(9)$ which are more readily checked. To see this, consider the $\mathfrak{e}_8$ Dynkin diagram, with Coxeter labels and extended root:



The horizontal line consists of the simple roots of $\mathfrak{sl}(9)$. In the standard way of embedding $\mathfrak{sl}(n)$ roots in $(n+1)$-dimensional space, an element in the Cartan algebra of $\mathfrak{sl}(9)$ (and, thereby, of $\mathfrak{e}_8$) can be written in an orthonormal basis as $M=(m_0,m_1-m_0,m_2-m_1,\ldots,m_7-m_6,-m_7)$. We have $\alpha_0=-\theta=-(2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+6\alpha_5+4\alpha_6+2\alpha_7+3\alpha_8)$. Solving for α_8 gives $\alpha_8=\frac{1}{3}(-1,-1,-1,-1,-1,-1,2,2,2)$ in the orthonormal basis.

Invariants under $\mathfrak{sl}(9)$ restricted to the CSA can be formed as $\mathrm{tr}M^n\equiv\sum_{i=1}^{9}(M_i)^n$ (i.e., the vector M above is thought of as the diagonal of a matrix M). They will all be automatically invariant under the Weyl group of $\mathfrak{sl}(9)$, generated by simple reflections permuting nearby components of the nine-dimensional vectors in the orthonormal basis. The only thing one has to check for invariance under the Weyl group of $\mathfrak{e}_8$ is invariance under reflection in the hyperplane orthogonal to the exceptional root α_8. As a (9×9)-matrix it is realized as

$$w(\alpha_8)=\mathbb{1}-\alpha_8^t\alpha_8=\frac{1}{9}\begin{bmatrix} 8&-1&-1&-1&-1&-1&2&2&2\\ -1&8&-1&-1&-1&-1&2&2&2\\ -1&-1&8&-1&-1&-1&2&2&2\\ -1&-1&-1&8&-1&-1&2&2&2\\ -1&-1&-1&-1&8&-1&2&2&2\\ -1&-1&-1&-1&-1&8&2&2&2\\ 2&2&2&2&2&2&5&-4&-4\\ 2&2&2&2&2&2&-4&5&-4\\ 2&2&2&2&2&2&-4&-4&5 \end{bmatrix}, \tag{B.1}$$

and acts on M as $w(\alpha_8)M=M+\frac{1}{3}m_5(-1,-1,-1,-1,-1,-1,2,2,2)$. A general ansatz for the restriction of the octic $\mathfrak{e}_8$ invariant to the CSA (using $\mathfrak{sl}(9)$ 'covariance') is $S(M)=\mathrm{tr}M^8+a\,\mathrm{tr}M^6\,\mathrm{tr}M^2+b\,\mathrm{tr}M^5\,\mathrm{tr}M^3+c(\mathrm{tr}M^4)^2+d\,\mathrm{tr}\,M^4(\mathrm{tr}M^2)^2+e(\mathrm{tr}M^3)^2\,\mathrm{tr}M^2+f(\mathrm{tr}M^2)^4$. The coefficient f is of course arbitrary, and will be left out. We demand that $S(w(\alpha_8)M)=S(M)$. A short Mathematica calculation then gives the values of the coefficients in the ansatz:

$$\begin{aligned} S(M)=\mathrm{tr}M^8&-\tfrac{28}{45}\,\mathrm{tr}M^6\,\mathrm{tr}M^2-\tfrac{28}{45}\,\mathrm{tr}M^5\,\mathrm{tr}M^3-\tfrac{7}{36}(\mathrm{tr}M^4)^2\\ &+\tfrac{7}{36}\,\mathrm{tr}M^4(\mathrm{tr}M^2)^2+\tfrac{7}{27}(\mathrm{tr}M^3)^2\,\mathrm{tr}M^2. \end{aligned} \tag{B.2}$$

This is the polynomial (in the symmetric (9×9)-matrix $\mathcal{P}$) we should look for in the R^4 terms if multiplied by a scalar automorphic form of E_8. It has to be the same formal expression already in terms of the P of $SO(8)$.

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Paper IV

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U-Duality and the compactified Gauss-Bonnet term

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Abstract: We present the complete toroidal compactification of the Gauss-Bonnet Lagrangian from D dimensions to $D-n$ dimensions. Our goal is to investigate the resulting action from the point of view of the "U-duality" symmetry $\mathrm{SL}(n+1,\mathbb{R})$ which is present in the tree-level Lagrangian when $D-n=3$. The analysis builds upon and extends the investigation of the paper [arXiv:0706.1183], by computing in detail the full structure of the compactified Gauss-Bonnet term, including the contribution from the dilaton exponents. We analyze these exponents using the representation theory of the Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ and determine which representation seems to be the relevant one for quadratic curvature corrections. By interpreting the result of the compactification as a leading term in a large volume expansion of an $\mathrm{SL}(n+1,\mathbb{Z})$-invariant action, we conclude that the overall exponential dilaton factor should not be included in the representation structure. As a consequence, all dilaton exponents correspond to weights of $\mathrm{sl}(n+1,\mathbb{R})$, which, nevertheless, remain on the positive side of the root lattice.

Keywords: Discrete and Finite Symmetries, Global Symmetries, M-Theory, String Duality.

Contents

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1. Introduction and summary

Dimensional reduction of supergravity theories is an efficient method of revealing symmetry structures which are "hidden" when the theories are formulated in maximal dimension. The first discovery of such a hidden symmetry was the so-called Ehlers symmetry of pure four-dimensional gravity compactified on a circle to three dimensions [1]. The global symmetry $\mathrm{GL}(1,\mathbb{R}) = \mathbb{R}$, corresponding to rescaling of the S^1, is in this case extended through dualisation of the Kaluza-Klein vector into a new scalar, revealing that the full global symmetry of the Lagrangian is, in fact, described by the group $\mathrm{SL}(2,\mathbb{R})$. The scalars in the theory parametrise the coset space $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$, where $\mathrm{SO}(2)$ is the maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$, playing the role of a local gauge symmetry. More generally, upon toroidal compactification of lowest order pure gravity in D spacetime dimensions on an n-torus, T^n, to three dimensions, the scalars parametrise the coset space $\mathrm{SL}(n+1,\mathbb{R})/\,\mathrm{SO}(n+$

1). The enhancement from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{SL}(n+1, \mathbb{R})$ is again due to the fact that in three dimensions all Kaluza-Klein vectors can be dualised to scalars.

Similar phenomena occur also for coupled gravity-dilaton-p-form theories, such as the bosonic sectors of the low-energy effective actions of string and M-theory. The most thoroughly investigated case is the toroidal compactification of eleven-dimensional supergravity on T^n to $d = 11 - n$ dimensions, for which the scalar sector parametrises the coset space $\mathcal{E}_{n(n)}/\mathcal{K}(\mathcal{E}_{n(n)})$, with $\mathcal{K}(\mathcal{E}_{n(n)})$ being the (locally realized) maximal compact subgroup of $\mathcal{E}_{n(n)}$ [2]. In particular, for reduction to three dimensions the global symmetry group is the split real form $\mathcal{E}_{8(8)}$, with maximal compact subgroup $\mathrm{Spin}(16)/\mathbb{Z}_2$. The global symmetry group $\mathcal{E}_{8(8)}$ is the U-duality group, which, from a string theory perspective, combines the non-perturbative S-duality group $\mathrm{SL}(2, \mathbb{R})$ of type IIB supergravity with the perturbative T-duality group $\mathrm{SO}(7, 7)$ [3].

These symmetries are present in the classical (tree-level) Lagrangian, but it is known from string theory that they must be broken by quantum effects. It has been conjectured that if $\mathcal{U}_d$ is the continuous symmetry group appearing upon compactification from D to $d = D - n$ dimensions, then a discrete subgroup $\mathcal{U}_d(\mathbb{Z}) \subset \mathcal{U}_d$ lifts to a symmetry of the full quantum theory [4−6].[1] The physical degrees of freedom of the scalar sector then parametrise the coset space $\mathcal{U}_d(\mathbb{Z})\backslash\mathcal{U}_d/\mathcal{K}(\mathcal{U}_d)$.

1.1 Non-Perturbative completion and automorphic forms

Recently, several authors [7−11] have initiated an investigation aimed at answering the question of whether or not the U-duality group $\mathcal{U}_3$ in three dimensions is preserved also if the tree-level Lagrangian is supplemented by higher order curvature corrections. The consensus has been that toroidal compactifications of quadratic and higher order corrections give rise to terms which are not $\mathcal{U}_3$-invariant.[2]

A nice example of a fairly well understood realisation of these mechanisms is the breaking of the classical $\mathrm{SL}(2, \mathbb{R})$ symmetry of the type IIB supergravity effective action down to the quantum S-duality group $\mathrm{SL}(2, \mathbb{Z})$ of the full type IIB string theory [12]. The next to leading order α'-corrections to the effective action are octic in derivatives of the metric, i.e., fourth order in powers of the Riemann tensor, and receives perturbative contributions only from tree-level and one-loop in the string genus expansion. However, this gives a scalar coefficient in front of the $\mathcal{R}^4$-terms in the effective action which is not $\mathrm{SL}(2, \mathbb{Z})$-invariant. This problem is resolved by noting that there are additional non-perturbative

[1]Strictly speaking, the name U-duality is reserved for the chain of exceptional discrete groups $\mathcal{E}_{n(n)}(\mathbb{Z})$, related to the toroidal compactification of M-theory (see [3] for a review). However, for convenience, we shall in this paper adopt a slight abuse of terminology and refer to any enhanced symmetry group $\mathcal{U}_d(\mathbb{Z})$ as a "U-duality" group. This then applies, for example, to the mapping class group $\mathrm{SL}(n+1, \mathbb{Z})$ of the internal torus in the reduction of pure gravity to three dimensions, and to the T-duality group $\mathrm{SO}(n, n, \mathbb{Z})$ appearing in the reduction of the coupled gravity-2-form system. Moreover, we shall refer to the continuous versions of these groups, $\mathcal{U}_d = \mathcal{U}_d(\mathbb{R})$, as "classical U-duality groups".

[2]One exception being ref. [10] in which the authors considered quadratic curvature corrections to pure gravity in four dimensions. In that special case, the most general correction can be related, through suitable field redefinitions, to the Gauss-Bonnet term which is topological in four dimensions and does not contribute to the dynamics. Hence, the $\mathrm{SL}(2, \mathbb{R})$-symmetry of the compactified Lagrangian is trivially preserved.

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contributions to the octic derivative terms arising from D-instantons ($D(-1)$-branes) [12]. This contribution can be seen as a "completion" of the coefficient to an $\mathrm{SL}(2,\mathbb{Z})$-invariant scalar function which is identified with a certain automorphic function, known as a non-holomorphic Eisenstein series. A weak-coupling (large volume) expansion of this function reproduces the perturbative tree-level and one-loop coefficients at lowest order.

In the scenario described above the completion to a U-duality invariant expression was achieved through the use of a scalar automorphic form, i.e., an automorphic function, which is completely $\mathrm{SL}(2,\mathbb{Z})$-invariant. More generally, one might find terms in the effective action whose non-perturbative completion requires automorphic forms transforming under the maximal compact subgroup $\mathcal{K}(\mathcal{U}_3)$. For example, this was found to be the case in [13], where interaction terms of sixteen fermions were analyzed. These terms transform under the maximal compact subgroup $\mathrm{U}(1) \subset \mathrm{SL}(2,\mathbb{R})$ and so the U-duality invariant completion requires in this case an automorphic form which transform with a $\mathrm{U}(1)$ weight that compensates for the transformation of the fermionic term, and thus renders the effective action invariant.

The need for automorphic forms which transform under the maximal compact subgroup $\mathcal{K}(\mathcal{U}_3)$ was also emphasized in [8], based on the observation that the dilaton exponents in compactified higher curvature corrections correspond to weights of the global symmetry group $\mathcal{U}_3$, implying that these terms transform non-trivially in some representation of $\mathcal{K}(\mathcal{U}_3)$. An explicit realisation of these arguments was found in [11] for the case of compactification on S^1 of the four-dimensional coupled Einstein-Liouville system, supplemented by a four-derivative curvature correction. The resulting effective action was shown to explicitly break the Ehlers $\mathrm{SL}(2,\mathbb{R})$-symmetry; however, an $\mathrm{SL}(2,\mathbb{Z})_{\mathrm{global}} \times \mathrm{U}(1)_{\mathrm{local}}$-invariant effective action was obtained by "lifting" the scalar coefficients to automorphic forms transforming with compensating $\mathrm{U}(1)$ weights. The non-perturbative completion implied by this lifting is in this case attributed to gravitational Taub-NUT instantons [11].

Similar conclusions were drawn in [9], in which compactifications of derivative corrections of second, third and fourth powers of the Riemann tensor were analyzed. Again, it was concluded that the $\mathcal{U}_3$-symmetry is explicitly broken by the correction terms. It was argued, in accordance with the type IIB analysis discussed above, that the result of the compactification – being inherently perturbative in nature — should be considered as the large volume expansion of a $\mathcal{U}_3(\mathbb{Z})$-invariant effective action. It was shown on general grounds that any term resulting from such a compactification can always be lifted to a U-duality invariant expression through the use of automorphic forms transforming in some representation of $\mathcal{K}(\mathcal{U}_3)$.

In this paper we extend some aspects of the analysis of [9]. In [9] only parts of the compactification of the Riemann tensor squared, $\hat{R}_{ABCD}\hat{R}^{ABCD}$, were presented. The terms which were analyzed were sufficient to show that the continuous symmetry was broken, and to argue for the necessity of introducing transforming automorphic forms to restore the U-duality symmetry $\mathcal{U}_3(\mathbb{Z})$. Moreover, the overall volume factor of the internal torus was neglected in the analysis.

We restrict our study to corrections quadratic in the Riemann tensor in order for a complete compactification to be a feasible task. More precisely, we shall focus on a four-derivative correction to the Einstein-Hilbert action in the form of the Gauss-Bonnet term $\hat{R}_{ABCD}\hat{R}^{ABCD} - 4\hat{R}_{AB}\hat{R}^{AB} + \hat{R}^2$. Modulo field equations, this is the only independent invariant quadratic in the Riemann tensor. We extend the investigations of [9] by giving the complete compactification on T^n of the Gauss-Bonnet term from D dimensions to $D-n$ dimensions. In the special case of compactifications to $D-n=3$ dimensions the resulting expression simplifies, making it amenable for a more careful analysis. In particular, one of the main points of this paper is to study the full structure of the dilaton exponents, with the purpose of determining the $\mathrm{sl}(n+1,\mathbb{R})$-representation structure associated with quadratic curvature corrections. In contrast to the general arguments of [7] we have here access to a complete expression after compactification, thus allowing us to perform an exhaustive analysis of the weight structure associated with all terms in the Lagrangian.

We note that effects of adding Gauss-Bonnet correction terms have recently been discussed in the contexts of black hole entropy (see [14] for a recent review and further references) and brane world scenarios (see, e.g., [15]).

1.2 A puzzle and a possible resolution

The research programme outlined above was initially inspired by recent results regarding the question of how curvature corrections in string and M-theory, analyzed close to a spacelike singularity (the "BKL-limit"), fit into the representation structure of the hyperbolic Kac-Moody algebra $E_{10(10)} = \mathrm{Lie}\ \mathcal{E}_{10(10)}$ [16, 17]. These authors found that generically such curvature corrections are associated with exponents which reside on the negative side of the root lattice of the algebra, indicating that correction terms fall into infinite-dimensional (non-integrable) lowest-weight representations of $E_{10(10)}$.[3] Moreover, it was shown that curvature corrections to eleven-dimensional supergravity match with the root lattice of $E_{10(10)}$ only for the special powers $3k+1,\ k=1,2,3,\ldots,$ of the Riemann tensor. This is in perfect agreement with explicit loop calculations, which reveal that the only correction terms with non-zero coefficients are $\mathcal{R}^4, \mathcal{R}^7, \ldots,$ etc. [18]. However, when reducing to ten-dimensions and repeating the analysis for type IIA and type IIB supergravity, the restriction on the curvature terms — obtained by requiring compatibility with the $E_{10(10)}$-root lattice — no longer match with known results from string calculations [17]. For example, the $E_{10(10)}$ analysis for type IIA predicts a correction term of order $\mathcal{R}^3$, which is known to be forbidden by supersymmetry. This implies that — even though correct for eleven-dimensional supergravity — the compatibility between higher derivative corrections and the root lattice of $E_{10(10)}$ is clearly not well-understood, and requires refinement.

These results are puzzling also in other respects, most notably because the weights that arise from curvature corrections are *negative* weights of $E_{10(10)}$; with the leading order term in a BKL-like expansion of the $\mathcal{R}^4$-terms being the lowest weight of the representation, and, in fact, corresponds to the negative of a dominant integral weight. This implies that the representation builds upwards and outwards from the interior of the negative fundamental Weyl chamber, rendering the representation non-integrable. From the point of view of

[3]The root lattice of $E_{10(10)}$ is self-dual, implying that the root lattice and the weight lattice coincide. The same is true for $E_{8(8)}$.

the nonlinear sigma model for $\mathcal{E}_{10(10)}/\mathcal{K}(\mathcal{E}_{10(10)})$ this result is also strange, because the correspondence with the tree-level Lagrangian in the BKL-limit requires the use of the Borel gauge, for which no negative weights appear in the Lagrangian [19] (see [20, 21] for reviews). The reason for these puzzling results is essentially due to the "lapse-function" N, representing the reparametrisation invariance in the timelike direction. At tree-level the powers of the lapse-function arising from the measure and from the Ricci scalar cancel, and the remaining exponents correspond to positive roots of $E_{10(10)}$. On the other hand, for terms of higher order in the Riemann tensor there are also higher powers of the lapse-function which "pushes" the exponents to the negative side of the root system.

From a different point of view, similar features have appeared in the analysis of [7]. These authors investigated the general structure of the dilaton exponents upon compactifications on T^8 of quartic curvature corrections to eleven-dimensional supergravity, emphasizing the importance of including the overall "volume factor", which parametrises the volume of the internal torus. Of course, in this case it is the Lie algebra $E_{8(8)} = \text{Lie}\ \mathcal{E}_{8(8)}$ which is the relevant one, rather than $E_{10(10)}$. However, the inclusion of the volume factor into the dilaton exponents when investigating the weight structure has precisely the same effect as the lapse-function had in the $E_{10(10)}$-case above, namely to push the exponents from the positive root lattice of $E_{8(8)}$ down to the negative root lattice, thus giving rise to negative weights of $E_{8(8)}$.

These results imply that one might use the simpler approach of compactification of curvature corrections to three dimensions in order to develop some intuition regarding the more difficult case of implementing the full $\mathcal{E}_{10(10)}$-symmetry in M-theory. Based on these considerations — and the results obtained in the present paper concerning the representation structure of the compactified Gauss-Bonnet term — we shall in fact argue that the overall volume factor should *not* be included in the analysis of the representation structure. This interpretation draws from the idea that the result of the compactification should be seen as the lowest order term in a large volume expansion of a manifestly U-duality invariant action. From this point of view the volume factor is then associated to the first term in an expansion of an automorphic form of $\mathcal{U}_3(\mathbb{Z})$, transforming in some representation of the maximal compact subgroup $\mathcal{K}(\mathcal{U}_3)$. Moreover, with this interpretation, the dilaton exponents of the compactified quadratic corrections exhibit a more natural structure in terms of representations of $\mathcal{U}_3$. It is our hope that these results can also be applied to the question of how higher derivative corrections to eleven-dimensional supergravity fit into $E_{10(10)}$.

1.3 Organisation of the paper

Our paper is organized as follows. In section 2 we present the result of the compactification of the Gauss-Bonnet term on T^n from D dimensions to $D - n = 3$ dimensions. The completely general action representing the compactification to arbitrary dimensions is given in appendix A. The result in three dimensions is given in section 2 after dualisation of all Kaluza-Klein vectors into scalars, which is the case of most interest from the U-duality point of view. We then proceed in section 3 with the analysis of the compactified Lagrangian. We analyze in detail the dilaton exponents in terms of the representation theory of $\mathrm{sl}(n+1, \mathbb{R})$, which is the enhanced symmetry group of the compactified tree-level Lagrangian. Finally, in

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section 4 we suggest a possible non-perturbative completion of the compactified Lagrangian into a manifestly U-duality invariant expression. We explain how this completion requires the lifting of the coefficients in the Lagrangian into automorphic forms transforming non-trivially under the maximal compact subgroup $\mathcal{K}(\mathcal{U}_3) \subset \mathcal{U}_3$. We interpret our results and provide a comparison with the existing literature. All calculational details are displayed in appendix A.

2. Compactification of the Gauss-Bonnet term

In this section we outline the derivation of the toroidal compactification of the Gauss-Bonnet term from D dimensions to $D-n$ dimensions. In eq. (A.22) of appendix A we give the full result for the compactification to arbitrary dimensions. Here we focus on the special case of $D-n=3$, which is the most relevant case for the questions we pursue in this paper.

2.1 The general procedure

The Gauss-Bonnet Lagrangian density is quadratic in the Riemann tensor and takes the explicit form

$$\mathcal{L}_{\mathrm{GB}} = \hat{e}\big[\hat{R}_{ABCD}\hat{R}^{ABCD} - 4\hat{R}_{AB}\hat{R}^{AB} + \hat{R}^2\big]. \tag{2.1}$$

The compactification of the D-dimensional Riemann tensor $\hat{R}^A_{BCD}$ on an n-torus, T^n, is done in three steps: first we perform a Weyl-rescaling of the total vielbein, followed by a splitting of the external and internal indices, and finally we define the parametrisation of the internal vielbein. In the following we shall always assume that the torsion vanishes.

Conventions and reduction Ansatz. Our index conventions are as follows. $M, N, \ldots$ denote D dimensional curved indices, and $A, B, \ldots$ denote D dimensional flat indices. Upon compactification we split the indices according to $M = (\mu, m)$, where $\mu, \nu, \ldots$ and $m, n, \ldots$ are curved external and internal indices, respectively. Similarly, the flat indices split into external and internal parts according to $A = (\alpha, a)$.

Our reduction Ansatz for the vielbein is

$$\hat{e}_M{}^A = e^{\varphi}\tilde{e}_M{}^A = e^{\varphi}\begin{pmatrix} e_\mu{}^\alpha & \mathcal{A}^m_\mu \tilde{e}_m{}^a \\ 0 & \tilde{e}_m{}^a \end{pmatrix}, \tag{2.2}$$

where the internal vielbein $\tilde{e}_m{}^a$ is an element of the isometry group $\mathrm{GL}(n,\mathbb{R})$ of the n-torus. Later on we shall parametrise $\tilde{e}_m{}^a$ in various ways. With this Ansatz, the line element becomes

$$ds^2_D = e^{2\varphi}\Big\{ds^2_{D-n} + \big[(dx^m + \mathcal{A}^m_{(1)})\tilde{e}_m{}^a\big]^2\Big\}. \tag{2.3}$$

Weyl-Rescaling. In order to obtain a Lagrangian in Einstein frame after dimensional reduction, we perform a Weyl-rescaling of the D-dimensional vielbein,

$$\hat{e}_M{}^A \longrightarrow \tilde{e}_M{}^A = e^{-\varphi}\hat{e}_M{}^A. \tag{2.4}$$

Note that all D-dimensional objects before rescaling are denoted $\hat{X}$, the Weyl-rescaled objects are denoted $\tilde{X}$, while the $d = (D-n)$-dimensional objects are written without any diacritics. After the Weyl-rescaling the Gauss-Bonnet Lagrangian, including the volume measure $\hat{e} = e^{D\varphi}\tilde{e}$, can be conveniently organized in terms of equations of motion and total derivatives. This is achieved using integration by parts, where $\tilde{\nabla}_{(A}\tilde{\partial}_{B)}\varphi$ does not appear explicitly. The resulting Lagrangian is (see appendix A):

$$\begin{aligned}\mathcal{L}_{\rm GB} = \tilde{e}e^{(D-4)\varphi}\Bigg\{&\tilde{R}^2_{\rm GB} - (D-3)(D-4)\bigg[2(D-2)(\tilde{\partial}\varphi)^2\tilde{\Box}\varphi + (D-2)(D-3)(\tilde{\partial}\varphi)^4\\ &+4\bigg(\tilde{R}_{AB} - \frac{1}{2}\eta_{AB}\tilde{R}\bigg)(\tilde{\partial}^A\varphi)(\tilde{\partial}^B\varphi)\bigg]\Bigg\}\\ &+2(D-3)\tilde{e}\tilde{\nabla}_A\Bigg\{e^{(D-4)\varphi}\bigg[(D-2)^2(\tilde{\partial}\varphi)^2\tilde{\partial}^A\varphi + 2(D-2)(\tilde{\Box}\varphi)\tilde{\partial}^A\varphi\\ &-(D-2)\tilde{\partial}^A(\tilde{\partial}\varphi)^2 + 4\bigg(\tilde{R}^{AB} - \frac{1}{2}\eta^{AB}\tilde{R}\bigg)\tilde{\partial}_B\varphi\bigg]\Bigg\},\end{aligned} \tag{2.5}$$

where $\tilde{R}^2_{GB}$ represents the rescaled Gauss-Bonnet combination. In $D = 4$ the Lagrangian is only altered by a total derivative, while in $D = 3$ the Lagrangian it is merely rescaled by a factor of $e^{-\varphi}$. The total derivative terms here will remain total derivatives even after the compactification. Along with the volume factor the Weyl-rescaling will determine the overall exponential dilaton factor, which shall play an important role in the analysis that follows.

2.2 Tree-level scalar coset symmetries

The internal vielbein $\hat{e}_m{}^a$ can be used to construct the internal metric $\hat{g}_{mn} = \hat{e}_m{}^a\hat{e}_n{}^b\delta_{ab}$, which is manifestly invariant under local SO(n) rotations in the reduced directions. Thus we are free to fix a gauge for the internal vielbein using the SO(n)-invariance. After compactification the volume measure becomes $\tilde{e} = e\tilde{e}_{\rm int}$, where e is the determinant of the spacetime vielbein and $\tilde{e}_{\rm int}$ is the determinant of the internal vielbein. Defining the Weyl-rescaling coefficient as $e^{-(D-2)\varphi} \equiv \tilde{e}_{\rm int}$ ensures that the reduced Lagrangian is in Einstein frame.

The GL($n,\mathbb{R}$) group element $\tilde{e}_m{}^a$ can now be parameterized in several ways, and we will discuss the two most natural choices here. The first choice is included for completeness, while it is the second choice which we shall subsequently employ in the compactification of the Gauss-Bonnet term.

First parametrisation — Making the symmetry manifest. First, there is the possibility of separating out the determinant of the internal vielbein according to $\tilde{e}_m{}^a = (\tilde{e}_{\rm int})^{1/n}\varepsilon_m{}^a = e^{-\frac{(D-2)}{n}\varphi}\varepsilon_m{}^a$, where $\varepsilon_m{}^a$ is an element of SL($n,\mathbb{R}$) in any preferred gauge. The line element takes the form

$$ds^2_D = e^{2\varphi}\left\{ds^2_{D-n} + e^{-2\frac{(D-2)}{n}\varphi}\left[(dx^m + \mathcal{A}^m_{(1)})\varepsilon_m{}^a\right]^2\right\}. \tag{2.6}$$

This Ansatz is nice for investigating the symmetry properties of the reduced Lagrangian because the GL($n,\mathbb{R}$)-symmetry of the internal torus is manifestly built into the formalism.

More precisely, the reduction of the tree-level Einstein-Hilbert Lagrangian, $\hat{e}\hat{R}$, to $d = D-n$ dimensions becomes,

$$\mathcal{L}^{[d]}_{\mathrm{EH}} = e\left[R - \frac{1}{4}e^{-2\frac{(D-2)}{n}\xi\rho}F_{c\alpha\beta}F^{c\alpha\beta} - \frac{1}{2}(\partial\rho)^2 - \mathrm{tr}(P_\alpha P^\alpha) - 2\xi\Box\rho\right], \tag{2.7}$$

where $F^c{}_{\alpha\beta} \equiv \varepsilon_m{}^a F^m{}_{\alpha\beta}$ and

$$P_\alpha{}^{bc} \equiv \varepsilon^{m(b}\partial_\alpha\varepsilon_m{}^{c)} = \tilde{P}_\alpha{}^{bc} + \frac{(D-2)}{n}\xi\partial_\alpha\rho\delta^{bc}. \tag{2.8}$$

Notice that $P_\alpha{}^{bc}$ is $\mathrm{sl}(n,\mathbb{R})$ valued and hence fulfills $\mathrm{tr}(P_\alpha) = 0$. To obtain eq. (2.7) we also performed a scaling $\varphi = \xi\rho$ with $\xi = \sqrt{\frac{n}{2(D-2)(D-n-2)}}$, so as to ensure that the scalar field ρ appears canonically normalized in the Lagrangian.

The $\mathrm{SL}(n,\mathbb{R})$-symmetry is manifest in this Lagrangian because the term $\mathrm{tr}(P_\alpha P^\alpha)$ is constructed using the invariant Killing form on $\mathrm{sl}(n,\mathbb{R})$. By dualising the two-form field strength $F_{(2)}$, the symmetry is enhanced to $\mathrm{SL}(n+1,\mathbb{R})$. With a slight abuse of terminology we call this the (classical) "U-duality" group. Since we are only investigating the pure gravity sector, this is of course only a subgroup of the full continuous U-duality group.

It was already shown in [9], that the tree-level symmetry $\mathrm{SL}(n+1,\mathbb{R})$ is not realized in the compactified Gauss-Bonnet Lagrangian. It was argued, however, that the quantum symmetry $\mathrm{SL}(n+1,\mathbb{Z})$ could be reinstated by "lifting" the result of the compactification through the use of automorphic forms. In this paper we take the same point of view, but since we now have access to the complete expression of the compactified Gauss-Bonnet Lagrangian we can here extend the analysis of [9] in some aspects. In order to do this we shall make use of a different parametrisation than the one displayed above, which illuminates the structure of the dilaton exponents in the Lagrangian. The dilaton exponents reveals the weight structure of the global symmetry group and so can give information regarding which representation of the U-duality group we are dealing with.

Second parametrisation — Revealing the root structure. The second natural choice of the internal vielbein is to parameterize it in triangular form by using dimension by dimension compactification [22–24]. Instead of extracting only the determinant of the vielbein, one dilaton scalar is pulled out for each compactified dimension according to $\tilde{e}_m{}^a = e^{-\frac{1}{2}\vec{f}_a\cdot\vec{\phi}}u_m{}^a$, where $\vec{\phi} = (\phi_1,\ldots,\phi_n)$ and

$$\vec{f}_a = 2(\alpha_1,\ldots,\alpha_{a-1},(D-n-2+a)\alpha_a,\underbrace{0,\ldots,0}_{n-a}), \tag{2.9}$$

with

$$\alpha_a = \frac{1}{\sqrt{2(D-n-2+a)(D-n-3+a)}}. \tag{2.10}$$

The internal vielbein is now the Borel representative of the coset $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$, with the diagonal degrees of freedom $e^{-\frac{1}{2}\vec{f}_a\cdot\vec{\phi}}$ corresponding to the Cartan generators and the upper triangular degrees of freedom

$$u_m{}^a = [(1-\mathcal{A}_{(0)})^{-1}]_m{}^a = [1 + \mathcal{A}_{(0)} + (\mathcal{A}_{(0)})^2 + \ldots]_m{}^a \tag{2.11}$$

corresponding to the positive root generators. The form of eq. (2.11) follows naturally from a step by step compactification, where the scalar potentials $(\mathcal{A}_{(0)})^i_j$, arising from the compactification of the graviphotons, are nonzero only when $i > j$. The sum of the vectors $\vec{f}_a$ can be shown to be

$$\sum_{a=1}^{n} \vec{f}_a = \frac{D-2}{3}\vec{g}, \tag{2.12}$$

$\vec{g} \equiv 6(\alpha_1, \alpha_2 \ldots, \alpha_n)$. In addition, $\vec{g}$ and $\vec{f}_a$ obey

$$\begin{aligned} \vec{g}\cdot\vec{g} &= \frac{18n}{(D-2)(D-n-2)}, \\ \vec{g}\cdot\vec{f}_a &= \frac{6}{D-n-2}, \\ \vec{f}_a\cdot\vec{f}_b &= 2\delta_{ab} + \frac{2}{D-n-2}, \end{aligned} \tag{2.13}$$

and

$$\sum_{a=1}^{n}(\vec{f}_a\cdot\vec{x})(\vec{f}_a\cdot\vec{y}) = 2(\vec{x}\cdot\vec{y}) + \frac{D-2}{9}(\vec{g}\cdot\vec{x})(\vec{g}\cdot\vec{y}). \tag{2.14}$$

These scalar products can naturally be used to define the Cartan matrix, once a set of simple root vectors are found. The line element becomes

$$ds_D^2 = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}}\Big\{ds_{D-n}^2 + \sum_{a=1}^{n} e^{-\vec{f}_a\cdot\vec{\phi}}\left[(dx^m + \mathcal{A}^m_{(1)})u_m{}^a\right]^2\Big\}, \tag{2.15}$$

yielding the corresponding Einstein-Hilbert Lagrangian in d dimensions

$$\mathcal{L}^{[d]}_{\mathrm{EH}} = e\left[R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{4}\sum_{a=1}^{n} e^{-\vec{f}_a\cdot\vec{\phi}} F_{a\beta\gamma}F^{a\beta\gamma} - \frac{1}{2}\sum_{\substack{b,c=1\\ b<c}}^{n} e^{(\vec{f}_b-\vec{f}_c)\cdot\vec{\phi}} G_{\alpha bc}G^{\alpha bc} - \frac{1}{3}\vec{g}\cdot\Box\vec{\phi}\right], \tag{2.16}$$

with $F^c{}_{\alpha\beta} \equiv u_m{}^a F^m{}_{\alpha\beta}$ and

$$G_\alpha{}^{bc} = u^{mb}\partial_\alpha u_m{}^c = e^{-\frac{1}{2}(\vec{f}_b-\vec{f}_c)\cdot\vec{\phi}}\left[\left(\tilde{P}_\alpha{}^{bc} + \frac{1}{2}\vec{f}_b\cdot\partial_\alpha\vec{\phi}\delta^{bc}\right) + Q_\alpha{}^{bc}\right]. \tag{2.17}$$

Here, no Einstein's summation rule is assumed for the flat internal indices. Notice also that $G_\alpha{}^{bc}$ is non-zero only when $b < c$.

We shall refer to the various exponents of the form $e^{\vec{x}\cdot\vec{\phi}}$ ($\vec{x}$ being some vector in $\mathbb{R}^n$) collectively as "dilaton exponents". If relevant, this also includes the contribution from the overall volume factor.

All the results obtained in this parametrisation can be converted to the first parametrisation simply by using the following identifications

$$\begin{aligned} \frac{1}{3}(\vec{g}\cdot\vec{\phi}) &= 2\xi\rho, \\ \vec{f}_a\cdot\vec{\phi} &= 2\frac{(D-2)}{n}\xi\rho, \qquad \forall a, \\ \vec{\phi}\cdot\vec{\phi} &= \rho^2, \end{aligned} \tag{2.18}$$

where one should keep in mind that $\xi = \sqrt{\frac{n}{2(D-2)(D-n-2)}}$. Notice also that our compactification procedure breaks down at $D-n=2$, in which case the scalar products in eq. (2.13) become ill-defined.

Even though proving the symmetry contained in the Lagrangian is somewhat more cumbersome compared to the first choice of parametrisation, since all the group actions have to be carried out adjointly in a formal manner, the second choice comes to its power when dealing with the exceptional symmetry groups of the supergravities for which no matrix representations exist. This parametrisation is particularly suitable for reading off the root vectors of the underlying symmetry algebra; they appear as exponential factors in front of each term in the Lagrangian. Identifying a complete set of root vectors in this way gives a necessary but not sufficient constraint on the underlying symmetry.

2.3 The Gauss-Bonnet lagrangian reduced to three dimensions

When reducing to $D-n=3$ dimensions, we can dualise the two-form field strength $\tilde{F}^a{}_{\alpha\beta} \equiv \tilde{e}_m{}^a F^m{}_{\alpha\beta}$ of the graviphoton $\mathcal{A}_{(1)}$ into the one-form $\tilde{H}_{a\alpha}$. More explicitly, we employ the standard dualisation

$$\delta_{ab}\tilde{F}^b{}_{\alpha\beta} = \epsilon_{\alpha\beta\gamma}\tilde{e}^m{}_a\partial^\gamma\chi_m \equiv \epsilon_{\alpha\beta\gamma}\tilde{H}_a{}^\gamma. \tag{2.19}$$

When we go to Einstein frame, the appearance of the inverse vielbein $\tilde{e}^m{}_a$ in the definition of the one-form $\tilde{H}_{a\alpha}$ implies there is a sign flip on its dilaton exponent in the Lagrangian after dualisation. The dualisation presented here follows from the tree-level Lagrangian, but in general receives higher order α'-corrections. However, these lead to terms of higher derivative order than quartic and so can be neglected in the present analysis [7, 9].

The compactification is performed according to the standard procedure by separating the indices; the detailed calculations can be found in appendix A. The final results are written in such way that the only explicit derivative terms appearing are divergences, total derivatives and first derivatives on the dilatons φ. The complete compactification of the Gauss-Bonnet Lagrangian on T^n to arbitrary dimensions $D-n$ is given in eq. (A.22) of appendix A.[4] This expression is rather messy and difficult to work with. However, by making use of all first order equations of motion, dualising all graviphotons to scalars, and restricting to $D-n=3$, the Lagrangian simplifies considerably. The end result reads

$$\begin{aligned}\mathcal{L}^{[3]}_{\mathrm{GB}} = \sqrt{|g|}e^{-2\varphi}\Big\{ &-\frac{1}{4}\tilde{H}_{a\gamma}\tilde{H}_b{}^\gamma\tilde{H}^a{}_\delta\tilde{H}^{b\delta} + \frac{1}{4}\tilde{H}^2\tilde{H}^2 - 4\tilde{H}^2(\partial\varphi)^2 + 2\tilde{H}^{c\alpha}\tilde{P}_{\alpha cd}\tilde{P}^{\beta de}\tilde{H}_{e\beta}\\ &-2\tilde{H}^{c\alpha}\tilde{P}_{\beta cd}\tilde{P}^{\beta de}\tilde{H}_{e\alpha} + 4\tilde{H}_{c\alpha}\tilde{P}^{\alpha cd}\tilde{H}_d{}^\beta\partial_\beta\varphi - 6\tilde{H}_{c\alpha}\tilde{P}^{\beta cd}\tilde{H}_d{}^\alpha\partial_\beta\varphi\\ &+2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\alpha\tilde{P}^\beta) + 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - (\tilde{P}^2)^2 + 8\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\beta)\partial^\alpha\varphi\\ &-4(D-2)\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\partial^\alpha\varphi\partial^\beta\varphi + 2(D+2)\tilde{P}^2(\partial\varphi)^2 + (D-2)(D-4)(\partial\varphi)^2(\partial\varphi)^2\Big\},\end{aligned} \tag{2.20}$$

[4]Kaluza-Klein reduction of quadratic curvature corrections has also been analyzed from a different point of view in [25].

where $\tilde{H}^2 \equiv \tilde{H}_{\alpha\beta}\tilde{H}^{\alpha\beta}$ and $\tilde{P}^2 \equiv \tilde{P}_{\alpha bc}\tilde{P}^{\alpha bc}$. Note that contributions from the boundary terms and terms proportional to the equations of motion have been ignored. The one-form $\tilde{P}_\alpha$ is the Maurer-Cartan form associated with the internal vielbein $\tilde{e}_m{}^a$, and so takes values in the Lie algebra $\mathrm{gl}(n,\mathbb{R}) = \mathrm{sl}(n,\mathbb{R}) \oplus \mathbb{R}$. Here, the abelian summand $\mathbb{R}$ corresponds to the "trace-part" of $\tilde{P}_\alpha$. Explicitly, we have $\mathrm{tr}(\tilde{P}_\alpha) = -(D-2)\partial_\alpha\varphi$. We shall discuss various properties of $\tilde{P}_\alpha$ in more detail below.

Finally, we note that the three-dimensional Gauss-Bonnet term is absent from the reduced Lagrangian because it vanishes identically in three dimensions:

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 = 0, \qquad (\alpha,\beta,\gamma,\delta = 1,2,3). \tag{2.21}$$

The remainder of this paper is devoted to a detailed analysis of the symmetry properties of eq. (2.20).

3. Algebraic structure of the compactified Gauss-Bonnet term

We have seen that the Ansatz presented in eq. (2.15) is particularly suitable for identifying the roots of the relevant symmetry algebra from the dilaton exponents associated with the diagonal components of the internal vielbein. Through this analysis one may deduce that for the lowest order effective action, the terms in the action are organized according to the adjoint representation of $\mathrm{sl}(n+1,\mathbb{R})$, for which the weights are the roots. The aim of this section is to extend the analysis to the Gauss-Bonnet Lagrangian. By general arguments [7, 8], it has been shown that the exponents no longer correspond to roots of the symmetry algebra but rather they now lie on the weight lattice. Here, however, we have access to the complete compactified Lagrangian and we may therefore present an explicit counting of the weights in the dilaton exponents and identify the relevant $\mathrm{sl}(n+1,\mathbb{R})$-representation.

An exhaustive analysis of the $\mathrm{sl}(4,\mathbb{R})$-representation structure of the Gauss-Bonnet term compactified from 6 to 3 dimensions on T^3 is performed. We do this in two alternative ways.

First, we neglect the contribution from the overall dilaton factor $e^{-2\varphi}$ in the representation structure. This is consistent before dualisation because this factor is $\mathrm{SL}(3,\mathbb{R})$-invariant. However, after dualisation this is no longer true and one must understand what role this factor plays in the algebraic structure. If one continues to neglect this factor then all the weights fit into the **84**-representation of $\mathrm{sl}(4,\mathbb{R})$ with Dynkin labels $[2,0,2]$.

On the other hand, including the overall exponential dilaton factor in the weight structure induces a shift on the weights so that the highest weight is associated with the **36**-representation of $\mathrm{sl}(4,\mathbb{R})$ instead, with Dynkin labels $[2,0,1]$. However, this representation is not "big enough" to incorporate all the weights in the Lagrangian. It turns out that there are additional weights outside of the **36** that fit into a **27**-representation of $\mathrm{sl}(3,\mathbb{R})$. Unfortunately there seems to be no obvious argument for which $\mathrm{sl}(4,\mathbb{R})$-representation those "extra" weights should belong to.

This indicates that the first approach, where the dilaton pre-factor is neglected, is the correct way to interpret the result of the compactification because then all weights are "unified" in a single representation of the U-duality group. A detailed demonstration of this follows below.

3.1 Kaluza-Klein reduction and $\mathrm{sl}(n,\mathbb{R})$-representations

We shall begin by rewriting the reduction Ansatz in a way which has a more firm Lie algebraic interpretation. Recall from eq. (2.15) that the standard Kaluza-Klein Ansatz for the metric is

$$ds_D^2 = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}} ds_d^2 + e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}} \sum_{i=1}^{n} e^{-\vec{f}_i\cdot\vec{\phi}} \left[\left(dx^m + \mathcal{A}_{(1)}^m \right) u_m{}^a \right]^2 . \tag{3.1}$$

The exponents in this Ansatz are linear forms on the space of dilatons. Let $\vec{e}_i,\ i = 1,\dots,n$, constitute an n-dimensional orthogonal basis of $\mathbb{R}^n$,

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}. \tag{3.2}$$

Since there is a non-degenerate metric on the space of dilatons (the Cartan subalgebra $\mathfrak{h} \subset \mathrm{sl}(n+1,\mathbb{R})$) we can use this to identify this space with its dual space of linear forms. Thus, we may express all exponents in the orthogonal basis $\vec{e}_i$ and the vectors $\vec{f}_i$ and $\vec{g}$ may then be written as

$$\begin{aligned} \vec{f}_i &= \sqrt{2}\vec{e}_i + \alpha\vec{g}, \\ \vec{g} &= \beta \sum_{i=1}^{n} \vec{e}_i, \end{aligned} \tag{3.3}$$

where the constants α and β are defined as

$$\begin{aligned} \alpha &= \frac{1}{3n}\left(D - 2 - \sqrt{(D-n-2)(D-2)} \right), \\ \beta &= \sqrt{\frac{18}{(D-n-2)(D-2)}}. \end{aligned} \tag{3.4}$$

Note here that the constant α is not the same as the α_a of eq. (2.9).

The combinations

$$\vec{f}_i - \vec{f}_j = \sqrt{2}\vec{e}_i - \sqrt{2}\vec{e}_j \tag{3.5}$$

span an $(n-1)$-dimensional lattice which can be identified with the root lattice of $A_{n-1} = \mathrm{sl}(n,\mathbb{R})$. For compactification of the pure Einstein-Hilbert action to three dimensions, the dilaton exponents precisely organize into the complete set of positive roots of $\mathrm{sl}(n,\mathbb{R})$, revealing that it is the adjoint representation which is the relevant one for the U-duality symmetries of the lowest order (two-derivative) action. After dualisation of the Kaluza-Klein one forms $\mathcal{A}_{(1)}$ the symmetry is lifted to the full adjoint representation of $\mathrm{sl}(n+1,\mathbb{R})$.

When we compactify higher derivative corrections to the Einstein-Hilbert action it is natural to expect that other representations of $\mathrm{sl}(n,\mathbb{R})$ and $\mathrm{sl}(n+1,\mathbb{R})$ become relevant. In order to pursue this question for the Gauss-Bonnet Lagrangian, we shall need some features of the representation theory of $\mathrm{sl}(n+1,\mathbb{R})$.

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Representation theory of $A_n = \mathrm{sl}(n+1,\mathbb{R})$. For the infinite class of simple Lie algebras A_n, it is possible to choose an embedding of the weight space $\mathfrak{h}^\star$ in $\mathbb{R}^{n+1}$ such that $\mathfrak{h}^\star$ is isomorphic to the subspace of $\mathbb{R}^{n+1}$ which is orthogonal to the vector $\sum_{i=1}^{n+1}\vec{e}_i$ (see, e.g., [26]). We can use this fact to construct an embedding of the $(n-1)$-dimensional weight space of $A_{n-1} = \mathrm{sl}(n,\mathbb{R})$ into the n-dimensional weight space of $A_n = \mathrm{sl}(n+1,\mathbb{R})$, in terms of the n basis vectors $\vec{e}_i$ of $\mathbb{R}^n$.

To this end we define the new vectors

$$\begin{aligned}\vec{\omega}_i &= \vec{f}_i - \left(\alpha + \frac{\sqrt{2}}{n\beta}\right)\vec{g} \\ &= \sqrt{2}\vec{e}_i - \frac{\sqrt{2}}{n}\sum_{j=1}^{n}\vec{e}_j,\end{aligned} \tag{3.6}$$

which have the property that

$$\vec{\omega}_i \cdot \vec{g} = \sqrt{2}\beta - \sqrt{2}\beta = 0. \tag{3.7}$$

This implies that the vectors $\vec{\omega}_i$ form a (non-orthogonal) basis of the $(n-1)$-dimensional subspace $U \subset \mathbb{R}^n$, orthogonal to $\vec{g}$. The space U is then isomorphic to the weight space $\mathfrak{h}^\star$ of $A_{n-1} = \mathrm{sl}(n,\mathbb{R})$. Since there are n vectors $\vec{\omega}_i$, this basis is overcomplete. However, it is easy to see that not all $\vec{\omega}_i$ are independent, but are subject to the relation

$$\sum_{i=1}^{n}\vec{\omega}_i = 0. \tag{3.8}$$

A basis of simple roots of $\mathfrak{h}^\star$ can now be written in three alternative ways

$$\vec{\alpha}_i = \vec{f}_i - \vec{f}_{i+1} = \vec{\omega}_i - \vec{\omega}_{i+1} = \sqrt{2}(\vec{e}_i - \vec{e}_{i+1}), \qquad (i = 1,\ldots,n-1). \tag{3.9}$$

What is the algebraic interpretation of the vectors $\vec{\omega}_i$? It turns out that they may be identified with the weights of the n-dimensional fundamental representation of $\mathrm{sl}(n,\mathbb{R})$. The condition $\sum_{i=1}^{n}\vec{\omega}_i = 0$ then reflects the fact that the generators of the fundamental representation are traceless.

In addition, we can use the weights of the fundamental representation to construct the fundamental weights $\vec{\Lambda}_i$, defined by

$$\vec{\alpha}_i \cdot \vec{\Lambda}_j = 2\delta_{ij}. \tag{3.10}$$

One finds

$$\vec{\Lambda}_i = \sum_{j=1}^{i}\vec{\omega}_j, \qquad (i = 1,\ldots,n-1), \tag{3.11}$$

which can be seen to satisfy eq. (3.10).

The relation, eq. (3.11), between the fundamental weights $\vec{\Lambda}_i$ and the weights of the fundamental representation $\vec{\omega}_i$ can be inverted to

$$\vec{\omega}_i = \vec{\Lambda}_i - \vec{\Lambda}_{i-1}, \qquad (i = 1,\ldots,n-1). \tag{3.12}$$

In addition, the n:th weight is

$$\vec{\omega}_n = -\vec{\Lambda}_{n-1}, \tag{3.13}$$

corresponding to the lowest weight of the fundamental representation.

We may now rewrite the Kaluza-Klein Ansatz in a way such that the weights $\vec{\omega}_i$ appear explicitly in the metric[5]

$$ds_D^2 = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}} ds_d^2 + e^{\gamma\vec{g}\cdot\vec{\phi}} \sum_{i=1}^{n} e^{-\vec{\omega}_i\cdot\vec{\phi}} \Big[\big(dx^m + \mathcal{A}^m_{(1)}\big) u_m{}^a \Big]^2, \tag{3.14}$$

with

$$\gamma = \frac{1}{3} - \alpha - \frac{\sqrt{2}}{n\beta}. \tag{3.15}$$

3.2 The algebraic structure of Gauss-Bonnet in three dimensions

We are interested in the dilaton exponents in the scalar part of the three-dimensional Lagrangian. For the Einstein-Hilbert action we know that these are of the forms

$$\vec{f}_a - \vec{f}_b \quad (b > a), \qquad \text{and} \qquad \vec{f}_a. \tag{3.16}$$

The first set of exponents $\vec{f}_a - \vec{f}_b$ correspond to the positive roots of $\mathrm{sl}(n, \mathbb{R})$ and the second set $\vec{f}_a$, which contributes to the scalar sector after dualisation, extends the algebraic structure to include all positive roots of $\mathrm{sl}(n+1, \mathbb{R})$. The highest weight $\vec{\lambda}^{\mathrm{hw}}_{\mathrm{ad},n}$ of the adjoint representation of $A_n = \mathrm{sl}(n+1, \mathbb{R})$ can be expressed in terms of the fundamental weights as

$$\vec{\lambda}^{\mathrm{hw}}_{\mathrm{ad},n} = \vec{\Lambda}_1 + \vec{\Lambda}_n, \tag{3.17}$$

corresponding to the Dynkin labels

$$[1, 0, \dots, 0, 1].$$

We see that before dualisation the highest weight of the adjoint representation of $\mathrm{sl}(n, \mathbb{R})$ arises in the dilaton exponents in the form $\vec{f}_1 - \vec{f}_n = \vec{\omega}_1 - \vec{\omega}_n = \vec{\Lambda}_1 + \vec{\Lambda}_{n-1} = \vec{\lambda}^{\mathrm{hw}}_{\mathrm{ad},n-1}$.

We proceed now to analyze the various dilaton exponents arising from the Gauss-Bonnet term after compactification to three dimensions. These can be extracted from each term in the Lagrangian eq. (2.20) by factoring out the diagonal components of the internal vielbein according to $\tilde{e}_m{}^a = e^{-\frac{1}{2}\vec{f}_a\cdot\vec{\phi}} u_m{}^a$. For example, before dualisation we have the manifestly $\mathrm{SL}(n, \mathbb{R})$-invariant term $\mathrm{tr}(\tilde{P}_\alpha \tilde{P}_\beta \tilde{P}^\alpha \tilde{P}^\beta)$. Expanding this gives (among others) the following types of terms

$$\begin{aligned} \mathrm{tr}\big(\tilde{P}_\alpha \tilde{P}_\beta \tilde{P}^\alpha \tilde{P}^\beta\big) \sim & \sum_{\substack{b\,<\,a,\,c \\ d\,<\,a,\,c}} e^{-(\vec{f}_a - \vec{f}_b + \vec{f}_c - \vec{f}_d)\cdot\vec{\phi}} G_{\alpha ba} G_\beta{}^{bc} G^\alpha{}_{dc} G^{\beta da} + \cdots \\ & + \sum_{a\,<\,c\,<\,d\,<\,b} e^{(\vec{f}_a - \vec{f}_b)\cdot\vec{\phi}} G_{\alpha ab} G_\beta{}^{ac} G^\alpha{}_{cd} G^{\beta db} + \cdots . \end{aligned} \tag{3.18}$$

[5] A similar construction was given in [27].

After dualisation, we need to take into account also terms containing $\tilde{H}^{\alpha}{}_{a}$. We have then, for example, the term

$$\tilde{H}^4 \sim \sum_{a,b} e^{(\vec{f}_a+\vec{f}_b)\cdot\vec{\phi}} H^4. \tag{3.19}$$

Many different terms in the Lagrangian might in this way give rise to the same dilaton exponents. As can be seen from eq. (3.18), the internal index contractions yield constraints on the various exponents. We list below all the "independent" exponents, i.e., those which are the least constrained. All other exponents follow as special cases of these. Before dualisation we find the following exponents:

$$\begin{aligned} \vec{f}_a - \vec{f}_b \qquad & (b > a), \\ \vec{f}_c + \vec{f}_d - \vec{f}_a - \vec{f}_b \qquad & (c < a,\ c < b,\ d < a,\ d < b), \\ \vec{f}_a + \vec{f}_b - \vec{f}_c - \vec{f}_d \qquad & (b < c,\ a < d), \end{aligned} \tag{3.20}$$

and after dualisation we also get contributions from

$$\begin{aligned} \vec{f}_a&, \\ \vec{f}_a + \vec{f}_b&, \\ \vec{f}_a + \vec{f}_b - \vec{f}_c&, \qquad (a < c,\ b < c). \end{aligned} \tag{3.21}$$

Let us first investigate the general weight structure of the dilaton exponents before dualisation. The highest weight arises from the terms of the form $\vec{f}_c + \vec{f}_d - \vec{f}_a - \vec{f}_b$ when $c = d = 1$ and $a = b = n$, i.e., for the dilaton vector $2\vec{f}_1 - 2\vec{f}_n$. This can be written in terms of the fundamental weights as follows

$$2\vec{f}_1 - 2\vec{f}_n = 2\vec{\omega}_1 - 2\vec{\omega}_n = 2\vec{\Lambda}_1 + 2\vec{\Lambda}_{n-1}, \tag{3.22}$$

which is the highest weight of the $[2, 0, \ldots, 0, 2]$-representation of $\mathrm{sl}(n, \mathbb{R})$.

3.3 Special case: compactification from $D = 6$ on T^3

In order to determine if this is indeed the correct representation for the Gauss-Bonnet term, we shall now restrict to the case of $n = 3$, i.e., compactification from $D = 6$ on T^3. We do this so that a complete counting of the weights in the Lagrangian is a tractable task.[6] Before dualisation we then expect to find the representation **27** of $\mathrm{sl}(3, \mathbb{R})$, with Dynkin labels $[2, 2]$. We will see that, after dualisation, this representation lifts to the representation **84** of $\mathrm{sl}(4, \mathbb{R})$, with Dynkin labels $[2, 0, 2]$.

It is important to realize that of course the Lagrangian will not display the complete set of weights in these representations, but only the *positive* weights, i.e., the ones that can be obtained by summing positive roots only. Let us begin by analyzing the weight structure before dualisation. From eq. (3.20) we find the weights

$$\begin{aligned} &\vec{f}_1 - \vec{f}_2, && \vec{f}_2 - \vec{f}_3, && \vec{f}_1 - \vec{f}_3, \\ &2(\vec{f}_1 - \vec{f}_2), && 2(\vec{f}_2 - \vec{f}_3), && 2(\vec{f}_1 - \vec{f}_3), \\ &2\vec{f}_1 - \vec{f}_2 - \vec{f}_3, && \vec{f}_1 + \vec{f}_2 - 2\vec{f}_3. && \end{aligned} \tag{3.23}$$

[6] The simpler case where one performs compactification from $D = 5$ on T^2 behaves in a similar fashion, where we find a level decomposition of $\mathrm{sl}(3, \mathbb{R})$ in terms of $\mathrm{sl}(2, \mathbb{R})$.

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The first three may be identified with the positive roots of $\mathrm{sl}(3,\mathbb{R})$, $\vec{\alpha}_1 = \vec{f}_1 - \vec{f}_2$, $\vec{\alpha}_2 = \vec{f}_2 - \vec{f}_3$ and $\vec{\alpha}_\theta = \vec{f}_1 - \vec{f}_3$. The second line then corresponds to $2\vec{\alpha}_1$, $2\vec{\alpha}_2$ and $2\vec{\alpha}_\theta$. The remaining weights are

$$\begin{aligned} \vec{f}_1 + \vec{f}_2 - 2\vec{f}_3 &= \vec{\alpha}_1 + 2\vec{\alpha}_2, \\ 2\vec{f}_1 - \vec{f}_2 - \vec{f}_3 &= 2\vec{\alpha}_1 + \vec{\alpha}_2. \end{aligned} \tag{3.24}$$

These weights are precisely the eight positive weights of the $\mathbf{27}$ representation of $\mathrm{sl}(3,\mathbb{R})$.

We now wish to see whether this representation lifts to any representation of $\mathrm{sl}(4,\mathbb{R})$, upon inclusion of the weights in eq. (3.21). As mentioned above, the natural candidate is an 84-dimensional representation of $\mathrm{sl}(4,\mathbb{R})$ with Dynkin labels $[2,0,2]$. It is illuminating to first decompose it in terms of representations of $\mathrm{sl}(3,\mathbb{R})$,

$$\mathbf{84} = \mathbf{27} \oplus \mathbf{15} \oplus \overline{\mathbf{15}} \oplus \mathbf{6} \oplus \overline{\mathbf{6}} \oplus \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{1}, \tag{3.25}$$

or, in terms of Dynkin labels,

$$[2,0,2] = [2,2] + [2,1] + [1,2] + [2,0] + [0,2] + [1,1] + [1,0] + [0,1] + [0,0]. \tag{3.26}$$

We may view this decomposition as a *level decomposition* of the representation $\mathbf{84}$, with the level ℓ being represented by the number of times the third simple root $\vec{\alpha}_3$ appears in each representation. From this point of view, and as we shall see in more detail shortly, the representations $\mathbf{27}, \mathbf{8}$ and $\mathbf{1}$ reside at $\ell = 0$, the representations $\mathbf{15}$ and $\mathbf{3}$ at $\ell = 1$, and the representation $\mathbf{6}$ at $\ell = 2$. The "barred" representations then reside at the associated negative levels. Knowing that we will only find the strictly positive weights in these representations, let us therefore start by listing these.

Firstly, we may neglect all representations at negative levels since these do not contain any positive weights. However, not all weights for $\ell \geq 0$ are positive. If we had decomposed the adjoint representation of $\mathrm{sl}(4,\mathbb{R})$ this problem would not have been present since all roots are either positive or negative, and hence all weights at positive level are positive and vice versa. In our case this is not true because for representations larger than the adjoint many weights are neither positive nor negative. It is furthermore important to realize that after dualisation it is the positive weights of $\mathrm{sl}(4,\mathbb{R})$ that we will obtain and not of $\mathrm{sl}(3,\mathbb{R})$. As can be seen in figure 1 the decomposition indeed includes weights which are negative weights of $\mathrm{sl}(3,\mathbb{R})$ but nevertheless positive weights of $\mathrm{sl}(4,\mathbb{R})$. An explicit counting reveals the following number of positive weights at each level (not counting weight multiplicities):

$$\begin{aligned} \ell = 0 &: 8, \\ \ell = 1 &: 8, \\ \ell = 2 &: 6. \end{aligned} \tag{3.27}$$

The eight weights at level zero are of course the positive weights of the $\mathbf{27}$ representation of $\mathrm{sl}(3,\mathbb{R})$ that we had before dualisation. In order to verify that we find all positive weights of $\mathbf{84}$ we must now check explicitly that after dualisation we get $8+6$ additional

Reps	ℓ	Positive Weights of sl$(4,\mathbb{R})$
3	1	$\vec{\alpha}_3,\ \vec{\alpha}_2+\vec{\alpha}_3,\ \vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3$
15	1	$2\vec{\alpha}_2+\vec{\alpha}_3,\ \vec{\alpha}_1+2\vec{\alpha}_2+\vec{\alpha}_3,\ 2\vec{\alpha}_1+2\vec{\alpha}_2+\vec{\alpha}_3,$ $2\vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3,\ (\vec{\alpha}_1+\vec{\alpha}_3)$
6	2	$2\vec{\alpha}_2+2\vec{\alpha}_3,\ \vec{\alpha}_1+2\vec{\alpha}_2+2\vec{\alpha}_3,\ 2\vec{\alpha}_1+2\vec{\alpha}_2+2\vec{\alpha}_3,$ $\vec{\alpha}_1+\vec{\alpha}_2+2\vec{\alpha}_3,\ 2\vec{\alpha}_3,\ \vec{\alpha}_2+2\vec{\alpha}_3$

Table 1: Positive weights at levels one and two.

positive weights. The total number of distinct weights of sl$(4,\mathbb{R})$ that should appear in the Lagrangian after compactification and dualisation is thus 22.

The lifting from sl$(3,\mathbb{R})$ to sl$(4,\mathbb{R})$ is done by adding the third simple root $\vec{\alpha}_3 \equiv \vec{f}_3$, from eq. (3.21). The complete set of new weights arising from eq. (3.21) is then

$$
\begin{aligned}
\ell = 1:\ & \vec{f}_1 = \vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3, && \vec{f}_2 = \vec{\alpha}_2+\vec{\alpha}_3,\\
& 2\vec{f}_1-\vec{f}_2 = 2\vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3, && 2\vec{f}_2-\vec{f}_3 = 2\vec{\alpha}_2+\vec{\alpha}_3,\\
& 2\vec{f}_1-\vec{f}_3 = 2\vec{\alpha}_1+2\vec{\alpha}_2+\vec{\alpha}_3, && \vec{f}_1+\vec{f}_2-\vec{f}_3 = \vec{\alpha}_1+2\vec{\alpha}_2+\vec{\alpha}_3,\\
& (\vec{f}_1+\vec{f}_3-\vec{f}_2 = \vec{\alpha}_1+\vec{\alpha}_3), && \vec{f}_3 = \vec{\alpha}_3,\\[1em]
\ell = 2:\ & 2\vec{f}_1 = 2\vec{\alpha}_1+2\vec{\alpha}_2+2\vec{\alpha}_3, && 2\vec{f}_2 = 2\vec{\alpha}_2+2\vec{\alpha}_3,\\
& 2\vec{f}_3 = 2\vec{\alpha}_3, && \vec{f}_1+\vec{f}_2 = \vec{\alpha}_1+2\vec{\alpha}_2+2\vec{\alpha}_3,\\
& \vec{f}_1+\vec{f}_3 = \vec{\alpha}_1+\vec{\alpha}_2+2\vec{\alpha}_3, && \vec{f}_2+\vec{f}_3 = \vec{\alpha}_2+2\vec{\alpha}_3.
\end{aligned}
\tag{3.28}
$$

In table 1 we indicate which representations these weights belong to and in figure 1 we give a graphical presentation of the level decomposition. The weight $\vec{\alpha}_1+\vec{\alpha}_3$ is put inside a parenthesis since terms giving this particular dilaton exponent in the Gauss-Bonnet combination are all absorbed into the equations of motion, and thus do not contribute according to our compactification procedure. However, generically it will contribute for a general second order curvature correction. We suspect the origin of this "missing" weight is connected to the mismatch in the multiplicity counting, which we will discuss briefly below. These results show that the Gauss-Bonnet term in $D=6$ compactified on T^3 to three dimensions gives rise to strictly positive weights that can all be fit into the **84**-representation of sl$(4,\mathbb{R})$.

Weight multiplicities. We have shown that the six-dimensional Gauss-Bonnet term compactified to three dimensions gives rise to positive weights of the **84**-representation of sl$(4,\mathbb{R})$. However, we have not yet addressed the issue of weight multiplicities. It is not clear how to approach this problem. Naively, one might argue that if k distinct terms in the Lagrangian are multiplied by the same dilaton exponential, corresponding to some weight $\vec{\lambda}$, then this weight has multiplicity k. Unfortunately, this type of counting does not seem to work, one of the reasons being that the notion of distinctness is not clearly defined.

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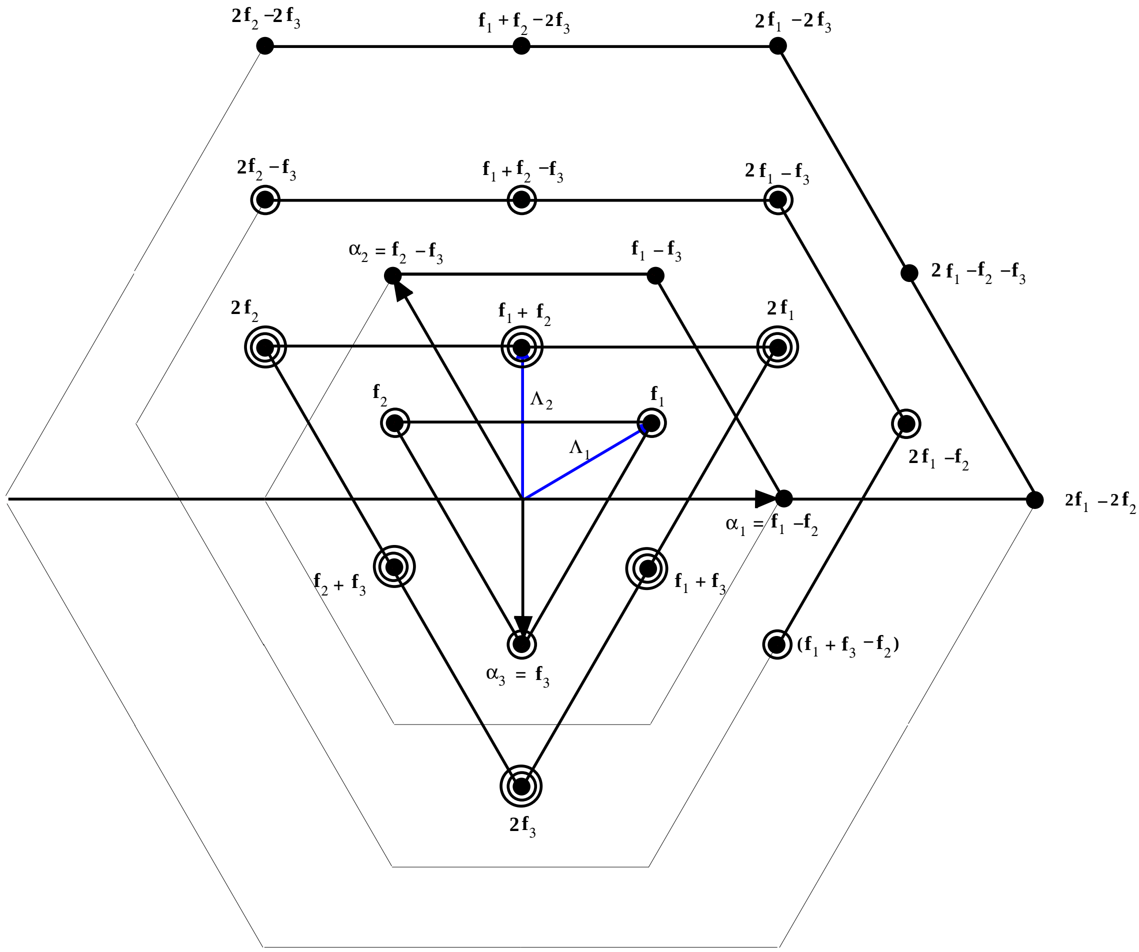

Figure 1: Graphical presentation of the representation structure of the compactified Gauss-Bonnet term. The black nodes arise from distinct dilaton exponents in the three-dimensional Lagrangian. The figure displays the level decomposition of the **84**-representation of $\mathrm{sl}(4,\mathbb{R})$ into representations of $\mathrm{sl}(3,\mathbb{R})$. Only positive levels are displayed. The black nodes correspond to positive weights of **84** of $\mathrm{sl}(4,\mathbb{R})$. Nodes with no rings represent the positive weights of the level zero representation **27**, nodes with one ring represent the positive weights of the level one representations **15** and **3**, while nodes with two rings represent the positive weights of the level two representation **6**. The shaded lines complete the representations with non-positive weights which are not displayed explicitly. The missing weight is put into a parenthesis.

Consider, for instance, the representations at $\ell = 1$. Both representations **15** and **3** contain the weights $\vec{f}_1$, $\vec{f}_2$ and $\vec{f}_3$. In **15** these have all multiplicity 2, while in **3** they have multiplicity 1. Thus, in total these weights have multiplicity 3 as weights of $\mathrm{sl}(3,\mathbb{R})$. Now, a detailed investigation reveals that the dilaton exponent $\vec{f}_a$ appears in the Gauss-Bonnet term accompanied with various different constraints on the index a, the no constraint case given in eq. (3.21) is merely the "most unconstrained" one. It can be easily shown that weights with lower value on index a have higher multiplicity. We therefore deduce that for all these weights there appears to be a mismatch in the multiplicity.

We suggest that the correct way to interpret this discrepancy in the weight multiplicities is as an indication of the need to introduce transforming automorphic forms in order to restore the $\mathrm{SL}(4,\mathbb{Z})$-invariance. This will be discussed more closely in section 4.

Including the Dilaton prefactor. We will now revisit the analysis from section 3.3, but here we include the contribution from the overall exponential factor $e^{-2\varphi}$ in the Lagrangian eq. (2.20). This factor arises as follows. The determinant of the D-dimensional vielbein is given by $\hat{e} = e^{D\varphi}\tilde{e}$, because of the Weyl-rescaling. Moreover, upon compactification the determinant of the rescaled vielbein splits according to $\tilde{e} = e\tilde{e}_{\mathrm{int}}$, where e represents the external vielbein and $\tilde{e}_{\mathrm{int}}$ the internal vielbein. The Weyl-rescaling is then chosen to be defined as $\tilde{e}_{\mathrm{int}} = e^{-(D-2)\varphi}$. This represents the volume of the n-torus, upon which we perform the reduction. Thus, the overall scaling contribution from the measure is $e^{D\varphi}e^{-(D-2)\varphi} = e^{2\varphi}$. In addition, we have a factor of $e^{-4\varphi}$ from Weyl-rescaling the Gauss-Bonnet term (see eq. (A.20) and eq. (A.21)). This gives a total overall dilaton prefactor of $e^{-2\varphi}$, which, after inserting $\varphi = \frac{1}{6}\vec{g}\cdot\vec{\phi}$, becomes $e^{-\frac{1}{3}\vec{g}\cdot\vec{\phi}}$.

The importance of the volume factor for compactified higher derivative terms was emphasized in [7], using the argument that after dualisation this factor is no longer invariant under the extended symmetry group $\mathrm{SL}(n+1,\mathbb{R})$ and so must be included in the weight structure. We shall see that the inclusion of this factor drastically modifies the previously presented structure.

The fundamental weights of $\mathrm{sl}(4,\mathbb{R})$. In order to perform this analysis, it is useful to first rewrite the simple roots and fundamental weights in a way which makes a comparison with [7] possible. We define arbitrary 3-vectors in $\mathbb{R}^3$ as follows

$$\hat{\vec{v}} = v_1\vec{\Lambda}_1 + v_2\vec{\Lambda}_2 + v_g\vec{g} = \big(\vec{v}, v_g\big) = \big(v_1, v_2, v_g\big), \tag{3.29}$$

where $\vec{\Lambda}_1$ and $\vec{\Lambda}_2$ are the fundamental weights of $\mathrm{sl}(3,\mathbb{R})$ and $\vec{g}$ is the basis vector taking us from the weight space $\mathbb{R}^2$ of $\mathrm{sl}(3,\mathbb{R})$ to the weight space $\mathbb{R}^3$ of $\mathrm{sl}(4,\mathbb{R})$. Note that

$$\vec{\Lambda}_1\cdot\vec{g} = \vec{\Lambda}_2\cdot\vec{g} = 0, \tag{3.30}$$

by virtue of eq. (3.7) and eq. (3.11), which implies

$$\hat{\vec{v}}\cdot\hat{\vec{u}} = \vec{v}\cdot\vec{u} + v_g u_g \vec{g}\cdot\vec{g}. \tag{3.31}$$

The scalar products may all be deduced using the orthonormal basis $\vec{e}_i$ of $\mathbb{R}^3$. Restricting to $D=6$ and $n=3$ gives

$$\vec{f}_a = \sqrt{2}\vec{e}_a + \frac{2}{9}\vec{g}, \tag{3.32}$$

and thus

$$\vec{\omega}_a = \vec{f}_a - \frac{4}{9}\vec{g}. \tag{3.33}$$

The relevant scalar products become

$$\begin{aligned}\vec{g}\cdot\vec{g} &= \frac{27}{2},\\ \vec{g}\cdot\vec{f}_a &= 6,\\ \vec{f}_a\cdot\vec{f}_b &= 2\delta_{ab}+2,\\ \vec{\omega}_a\cdot\vec{\omega}_b &= 2\delta_{ab}-\frac{2}{3}.\end{aligned} \tag{3.34}$$

The simple roots of $\mathrm{sl}(3,\mathbb{R})$ may now be written as

$$\begin{aligned}\hat{\vec{\alpha}}_1 &= \big(\vec{\alpha}_1,0\big)=\big(2,-1,0\big),\\ \hat{\vec{\alpha}}_2 &= \big(\vec{\alpha}_2,0\big)=\big(-1,2,0\big),\end{aligned} \tag{3.35}$$

and the third simple root becomes

$$\hat{\vec{\alpha}}_3=\vec{f}_3=\vec{\omega}_3+\frac{4}{9}\vec{g}=-\vec{\Lambda}_2+\frac{4}{9}\vec{g}=\left(0,-1,\frac{4}{9}\right). \tag{3.36}$$

In addition, the associated fundamental weights $\hat{\vec{\Lambda}}_i$, $i=1,2,3$, of $\mathrm{sl}(4,\mathbb{R})$, defined by

$$\hat{\vec{\alpha}}_i\cdot\hat{\vec{\Lambda}}_j=2\delta_{ij}, \tag{3.37}$$

become

$$\hat{\vec{\Lambda}}_1=\left(1,0,\frac{1}{9}\right),\quad \hat{\vec{\Lambda}}_2=\left(0,1,\frac{2}{9}\right),\quad \hat{\vec{\Lambda}}_3=\left(0,0,\frac{1}{3}\right). \tag{3.38}$$

Let us check that these indeed correspond to the fundamental weights of $\mathrm{sl}(4,\mathbb{R})$, by computing the highest weight $2\hat{\vec{\Lambda}}_1+2\hat{\vec{\Lambda}}_3$ explicitly,

$$\begin{aligned}2\hat{\vec{\Lambda}}_1+2\hat{\vec{\Lambda}}_3 &= 2\vec{\Lambda}_1+\frac{2}{9}\vec{g}+\frac{2}{3}\vec{g}\\ &= 2\left(\vec{\omega}_1+\frac{4}{9}\vec{g}\right)\\ &= 2\vec{f}_1\\ &= 2\hat{\vec{\alpha}}_1+2\hat{\vec{\alpha}}_2+2\hat{\vec{\alpha}}_3.\end{aligned} \tag{3.39}$$

This result is consistent with being the highest weight of the **84** representation of $\mathrm{sl}(4,\mathbb{R})$ as can be seen in figure 1.

Dualisation and the overall Dilaton factor. Let us now include the dilaton prefactor in the analysis. In terms of $\mathrm{sl}(4,\mathbb{R})$-vectors the volume factor can be identified with a negative shift in $\hat{\vec{\Lambda}}_3$, i.e.,

$$e^{-\frac{1}{3}\vec{g}\cdot\vec{\phi}}=e^{-\hat{\vec{\Lambda}}_3\cdot\vec{\phi}}. \tag{3.40}$$

As already mentioned above, this factor is irrelevant before dualisation because $\vec{g}\cdot\vec{\phi}$ is invariant under $\mathrm{SL}(3,\mathbb{R})$. Thus, before dualisation the manifest $\mathrm{SL}(3,\mathbb{R})$-symmetry of the compactified Gauss-Bonnet term is associated with the **27**-representation of $\mathrm{sl}(3,\mathbb{R})$.

After dualisation, all the dilaton exponents in eq. (3.20) and eq. (3.21) become shifted by a factor of $-\hat{\vec{\Lambda}}_3$. In particular, the new highest weight is

$$\left(2\hat{\vec{\Lambda}}_1 + 2\hat{\vec{\Lambda}}_3\right) - \hat{\vec{\Lambda}}_3 = 2\hat{\vec{\Lambda}}_1 + \hat{\vec{\Lambda}}_3, \tag{3.41}$$

corresponding to the **36** representation of $\mathrm{sl}(4,\mathbb{R})$, with Dynkin labels $[2,0,1]$. This is consistent with the general result of [7] that a generic curvature correction to pure Einstein gravity of order $l/2$ should be associated with an $\mathrm{sl}(n+1,\mathbb{R})$-representation with highest weight $\frac{l}{2}\hat{\vec{\Lambda}}_1 + \hat{\vec{\Lambda}}_n$.

However, this is not the full story. A more careful examination in fact reveals that the **36** representation cannot incorporate all the dilaton exponents appearing in the Lagrangian, in contrast to the **84**-representation of figure 1. To see this, let us decompose **36** in terms of representations of $\mathrm{sl}(3,\mathbb{R})$. The result is:

$$\begin{aligned} \mathbf{36} &= \mathbf{15} \oplus \mathbf{8} \oplus \mathbf{6} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}, \\ [2,0,1] &= [2,1] + [1,1] + [2,0] + [1,0] + [0,1] + [0,0]. \end{aligned} \tag{3.42}$$

Comparing this with eq. (3.25), we see that the representations $\mathbf{27}, \overline{\mathbf{15}}$ and $\bar{\mathbf{6}}$ are no longer present. For the latter two this is not a problem since they were never present in the previous analysis. What happens is that the **6** of **84** gets shifted "downwards" and becomes the **6** of **36**. Similarly, the **15** and **3** of **84** become the **15** and **3** of **36**. This takes into account all the shifted dilaton exponents arising from the dualisation process. However, since there is not enough "room" for the **27** of $\mathrm{sl}(3,\mathbb{R})$ in eq. (3.42), some of the dilaton exponents (the ones corresponding to $2\vec{f}_2 - 2\vec{f}_3,\ \vec{f}_1 + \vec{f}_2 - 2\vec{f}_3,\ 2\vec{f}_1 - 2\vec{f}_3, 2\vec{f}_1 - \vec{f}_2 - \vec{f}_3$ and $2\vec{f}_1 - 2\vec{f}_2$) arising from the pure $\tilde{P}$-terms remain outside of **36**. In fact, due to the shift of $-\hat{\vec{\Lambda}}_3$ these have now become *negative* weights of $\mathrm{sl}(4,\mathbb{R})$, because they are below the hyperplane defined by $\vec{g}\cdot\vec{x} = 0$. Although we know that these weights still correspond to positive weights of the **27** representation of $\mathrm{sl}(3,\mathbb{R})$, we are not able to determine which representation of $\mathrm{sl}(4,\mathbb{R})$ they belong to.

By a straightforward generalisation of this analysis to compactifications of quadratic curvature corrections from arbitrary dimensions D, we may conclude that the highest weight $2\hat{\vec{\Lambda}}_1 + \hat{\vec{\Lambda}}_n$, can never incorporate the dilaton exponents associated with the $[2,0,\ldots,0,2]$-representation of $\mathrm{sl}(n,\mathbb{R})$ before dualisation.

4. Discussion and conclusions

It is clear from the analysis in the previous section that the overall dilaton factor $e^{-\hat{\vec{\Lambda}}_3\cdot\vec{\phi}}$ (or, more generally, $e^{-\hat{\vec{\Lambda}}_n\cdot\vec{\phi}}$) complicates the interpretation of the dilaton exponents in terms of $\mathrm{sl}(n+1,\mathbb{R})$-representations. A similar problem has arisen in attempts at incorporating the representation structure of the hyperbolic Kac-Moody algebra $E_{10(10)}$ into curvature corrections to string and M-theory [16, 17]. There it is the "lapse function" N which plays the role of the volume factor. Similarly to our findings, the work of [16, 17] reveals that curvature corrections to, e.g., eleven-dimensional supergravity, fit into *negative* weights of $E_{10(10)}$

if the contribution from the lapse function is included. In addition, there are indications that the relevant representations of $E_{10(10)}$ are so-called *non-integrable* representations, which are not well understood.

Given these considerations, it would be desirable to have an alternative interpretation of the results where one neglects the overall volume factor (or, in the $E_{10(10)}$-case, the lapse function) in the analysis of the weight structure.

First, what information does the weight structure contain? Apart from the overall dilaton factor, the reduction of any higher derivative term $\sim \mathcal{R}^p$ will give rise to terms with $\mathcal{P}^{2p}$ (and terms with more derivatives and fewer $\mathcal{P}$'s), where $\mathcal{P}$ represents any of the "building blocks" P, H and $\partial\phi$ (we suppress all 3-dimensional indices). The appearance of weights of $\mathrm{sl}(n+1,\mathbb{R})$ (without the uniform shift from the overall dilaton factor) reflects the fact that we use fields which are components of the symmetric part of the left-invariant Maurer-Cartan form $\mathcal{P}$ of $\mathrm{sl}(n+1,\mathbb{R})$. Moreover, the dilaton factor contains information about the number of such fields. A term $\mathcal{R}^{l/2}$ will generically give weights in the weight space of the representation $[l/2, 0, \ldots, 0, l/2]$ of $\mathrm{sl}(n+1,\mathbb{R})$, and fill out the positive part of this weight space.[7] This much is clear from the observation that the overall dilaton factor really is "overall".

The presence of the overall dilaton factor shifts this weight space uniformly in a negative direction. This shift happens to be by a vector in the weight lattice of $\mathrm{sl}(n+1,\mathbb{R})$ for any value of p. However, we emphasize that the dilaton exponents still lie in the weight space of the representation $[l/2, 0, \ldots, 0, l/2]$, albeit shifted "downwards". From this point of view, the weight space of the representation with the shifted highest weight of $[l/2, 0, \ldots, 0, l/2]$ as highest weight – for example, the representation $[2, 0, 1]$ in the case discussed above — does not contain all the weights that appear in the reduced Lagrangian, and therefore does not appear to be relevant.

4.1 An $\mathrm{SL}(n+1,\mathbb{Z})$-invariant effective action

Consider now the fact that it is really the discrete "U-duality" group $\mathrm{SL}(n+1,\mathbb{Z}) \subset \mathrm{SL}(n+1,\mathbb{R})$ which is expected to be a symmetry of the complete effective action. Therefore, the compactified action should be seen as a remnant of the full U-duality invariant action, arising from a "large volume expansion" of certain automorphic forms.

Schematically, a generic, quartic, scalar term in the action after compactification of the Gauss-Bonnet term is of the form

$$\int d^3x \sqrt{|g|} e^{-\hat{\vec{\Lambda}}_n \cdot \vec{\phi}} F(\mathcal{P}), \tag{4.1}$$

where $F(\mathcal{P})$ is a quartic polynomial in the components of the Maurer-Cartan form mentioned above. F will be invariant under $\mathrm{SO}(n)$ by construction, but generically not under $\mathrm{SO}(n+1)$.

To obtain an action which is a scalar under $\mathrm{SO}(n+1)$ we must first "lift" the result of the compactification to a globally $\mathrm{SL}(n+1,\mathbb{Z})$-invariant expression. This can be done

[7] We note that the representation structure encountered here is of the same type as for the lattice of BPS charges in string theory on T^n [28].

by replacing $e^{-\hat{\vec{\Lambda}}_n\cdot\vec{\phi}}F(\mathcal{P})$ by a suitable automorphic form contracted with four $\mathcal{P}$'s:

$$\Psi_{I_1\ldots I_8}(X)\mathcal{P}^{I_1I_2}\mathcal{P}^{I_3I_4}\mathcal{P}^{I_5I_6}\mathcal{P}^{I_7I_8}, \tag{4.2}$$

where the I's are vector indices of $\mathrm{SO}(n+1)$. Here, $\Psi(X)$ is an automorphic form transforming in some representation of $\mathrm{SO}(n+1)$, and is constructed as an Eisenstein series, following, e.g., refs. [8, 9]. We must demand that when the large volume limit, $\hat{\vec{\Lambda}}_n\cdot\vec{\phi}\to-\infty$, is imposed, the leading behaviour is

$$\Psi_{I_1\ldots I_8}(X)\mathcal{P}^{I_1I_2}\mathcal{P}^{I_3I_4}\mathcal{P}^{I_5I_6}\mathcal{P}^{I_7I_8} \quad\longrightarrow\quad e^{-\hat{\vec{\Lambda}}_n\cdot\vec{\phi}}F(\mathcal{P}). \tag{4.3}$$

This limit was taken explicitly in [8, 9]. This gives conditions on which irreducible $\mathrm{SO}(n+1)$ representations the automorphic forms transform under (from the tensor structure), as well as a single condition on the "weights" of the automorphic forms (from the matching of the overall dilaton factor). Automorphic forms exist for continuous values of the weight (unlike holomorphic Eisenstein series) above some minimal value derived from convergence of the Eisenstein series. It was proven in [9] that any $\mathrm{SO}(n)$-covariant tensor structure can be reproduced as the large volume limit of some automorphic form, and that the weight dictated by the overall dilaton factor is consistent with the convergence criterion.

Under the assumption that these arguments are valid, we may conclude that the representation theoretic structure of the dilaton exponents in the polynomial F should be analyzed without inclusion of the volume factor $e^{-\hat{\vec{\Lambda}}_n\cdot\vec{\phi}}$, and hence, for the Gauss-Bonnet term ($l=4$), it is the $[2,0,\ldots,0,2]$-representation which is the relevant one (in the sense above, that we are dealing with products of four Maurer-Cartan forms), and *not* the representation $[2,0,\ldots,0,1]$. Another indication for why the representation with highest weight $2\hat{\vec{\Lambda}}_1+\hat{\vec{\Lambda}}_n$ cannot be the relevant one is that it is not contained in the tensor product of the adjoint representation $[1,0,\ldots,0,1]$ of $\mathrm{sl}(n+1,\mathbb{R})$ with itself.

The present point of view also suggest a possible explanation for the discrepancy of the weight multiplicities observed in the previous section. In the complete $\mathrm{SL}(n+1,\mathbb{Z})$-invariant four-derivative effective action the multiplicities of the weights in the $[2,0,\ldots,0,2]$-representation necessarily match because the action is constructed directly from the $\mathrm{sl}(n+1,\mathbb{R})$-valued building block $\mathcal{P}$. When taking the large volume limit, eq. (4.3), a lot of information is lost (see, e.g., [9]) and it is therefore natural that the result of the compactification does not display the correct weight multiplicities. Thus, it is only after taking the non-perturbative completion, eq. (4.2), that we can expect to reproduce correctly the weight multiplicities of the representation $[2,0,\ldots,0,2]$.

4.2 Algebraic constraints on curvature corrections

Our results have additional implications for the interpretations of the weight structure laid forward in [7]. In the analysis of the compactification of eleven-dimensional supergravity to three dimensions these authors include the volume factor when investigating the weight structure of $E_{8(8)}$. This implies that an arbitrary weight for the $l/2$:th order correction terms contains a factor of $(\frac{1}{3}-\frac{l}{6})\hat{\vec{\Lambda}}_8$. In our example above this precisely corresponds to the volume factor $\hat{\tilde{\Lambda}}_n$. Including this factor and demanding that all dilaton exponents should be on the weight lattice of $E_{8(8)}$ gives the constraint

$$\frac{1}{3} - \frac{l}{6} \in \mathbb{Z} \quad \Longleftrightarrow \quad l = 6k+2, \quad (k = 0, 1, 2, \dots). \tag{4.4}$$

This implies that these can only be on the weight lattice of $E_{8(8)}$ if the orders of the curvature correction are the celebrated powers $\frac{l}{2} = 3k+1$, $k = 0, 1, 2, \dots$. However, if our interpretation is correct, the volume factor should be left outside of the representation structure and so this argument about the restrictions on l does not seem to be applicable from a purely mathematical point of view, since also intermediate values can be reproduced by automorphic forms with some (continuous) weight.[8]

This, of course, does not mean that the result itself is incorrect (it is well known, e.g., that the first higher-derivative correction allowed by supersymmetry is of order $\mathcal{R}^4$, as is the first correction obtained by superstring calculations), only that the arguments used in order to reach it have to be refined. In order to obtain the result in the present context, one would need information restricting the weights of the automorphic forms that may enter to some discrete values. Real automorphic forms defined by Eisenstein series, unlike the holomorphic ones of $\mathrm{SL}(2,\mathbb{R})$ (or $\mathrm{Sp}(2n)$ in general), are defined for continuous values of the weight, bounded from below only by the convergence of the series. When one-loop calculations in eleven-dimensional supergravity have been used to derive automorphic forms occurring in $d = 9$ [12], it is clear how well-defined values of the weights arise. The corresponding picture for compactification to lower dimensions is less clear, due to the presence of membrane and 5-brane instantons [28, 29], but there is no doubt a corresponding mechanism at play, although we lack enough insight into the microscopic degrees of freedom to make a clear statement about it.

We suspect that a reasoning along similar lines may be used for the case of $E_{10(10)}$, and that it may again lead to the conclusion that the shifted highest weight should not be interpreted as the highest weight of a new (non-integrable) representation. Instead, it may be possible to deal with automorphic forms transforming in some integrable representations of the maximal compact subgroup of $E_{10(10)}$.

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[8] The fact that $E_{8(8)}$-invariant terms which do not arise from the compactification of $\mathcal{R}^{3k+1}$ curvature corrections can exist in $D = 3$ follows also from the work of [8], which however emphasizes a different role of the dilaton pre-factors compared to the one suggested here. We thank the authors of [8] for correspondence on this issue.

A. Squared curvature terms

Here we present all the detailed computations of the compactification.

A.1 Weyl-rescaling

Weyl-rescaling the D-dimensional metric by a factor $e^{2\varphi}$:

$$\hat{g}_{MN} = e^{2\varphi}\tilde{g}_{MN}, \tag{A.1}$$

yields the rescaled Riemann tensor

$$\begin{aligned}\hat{R}_{ABCD} = e^{-2\varphi}\big[&\tilde{R}_{ABCD} - 2\big(\eta_{[A|C|}\tilde{\nabla}_{B]}\tilde{\partial}_D\varphi - \eta_{[A|D|}\tilde{\nabla}_{B]}\tilde{\partial}_C\varphi\big)\\ &+2\big(\eta_{[A|C|}\tilde{\partial}_{B]}\varphi\tilde{\partial}_D\varphi - \eta_{[A|D|}\tilde{\partial}_{B]}\varphi\tilde{\partial}_C\varphi\big) - 2\eta_{[A|C|}\eta_{B]D}(\tilde{\partial}\varphi)^2\big],\end{aligned} \tag{A.2}$$

Ricci tensor

$$\hat{R}_{AB} = e^{-2\varphi}\big[\tilde{R}_{AB} - \eta_{AB}\tilde{\Box}\varphi - (D-2)\tilde{\nabla}_A\tilde{\partial}_B\varphi + (D-2)\tilde{\partial}_A\varphi\tilde{\partial}_B\varphi - (D-2)\eta_{AB}(\tilde{\partial}\varphi)^2\big], \tag{A.3}$$

and curvature scalar

$$\hat{R} = e^{-2\varphi}\big[\tilde{R} - (D-1)(D-2)(\tilde{\partial}\varphi)^2 - 2(D-1)\tilde{\Box}\varphi\big]. \tag{A.4}$$

Squaring the curvature terms we find

$$\begin{aligned}(\hat{R}_{ABCD})^2 = e^{-4\varphi}\Bigg[&(\tilde{R}_{ABCD})^2 + 8\left(\tilde{R}_{AB} - \frac{1}{2}\eta_{AB}\tilde{R}\right)\tilde{\partial}^A\varphi\tilde{\partial}^B\varphi - 8\tilde{R}_{AB}\tilde{\nabla}^A\tilde{\partial}^B\varphi + 4(\tilde{\Box}\varphi)^2\\ &+4(D-2)(\tilde{\nabla}_A\tilde{\partial}_B\varphi)(\tilde{\nabla}^A\tilde{\partial}^B\varphi) + 8(D-2)(\tilde{\partial}\varphi)^2\tilde{\Box}\varphi\\ &-8(D-2)\tilde{\partial}^A\varphi\tilde{\partial}^B\varphi\tilde{\nabla}_A\tilde{\partial}_B\varphi + 2(D-1)(D-2)(\tilde{\partial}\varphi)^2(\tilde{\partial}\varphi)^2\Bigg],\end{aligned} \tag{A.5}$$

$$\begin{aligned}(\hat{R}_{AB})^2 = e^{-4\varphi}\big[&(\tilde{R}_{AB})^2 - 2\tilde{R}\tilde{\Box}\varphi - 2(D-2)\tilde{R}_{AB}\tilde{\nabla}^A\tilde{\partial}^B\varphi + (3D-4)(\tilde{\Box}\varphi)^2\\ &+2(D-2)\big(\tilde{R}_{AB} - \eta_{AB}\tilde{R}\big)(\tilde{\partial}^A\varphi)(\tilde{\partial}^B\varphi) + (D-2)^2(\tilde{\nabla}_A\tilde{\partial}_B\varphi)(\tilde{\nabla}^A\tilde{\partial}^B\varphi)\\ &+(D-1)(D-2)^2(\tilde{\partial}\varphi)^4 + 2(D-2)(2D-3)(\tilde{\Box}\varphi)(\tilde{\partial}\varphi)^2\\ &-2(D-2)^2(\tilde{\nabla}_A\tilde{\partial}_B\varphi)(\tilde{\partial}^A\varphi)(\tilde{\partial}^B\varphi)\big],\end{aligned} \tag{A.6}$$

$$\begin{aligned}\hat{R}^2 = e^{-4\varphi}\big[&\tilde{R}^2 - 4(D-1)\tilde{R}\tilde{\Box}\varphi - 2(D-1)(D-2)\tilde{R}(\tilde{\partial}\varphi)^2 + 4(D-1)^2(\tilde{\Box}\varphi)^2\\ &+4(D-1)^2(D-2)(\tilde{\Box}\varphi)(\tilde{\partial}\varphi)^2 + (D-1)^2(D-2)^2(\tilde{\partial}\varphi)^4\big].\end{aligned} \tag{A.7}$$

Combining these, the Gauss-Bonnet combination can be written as

$$\begin{aligned}\hat{R}^2_{\text{GB}} = e^{-4\varphi}\Bigg\{&\tilde{R}^2_{\text{GB}} + (D-3)\Bigg[8\left(\tilde{R}_{AB} - \frac{1}{2}\eta_{AB}\tilde{R}\right)\tilde{\nabla}^A\tilde{\partial}^B\varphi - 8\tilde{R}_{AB}\tilde{\partial}^A\varphi\tilde{\partial}^B\varphi\\ &-2(D-4)\tilde{R}(\tilde{\partial}\varphi)^2 + 4(D-2)(D-3)(\tilde{\partial}\varphi)^2\tilde{\Box}\varphi + 8(D-2)(\tilde{\nabla}_A\tilde{\partial}_B\varphi)\tilde{\partial}^A\varphi\tilde{\partial}^B\varphi\\ &+4(D-2)[(\tilde{\Box}\varphi)^2 - (\tilde{\nabla}_A\tilde{\partial}_B\varphi)(\tilde{\nabla}^A\tilde{\partial}^B\varphi)] + (D-1)(D-2)(D-4)(\tilde{\partial}\varphi)^4\Bigg]\Bigg\}.\end{aligned} \tag{A.8}$$

The Gauss-Bonnet Lagrangian, including the measure $\hat{e} = e^{D\varphi}\tilde{e}$, can now be conveniently grouped in terms of equations of motion and total derivatives. This is achieved using integrations by parts, where no explicit appearance of $\tilde{\nabla}_{(A}\tilde{\partial}_{B)}\varphi$ is required. The resulting Lagrangian is

$$\begin{aligned}\mathcal{L}_{\rm GB} = \tilde{e}e^{(D-4)\varphi}\Big\{&\tilde{R}^2_{\rm GB} - (D-3)(D-4)\Big[2(D-2)(\tilde{\partial}\varphi)^2\tilde{\Box}\varphi + (D-2)(D-3)(\tilde{\partial}\varphi)^4\\&+4\Big(\tilde{R}_{AB} - \frac{1}{2}\eta_{AB}\tilde{R}\Big)(\tilde{\partial}^A\varphi)(\tilde{\partial}^B\varphi)\Big]\Big\}\\&+2(D-3)\tilde{e}\tilde{\nabla}_A\Big\{e^{(D-4)\varphi}\Big[(D-2)^2(\tilde{\partial}\varphi)^2\tilde{\partial}^A\varphi + 2(D-2)(\tilde{\Box}\varphi)\tilde{\partial}^A\varphi\\&-(D-2)\tilde{\partial}^A(\tilde{\partial}\varphi)^2 + 4\Big(\tilde{R}^{AB} - \frac{1}{2}\eta^{AB}\tilde{R}\Big)\tilde{\partial}_B\varphi\Big]\Big\}. \qquad \text{(A.9)}\end{aligned}$$

Notice that the total derivative terms in this expression will remain total derivatives after the compactification as well.

A.2 Compactification

In compactification of gravity from D to $d = (D-n)$ dimensions the vielbein one-form is given by

$$\tilde{e}^A = (\tilde{e}^\alpha, \tilde{e}^a) = \big(e^\alpha, [dx^m + \mathcal{A}^m_{(1)}]\tilde{e}_m{}^a\big), \qquad \text{(A.10)}$$

with the determinant denoted by $\tilde{e} = e\tilde{e}_{\rm int}$. Dropping all dependence on the torus coordinates, i.e., $\tilde{d} = d = dx^\mu\partial_\mu$, the compactified spin connection one-form is found to be

$$\begin{aligned}\tilde{\omega}^\alpha{}_\beta &= \omega^\alpha{}_\beta - \frac{1}{2}\tilde{e}^c\tilde{F}_c{}^\alpha{}_\beta,\\ \tilde{\omega}^\alpha{}_b &= \frac{1}{2}e^\gamma\tilde{F}_{b\gamma}{}^\alpha - \tilde{e}^c\tilde{P}^\alpha{}_{cb},\\ \tilde{\omega}^a{}_b &= e^\gamma Q_\gamma{}^a{}_b, \qquad \text{(A.11)}\end{aligned}$$

where $\tilde{P}_\alpha{}^{bc} = \tilde{e}^{m(b}\partial_\alpha\tilde{e}_m{}^{c)}$, $Q_\alpha{}^{bc} = \tilde{e}^{m[b}\partial_\alpha\tilde{e}_m{}^{c]}$ and $\tilde{F}^a{}_{\beta\gamma} = 2\tilde{e}_m{}^a e^\mu{}_{[\beta}e^\nu{}_{\gamma]}\partial_\mu\mathcal{A}^m_\nu$.

Using the spin connection it is now straightforward to compute the compactified Riemann tensor

$$\begin{aligned}\tilde{R}_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} - \frac{1}{2}\big(\tilde{F}_{c\alpha[\gamma}\tilde{F}^c{}_{|\beta|\delta]} + \tilde{F}_{c\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\big),\\ \tilde{R}_{\alpha\beta\gamma d} &= D_{[\alpha}\tilde{F}_{|d|\beta]\gamma} - \tilde{F}^c{}_{\alpha\beta}\tilde{P}_{\gamma cd},\\ \tilde{R}_{\alpha\beta cd} &= \frac{1}{2}\tilde{F}_{[c|\alpha|}{}^\gamma\tilde{F}_{d]\gamma\beta} - 2\tilde{P}_\alpha{}^e{}_{[c}\tilde{P}_{|\beta|d]e},\\ \tilde{R}_{\alpha b\gamma d} &= -D_\alpha\tilde{P}_{\gamma bd} - \tilde{P}_{\alpha b}{}^e\tilde{P}_{\gamma de} + \frac{1}{4}\tilde{F}_{b\gamma\epsilon}\tilde{F}_{d\alpha}{}^\epsilon,\\ \tilde{R}_{ab\gamma d} &= \tilde{F}_{[a|\gamma\epsilon|}\tilde{P}^\epsilon{}_{b]d},\\ \tilde{R}_{abcd} &= -2\tilde{P}_{\epsilon a[c}\tilde{P}^\epsilon{}_{|b|d]}, \qquad \text{(A.12)}\end{aligned}$$

which contracted yields the Ricci tensor

$$
\begin{aligned}
\tilde{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}\tilde{F}_{c\epsilon\alpha}\tilde{F}^{c\epsilon}{}_{\beta} - \tilde{P}_{\alpha cd}\tilde{P}_{\beta}{}^{cd} - \mathrm{tr}(D_\alpha \tilde{P}_\beta), \\
\tilde{R}_{\alpha b} &= \frac{1}{2}\big(D_\epsilon \tilde{F}_{b\alpha}{}^{\epsilon} + \tilde{F}_{c\alpha\delta}\tilde{P}^{\delta c}{}_{b} + \tilde{F}_{b\alpha\epsilon}\mathrm{tr}\tilde{P}^{\epsilon}\big), \\
\tilde{R}_{ab} &= -D^{\epsilon}\tilde{P}_{\epsilon ab} - \tilde{P}_{\epsilon ab}\mathrm{tr}\tilde{P}^{\epsilon} + \frac{1}{4}\tilde{F}_{a\gamma\delta}\tilde{F}_{b}{}^{\gamma\delta},
\end{aligned} \tag{A.13}
$$

and the curvature scalar

$$
\tilde{R} = R - \frac{1}{4}\tilde{F}^2 - \tilde{P}^2 - (\mathrm{tr}\tilde{P})^2 - 2\mathrm{tr}(D_\epsilon \tilde{P}^\epsilon). \tag{A.14}
$$

The trace is always taken over the internal indices, also $\tilde{F}^2 \equiv \tilde{F}_{a\beta\gamma}\tilde{F}^{a\beta\gamma}$ and $\tilde{P}^2 \equiv \tilde{P}_{\alpha bc}\tilde{P}^{\alpha bc}$. The covariant derivative D is defined as $D \equiv \nabla + Q \equiv \partial + \omega + Q$, where $\omega_{\alpha\beta\gamma}$ is the spacetime spin connection and $Q_{\alpha bc}$ can be thought of as a gauge connection for the SO(n)-symmetry.

Squaring the curvature tensor components we find:

$$
\begin{aligned}
(\tilde{R}_{\alpha\beta\gamma\delta})^2 &= R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{3}{2}R_{\alpha\beta\gamma\delta}\tilde{F}_{e}{}^{\alpha\beta}\tilde{F}^{e\gamma\delta} + \frac{3}{8}\tilde{F}_{c\alpha\beta}\tilde{F}_{d}{}^{\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}^{d\gamma\delta} \\
&\quad + \frac{3}{8}\tilde{F}_{c\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}_{d}{}^{\alpha\gamma}\tilde{F}^{d\beta\delta}, \\
(\tilde{R}_{\alpha\beta\gamma d})^2 &= (D_{[\alpha}\tilde{F}_{|d|\beta]\gamma})(D^{\alpha}\tilde{F}^{d\beta\gamma}) - 2(D_\alpha \tilde{F}_{d\beta\gamma})\tilde{F}^{c\alpha\beta}\tilde{P}^{\gamma d}{}_{c} + \tilde{F}^{a}{}_{\gamma\delta}\tilde{P}_{\epsilon ab}\tilde{P}^{\epsilon bc}\tilde{F}_{c}{}^{\gamma\delta}, \\
(\tilde{R}_{\alpha\beta cd})^2 &= 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}^\alpha\tilde{P}_\beta\tilde{P}^\beta) - 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\alpha\tilde{P}^\beta) + \frac{1}{8}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta} \\
&\quad - \frac{1}{8}\tilde{F}_{c\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}_{d}{}^{\alpha\gamma}\tilde{F}^{d\beta\delta} - \tilde{F}^{c}{}_{\beta\gamma}\tilde{P}_{\delta ce}\tilde{P}^{\gamma ed}\tilde{F}_{d}{}^{\beta\delta} + \tilde{F}^{c}{}_{\beta\gamma}\tilde{P}^{\gamma}{}_{ce}\tilde{P}_{\delta}{}^{ed}\tilde{F}_{d}{}^{\beta\delta}, \\
(\tilde{R}_{\alpha b\gamma d})^2 &= (D_\alpha\tilde{P}_{\gamma bd})(D^\alpha\tilde{P}^{\gamma bd}) + 2(D_\alpha\tilde{P}_{\gamma bd})\tilde{P}^{\alpha be}\tilde{P}^{\gamma d}{}_{e} - \frac{1}{2}(D_\alpha\tilde{P}_{\gamma bd})\tilde{F}^{b\alpha\epsilon}\tilde{F}^{d\gamma}{}_{\epsilon} \\
&\quad + \mathrm{tr}(\tilde{P}_\alpha\tilde{P}^\alpha\tilde{P}_\gamma\tilde{P}^\gamma) + \frac{1}{16}\tilde{F}_{c\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}_{d}{}^{\alpha\gamma}\tilde{F}^{d\beta\delta} - \frac{1}{2}\tilde{F}^{b}{}_{\beta\gamma}\tilde{P}_{\delta be}\tilde{P}^{\gamma ed}\tilde{F}_{d}{}^{\beta\delta}, \\
(\tilde{R}_{ab\gamma d})^2 &= \frac{1}{2}\big[\tilde{F}_{c}{}^{\alpha}{}_{\delta}\tilde{F}^{c\beta\delta}\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta) - \tilde{F}^{a}{}_{\beta\gamma}\tilde{P}_{\delta ad}\tilde{P}^{\gamma db}\tilde{F}_{b}{}^{\beta\delta}\big], \\
(\tilde{R}_{abcd})^2 &= 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\alpha\tilde{P}^\beta).
\end{aligned} \tag{A.15}
$$

The compactified Ricci tensor and curvature scalar squared are

$$
\begin{aligned}
(\tilde{R}_{\alpha\beta})^2 &= R_{\alpha\beta}R^{\alpha\beta} - R_{\alpha\beta}\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta} - 2R_{\alpha\beta}\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - 2R_{\alpha\beta}\mathrm{tr}(D^\alpha\tilde{P}^\beta) \\
&\quad + \mathrm{tr}(D_\alpha\tilde{P}_\beta)\mathrm{tr}(D^\alpha\tilde{P}^\beta) + 2\mathrm{tr}(D_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) + \mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) \\
&\quad + \mathrm{tr}(D_\alpha\tilde{P}_\beta)\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta} + \mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta} + \frac{1}{4}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta}, \\
(\tilde{R}_{\alpha b})^2 &= \frac{1}{4}\big[(D_\gamma\tilde{F}_{b\alpha}{}^{\gamma})(D_\delta\tilde{F}^{b\alpha\delta}) + 2(D_\gamma\tilde{F}_{b\alpha}{}^{\gamma})\tilde{F}^{b\alpha\beta}\mathrm{tr}\tilde{P}_\beta + 2(D_\gamma\tilde{F}_{c\alpha}{}^{\gamma})\tilde{P}_{\beta}{}^{cd}\tilde{F}_{d}{}^{\alpha\beta} \\
&\quad + \tilde{F}^{c}{}_{\beta\gamma}\tilde{P}^{\gamma}{}_{ce}\tilde{P}_{\delta}{}^{ed}\tilde{F}_{d}{}^{\beta\delta} + 2\tilde{F}_{b\alpha\gamma}\tilde{P}_{\beta}{}^{bc}\tilde{F}_{c}{}^{\alpha\beta}\mathrm{tr}\tilde{P}^{\gamma} + \tilde{F}_{b\alpha\gamma}\tilde{F}^{b\alpha}{}_{\delta}(\mathrm{tr}\tilde{P}^{\gamma})(\mathrm{tr}\tilde{P}^{\delta})\big], \\
(\tilde{R}_{ab})^2 &= \mathrm{tr}(D_\alpha\tilde{P}^\alpha D_\beta\tilde{P}^\beta) + 2(D_\alpha\tilde{P}^{\alpha bc})\tilde{P}_{\beta bc}\mathrm{tr}\tilde{P}^\beta - \frac{1}{2}(D_\alpha\tilde{P}^{\alpha bc})\tilde{F}_{b\gamma\delta}\tilde{F}_{c}{}^{\gamma\delta} \\
&\quad + \mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)(\mathrm{tr}\tilde{P}^\alpha)(\mathrm{tr}\tilde{P}^\beta) - \frac{1}{2}\tilde{F}_{b\alpha\beta}\tilde{P}_{\gamma}{}^{bc}\tilde{F}_{c}{}^{\alpha\beta}\mathrm{tr}\tilde{P}^{\gamma} + \frac{1}{16}\tilde{F}_{c\alpha\beta}\tilde{F}_{d}{}^{\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}^{d\gamma\delta},
\end{aligned} \tag{A.16}
$$

and

$$\begin{aligned}\tilde{R}^2 &= R^2 - \frac{1}{2}R\tilde{F}^2 - 4R\mathrm{tr}(D_\alpha\tilde{P}^\alpha) - 2R\tilde{P}^2 - 2R(\mathrm{tr}\tilde{P})^2 + \frac{1}{16}(\tilde{F}^2)^2 + \tilde{F}^2\mathrm{tr}(D_\alpha\tilde{P}^\alpha)\\ &+\frac{1}{2}\tilde{F}^2\tilde{P}^2 + \frac{1}{2}\tilde{F}^2(\mathrm{tr}\tilde{P})^2 + 4[\mathrm{tr}(D_\alpha\tilde{P}^\alpha)]^2 + 4\mathrm{tr}(D_\alpha\tilde{P}^\alpha)\tilde{P}^2 + 4\mathrm{tr}(D_\alpha\tilde{P}^\alpha)(\mathrm{tr}\tilde{P})^2\\ &+(\tilde{P}^2)^2 + 2\tilde{P}^2(\mathrm{tr}\tilde{P})^2 + ((\mathrm{tr}\tilde{P})^2)^2. \end{aligned} \tag{A.17}$$

Choosing a basis where all explicit derivatives appearing are either divergences or total derivatives, we can rewrite three of the quadratic curvature components as

$$\begin{aligned}(\tilde{R}_{\alpha\beta\gamma d})^2 &= \frac{1}{2}\big[R_{\alpha\beta\gamma\delta}\tilde{F}_e{}^{\alpha\gamma}\tilde{F}^{e\beta\delta} - R_{\alpha\beta}\tilde{F}_{e\gamma}{}^{\alpha}\tilde{F}^{e\gamma\beta} + (D_\gamma\tilde{F}_{b\alpha}{}^{\gamma})(D_\delta\tilde{F}^{b\alpha\delta})\\ &-\tilde{F}_{c\alpha\beta}(D_\gamma\tilde{P}^{\gamma cd})\tilde{F}_d{}^{\alpha\beta} + \tilde{F}^a{}_{\gamma\delta}\tilde{P}_{\epsilon ab}\tilde{P}^{\epsilon bc}\tilde{F}_c{}^{\gamma\delta} - \tilde{F}^c{}_{\beta\gamma}\tilde{P}^\gamma{}_{ce}\tilde{P}_\delta{}^{ed}\tilde{F}_d{}^{\beta\delta}\\ &+3\tilde{F}^c{}_{\beta\gamma}\tilde{P}_{\delta ce}\tilde{P}^{\gamma ed}\tilde{F}_d{}^{\beta\delta}\big] + \frac{1}{2}\nabla_\alpha\big[(D_\gamma\tilde{F}_{e\beta}{}^{\alpha})\tilde{F}^{e\beta\gamma} - (D_\gamma\tilde{F}_{e\beta}{}^{\gamma})\tilde{F}^{e\beta\alpha}\\ &+\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}\big],\\ (\tilde{R}_{\alpha b\gamma d})^2 &= \frac{1}{2}(D_\alpha\tilde{F}_{c\beta}{}^{\alpha})\tilde{P}_\gamma{}^{cd}\tilde{F}_d{}^{\beta\gamma} - \frac{1}{8}\tilde{F}_{c\alpha\beta}(D_\gamma\tilde{P}^{\gamma cd})\tilde{F}_d{}^{\alpha\beta} + \mathrm{tr}[(D_\alpha\tilde{P}^\alpha)(D_\beta\tilde{P}^\beta)]\\ &-\mathrm{tr}[(D_\alpha\tilde{P}^\alpha)\tilde{P}_\beta\tilde{P}^\beta] - R_{\alpha\beta}\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) + \frac{1}{16}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta}\\ &-\frac{1}{4}\tilde{F}^a{}_{\gamma\delta}\tilde{P}_{\epsilon ab}\tilde{P}^{\epsilon bc}\tilde{F}_c{}^{\gamma\delta} + 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\alpha\tilde{P}^\beta) - \mathrm{tr}(\tilde{P}_\alpha\tilde{P}^\alpha\tilde{P}_\beta\tilde{P}^\beta) + \nabla_\alpha\Bigg[\mathrm{tr}(\tilde{P}^\alpha\tilde{P}_\beta\tilde{P}^\beta)\\ &+\mathrm{tr}(\tilde{P}_\beta D^\beta\tilde{P}^\alpha - \tilde{P}^\alpha D^\beta\tilde{P}_\beta) + \frac{1}{8}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma} - \frac{1}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\gamma cd}\tilde{F}_d{}^{\beta\alpha}\Bigg],\\ (\tilde{R}_{\alpha\beta})^2 &= R_{\alpha\beta}R^{\alpha\beta} - R_{\alpha\beta}\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta} - 2R_{\alpha\beta}\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - R_{\alpha\beta}\mathrm{tr}\tilde{P}^\alpha\mathrm{tr}\tilde{P}^\beta - R\mathrm{tr}(D_\alpha\tilde{P}^\alpha)\\ &-(D_\alpha\tilde{F}_{c\delta}{}^{\alpha})\tilde{F}^{c\delta\beta}\mathrm{tr}\tilde{P}_\beta - 2\mathrm{tr}[(D_\alpha\tilde{P}^\alpha)\tilde{P}^\beta]\mathrm{tr}\tilde{P}_\beta + \mathrm{tr}(D_\alpha\tilde{P}^\alpha)\mathrm{tr}(D_\beta\tilde{P}^\beta)\\ &+\frac{1}{4}\mathrm{tr}(D_\alpha\tilde{P}^\alpha)\tilde{F}^2 + \mathrm{tr}(D_\alpha\tilde{P}^\alpha)\mathrm{tr}(\tilde{P}_\beta\tilde{P}^\beta) + \frac{1}{4}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta}\\ &+\frac{1}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}\mathrm{tr}\tilde{P}_\alpha - \tilde{F}_{c\beta\gamma}\tilde{P}^{\gamma cd}\tilde{F}_d{}^{\beta\alpha}\mathrm{tr}\tilde{P}_\alpha + \tilde{F}_{c\delta\alpha}\tilde{F}^{c\delta}{}_{\beta}\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta)\\ &+\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) + \nabla_\alpha\big[\mathrm{tr}(D^\alpha\tilde{P}_\beta)\mathrm{tr}\tilde{P}^\beta - \mathrm{tr}(D^\beta\tilde{P}_\beta)\mathrm{tr}\tilde{P}^\alpha - \frac{1}{4}\tilde{F}^2\mathrm{tr}\tilde{P}^\alpha\\ &-2(R^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}R)\mathrm{tr}\tilde{P}_\beta + \tilde{F}^{c\delta\alpha}\tilde{F}_{c\delta}{}^{\beta}\mathrm{tr}\tilde{P}_\beta + 2\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta)\mathrm{tr}\tilde{P}_\beta - \tilde{P}^2\mathrm{tr}\tilde{P}^\alpha\big]. \end{aligned} \tag{A.18}$$

This choice of basis is only possible for the curvature terms squared; for general powers of the Riemann tensor no such basis exists. The square of the uncompactified curvature terms can now be written as a sum of the quadratic compactified curvature components

$$\begin{aligned}(\tilde{R}_{ABCD})^2 &= \tilde{R}_{\alpha\beta\gamma\delta}\tilde{R}^{\alpha\beta\gamma\delta} + 4\tilde{R}_{\alpha\beta\gamma d}\tilde{R}^{\alpha\beta\gamma d} + 2\tilde{R}_{\alpha\beta cd}\tilde{R}^{\alpha\beta cd} + 4\tilde{R}_{\alpha b\gamma d}\tilde{R}^{\alpha b\gamma d}\\ &+4\tilde{R}_{ab\gamma d}\tilde{R}^{ab\gamma d} + \tilde{R}_{abcd}\tilde{R}^{abcd},\\ (\tilde{R}_{AB})^2 &= \tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta} + 2\tilde{R}_{\alpha b}\tilde{R}^{\alpha b} + \tilde{R}_{ab}\tilde{R}^{ab}. \end{aligned} \tag{A.19}$$

Since the total volume measure is $\hat{e} = e^{D\varphi}e\tilde{e}_{\mathrm{int}}$, the factor $e^{D\varphi}\tilde{e}_{\mathrm{int}}$ has to be moved inside the total derivatives using integration by parts. The Riemann tensor squared will then be given by

$$
\begin{aligned}
&\hat{e}e^{-4\varphi}\big(\tilde{R}_{ABCD}\big)^2\\
&\quad= ee^{(D-4)\varphi}\tilde{e}_{\rm int}\bigg\{R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-\frac{1}{2}R_{\alpha\beta\gamma\delta}\tilde{F}_e{}^{\alpha\beta}\tilde{F}^{e\gamma\delta}-2R_{\alpha\beta}\big[\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta}+2{\rm tr}(\tilde{P}^\alpha\tilde{P}^\beta)\big]\\
&\qquad+2D_\alpha\tilde{F}_{c\delta}{}^{\alpha}\big[D_\beta\tilde{F}^{c\delta\beta}+\tilde{P}_\beta{}^{cd}\tilde{F}_d{}^{\delta\beta}+{\rm tr}\tilde{P}_\beta\tilde{F}^{c\delta\beta}+(D-4)\partial_\beta\varphi\tilde{F}^{c\delta\beta}\big]+4{\rm tr}(D_\alpha\tilde{P}^\alpha D_\beta\tilde{P}^\beta)\\
&\qquad-4{\rm tr}[(D_\alpha\tilde{P}^\alpha)\tilde{P}_\beta\tilde{P}^\beta]-\frac{5}{2}\tilde{F}_{c\alpha\beta}(D_\gamma\tilde{P}^{\gamma cd})\tilde{F}_d{}^{\alpha\beta}+4{\rm tr}(\tilde{P}_\alpha D_\beta\tilde{P}^\beta)\big[{\rm tr}\tilde{P}^\alpha+(D-4)\partial^\alpha\varphi\big]\\
&\qquad+\frac{1}{2}{\rm tr}(D_\alpha\tilde{P}^\alpha)\big[\tilde{F}^2+4\tilde{P}^2\big]+\frac{3}{8}\tilde{F}_{c\alpha\beta}\tilde{F}_d{}^{\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\tilde{F}^{d\gamma\delta}+\frac{1}{2}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_\beta\tilde{F}_{d\delta}{}^\alpha\tilde{F}^{d\delta\beta}\\
&\qquad+\frac{1}{8}\tilde{F}_{c\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\tilde{F}_d{}^{\alpha\gamma}\tilde{F}^{d\beta\delta}+2\tilde{F}^c{}_{\beta\gamma}\tilde{P}_{\delta ce}\tilde{P}^{\gamma ed}\tilde{F}_d{}^{\beta\delta}+\tilde{F}^a{}_{\gamma\delta}\tilde{P}_{\epsilon ab}\tilde{P}^{\epsilon bc}\tilde{F}_c{}^{\gamma\delta}+2{\rm tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\alpha\tilde{P}^\beta)\\
&\qquad+2{\rm tr}(\tilde{P}_\alpha\tilde{P}_\beta)\tilde{F}_{c\delta}{}^\alpha\tilde{F}^{c\delta\beta}-\frac{3}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}\big[{\rm tr}\tilde{P}_\alpha+(D-4)\partial_\alpha\varphi\big]+2{\rm tr}(\tilde{P}_\alpha\tilde{P}_\beta){\rm tr}(\tilde{P}^\alpha\tilde{P}^\beta)\\
&\qquad+\frac{1}{2}\big(\tilde{F}^2+4\tilde{P}^2\big)\big[({\rm tr}\tilde{P})^2+2(D-4){\rm tr}\tilde{P}_\alpha\partial^\alpha\varphi+(D-4)^2(\partial\varphi)^2+(D-4)\Box\varphi\big]\\
&\qquad-4{\rm tr}(\tilde{P}_\alpha\tilde{P}^\alpha\tilde{P}_\beta)\big[{\rm tr}\tilde{P}^\beta+(D-4)\partial^\beta\varphi\big]\big\}+e\nabla_\alpha\big\{e^{(D-4)\varphi}\tilde{e}_{\rm int}\big[-2(D_\beta\tilde{F}_{c\delta}{}^\beta)\tilde{F}^{c\delta\alpha}\\
&\qquad-4{\rm tr}(\tilde{P}^\alpha D_\beta\tilde{P}^\beta)+\frac{3}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}+4{\rm tr}(\tilde{P}^\alpha\tilde{P}_\beta\tilde{P}^\beta)-\big[{\rm tr}\tilde{P}^\alpha+(D-4)\partial^\alpha\varphi\big](\tilde{F}^2+4\tilde{P}^2)\big]\\
&\qquad+D^\alpha\left[e^{(D-4)\varphi}\tilde{e}_{\rm int}\left(\frac{1}{2}\tilde{F}^2+2\tilde{P}^2\right)\right]\bigg\}
\end{aligned}
\tag{A.20}
$$

and the Ricci tensor squared is given by

$$
\begin{aligned}
&\hat{e}e^{-4\varphi}\big(\tilde{R}_{AB}\big)^2\\
&\quad= ee^{(D-4)\varphi}\tilde{e}_{\rm int}\bigg\{R_{\alpha\beta}\big[R^{\alpha\beta}-\tilde{F}_{c\delta}{}^\alpha\tilde{F}^{c\delta\beta}-2{\rm tr}(\tilde{P}^\alpha\tilde{P}^\beta)+{\rm tr}\tilde{P}^\alpha{\rm tr}\tilde{P}^\beta+2(D-4){\rm tr}\tilde{P}^\alpha\partial^\beta\varphi\big]\\
&\qquad-R\big[{\rm tr}(D_\alpha\tilde{P}^\alpha)+({\rm tr}\tilde{P})^2+(D-4){\rm tr}\tilde{P}_\alpha\partial^\alpha\varphi\big]+(D_\alpha\tilde{F}_{b\gamma}{}^\alpha)\left[\frac{1}{2}D_\beta\tilde{F}^{b\gamma\beta}+\tilde{P}_\delta{}^{bc}\tilde{F}_c{}^{\gamma\delta}\right]\\
&\qquad+(D_\alpha\tilde{P}^\alpha{}_{bc})\left[D_\beta\tilde{P}^{\beta bc}-\frac{1}{2}\tilde{F}^b{}_{\gamma\delta}\tilde{F}^{c\gamma\delta}\right]+\frac{1}{4}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_\beta\tilde{F}_{d\delta}{}^\alpha\tilde{F}^{d\delta\beta}+\frac{1}{16}\tilde{F}_{c\alpha\beta}\tilde{F}_d{}^{\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\tilde{F}^{d\gamma\delta}\\
&\qquad+{\rm tr}(D_\alpha\tilde{P}^\alpha)\left[{\rm tr}(D_\beta\tilde{P}^\beta)+\frac{1}{4}\tilde{F}^2+\tilde{P}^2+\frac{3}{2}({\rm tr}\tilde{P})^2+(D-4){\rm tr}\tilde{P}_\beta\partial^\beta\varphi\right]\\
&\qquad+\frac{1}{2}\tilde{F}^c{}_{\beta\gamma}\tilde{P}^\gamma{}_{ce}\tilde{P}_\delta{}^{ed}\tilde{F}_d{}^{\beta\delta}+\tilde{F}_{c\delta\alpha}\tilde{F}^{c\delta}{}_\beta\left[{\rm tr}(\tilde{P}^\alpha\tilde{P}^\beta)-\frac{1}{2}{\rm tr}\tilde{P}^\alpha{\rm tr}\tilde{P}^\beta-(D-4){\rm tr}\tilde{P}^\alpha\partial^\beta\varphi\right]\\
&\qquad+\frac{1}{4}\tilde{F}^2\big[({\rm tr}\tilde{P})^2+(D-4){\rm tr}\tilde{P}_\alpha\partial^\alpha\varphi\big]+\tilde{P}^2\big[({\rm tr}\tilde{P})^2+(D-4){\rm tr}\tilde{P}_\alpha\partial^\alpha\varphi\big]\\
&\qquad+{\rm tr}(\tilde{P}_\alpha\tilde{P}_\beta)\big[{\rm tr}(\tilde{P}^\alpha\tilde{P}^\beta)-{\rm tr}\tilde{P}^\alpha{\rm tr}\tilde{P}^\beta-2(D-4){\rm tr}\tilde{P}^\alpha\partial^\beta\varphi\big]\\
&\qquad+\frac{1}{2}({\rm tr}\tilde{P})^2\big[({\rm tr}\tilde{P})^2+2(D-4){\rm tr}\tilde{P}_\alpha\partial^\alpha\varphi+(D-4)^2(\partial\varphi)^2+(D-4)\Box\varphi\big]\big\}\\
&\qquad+e\nabla_\alpha\big\{e^{(D-4)\varphi}\tilde{e}_{\rm int}\Big[\big(-2R^{\alpha\beta}+\eta^{\alpha\beta}R+\tilde{F}_{c\delta}{}^\alpha\tilde{F}^{c\delta\beta}+2{\rm tr}(\tilde{P}^\alpha\tilde{P}^\beta)\\
&\qquad-(D-4)\partial^\alpha\varphi{\rm tr}\tilde{P}^\beta\big){\rm tr}\tilde{P}_\beta-\left(\frac{1}{4}\tilde{F}^2+{\rm tr}(D_\beta\tilde{P}^\beta)+{\rm tr}(\tilde{P}_\beta\tilde{P}^\beta)+({\rm tr}\tilde{P})^2\right){\rm tr}\tilde{P}^\alpha\Big]\\
&\qquad+\frac{1}{2}D^\alpha\big[e^{(D-4)\varphi}\tilde{e}_{\rm int}({\rm tr}\tilde{P})^2\big]\bigg\}.
\end{aligned}
\tag{A.21}
$$

Using also $\tilde{\Box}\varphi = \Box\varphi + \partial_\alpha\varphi \mathrm{tr}\tilde{P}^\alpha$ and $(\tilde{\partial}\varphi)^2 = (\partial\varphi)^2$ we have all the ingredients to extract the compactified Gauss-Bonnet Lagrangian, eq. (A.9). Notice that $\hat{e}(\hat{\nabla}_A\hat{X}^A) = \hat{\partial}_M(\hat{e}\hat{X}^M) = \partial_\mu(\hat{e}\hat{X}^\mu)$ holds after the compactification as well, implying that the total derivative terms in eq. (A.9) will still be total derivatives even after the compactification. Together with the result from the Weyl-rescaling, eq. (A.9), the complete result after the compactification is

$$
\begin{aligned}
&\hat{e}\hat{R}^2_{\mathrm{GB}} = \sqrt{|g|}\tilde{e}_{\mathrm{int}}e^{(D-4)\varphi}\Bigg\{[R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2] - \frac{1}{2}[R_{\alpha\beta\gamma\delta}\tilde{F}_c{}^{\alpha\beta}\tilde{F}^{c\gamma\delta} + R\tilde{F}^2 \\
&-4R_{\alpha\beta}\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta}] + \frac{1}{8}\tilde{F}_{c\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\tilde{F}_d{}^{\alpha\gamma}\tilde{F}^{d\beta\delta} - \frac{1}{2}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta} + \frac{(D-n)}{16(D-n-2)}(\tilde{F}^2)^2 \\
&+2\tilde{F}^c{}_{\beta\gamma}\tilde{P}_{\delta ce}\tilde{P}^{\gamma ed}\tilde{F}_d{}^{\beta\delta} + \tilde{F}^2\left(\frac{1}{2}(\mathrm{tr}\tilde{P})^2 + (D-4)\mathrm{tr}\tilde{P}_\beta\partial^\beta\varphi + \frac{(D-4)^2}{2}(\partial\varphi)^2\right) \\
&-\frac{1}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}(\mathrm{tr}\tilde{P}_\alpha + (D-4)\partial_\alpha\varphi) - 2\tilde{F}_{c\beta\gamma}\tilde{P}^{\gamma cd}\tilde{F}_d{}^{\beta\alpha}(\mathrm{tr}\tilde{P}_\alpha + (D-4)\partial_\alpha\varphi) \\
&+2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\alpha\tilde{P}^\beta) + 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - (\tilde{P}^2)^2 - 4(D-2)\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\partial^\alpha\varphi\partial^\beta\varphi \\
&-4\mathrm{tr}(\tilde{P}_\alpha\tilde{P}^\alpha\tilde{P}^\beta)(\mathrm{tr}\tilde{P}_\beta + (D-4)\partial_\beta\varphi) - 4(D-4)(\mathrm{tr}\tilde{P})^2\mathrm{tr}\tilde{P}_\alpha\partial^\alpha\varphi \\
&+2\tilde{P}^2((\mathrm{tr}\tilde{P})^2 + 2(D-4)\mathrm{tr}\tilde{P}_\alpha\partial^\alpha\varphi + (D^2-7D+14)(\partial\varphi)^2) - (\mathrm{tr}\tilde{P})^2(\mathrm{tr}\tilde{P})^2 \\
&-2(D^2-7D+14)(\mathrm{tr}\tilde{P})^2(\partial\varphi)^2 - 4(D^2-8D+14)(\mathrm{tr}\tilde{P}_\alpha\partial^\alpha\varphi)^2 \\
&-4(D-4)(D^2-7D+11)\mathrm{tr}\tilde{P}_\alpha\partial^\alpha\varphi(\partial\varphi)^2 - (D-2)(D-3)(D-4)(D-5)(\partial\varphi)^2(\partial\varphi)^2 \\
&+\Bigg[R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R - \frac{1}{2}\tilde{F}_{c\delta\alpha}\tilde{F}^{c\delta}{}_\beta + \frac{1}{8}\eta_{\alpha\beta}\tilde{F}^2 - \mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta) + \frac{1}{2}\eta_{\alpha\beta}\tilde{P}^2 + (D-2)\partial_\alpha\varphi\partial_\beta\varphi \\
&-\frac{(D-2)}{2}\eta_{\alpha\beta}(\partial\varphi)^2\Bigg](4\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - 4\mathrm{tr}\tilde{P}^\alpha\mathrm{tr}\tilde{P}^\beta - 8(D-4)\mathrm{tr}\tilde{P}^\alpha\partial^\beta\varphi \\
&-4(D-3)(D-4)\partial^\alpha\varphi\partial^\beta\varphi) \\
&+[D_\alpha\tilde{F}_{c\delta}{}^{\alpha} + \tilde{P}_{\alpha c}{}^{e}\tilde{F}_{e\delta}{}^{\alpha}](-2\tilde{F}_d{}^{\delta\gamma}\tilde{P}_\gamma{}^{cd} + 2\tilde{F}^{c\delta\gamma}\mathrm{tr}\tilde{P}_\gamma + 2(D-4)\tilde{F}^{c\delta\gamma}\partial_\gamma\varphi) \\
&+\Bigg[D_\alpha\tilde{P}^\alpha{}_{cd} - \frac{1}{4}\tilde{F}_{c\alpha\beta}\tilde{F}_d{}^{\alpha\beta} - \frac{1}{(D-2)}\delta_{cd}\mathrm{tr}(D_\alpha\tilde{P}^\alpha)\Bigg]\Bigg(- \frac{1}{2}\tilde{F}^c{}_{\gamma\delta}\tilde{F}^{d\gamma\delta} - 4\tilde{P}_{\gamma e}{}^{c}\tilde{P}^{\gamma ed} \\
&+4\tilde{P}^{\gamma cd}\mathrm{tr}\tilde{P}_\gamma + 4(D-4)\tilde{P}^{\gamma cd}\partial_\gamma\varphi\Bigg) \\
&+\frac{1}{(D-2)}\Bigg[\mathrm{tr}(D_\alpha\tilde{P}^\alpha) - \frac{(D-2)}{4(D-n-2)}\tilde{F}^2\Bigg]\Bigg(\frac{(D-3)}{2}\tilde{F}^2 + 2(D-4)\tilde{P}^2 - 2(D-4)(\mathrm{tr}\tilde{P})^2 \\
&-4(D-3)(D-4)\mathrm{tr}\tilde{P}_\beta\partial^\beta\varphi - 2(D-2)(D-3)(D-4)(\partial\varphi)^2\Bigg) \\
&+2(D-4)\Bigg[\Box\varphi + \frac{1}{4(D-n-2)}\tilde{F}^2\Bigg]\Bigg(\frac{1}{4}\tilde{F}^2 + \tilde{P}^2 - (\mathrm{tr}\tilde{P})^2 - 2(D-3)\mathrm{tr}\tilde{P}_\beta\partial^\beta\varphi \\
&-(D-2)(D-3)(\partial\varphi)^2\Bigg)\Bigg\} \\
&+\mathcal{L}_{TD}, \qquad\qquad (\mathrm{A.22})
\end{aligned}
$$

where the last term, $\mathcal{L}_{TD}$, is a total derivative which is given explicitly by

$$\mathcal{L}_{TD} = \sqrt{|g|}D^\alpha\Big\{D_\alpha\Big[\tilde{e}_{\rm int}e^{(D-4)\varphi}\Big(\frac{1}{2}\tilde{F}^2+2\tilde{P}^2-2({\rm tr}\tilde{P})^2\Big)\Big]+\tilde{e}_{\rm int}e^{(D-4)\varphi}\Big[$$
$$+8\Big[R_{\alpha\beta}-\frac{1}{2}\eta_{\alpha\beta}R-\frac{1}{2}\tilde{F}_{c\delta\alpha}\tilde{F}^{c\delta}{}_\beta+\frac{1}{8}\eta_{\alpha\beta}\tilde{F}^2-{\rm tr}(\tilde{P}_\alpha\tilde{P}_\beta)+\frac{1}{2}\eta_{\alpha\beta}\tilde{P}^2+(D-2)\partial_\alpha\varphi\partial_\beta\varphi$$
$$-\frac{(D-2)}{2}\eta_{\alpha\beta}(\partial\varphi)^2\Big]{\rm tr}\tilde{P}^\beta-4\Big[D_\beta\tilde{P}^{\beta cd}-\frac{1}{4}\tilde{F}^c{}_{\beta\gamma}\tilde{F}^{d\beta\gamma}-\frac{1}{(D-2)}\delta^{cd}{\rm tr}(D_\beta\tilde{P}^\beta)\Big]\tilde{P}_{\alpha cd}$$
$$-2[D_\gamma\tilde{F}^{c\delta\gamma}+\tilde{P}_\gamma{}^{ce}\tilde{F}_e{}^{\delta\gamma}]\tilde{F}_{c\delta\alpha}+4\frac{(D-3)}{(D-2)}\Big[{\rm tr}(D_\beta\tilde{P}^\beta)-\frac{(D-2)}{4(D-n-2)}\tilde{F}^2\Big]{\rm tr}\tilde{P}_\alpha$$
$$+\frac{1}{2}\tilde{F}_{c\beta\gamma}\tilde{P}_\alpha{}^{cd}\tilde{F}_d{}^{\beta\gamma}+2\tilde{F}_{c\beta\alpha}\tilde{P}_\gamma{}^{cd}\tilde{F}_d{}^{\beta\gamma}+\frac{(n-1)}{(D-n-2)}\tilde{F}^2{\rm tr}\tilde{P}_\alpha-(D-4)\tilde{F}^2\partial_\alpha\varphi$$
$$+4{\rm tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\beta)+4(({\rm tr}\tilde{P})^2-{\rm tr}(\tilde{P}_\beta\tilde{P}^\beta))({\rm tr}\tilde{P}_\alpha+(D-4)\partial_\alpha\varphi)$$
$$+(D^2-9D+16)(4\partial_\alpha\varphi{\rm tr}\tilde{P}_\beta-2{\rm tr}\tilde{P}_\alpha\partial_\beta\varphi)\partial^\beta\varphi\Big]\Big\}. \qquad (A.23)$$

All terms are thus grouped according to equations of motion and total derivatives. The first two square parenthesis in eq. (A.22) — containing terms involving only the Riemann tensor and $\tilde{F}$ — will vanish identically when compactifying to three dimensions.

Varying the compactified Einstein-Hilbert action, $\mathcal{L}_{\rm EH}=\hat{e}\hat{R}$, the tree-level equations of motion are found to be

$$\begin{aligned}0 &= R_{\alpha\beta}-\tilde{P}_{\alpha cd}\tilde{P}_\beta{}^{cd}-\frac{1}{2}\tilde{F}_{c\alpha\delta}\tilde{F}^c{}_\beta{}^\delta+\frac{1}{4(D-n-2)}\eta_{\alpha\beta}\tilde{F}^2+(D-2)\partial_\alpha\varphi\partial_\beta\varphi,\\ 0 &= D_\gamma\tilde{F}_a{}^{\beta\gamma}+\tilde{P}_{\gamma ad}\tilde{F}^{d\beta\gamma},\\ 0 &= D_\gamma\tilde{P}^\gamma{}_{ab}-\frac{1}{4}\tilde{F}_{a\gamma\delta}\tilde{F}_b{}^{\gamma\delta}-\frac{1}{4(D-n-2)}\delta_{ab}\tilde{F}^2.\end{aligned} \qquad (A.24)$$

Notice that tracing the last equation in eq. (A.24), one finds $\Box\varphi+\frac{1}{4(D-n-2)}\tilde{F}^2=0$ for the dilatons. Except for the equations of motion, the fields will also obey the Bianchi identities

$$\nabla_{[a}\tilde{F}^m{}_{\beta\gamma]}=0, \qquad (A.25)$$

and the Maurer-Cartan equations

$$\begin{aligned}0 &= D_{[\alpha}\tilde{P}_{\beta]cd},\\ 0 &= \nabla_{[\alpha}Q_{\beta]cd}-Q_{[\alpha|c|}{}^eQ_{\beta]de}+\tilde{P}_{[\alpha|c|}{}^e\tilde{P}_{\beta]de}.\end{aligned} \qquad (A.26)$$

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Paper V

Instanton Corrections to the Universal Hypermultiplet and Automorphic Forms on $SU(2,1)$

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Abstract: The tree-level hypermultiplet moduli space in Type IIA string theory compactified on a rigid Calabi-Yau threefold is the symmetric space $SU(2,1)/SU(2) \times U(1)$. An outstanding problem is to determine the exact metric on this moduli space after including all perturbative or non-perturbative quantum corrections. To gain ground on this issue, we posit that quantum corrections preserve a discrete subgroup of the continuous isometry group given by the Picard modular group $SU(2,1;\mathbb{Z}[i])$. Based on this assumption, we construct an $SU(2,1;\mathbb{Z}[i])$-invariant, non-holomorphic Eisenstein series, and show that its non-Abelian Fourier coefficients have the expected form for instanton corrections due to Euclidean D2-branes wrapping the canonical 3-cycles $\mathcal{A}$ and $\mathcal{B}$, as well as Euclidean NS5-branes wrapping the entire Calabi-Yau manifold. We relate our results to recent advances in understanding deformations of quaternionic-Kähler manifolds $\mathcal{M}$, by uplifting to the twistor space $\mathcal{Z}_{\mathcal{M}}$, a $\mathbb{C}P^1$ bundle over $\mathcal{M}$. In particular, we conjecture the non-perturbative completion of the contact potential $e^{\Phi(x^\mu,z)}$ at the north pole $z=0$ of the twistor space $\mathcal{Z}_{\mathcal{M}_{\rm UH}}$.

Keywords: Discrete and Finite Symmetries, String Duality, Instantons.

Contents

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1. Introduction and Summary

String theory compactified on a compact manifold $\mathcal{X}$ typically leads to a low energy effective action with an often large number of massless scalar fields valued in a moduli space $\mathcal{M}$. In general, the Riemannian metric on $\mathcal{M}$ is deformed by perturbative and non-perturbative quantum corrections, making it very difficult to determine the exact form of the quantum effective action. In this paper we study the particular case of compactifications of type IIA string theory on a rigid Calabi-Yau threefold $\mathcal{X}$ (i.e. with Betti number $h_{2,1}(\mathcal{X}) = 0$). In this case, the hypermultiplet part of the moduli space $\mathcal{M}$ is known to be described, at tree-level in the string perturbative expansion, by the symmetric space $\mathcal{M}_{\text{UH}} = SU(2,1)/(SU(2) \times U(1))$. We analyze the quantum corrections to this classical geometry, and in particular conjecture the form of D2-brane and NS5-brane instanton contributions. Before entering into details, we begin by discussing some of the ideas leading up to our proposal.

1.1 Rigid Moduli Spaces for $\mathcal{N} \geq 4$ and Eisenstein series

For compactifications preserving $\mathcal{N} \geq 4$ supersymmetry in $D = 4$, the moduli space is always locally a symmetric space $\mathcal{M} = G/K$, with G being a global symmetry and K, the maximal compact subgroup of G, being a local R-symmetry. In particular, $\mathcal{M}$ has restricted holonomy group K and is rigid (see, e.g., [1] for a nice discussion). Quantum corrections are encoded in the global structure of $\mathcal{M}$, given by a double coset

$$\mathcal{M}_{\text{exact}} = G(\mathbb{Z})\backslash G/K \tag{1.1}$$

where $G(\mathbb{Z})$ is typically an arithmetic subgroup of G, known as the S-, T- or U-duality group, depending on the context [2–4]. For example, M-theory compactified on T^7 (or type IIA/B on T^6), gives rise to $\mathcal{N} = 8$ supergravity in four dimensions, whose exact moduli space is conjectured to be $E_{7(7)}(\mathbb{Z})\backslash E_{7(7)}/(SU(8)/\mathbb{Z}_2)$ [2]. In such cases, the quantum effective action is expected to be invariant under $G(\mathbb{Z})$, which gives a powerful constraint on possible quantum corrections.

This idea was exploited with great success in the seminal work [5] in the context of type IIB supergravity in ten dimensions, where the higher-derivative $\mathcal{R}^4$-type corrections were proposed to be given by a non-holomorphic Eisenstein series $\mathcal{E}^{SL(2,\mathbb{Z})}_{3/2}(\tau,\bar{\tau})$ as a function of the "axio-dilaton" $\tau = C_{(0)} + ie^{-\phi}$ valued on the Poincaré upper half plane

$\mathcal{M} = SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$. This proposal reproduced the known tree-level and one-loop corrections [6, 7], predicted the absence of higher loop corrections, later verified by an explicit two-loop computation [8], and suggested the exact form of D(-1)-instanton contributions, later corroborated by explicit matrix model computations [9, 10]. From the mathematical point of view, perturbative corrections and instanton contributions correspond to the constant terms and Fourier coefficients of the automorphic form $\mathcal{E}_{3/2}^{SL(2,\mathbb{Z})}$. This work was extended for toroidal compactifications of M-theory on T^n, where the R^4-type corrections were argued to be given by Eisenstein series of the respective U-duality group [11–13], predicting the contributions of Euclidean Dp-brane instantons, and, when $n \geq 6$, NS5-branes. Unfortunately, extracting the constant terms and Fourier coefficients of Eisenstein series is not an easy task, and it has been difficult to put the conjecture to the test. Part of our motivation is to develop the understanding of Eisenstein series beyond the relatively well understood case of $G(\mathbb{Z}) = SL(n,\mathbb{Z})$.

1.2 The Hypermultiplet Moduli Space of $\mathcal{N} = 2$ Supergravity

Compactifications with fewer unbroken supersymmetries ($\mathcal{N} \leq 2$ in $D = 4$) lead to moduli spaces which are generically not symmetric spaces. An interesting example, which we will be mainly concerned with in this paper, is type IIA string theory compactified on a Calabi-Yau threefold X, leading to $\mathcal{N} = 2$ supergravity in four dimensions coupled to $h_{1,1}$ vector multiplets and $h_{2,1} + 1$ hypermultiplets. The moduli space locally splits into a direct product $\mathcal{M} = \mathcal{M}_{\rm V} \times \mathcal{M}_{\rm H}$, where $\mathcal{M}_{\rm V}$ is a $2h_{1,1}$-dimensional special Kähler manifold, and $\mathcal{M}_{\rm H}$ a $4(h_{2,1}+1)$-dimensional quaternionic-Kähler manifold, respectively. $\mathcal{M}_{\rm V}$ encodes the (complexified) Kähler structure of X, while $\mathcal{M}_{\rm H}$ encodes deformations of the complex structure. $\mathcal{M}_{\rm V}$ is exact at tree-level in the perturbative string expansion, and well understood thanks to classical mirror symmetry (see e.g. [14] for an extensive introduction). In this paper we focus on the less understood hypermultiplet moduli space $\mathcal{M}_{\rm H}$. Note however that upon further compactification on a circle, $\mathcal{M}_{\rm V}$ is extended to a $4(h_{1,1} + 1)$-dimensional quaternionic-Kähler manifold by the c-map, and the vector and hypermultiplet moduli spaces become equally complicated, being exchanged under T-duality along the circle [15].

Contrary to $\mathcal{M}_{\rm V}$, the hypermultiplet moduli space $\mathcal{M}_{\rm H}$ receives perturbative and non-perturbative corrections in the string perturbative expansion [16, 18–21]. The non-perturbative corrections are due to Euclidean D2-branes wrapping special Lagrangian submanifolds in X, as well as Euclidean NS5-branes wrapping the entire Calabi-Yau threefold [16].[1] It has been an outstanding problem to understand how these effects modify the geometry of the moduli space $\mathcal{M}_{\rm H}$, mainly due to the fact that quaternionic-Kähler geometry is much more complicated than special Kähler geometry. Recently, however, it has become apparent that twistor techniques can be efficiently applied to quaternionic-Kähler geometry. In particular, deformations of the quaternionic-Kähler geometry of $\mathcal{M}_{\rm H}$ are in one-to-one correspondance with deformations of its twistor space $\mathcal{Z}_{\mathcal{M}_{\rm H}}$, a $\mathbb{C}P^1$ bundle over $\mathcal{M}_{\rm H}$ [22–24] (see [25–29] for a physics realization of this equivalence). One virtue of this approach is that, contrary to $\mathcal{M}_{\rm H}$, the twistor space $\mathcal{Z}_{\mathcal{M}_{\rm H}}$ is Kähler, and therefore quantum

[1] See [17] for a recent analysis of these effects in the heterotic framework.

corrections to $\mathcal{M}_{\rm H}$ can in principle be described in terms of the Kähler potential on its twistor space $\mathcal{Z}_{\mathcal{M}_{\rm H}}$. Furthermore, $\mathcal{Z}_{\mathcal{M}_{\rm H}}$ being a complex contact manifold, it can be described by holomorphic data, corresponding to the complex symplectomorphisms between complex Darboux coordinate patches.

Using these techniques, much headway has been made in summing up part of the instanton corrections to hypermultiplet moduli spaces in both type IIA and IIB string theory [30–32, 28, 29, 33, 34]. These techniques were combined with the $SL(2,\mathbb{Z})$-invariance of the four-dimensional effective action in [30], to obtain the quantum corrections to geometry of $\mathcal{M}_{\rm H}^{\rm IIB}$ due to D(-1), F1 and D1 instantons[2]. In this context, the Eisenstein series $\mathcal{E}_{3/2}^{SL(2,\mathbb{Z})}(\tau,\bar{\tau})$, discussed above, reappears as the D(-1) instanton contribution to the Kähler potential on the twistor space of $\mathcal{M}_{\rm H}^{\rm IIB}$. This result was then mapped over to the IIA side using mirror symmetry [31], resulting in a description of the corrected geometry of $\mathcal{M}_{\rm H}$ due to Euclidean D2-branes wrapping A-cycles in X. Subsequently, in [28], also the contribution from D2-branes wrapping B-cycles was obtained by covariantizing the result of [31] under "electric-magnetic duality" between A- and B-cycles in the Calabi-Yau. By the T-duality argument mentioned above, this also provides the contributions of 4D BPS black holes to the vector multiplet moduli space in type IIA or IIB string theory compactified on $\mathcal{X}\times S_1$. However, the NS5-brane contributions (or the Kaluza-Klein monopole contributions on the vector multiplet side) have so far proven to be considerably more elusive, although they can be in principle reached following the "roadmap" proposed in [31]. By postulating invariance under a larger discrete group $SL(3,\mathbb{Z})$, a subset of the NS5-brane contributions corresponding to the "extended universal hypermultiplet" was conjectured in [35]. This analysis (and presumably also the analysis in [30]) breaks down for rigid Calabi-Yau threefolds, the sector which we address in this work.

1.3 Rigid Calabi-Yau threefolds and the Picard Modular Group

In the present paper, we study the hypermultiplet moduli space $\mathcal{M}_{\rm H}$ in a restricted setting, namely for type IIA string theory compactified on a *rigid* Calabi-Yau threefold $\mathcal{X}$ i.e. such $h_{2,1}(\mathcal{X})=0$. By the T-duality argument indicated above, our analysis applies equally to the vector multiplet moduli space in type IIB string theory compactified on $\mathcal{X}\times S_1$. Rigid Calabi-Yau threefolds are rather rare, but examples can be found in the mathematics (see, e.g., [36]) and the physics literature (see, e.g., [37–40]). One of their peculiarities is that they do not admit a mirror Calabi-Yau threefold in the usual sense, since $h_{1,1}(\mathcal{X})$ is always greater than one[3]. Thus, it is no longer clear that $\mathcal{M}_{\rm H}$ should admit an isometric action of $SL(2,\mathbb{Z})$. Moreover, rigid Calabi-Yau threefolds do not admit a K3 fibration, so heterotic/type II duality cannot be applied [41].

[2]The twistorial realization of $SL(2,\mathbb{Z})$ has been recently clarified in [34].

[3]It is possible that the superconformal field theory on $\mathcal{X}$ admits a mirror description as a Landau-Ginzburg model LG, but it is not obvious that this equivalence should extend at the non-perturbative level. Put differently, it is unclear whether type IIA on LG can still be lifted to M-theory, or whether type IIB on LG still exhibits $SL(2,\mathbb{Z})$ symmetry.

For such Calabi-Yau threefolds then, the hypermultiplet sector consists solely of the "universal hypermultiplet". It is given at tree-level by the quaternionic-Kähler[4] symmetric space $\mathcal{M}_{\mathrm{UH}}(\mathcal{X}) = SU(2,1)/\big(SU(2)\times U(1)\big)$ [15], with left-invariant metric

$$ds^2_{\mathcal{M}_{\mathrm{UH}}} = 2\Big(d\phi^2 + e^{2\phi}\big(d\chi^2 + d\tilde{\chi}^2\big) + e^{4\phi}\big(d\psi + \chi d\tilde{\chi} - \tilde{\chi} d\chi\big)^2\Big), \tag{1.2}$$

where e^{ϕ} is the four-dimensional dilaton, $\chi + i\tilde{\chi}$ is the component of the ten-dimensional Ramond-Ramond 3-form $C_{(3)}$ on $H^{3,0}$, and ψ is the NS-NS axion, dual to the 2-form $B_{\mu\nu}$ in $D=4$. From the equivalent point of view of type IIB string theory on $\mathcal{X}\times S_1$, e^{ϕ} is instead the inverse radius of the circle in 4D Planck units, while $\chi + i\tilde{\chi}$ is the component of the ten-dimensional Ramond-Ramond 4-form $C_{(4)}$ on $H^{3,0}\times S^1$. The fate of the global symmetry $SU(2,1)$ when higher derivative corrections in $D=4$ are included in this set up will be analyzed in a follow-up paper [42].

At this stage it should be emphasized that even though the scalar fields just mentioned occur universally in any Calabi-Yau manifold, it is not true that the "universal hypermultiplet manifold" (1.2) is a universal subsector of the hypermultiplet moduli space $\mathcal{M}_{\mathrm{H}}(X)$ when X is non rigid [1].[5] (this is in contrast to the generalized universal hypermultiplet sector introduced in [35]). However, in cases where $\mathcal{M}$ is a symmetric space, it can often be written as a fiber bundle over $\mathcal{M}_{\mathrm{UH}}$. One example is type II string theory compactified on T^7, where the moduli space can be written as the fiber bundle [15, 1]

$$\frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2} \to \frac{SU(2,1)}{SU(2)\times U(1)} \times \frac{SL(2,\mathbb{R})}{SO(2)}\,. \tag{1.3}$$

A similar decomposition occurs for very special $\mathcal{N}=2$ supergravity theories, where the second factor on the r.h.s. is replaced by a non-compact version of the 5-dimensional U-duality group REF?. Thus, it is possible that the construction in this paper generalizes beyond the case of rigid Calabi Yau threefolds.

While quantum corrections are bound to break the continuous isometric action of $G = SU(2,1)$, we posit that they preserve a discrete arithmetic subgroup $G(\mathbb{Z})$ (note however that we do not assume that $\mathcal{M}^{\mathrm{exact}}_{\mathrm{UH}}$ is a double coset $G(\mathbb{Z})\backslash G/K$). This subgroup should contain

1. A discrete Heisenberg group $N(\mathbb{Z})$, acting by discrete (Peccei-Quinn) shift symmetries on the axions $\chi, \tilde{\chi}$ and ψ [43]:

$$\begin{aligned} \chi &\longmapsto \chi + a\,, \\ \tilde{\chi} &\longmapsto \tilde{\chi} + b\,, \\ \psi &\longmapsto \psi + \frac{1}{2}c - a\tilde{\chi} + b\chi, \end{aligned} \tag{1.4}$$

   where $a, b, c \in \mathbb{Z}$, while leaving the dilaton invariant. The breaking of the continuous shifts of χ and $\tilde{\chi}$ are due to D2-brane instantons, while the breaking of the shift of ψ is due to NS5-brane instantons. The factor $1/2$ appearing in front of c is in agreement with the quantisation condition on the NS5-brane instantons derived in [43].

[4] Actually, the classical $\mathcal{M}_{\mathrm{UH}}$ is also Kähler, an exception among quaternionic-Kähler manifolds.

[5] We thank Nick Halmagyi for discussions on this issue.

2. The "electric-magnetic duality" R which interchanges the R-R scalars χ and $\tilde{\chi}$ [43]:

$$R\;:\;(\chi,\tilde{\chi})\;\longmapsto\;(-\tilde{\chi},\chi). \tag{1.5}$$

Microscopically, this corresponds to a phase shift on the holomorphic 3-form $\Omega_3 \sim dz^1 \wedge dz^2 \wedge dz^3$ of $\mathcal{X}$.

3. Finally and as the most significant hypothesis in this work, we assume that a discrete subgroup $SL(2,\mathbb{Z})$ of the four-dimensional S-duality (or, on the type IIB side, Ehlers symmetry), acting in the standard non-linear way on the complex parameter $\tau = \chi + ie^{-\phi}$ on the slice $\tilde{\chi} = \psi = 0$, is left unbroken by quantum corrections. As in earlier endeavours [44–48], it is difficult to justify this assumption rigorously, but the fact, demonstrated herein, that it leads to physically sensible results can be taken as support for this assumption.

Based on these assumptions, we show that $G(\mathbb{Z})$ must be the Picard modular group[6] $SU(2,1;\mathbb{Z}[i])$, defined as the intersection (see e.g. [49])

$$SU(2,1;\mathbb{Z}[i]) := SU(2,1)\cap SL(3,\mathbb{Z}[i]), \tag{1.6}$$

where $\mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z}$ denotes the Gaussian integers. The fact that a discrete subgroup of $SU(2,1)$ satisfying these physical requirements exists at all is already remarkable and by no means trivial.

1.4 Eisenstein series for the Picard Modular Group

Having identified (1.6) as a candidate symmetry group, we apply standard machinery to construct automorphic forms of $SU(2,1;\mathbb{Z}[i])$. For the sake of completeness, and because they mutually enlighten each other, we shall present three equivalent constructions of Eisenstein series for the Picard group:

- First, we generalize the construction in [13] of non-holomorphic Eisenstein series for real classical groups over $\mathbb{Z}$ to unitary groups over $\mathbb{Z}[i]$. This is a representation in terms of a sum over a constrained lattice and leads to the Eisenstein series

$$\mathcal{E}_s(\mathcal{K}) := \sum_{\vec{\omega}\in\mathbb{Z}[i]^3,\vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega}=0}{}^{\!\!\prime}\; \Big[\vec{\omega}^\dagger\cdot\mathcal{K}\cdot\vec{\omega}\Big]^{-s}, \tag{1.7}$$

  where $\mathcal{K}$ is a matrix parametrizing the coset space $SU(2,1)/\big(SU(2)\times U(1)\big)$, and the sum runs over non-zero triplets of Gaussian integers $\vec{\omega}$ subject to a certain quadratic constraint (3.3). The prime on the sum indicates that the point where all summation variables are equal to zero is to be excluded from the summation. We will use this notation throughout the paper.

[6] The nomenclature "Picard group" is not unique, in fact our Picard group is a member of a family of similar groups $PSU(1,n+1;\mathbb{Z}[i])$ of which the case $n=0$, corresponding to $PSL(2,\mathbb{Z}[i])$ is also often called *the* Picard group. In this paper we will always mean $SU(2,1;\mathbb{Z}[i])$ when speaking of the Picard group.

- Second, we construct the $SU(2,1;\mathbb{Z}[i])$-invariant Poincaré series

$$\mathcal{P}_s(\mathcal{Z}) := \sum_{\gamma\in N(\mathbb{Z})\backslash SU(2,1;\mathbb{Z}[i])} \mathcal{F}(\gamma\cdot\mathcal{Z})^s, \tag{1.8}$$

  where $\mathcal{F}(\mathcal{Z})$ is a function on $SU(2,1)/\big(SU(2)\times U(1)\big)$ which is manifestly invariant under the Heisenberg subgroup $N(\mathbb{Z})\subset SU(2,1;\mathbb{Z}[i])$.

- Third, in Appendix A we apply the general adelic method for constructing automorphic forms as explained [50–52] to obtain

$$\Psi(\mathcal{V}) := \sum_{(C_1,C_2)\in\mathbb{Q}[i]^2}{}^{\prime} \delta\Big(|C_1|^2 - 2\Im(C_2)\Big)\Big[\prod_{p<\infty} f_p(C_1,C_2)\Big]\rho(\mathcal{V})\cdot f_K(C_1,C_2), \tag{1.9}$$

  where $\mathcal{V}$ is a representative for the coset space $SU(2,1)/\big(SU(2)\times U(1)\big)$, ρ is a linear representation of $SU(2,1)$ in the principal continuous series, f_K is the $SU(2)\times U(1)$-invariant spherical vector from [53], f_p is its p-adic analog, and the infinite product runs over all prime numbers.

As we show in Section 3, the first two methods produce the same automorphic form, up to normalization, and in Appendix A we also verify that the same Eisenstein series can be constructed using techniques from p-adic number theory.

We work out explicitly the Fourier expansion of $\mathcal{E}_s$ in Section 4, including the constant terms and (non-)Abelian Fourier coefficients, and find that it takes the form

$$\begin{aligned}\mathcal{E}_s(\phi,\chi,\tilde{\chi},\psi) &= 4\zeta_{\mathbb{Q}[i]}(s)\left[e^{-2s\phi} + \frac{\mathfrak{Z}(2-s)}{\mathfrak{Z}(s)}e^{-2(2-s)\phi}\right]\\ &\quad + e^{-2\phi}\sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2}{}^{\prime} C^{(\mathrm{A})}_{\ell_1,\ell_2}(s)K_{2s-2}\Big(2\pi e^{-\phi}\sqrt{\ell_1^2+\ell_2^2}\Big)e^{2\pi i(\ell_1\chi+\ell_2\tilde{\chi})}\\ &\quad + \sum_{k\in\mathbb{Z}}{}^{\prime}\sum_{\ell\in\mathbb{Z}/(4k\mathbb{Z})}\sum_{n\in\mathbb{Z}+\frac{\ell}{4k}} \mathfrak{C}^{(\mathrm{NA})}_{k,\ell}(\phi,\tilde{\chi}-n;s)e^{-8\pi ikn\chi+4\pi ik(\psi+\chi\tilde{\chi})},\end{aligned} \tag{1.10}$$

where the first line represents the constant terms, corresponding to the leading terms in an expansion about the cusp $e^{\phi}\to 0$. Here $\zeta_{\mathbb{Q}[i]}(s)$ is the Dedekind zeta function and $\mathfrak{Z}(s)$ is defined in (4.60). The second line is the Abelian contribution, corresponding to an expansion with respect to the Abelian part of the Heisenberg group $N\subset SU(2,1)$. The numerical Fourier coefficients $C^{(\mathrm{A})}_{\ell_1,\ell_2}(s)$ are computed in Section 4.5 and K_s denotes the modified Bessel function. Finally, the third line is the non-Abelian contribution, representing the expansion of $\mathcal{E}_s(\phi,\chi,\tilde{\chi},\psi)$ with respect to the non-Abelian part of the Heisenberg group. The non-Abelian coefficients $\mathfrak{C}^{(\mathrm{NA})}_{k,\ell}(\phi,\tilde{\chi}-n;s)$ are discussed in more detail in Section 4.

1.5 Eisenstein Series and the Exact Universal Hypermultiplet Geometry

We propose that the $SU(2,1;\mathbb{Z}[i])$-invariant Eisenstein series $\mathcal{E}_s(\phi,\chi,\tilde{\chi},\psi)$ for $s=3/2$ gives a non-perturbative completion of the contact potential $e^{\Phi(x^\mu,z)}$ on the north pole $z=0$

of the twistor space $\mathcal{Z}_{\mathcal{M}_{\rm UH}}$ of the universal hypermultiplet. In this context, the constant terms of the Fourier expansion (1.10) represent the classical and one-loop perturbative correction to the metric on the universal hypermultiplet moduli space. The Abelian term then contains the effects from D2-brane instantons arising from charge (ℓ_1, ℓ_2) Euclidean D2-branes wrapping supersymmetric 3-cycles in the rigid Calabi-Yau manifold $\mathcal{X}$. The Abelian Fourier coefficients $C^{(\rm A)}_{\ell_1,\ell_2}(3/2)$ are related via (4.72) to the instanton measure $\mu_{3/2}(\ell_1, \ell_2)$ which counts homology classes $\ell_1\mathcal{A}+\ell_2\mathcal{B} \in H_3(\mathcal{X})$, and generalizes the familiar D(-1) instanton measure $\mu_{3/2}(N)$ of [5] which is also known to capture the effects of pure charge N A-type D2-brane instantons [31] (corresponding to $\ell_2 = 0$). The non-Abelian term also encodes the contribution from pure charge k NS5-brane instanton corrections, together with effects of bound states of D2- and NS5-brane instantons. Thus, in this special example of rigid Calabi-Yau compactifications, we are able to evade the difficulties arising due to NS5-branes in generic Calabi-Yau compactifications.

1.6 Outline

In Section 2 we give a detailed description of the group $SU(2, 1)$, the symmetric space $SU(2, 1)/(SU(2) \times U(1))$ and the Picard modular group $SU(2, 1; \mathbb{Z}[i])$. In Section 3 we use two different methods to construct an $SU(2, 1; \mathbb{Z}[i])$-invariant Eisenstein series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$ in the principal continuous series of $SU(2, 1)$. We then proceed in Section 4 to compute in detail the Fourier expansion of $\mathcal{E}_s$, extracting explicit forms for the constant terms as well as the Abelian and non-Abelian Fourier coefficients. Finally, in Section 5, we use the automorphic form $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$ at order $s = 3/2$ to conjecture the exact form of the D2-brane and NS5-brane instanton corrections to the universal hypermultiplet moduli space $\mathcal{M}_{\rm UH}$. For completeness, we give in Appendix A a third construction of the Eisenstein series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$ using results from p-adic number theory. This construction generalizes the analysis of [53] to the automorphic setting. In Appendix B we also give some number-theoretic details on the derivation of the Abelian instanton measure.

2. On the Picard Modular Group $SU(2, 1; \mathbb{Z}[i])$

As indicated in the introduction, an important role in this paper is played by the symmetric space $SU(2, 1)/(SU(2)\times U(1))$. This space describes the tree-level geometry of the universal hypermultiplet in type IIA string theory compactified on a rigid Calabi-Yau threefold, and also as the scalar coset manifold of the Einstein–Maxwell system when dimensionally reduced from $D = 4$ to $D = 3$ dimensions on a spacelike circle. In this section, we set up notations for the group $SU(2, 1)$, give two equivalent descriptions of the symmetric space $SU(2, 1)/(SU(2) \times U(1))$, and introduce the Picard modular group $SU(2, 1; \mathbb{Z}[i])$.

2.1 The Group $SU(2, 1)$ and its Lie Algebra $\mathfrak{su}(2, 1)$

The Lie group $SU(2, 1)$ is defined as a subgroup of the group $GL(3, \mathbb{C})$ of invertible (3×3) complex matrices via

$$SU(2,1) = \Big\{ g \in GL(3,\mathbb{C}) \,:\, g^\dagger \eta g = \eta \;\text{ and } \det(g) = 1 \Big\} \,. \tag{2.1}$$

Here, the defining metric η is given by

$$\eta = \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{2.2}$$

and has signature $(++-)$. We note that the condition $g^\dagger \eta g = \eta$ already implies $|\det(g)| = 1$ and so we can also think of $SU(2,1)$ as the set of unitary matrices $U(2,1)$ modulo a pure phase, $SU(2,1) \cong PU(2,1)$, with the projectivization P referring to the equivalence relation $g \sim g e^{i\alpha}$ for $\alpha \in [0, 2\pi)$. The diagonal matrices $e^{i\alpha}\mathrm{diag}(1,1,1)$ form the center of the group $U(2,1)$.

The Lie group $SU(2,1)$ as defined in (2.1) has as Lie algebra of real dimension 8

$$\mathfrak{su}(2,1) = \left\{ X \in \mathfrak{gl}(3,\mathbb{C}) \,:\, X^\dagger \eta + \eta X = 0 \text{ and } \mathrm{tr}(X) = 0 \right\} . \tag{2.3}$$

It consists of four compact and four non-compact generators, the maximal real torus is one-dimensional. Since we will have ample opportunity to refer to specific generators we define the non-compact and compact Cartan generators

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad J = \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix} , \tag{2.4}$$

the positive step operators

$$X_1 = \begin{pmatrix} 0 & -1+i & 0 \\ 0 & 0 & 1-i \\ 0 & 0 & 0 \end{pmatrix} , \quad \tilde{X}_1 = \begin{pmatrix} 0 & 1+i & 0 \\ 0 & 0 & 1+i \\ 0 & 0 & 0 \end{pmatrix} , \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{2.5}$$

and the negative step operators

$$Y_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1+i & 0 & 0 \\ 0 & -1-i & 0 \end{pmatrix} , \quad \tilde{Y}_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -1+i & 0 & 0 \\ 0 & -1+i & 0 \end{pmatrix} , \quad Y_{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} . \tag{2.6}$$

The subscript refers to the eigenvalue under the adjoint action of the non-compact Cartan generator H, e.g. $[H, X_1] = X_1$ — the adjoint action of the compact Cartan generator J is not diagonalisable over the real numbers. Furthermore, the generators satisfy

$$\left[X_1, \tilde{X}_1\right] = -4X_2 \,, \tag{2.7}$$

such that the positive step operators form a Heisenberg algebra. Furthermore, the negative step operators Y are minus the Hermitian conjugate of the positive step operator X.

The Lie algebra $\mathfrak{su}(2,1)$ has a natural five grading by the generator H as a direct sum of vector spaces

$$\mathfrak{su}(2,1) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \,, \tag{2.8}$$

with

$$\mathfrak{g}_{-2} = \mathbb{R}Y_{-2}\,,\ \mathfrak{g}_{-1} = \mathbb{R}Y_{-1} \oplus \mathbb{R}\tilde{Y}_{-1}\,,\ \mathfrak{g}_0 = \mathbb{R}H \oplus \mathbb{R}J\,,\ \mathfrak{g}_1 = \mathbb{R}X_1 \oplus \mathbb{R}\tilde{X}_1\,,\ \mathfrak{g}_2 = \mathbb{R}X_2\,. \tag{2.9}$$

One sees that the H-eigenspaces with eigenvalue ± 1 are degenerate. This is a characteristic feature of the reduced root system BC_1 underlying the real form $\mathfrak{su}(2,1)$ of $\mathfrak{sl}(3,\mathbb{C})$. There is a single root α since the real rank of $\mathfrak{su}(2,1)$ is one, and there are non-trivial root spaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$ corresponding to α and 2α, respectively.[7] The $\mathfrak{sl}(2,\mathbb{R})$ subalgebra associated with the 2α root space is canonically normalised and can be given a standard basis for example with H, $E = X_2$ and $F = -Y_{-2}$, so that $[E,F] = H$. The corresponding $SL(2,\mathbb{R})$ subgroup of $SU(2,1)$ is given by matrices of the form

$$\left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \ :\ a,b,c,d \in \mathbb{R} \text{ and } ad - bc = 1 \right\} \subset SU(2,1)\,. \tag{2.10}$$

Under this embedding, the fundamental representation of $SU(2,1)$ decomposes as $3 = 2 \oplus 1$. There exists a second, non regular embedding of $SL(2,\mathbb{R})$ inside $SU(2,1)$, consisting of matrices of the form

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a^2 & (-1+i)ab & ib^2 \\ (-1-i)ac & ad+bc & (1-i)bd \\ -ic^2 & (1+i)cd & d^2 \end{pmatrix} \ :\ a,b,c,d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}. \tag{2.11}$$

Under this embedding, the fundamental representation of $SU(2,1)$ remains irreducible. The two subgroups (2.10) and (2.11) together generate the whole of $SU(2,1)$.

The Iwasawa decomposition of the Lie algebra $\mathfrak{su}(2,1)$ reads

$$\mathfrak{su}(2,1) = \mathfrak{n}_+ \oplus \mathfrak{a} \oplus \mathfrak{k}, \tag{2.12}$$

where the non-compact (Abelian) Cartan subalgebra $\mathfrak{a} = \mathbb{R}H$ while the nilpotent subspace $\mathfrak{n}_+ = \mathbb{R}X_1 \oplus \mathbb{R}\tilde{X}_1 \oplus \mathbb{R}X_2$ is spanned by the positive step operators. The compact subalgebra of $\mathfrak{su}(2,1)$ is $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ as a direct sum of Lie algebras.[8] The generators of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ are given explicitly by the anti-Hermitian matrices

$$\begin{aligned} \hat{K}_1 &= \frac{1}{4}\left(X_1 + Y_{-1}\right)\,, \quad \hat{K}_2 = \frac{1}{4}\left(\tilde{X}_1 + \tilde{Y}_{-1}\right)\,, \quad \hat{K}_3 = \frac{1}{4}\left(X_2 + Y_{-2} + J\right)\,, \\ \hat{J} &= \frac{3}{4}\left(X_2 + Y_{-2}\right) - \frac{1}{4}J\,. \end{aligned} \tag{2.13}$$

These satisfy $\left[\hat{J}, \hat{K}_i\right] = 0$ and $\left[\hat{K}_i, \hat{K}_j\right] = -\epsilon_{ijk}\hat{K}_k$. The Weyl group of the reduced root system BC_1 is

$$\mathcal{W}(\mathfrak{su}(2,1)) = \mathcal{W}(BC_1) \cong \mathbb{Z}_2\,, \tag{2.14}$$

corresponding to the Weyl reflection with respect to α.

[7] A discussion of the restricted root system can for example be found in [54].

[8] By contrast, the Iwasawa decomposition (2.12) is only a direct sum of vector spaces and not of Lie algebras.

2.2 Complex Hyperbolic Space

The group $SU(2,1)$ acts transitively and isometrically on the complex two-dimensional space[9]

$$\mathbb{CH}^2 = \left\{ \mathcal{Z} = (z_1, z_2) \in \mathbb{C}^2 \,:\, \mathcal{F}(\mathcal{Z}) > 0 \right\} , \tag{2.15}$$

equipped with the Kähler metric

$$ds^2 = \frac{1}{4}\mathcal{F}^{-2}\left[dz_1 d\bar{z}_1 + iz_2 dz_1 d\bar{z}_2 - i\bar{z}_2 dz_2 d\bar{z}_1 + 2\Im(z_1) dz_2 d\bar{z}_2\right] . \tag{2.16}$$

The "height function" $\mathcal{F} : \mathbb{C}^2 \to \mathbb{R}$ is defined by

$$\mathcal{F}(\mathcal{Z}) := \Im(z_1) - \frac{1}{2}|z_2|^2 > 0 \,, \tag{2.17}$$

and provides a Kähler potential for the metric (2.16),

$$K_{\mathbb{CH}^2}(\mathcal{Z}) = -\log \mathcal{F}(\mathcal{Z}) \,. \tag{2.18}$$

The action of $SU(2,1)$ on $\mathcal{Z} \in \mathbb{CH}^2$ is via fractional linear transformations

$$g \cdot \mathcal{Z} = \frac{A\mathcal{Z} + B}{C\mathcal{Z} + D} \qquad \text{for} \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{2.19}$$

where the blocks A, B, C and D have the sizes (2×2), (2×1), (1×2) and (1×1), respectively, so that the denominator is a complex number. Since the height function transforms as

$$\mathcal{F}(g \cdot \mathcal{Z}) = \frac{\mathcal{F}(\mathcal{Z})}{|C\mathcal{Z} + D|^2} \,, \tag{2.20}$$

the condition $\mathcal{F}(\mathcal{Z}) > 0$ is preserved and the action is isometric. In fact, when verifying (2.20) one only requires the condition $g^\dagger \eta g = \eta$ so that (2.19) defines an action of all of $U(2,1)$ on complex hyperbolic two-space. Since elements from the center act trivially, one can restrict to $PU(2,1) \cong SU(2,1)$ to obtain a simply transitive action. We will refer to the space $\mathbb{CH}^2$ defined in (2.15) as the *complex hyperbolic space*, or the *complex upper half plane*. The slice $z_2 = 0, \Im(z_1) > 0$ inside $\mathbb{CH}^2$ is preserved the action of the $SL(2,\mathbb{R})$ subgroup in (2.10), and gives an embedding of the standard Poincaré upper half plane inside $\mathbb{CH}^2$.

2.3 Relation to the Scalar Coset Manifold $SU(2,1)/(SU(2) \times U(1))$

The complex hyperbolic upper half plane is isomorphic to (a connected component of) the Hermitian symmetric space

$$\mathbb{CH}^2 \cong SU(2,1)/(SU(2) \times U(1)) \,, \tag{2.21}$$

where the right hand side should properly be restricted to the connected component of the identity. The Hermitian symmetric space is of real dimension four and can be parametrized

[9] This is referred to as the unbounded hyperquadric model in [49].

by four real variables $\{\phi, \chi, \tilde{\chi}, \psi\}$ in triangular gauge, using the Iwasawa decomposition (2.12), as

$$\mathcal{V} = e^{\chi X_1 + \tilde{\chi}\tilde{X}_1 + 2\psi X_2} e^{-\phi H} = \begin{pmatrix} e^{-\phi} & \tilde{\chi} - \chi + i(\chi + \tilde{\chi}) & e^{\phi}\left(2\psi + i(\chi^2 + \tilde{\chi}^2)\right) \\ 0 & 1 & e^{\phi}\left(\chi + \tilde{\chi} + i(\tilde{\chi} - \chi)\right) \\ 0 & 0 & e^{\phi} \end{pmatrix} . \tag{2.22}$$

The symmetric space is a right coset in our conventions and the element $\mathcal{V}$ transforms as $\mathcal{V} \to g\mathcal{V}k^{-1}$ with $g \in SU(2,1)$ and $k \in SU(2) \times U(1)$. The four scalar fields can take arbitrary real values.

It is convenient to define the Hermitian matrix

$$\mathcal{K} = \mathcal{V}\mathcal{V}^{\dagger} \tag{2.23}$$

that transforms as $\mathcal{K} \to g\mathcal{K}g^{\dagger}$ under the action of $g \in SU(2,1)$. Explicitly, this matrix reads

$$\mathcal{K} = \begin{pmatrix} e^{-2\phi} + |\lambda|^2 + e^{2\phi}|\gamma|^2 & i\bar{\lambda} + e^{2\phi}\bar{\lambda}\gamma & e^{2\phi}\gamma \\ -i\lambda + e^{2\phi}\lambda\bar{\gamma} & 1 + e^{2\phi}|\lambda|^2 & e^{2\phi}\lambda \\ e^{2\phi}\bar{\gamma} & e^{2\phi}\bar{\lambda} & e^{2\phi} \end{pmatrix}, \tag{2.24}$$

where, for later convenience, we defined the complex variables

$$\lambda := \chi + \tilde{\chi} + i(\tilde{\chi} - \chi), \qquad \gamma := 2\psi + \frac{i}{2}|\lambda|^2. \tag{2.25}$$

From $\mathcal{K}$ one can define the metric on the symmetric space via

$$ds^2 = -\frac{1}{8}\mathrm{tr}\left(d\mathcal{K}\, d(\mathcal{K}^{-1})\right) = \frac{1}{8}\mathrm{tr}\left(\mathcal{V}^{-1}d\mathcal{V} + (\mathcal{V}^{-1}d\mathcal{V})^{\dagger}\right)^2 . \tag{2.26}$$

Working this out for the coset element (2.22) one finds the following $SU(2,1)$ invariant metric

$$ds^2 = d\phi^2 + e^{2\phi}(d\chi^2 + d\tilde{\chi}^2) + e^{4\phi}(d\psi + \chi d\tilde{\chi} - \tilde{\chi}d\chi)^2 . \tag{2.27}$$

Comparing (2.27) to (2.16) leads to the identification

$$\begin{aligned} z_1 &= 2\psi + i\left(e^{-2\phi} + \frac{1}{2}|z_2|^2\right) = 2\psi + i\left(e^{-2\phi} + \chi^2 + \tilde{\chi}^2\right) , \\ z_2 &= \chi + \tilde{\chi} + i(\tilde{\chi} - \chi) . \end{aligned} \tag{2.28}$$

Note that $z_1 = \gamma + ie^{-2\phi}, z_2 = \lambda$, and the condition $\mathcal{F}(\mathcal{Z}) > 0$ is automatically satisfied. In the variables $\mathcal{Z} = (z_1, z_2)$ given by (2.28), the matrix $\mathcal{K}$ of (2.23) takes the simple form

$$\mathcal{K} = \tilde{\mathcal{K}} + \eta , \tag{2.29}$$

where η is the defining matrix of $SU(2,1)$ given in (2.2) and

$$\tilde{\mathcal{K}} = e^{2\phi}\begin{pmatrix} |z_1|^2 & z_1\bar{z}_2 & z_1 \\ \bar{z}_1 z_2 & |z_2|^2 & z_2 \\ \bar{z}_1 & \bar{z}_2 & 1 \end{pmatrix} , \tag{2.30}$$

bearing in mind that $e^{2\phi} = \mathcal{F}(\mathcal{Z})^{-1}$. The relations (2.28) together with (2.19) allow to determine the action of an element of $SU(2,1)$ in the real coordinates $\phi, \chi, \tilde{\chi}, \psi$. In particular, one may check that on the slice $\tilde{\chi} = \psi = 0$, the $SL(2,\mathbb{R})$ subgroup (2.11) acts by fractional linear transformations on the complex modulus $\tau = \chi + ie^{-\phi}$. This action is the remnant of the $SL(2,\mathbb{R})$ S-duality in ten-dimensional type IIB string theory.

In Section 3 it will prove convenient to use the complex variable γ rather than z_1.

2.4 Coset Transformations and Subgroups of $SU(2,1)$

We now study the effect of some particular elements of $SU(2,1)$ on complex hyperbolic two-space. The specific transformations we investigate are the ones with an immediate physical interpretation.

Heisenberg Translations

Let N denote the exponential of the nilpotent algebra of positive step operators $\mathfrak{n}_+$. We define the following elements of N

$$T_1 = \begin{pmatrix} 1 & -1+i & i \\ 0 & 1 & 1-i \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} 1 & 1+i & i \\ 0 & 1 & 1+i \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.31}$$

These are such that $T_1 = \exp(X_1)$ etc. Any element $n \in N$ can be written as

$$\begin{aligned} n &= (T_1)^a(\tilde{T}_1)^b(T_2)^{c+2ab} = e^{aX_1 + b\tilde{X}_1 + cX_2} \\ &= \begin{pmatrix} 1 & a(-1+i)+b(1+i) & c+i(a^2+b^2) \\ 0 & 1 & a(1-i)+b(1+i) \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{2.32}$$

for $a, b, c \in \mathbb{R}$. The effect of this transformation on $\mathcal{Z} = (z_1, z_2)$ is

$$\begin{aligned} z_1 &\longmapsto z_1 + \big[a(-1+i)+b(1+i)\big]z_2 + c + i(a^2+b^2)\,, \\ z_2 &\longmapsto z_2 + a(1-i) + b(1+i)\,, \end{aligned} \tag{2.33}$$

or in terms of the four scalars fields of (2.22)

$$\begin{aligned} \phi &\longmapsto \phi\,, \\ \chi &\longmapsto \chi + a\,, \\ \tilde{\chi} &\longmapsto \tilde{\chi} + b\,, \\ \psi &\longmapsto \psi + \frac{1}{2}c - a\tilde{\chi} + b\chi\,. \end{aligned} \tag{2.34}$$

The appearance of the shift parameters a and b in the transformation of ψ is due to the non-Abelian structure of $\mathfrak{n}_+$ given by the Heisenberg algebra (2.7). This effect is also evident in the first line of the expression (2.32) for the general element of N. From the point of view of the coset, the Heisenberg translations do not require any compensating transformation as they preserve the Borel gauge.

Rotations

Rotations are generated by the compact Cartan element J of $\mathfrak{su}(2,1)$ given in (2.4). Let

$$R = \exp(\pi J/2) = \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}, \tag{2.35}$$

then the most general transformation of this type is given by R^σ, for $\sigma \in [0,4)$, and acts on $\mathcal{Z} = (z_1, z_2)$ via

$$z_1 \to z_1 , \quad z_2 \to e^{i\pi\sigma/2} z_2 . \tag{2.36}$$

In terms of the four scalar fields this transformation reads

$$\begin{aligned}
\phi &\longmapsto \phi , \\
\chi &\longmapsto \cos(\pi\sigma/2)\chi - \sin(\pi\sigma/2)\tilde{\chi} , \\
\tilde{\chi} &\longmapsto \sin(\pi\sigma/2)\chi + \cos(\pi\sigma/2)\tilde{\chi} , \\
\psi &\longmapsto \psi
\end{aligned} \tag{2.37}$$

and so rotates the two scalars χ and $\tilde{\chi}$ among each other while leaving the other two invariant. The compensating transformation to restore the Borel gauge for the coset element (2.22) is $k = R^\sigma$.

Involution

The last transformation of interest is the following involution

$$S = \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tag{2.38}$$

which acts on $\mathcal{Z} = (z_1, z_2)$ according to

$$z_1 \mapsto -\frac{1}{z_1} , \quad z_2 \mapsto -i\frac{z_2}{z_1} , \tag{2.39}$$

corresponding to the non-trivial generator in the Weyl group (2.14). For the real scalars themselves we find the following transformation

$$\begin{aligned}
\phi &\longmapsto -\frac{1}{2}\ln\left[\frac{e^{-2\phi}}{4\psi^2 + [e^{-2\phi} + \chi^2 + \tilde{\chi}^2]^2}\right] , \\
\chi &\longmapsto \frac{2\psi\tilde{\chi} - (e^{-2\phi} + \chi^2 + \tilde{\chi}^2)\chi}{4\psi^2 + [e^{-2\phi} + \chi^2 + \tilde{\chi}^2]^2} , \\
\tilde{\chi} &\longmapsto \frac{2\psi\chi + (e^{-2\phi} + \chi^2 + \tilde{\chi}^2)\tilde{\chi}}{4\psi^2 + [e^{-2\phi} + \chi^2 + \tilde{\chi}^2]^2} , \\
\psi &\longmapsto -\frac{\psi}{4\psi^2 + [e^{-2\phi} + \chi^2 + \tilde{\chi}^2]^2} .
\end{aligned} \tag{2.40}$$

It is straightforward to check that the required compensating transformation in this case indeed belongs to the maximal compact subgroup $SU(2) \times U(1)$.

2.5 The Picard Modular Group

We finally discuss the Picard modular group $SU(2,1;\mathbb{Z}[i])$. This group can be defined as the intersection [49]

$$SU(2,1;\mathbb{Z}[i]) := SU(2,1) \cap SL(3,\mathbb{Z}[i]), \tag{2.41}$$

where $\mathbb{Z}[i]$ denotes the Gaussian integers

$$\mathbb{Z}[i] = \{z \in \mathbb{C} \,:\, \Re(z), \Im(z) \in \mathbb{Z}\} = \{m_1 + im_2 \,:\, m_1, m_2 \in \mathbb{Z}\}\,. \tag{2.42}$$

This definition implies that any element $g \in SU(2,1)$ which has only Gaussian integer matrix entries belongs to $SU(2,1;\mathbb{Z}[i])$. In view of the discussion of $PU(2,1) \cong SU(2,1)$ the Picard modular group can also be called $PU(2,1;\mathbb{Z}[i])$.

Let us now examine the particular $SU(2,1)$-transformations of the previous subsection to check whether they belong to the Picard group. The Heisenberg group $N \subset SU(2,1)$ contains a subgroup $N(\mathbb{Z}) := N \cap SU(2,1;\mathbb{Z}[i])$. By inspection of Eq. (2.32) we see that $N(\mathbb{Z})$ must be of the form

$$N(\mathbb{Z}) = \left\{e^{aX_1 + b\tilde{X}_1 + cX_2} \,:\, a, b, c \in \mathbb{Z}\right\}. \tag{2.43}$$

In view of (2.32), a natural set of generators for $N(\mathbb{Z})$ is given by the three matrices in (2.31) T_1, $\tilde{T}_1$ and T_2. The action of these discrete shifts are then as given in (2.34) with parameters $a, b, c \in \mathbb{Z}$. The translations (2.31) are of infinite order in the Picard modular group.

The rotation R^σ defined in (2.35) only is an element for the discrete values of the exponent $\sigma = 0, 1, 2, 3$, and R is an element of order 4 in the Picard modular group. The action of R on the scalar fields is

$$R: \quad (\chi, \tilde{\chi}) \longmapsto (-\tilde{\chi}, \chi). \tag{2.44}$$

Physically speaking, this corresponds to electric-magnetic duality, which is expected to be preserved in the quantum theory [43].

Finally, we will examine the involution S in Eq. (2.38). Clearly, the involution is an element of the Picard modular group. The involution S is of order 2 in the Picard modular group. As already noted above, the involution (2.38) corresponds to the Weyl reflection of the restricted root system BC_1 of the non-split real form $\mathfrak{su}(2,1)$. The Weyl reflection is associated with the (long) root 2α. We can also give an interpretation to the rotation R. This is a transformation that rotates within the degenerate, two-dimensional α root space, spanned by the generators X_1 and $\tilde{X}_1$.

The Picard modular group acts discontinuously on the complex hyperbolic space $\mathbb{CH}^2$. A fundamental domain for its action has been given by Francsics and Lax in [49]. Recently, they have also proven that the Picard modular group $SU(2,1;\mathbb{Z}[i])$ is generated by the translations T_1 and T_2, together with the rotation R and the involution S [55].[10]

Since the two translations T_1 and $\tilde{T}_1$ are related through "electric-magnetic duality" by $\tilde{T}_1 = RT_1R^{-1}$, one may equivalently choose either of the translations T_1 or $\tilde{T}_1$ associated

[10] We are very grateful to G. Francsics and P. Lax for communicating this result to us prior to publication.

with the α root space in the theorem. Since all three translations T_1, $\tilde{T}_1$ and T_2 will turn out to have a clear physical interpretation we present the Picard modular group as generated (non-minimally) by the following five elements:

$$T_1 = \begin{pmatrix} 1 & -1+i & i \\ 0 & 1 & 1-i \\ 0 & 0 & 1 \end{pmatrix}, \tilde{T}_1 = \begin{pmatrix} 1 & 1+i & i \\ 0 & 1 & 1+i \\ 0 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$R = \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix}. \tag{2.45}$$

In accordance with the $SL(2,\mathbb{R})$ subgroup identified in (2.10) we note that there is an $SL(2,\mathbb{Z}) \subset SU(2,1;\mathbb{Z}[i])$ that acts on the slice $z_2 = 0$ of complex hyperbolic space as the usual modular group on the remaining variable z_1.

3. Eisenstein Series for the Picard Modular Group

In this section we shall construct Eisenstein series for the Picard modular group in the principal continuous series representation of $SU(2,1)$. We shall give three different constructions, which despite being equivalent mutually enlighten each other. In Section 3.1 we construct a manifestly $SU(2,1;\mathbb{Z}[i])$-invariant function on $SU(2,1)/(SU(2)\times U(1))$ by summing over points in the three-dimensional Gaussian lattice $\mathbb{Z}[i]^3$. This produces a non-holomorphic Eisenstein series $\mathcal{E}_s$, parametrized by s, which will be the central object of study in the remainder of this paper. In Section 3.2, we use the isomorphism between the coset space $SU(2,1)/(SU(2)\times U(1))$ and the complex upper half plane $\mathbb{CH}^2$ to construct a Poincaré series $\mathcal{P}_s$ on $\mathbb{CH}^2$. This turns out to be the same to $\mathcal{E}_s$ up to an s-dependent Dedekind zeta function factor. For completeness, in Appendix A we give a third construction using standard adelic techniques, which illuminates the representation-theoretic nature of $\mathcal{E}_s$.

3.1 Lattice Construction and Quadratic Constraint

Following [13], a non-holomorphic function on the double quotient

$$SU(2,1;\mathbb{Z}[i])\backslash SU(2,1)/(SU(2)\times U(1)) \tag{3.1}$$

can be constructed as the Eisenstein series

$$\mathcal{E}_s(\mathcal{K}) := \sideset{}{'}\sum_{\vec{\omega}\in\mathbb{Z}[i]^3,\ \vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega}=0} \Big[\vec{\omega}^\dagger \cdot \mathcal{K} \cdot \vec{\omega}\Big]^{-s}, \qquad \vec{\omega} = \begin{pmatrix} \bar{\omega}_3 \\ \bar{\omega}_2 \\ \bar{\omega}_1 \end{pmatrix}, \tag{3.2}$$

where $\mathcal{K} = \mathcal{V}\mathcal{V}^\dagger$ is the "generalized metric" which was constructed explicitly in Eq. (2.24), and the sum runs over 3-vectors of Gaussian integers $\vec{\omega} \neq (0,0,0)$ subject to the quadratic constraint

$$\vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = |\omega_2|^2 - 2\Im(\omega_1\bar{\omega}_3) = 0 \tag{3.3}$$

and the point $\vec{\omega} = 0$ is excluded from the sum as indicated by the prime. The Eisenstein series defined in (3.2) converges absolutely for $\Re(s) > 2$. Writing out Eq. (3.2) yields[11]

$$\mathcal{E}_s(\phi,\lambda,\gamma) = \sum_{\vec{\omega}\in\mathbb{Z}[i]^3,\ \vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega}=0}^{\prime} e^{-2s\phi}\Big[|\omega_1+\omega_2\lambda+\omega_3\gamma|^2+e^{-2\phi}|\omega_2+i\omega_3\bar{\lambda}|^2+e^{-4\phi}|\omega_3|^2\Big]^{-s}. \quad (3.4)$$

The variables λ and γ were defined as functions of $\mathcal{Z} = (z_1, z_2)$ in (2.25). To explain the role the quadratic constraint (3.3), it is convenient to the isomorphism between the coset space $SU(2,1)/(SU(2)\times U(1))$ and the complex hyperbolic space $\mathbb{CH}^2$, as discussed in Section 2.3. We recall from (2.30) that in terms of the variable $\mathcal{Z} = (z_1, z_2) \in \mathbb{CH}^2$, the matrix $\mathcal{K}$ reads

$$\mathcal{K} = \tilde{\mathcal{K}} + \eta, \quad (3.5)$$

where η is the $SU(2,1)$-invariant metric, Eq. (2.2), and the matrix $\tilde{\mathcal{K}}$ is given by

$$\tilde{\mathcal{K}} = e^{2\phi}\begin{pmatrix} |z_1|^2 & z_1\bar{z}_2 & z_1 \\ \bar{z}_1 z_2 & |z_2|^2 & z_2 \\ \bar{z}_1 & \bar{z}_2 & 1 \end{pmatrix} = \tilde{\mathcal{V}}\tilde{\mathcal{V}}^\dagger \quad \text{for} \quad \tilde{\mathcal{V}} = e^{\phi}\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.6)$$

In this new parametrization, the Eisenstein series becomes

$$\mathcal{E}_s(\mathcal{Z}) = \sum_{\vec{\omega}\in\mathbb{Z}[i]^3,\ \vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega}=0}^{\prime} \Big[\vec{\omega}^\dagger\cdot\tilde{\mathcal{K}}\cdot\vec{\omega}+\vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega}\Big]^{-s} = \sum_{\vec{\omega}\in\mathbb{Z}[i]^3,\ \vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega}=0}^{\prime} e^{-2s\phi}|\omega_1+\omega_2 z_2+\omega_3 z_1|^{-2s}. \quad (3.7)$$

The constraint (3.3) can now be motivated as follows [13]. Since the coset representative $\mathcal{V} \in SU(2,1)/(SU(2)\times U(1))$ transforms in the fundamental representation $\mathcal{R}$ of $SU(2,1)$, the generalized metric $\mathcal{K} = \mathcal{V}\mathcal{V}^\dagger$ transforms in the symmetric tensor product $\mathcal{R}\otimes_s\mathcal{R}$. As reflected in (3.5), this tensor product is not irreducible. In order for $\mathcal{E}_s$ to be an eigenfunction of the Laplacian on $\mathbb{CH}^2$, it is necessary to project out the singlet component in (3.5), hence to enforce the constraint (3.3) in the sum. To be specific, the Laplacian on the coset space $\mathbb{CH}^2$, written in terms of the real variables $\{y = e^{-2\phi}, \chi, \tilde{\chi}, \psi\}$, is given by

$$\Delta_{\mathbb{CH}^2} = \frac{1}{4}y(\partial_\chi^2+\partial_{\tilde{\chi}}^2)+\frac{1}{4}(y^2+y(\chi^2+\tilde{\chi}^2))\partial_\psi^2+\frac{1}{2}y(\tilde{\chi}\partial_\chi-\chi\partial_{\tilde{\chi}})\partial_\psi+y^2\partial_y^2-y\partial_y. \quad (3.8)$$

Taking into account the quadratic constraint (3.3), it is straightforward to check that $\mathcal{E}_s$ is an eigenfunction of the Laplacian,

$$\Delta_{\mathbb{CH}^2}\mathcal{E}_s(\phi,\lambda,\gamma) = s(s-2)\,\mathcal{E}_s(\phi,\lambda,\gamma)\,. \quad (3.9)$$

Since $SU(2,1)$ admits two Casimir operators of degree 2 and 3, and since $\Delta_{\mathbb{CH}^2}$ represents the action of the quadratic Casimir on the space of (square-integrable) functions on $SU(2,1)/(SU(2)\times U(1))$, one may ask whether $\mathcal{E}_s$ is also an eigenvector of an invariant differential operator of degree 3. It turns out however, as already noticed in [53], that the representation of the cubic Casimir in the space of functions on $SU(2,1)/(SU(2)\times U(1))$

[11] We note that the same summand and constraint appear in the analysis of [56].

vanishes identically. In terms of the parametrization of the Casimir eigenvalues by the complex variables (p, q) used in [57,53], the Eisenstein series $\mathcal{E}_s$ is attached to the principal spherical representation with $p = q = s - 2$ (see Appendix A for more details).

Let us also comment on the functional dimension of the representation associated to $\mathcal{E}_s$. The summation ranges over the lattice $\mathbb{Z}[i]^3 \sim \mathbb{Z}^6$. Since both the summand and the constraint are homogeneous in $\vec{\omega}$ one can factor out an overall common Gaussian integer. Among the remaining four real integers the (real) quadratic constraint $|\omega_2|^2 - 2\Im(\omega_1\bar{\omega}_3) = 0$ eliminates one of the summation variables, so that we are effectively summing over three integers only. This is consistent with the functional dimension of the continuous series representation mentioned in the previous paragraph, and also with the expected number of instanton charges.

3.2 Poincaré Series on the Complex Upper Half Plane

In the mathematical literature, a standard way of constructing non-holomorphic Eisenstein series on a symmetric space G/K is in terms of *Poincaré series*. For the case of the coset space $SL(2,\mathbb{R})/SO(2)$, parametrized by a complex coordinate τ, such a Poincaré series is obtained by summing the function $\Im(\gamma \cdot \tau)^s$ over the orbit $\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})$, where Γ_∞ is generated by $T : \tau \mapsto \tau + 1$. This indeed produces a non-holomorphic Eisenstein series on the double quotient $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$ with eigenvalue $s(s-1)$ under the Laplacian on $SL(2,\mathbb{R})/SO(2)$ (for a very nice treatment, see [58]).

Here we would like to generalize this construction to a Poincaré series on the complex upper half plane $\mathbb{CH}^2$, parametrized by the variable $\mathcal{Z} = (z_1, z_2)$. The generalization of $\Im(\tau)$ is then given by the $N(\mathbb{Z})$-invariant function $\mathcal{F}(\mathcal{Z})$, constructed in (2.17) [59].[12] The invariance of $\mathcal{F}(\mathcal{Z})$ under $N(\mathbb{Z})$ can be checked by direct substitution of the Heisenberg translations in Eq. (2.33). As we have seen in Section 2, the Picard modular group $SU(2,1;\mathbb{Z}[i])$ acts by fractional transformations on $\mathcal{Z} \in \mathbb{CH}^2$ such that the function $\mathcal{F}(\mathcal{Z})$ transforms as

$$\mathcal{F}(\gamma \cdot \mathcal{Z}) = \frac{\mathcal{F}(\mathcal{Z})}{|C\mathcal{Z} + D|^2}, \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,1;\mathbb{Z}[i]). \tag{3.10}$$

A Poincaré series for the Picard group may now be constructed as follows

$$\mathcal{P}_s(\mathcal{Z}) := \sum_{\gamma \in N(\mathbb{Z})\backslash SU(2,1;\mathbb{Z}[i])} \mathcal{F}(\gamma \cdot \mathcal{Z})^s = \sum_{\gamma \in N(\mathbb{Z})\backslash SU(2,1;\mathbb{Z}[i])} \left(\frac{\mathcal{F}(\mathcal{Z})}{|C\mathcal{Z} + D|^2}\right)^s. \tag{3.11}$$

Taking $C \equiv (\omega_3, \omega_2) \in \mathbb{Z}[i]^2$ and $D \equiv \omega_1 \in \mathbb{Z}[i]$, and recalling $\mathcal{F}(\mathcal{Z}) = e^{-2\phi}$, then reproduces the same form of the Eisenstein series as in Eq. (3.7), i.e.

$$\mathcal{P}_s(\mathcal{Z}) = \sum_{\gamma \in N(\mathbb{Z})\backslash SU(2,1;\mathbb{Z}[i])} e^{-2s\phi}|\omega_1 + \omega_2 z_2 + \omega_3 z_1|^{-2s}. \tag{3.12}$$

The sum over orbits in $N(\mathbb{Z})\backslash SU(2,1;\mathbb{Z}[i])$ is equivalent to the sum over the Gaussian lattice $\mathbb{Z}[i]^3$ modulo the constraint $\vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = 0$, and with a coprime condition on the

[12] We are grateful to Genkai Zhang for helpful discussions on this construction.

summation variables $\vec{\omega}$ [60]:

$$\mathcal{P}_s(\mathcal{Z}) = \sum_{\substack{\vec{\omega}' \in \mathbb{Z}[i]^3,\, \vec{\omega}'^{\dagger} \cdot \eta \cdot \vec{\omega}' = 0 \\ (\omega'_1, \omega'_2, \omega'_3) = 1}}' |\omega'_1 + \omega'_2 z_2 + \omega'_3 z_1|^{-2s}. \tag{3.13}$$

Defining $\vec{\omega} = \vec{\omega}'\beta$ with $\beta = \gcd(\omega_1, \omega_2, \omega_3) \in \mathbb{Z}[i]$ and inserting this into (3.4) we then have the relation

$$\mathcal{E}_s(\phi, \lambda, \gamma) = 4\zeta_{\mathbb{Q}[i]}(s)\mathcal{P}_s(\mathcal{Z}), \tag{3.14}$$

where $\zeta_{\mathbb{Q}[i]}(s)$ is the Dedekind zeta function for the quadratic extension $\mathbb{Q}[i]$ of the rational numbers. This will be discussed in more detail in Section 4.2 (see Eq. (4.19)).

4. Fourier Expansion of $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$

In this section we shall compute the Fourier expansion of the Eisenstein series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$. We begin by describing the general structure of the expansion which depends only on the Heisenberg subgroup $N \subset SU(2,1)$, after which we use Poisson resummation to compute the explicit form of the Fourier coefficients which contain the arithmetic data of the group $SU(2,1;\mathbb{Z}[i])$.

4.1 General Considerations

We recall from Eq. (3.4) that the Eisenstein series takes the form

$$\mathcal{E}_s(\phi, \lambda, \gamma) = \sum_{\vec{\omega} \in \mathbb{Z}[i]^3,\; \vec{\omega}^{\dagger} \cdot \eta \cdot \vec{\omega} = 0}' e^{-2s\phi} \Big[|\omega_1 + \omega_2 \lambda + \omega_3 \gamma|^2 + e^{-2\phi} |\omega_2 + i\omega_3 \bar{\lambda}|^2 + e^{-4\phi} |\omega_3|^2 \Big]^{-s}, \tag{4.1}$$

where

$$\lambda = \chi + \tilde{\chi} + i(\tilde{\chi} - \chi), \qquad \gamma = 2\psi + \frac{i}{2}|\lambda|^2. \tag{4.2}$$

The summation is over the Gaussian integers

$$\omega_1 = m_1 + im_2, \qquad \omega_2 = n_1 + in_2, \qquad \omega_3 = p_1 + ip_2, \tag{4.3}$$

excluding the term $\vec{\omega} = (\omega_1, \omega_2, \omega_3) = (0,0,0)$, and subject to the constraint

$$\vec{\omega}^{\dagger} \cdot \eta \cdot \vec{\omega} = |\omega_2|^2 - 2\Im(\omega_1 \bar{\omega}_3) = n_1^2 + n_2^2 + 2m_1 p_2 - 2m_2 p_1 = 0, \tag{4.4}$$

where η is the $SU(2,1)$-invariant metric defined in Eq. (2.2). The main complication of the Fourier expansion is the fact that the nilpotent group $N \subset SU(2,1)$ is non-Abelian, as is clear from (2.7). It is isomorphic to a three-dimensional Heisenberg group, where the center $Z = [N,N]$ is parametrized by ψ. The Fourier expansion therefore splits into an *Abelian part* and a *non-Abelian* part. The Abelian term corresponds to an expansion with respect to the abelianized group N/Z, while the non-Abelian terms represent the expansion with respect to the center Z. This general structure of the Fourier expansion of automorphic forms for the Picard modular group is discussed in detail by Ishikawa [61], to which we refer the interested reader. A similar discussion may also be found in the mathematics [62, 63] and physics [35] literature for the case of automorphic forms on $SL(3,\mathbb{R})/SO(3)$.

We have seen in Section 2 that the action of $N(\mathbb{Z}) = N \cap SU(2,1;\mathbb{Z}[i])$ on $\chi, \tilde{\chi}$ and ψ is given by:

$$\begin{aligned} N : \chi &\longmapsto \chi + a, \\ \tilde{\chi} &\longmapsto \tilde{\chi} + b, \\ \psi &\longmapsto \psi + \frac{1}{2}c - a\tilde{\chi} + b\chi \end{aligned} \tag{4.5}$$

for $a, b, c \in \mathbb{Z}$. We denote an arbitrary element of the Heisenberg group $N(\mathbb{Z})$ by $\mathcal{U}_{a,b;c}$. Since the Eisenstein series (4.1) is in particular invariant under $N(\mathbb{Z})$ we can organize the Fourier expansion by diagonalizing different subgroups of the non-Abelian Heisenberg group $N(\mathbb{Z})$.

Explicitly, we write the general form of the Fourier expansion as

$$\mathcal{E}_s(\phi,\chi,\tilde{\chi},\psi) = \mathcal{E}_s^{(\mathrm{const})}(\phi) + \mathcal{E}_s^{(\mathrm{A})}(\phi,\chi,\tilde{\chi}) + \mathcal{E}_s^{(\mathrm{NA})}(\phi,\chi,\tilde{\chi},\psi), \tag{4.6}$$

where $\mathcal{E}_s^{(\mathrm{const})}(\phi)$ is the constant term and

$$\begin{aligned} \mathcal{E}_s^{(\mathrm{A})}(\phi,\chi,\tilde{\chi}) &= \sideset{}{'}\sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} \mathfrak{C}^{(\mathrm{A})}_{\ell_1,\ell_2}(\phi;s) e^{-2\pi i(\ell_1\chi+\ell_2\tilde{\chi})}, \\ \mathcal{E}_s^{(\mathrm{NA})}(\phi,\chi,\tilde{\chi},\psi) &= \sideset{}{'}\sum_{k\in\mathbb{Z}} \mathfrak{C}^{(\mathrm{NA})}_{k}(\phi,\chi,\tilde{\chi};s) e^{-4\pi i k\psi} \end{aligned} \tag{4.7}$$

are called the Abelian and non-Abelian terms, respectively. Following [61, 35], we proceed to extract an additional phase factor in the non-Abelian term which accounts for the shifts of ψ along the non-central directions. This yields the following structure of the non-Abelian term

$$\mathcal{E}_s^{(\mathrm{NA})}(\phi,\chi,\tilde{\chi},\psi) = \sideset{}{'}\sum_{k\in\mathbb{Z}} \sum_{\ell=0}^{4|k|-1} \sum_{n\in\mathbb{Z}+\frac{\ell}{4|k|}} \mathfrak{C}^{(\mathrm{NA})}_{k,\ell}(\phi,\tilde{\chi}-n;s) e^{8\pi i k n\chi - 4\pi i k(\psi+\chi\tilde{\chi})} \tag{4.8}$$

The Abelian term is manifestly invariant under shifts of the form $\mathcal{U}_{a,b;0} \in N(\mathbb{Z})/Z$. In the non-Abelian term, invariance under

$$\begin{aligned} \mathcal{U}_{1,0;0} : \chi &\longmapsto \chi + 1 \\ : \psi &\longmapsto \psi - \tilde{\chi} \end{aligned} \tag{4.9}$$

is manifest since $4kn \in \mathbb{Z}$. On the other hand, the transformation

$$\begin{aligned} \mathcal{U}_{0,1;0} : \tilde{\chi} &\longmapsto \tilde{\chi} + 1 \\ : \psi &\longmapsto \psi + \chi \end{aligned} \tag{4.10}$$

requires a compensating shift $n \mapsto n+1$ on the summation, under which the variation of the total phase cancels. Note also the restricted dependence on $\tilde{\chi}$ in the Fourier coefficient;

upon shifting $\tilde{\chi} \mapsto \tilde{\chi} + 1$ and compensating $n \mapsto n + 1$ the coefficient is indeed invariant. Finally, invariance under $\mathcal{U}_{0,0;1}$ is manifest since this gives an overall phase $e^{-4\pi ik/2} = 1$.

Note that in writing the non-Abelian term (4.8) we have made an explicit choice of *polarization*, in the sense that we have manifestly diagonalized the action of Heisenberg shifts of the restricted form $\mathcal{U}_{a,0;c}$. We could have chosen the other polarization in which we instead diagonalize the action of $\mathcal{U}_{0,b;c}$. In this case, the non-Abelian term reads

$$\mathcal{E}_s^{(\mathrm{NA})}(\phi, \chi, \tilde{\chi}, \psi) = \sum_{k\in\mathbb{Z}}{}' \sum_{\ell'=0}^{4|k|-1} \sum_{\tilde{n}\in\mathbb{Z}+\frac{\ell'}{4|k|}} \tilde{\mathfrak{C}}_{k,\ell'}^{(\mathrm{NA})}(\phi, \chi - n; s) e^{-8\pi ik\tilde{n}\tilde{\chi} - 4\pi ik(\psi - \chi\tilde{\chi})}. \tag{4.11}$$

The Fourier coefficients $\mathfrak{C}_{k,\ell}^{(\mathrm{NA})}$ and $\tilde{\mathfrak{C}}_{k,\ell'}^{(\mathrm{NA})}$ in the two different polarizations are related via a Fourier transform (see [35]). In what follows we shall for definiteness choose to work within the first polarization above.

Besides invariance under the Heisenberg group we can also use invariance under the electric-magnetic duality transformation $R : (\chi, \tilde{\chi}) \mapsto (-\tilde{\chi}, \chi)$ of (2.44). On the Abelian term this implies that the coefficient $\mathfrak{C}_{\ell_1,\ell_2}^{(\mathrm{A})}$ is invariant under $\pi/2$ rotations of (ℓ_1, ℓ_2). On the non-Abelian term (4.8) application of R leads to

$$\mathcal{E}_s^{(\mathrm{NA})}(\phi, \chi, \tilde{\chi}, \psi) = \sum_{k\in\mathbb{Z}}{}' \sum_{\ell=0}^{4|k|-1} \sum_{n\in\mathbb{Z}+\frac{\ell}{4|k|}} \mathfrak{C}_{k,\ell}^{(\mathrm{NA})}(\phi, \chi - n; s) e^{-8\pi ikn\tilde{\chi} - 4\pi ik(\psi - \chi\tilde{\chi})} \tag{4.12}$$

and hence we have $\mathfrak{C}_{k,\ell}^{(\mathrm{NA})} = \tilde{\mathfrak{C}}_{k,\ell'}^{(\mathrm{NA})}$, relating the two choices of polarization as to be expected from electric-magnetic duality. Applying R again shows that $\mathfrak{C}_{k,\ell}^{(\mathrm{NA})}(\phi, \tilde{\chi} - n; s)$ must be an *even* function of $\tilde{\chi} - n$.

Finally, we can use the Laplacian condition on the Eisenstein series $\mathcal{E}_s$ (see Eq. (3.9)) to further constrain the Fourier coefficients $\mathfrak{C}_{\ell_1,\ell_2}^{(\mathrm{A})}$ and $\mathfrak{C}_{k,\ell}^{(\mathrm{NA})}$ and determine their functional dependence on the moduli. In all cases, we require normalizability of the solution, which physically means a well-behaved 'weak-coupling' limit $e^{\phi} \to 0$. Plugging in the abelian term $\mathcal{E}_s^{(\mathrm{A})}$ into the eigenvalue equation (3.9) yields an equation for the ϕ-dependence of the coefficients which is solved by a modified Bessel function. More precisely, we find that the Abelian term in the expansion takes the form

$$\mathcal{E}_s^{(\mathrm{A})}(\phi, \chi, \tilde{\chi}, \psi) = e^{-2\phi} \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2}{}' C_{\ell_1,\ell_2}^{(\mathrm{A})}(s) K_{2s-2}\Big(2\pi e^{-\phi}\sqrt{\ell_1^2 + \ell_2^2}\Big) e^{-2\pi i(\ell_1\chi + \ell_2\tilde{\chi})}, \tag{4.13}$$

where the remaining coefficients $C_{\ell_1,\ell_2}^{(\mathrm{A})}(s)$ are now independent of ϕ and encode the arithmetic information of the group $SU(2,1;\mathbb{Z}[i])$. The precise form of these numerical coefficients will be computed in Section 4.5 below. Turning to the non-Abelian term (4.8), the Laplacian condition on the coefficient separates into a harmonic oscillator equation in the variable $x = \tilde{\chi} - n$, with solution given by a Hermite polynomial H, as well as a hypergeometric equation in the variable $y = e^{-2\phi}$ whose solution can be written in terms of a Whittaker function W. The separation of variables induces a sum over the eigenvalues of the Harmonic oscillator and we finally find the following structure of the non-Abelian term:

$$
\begin{aligned}
\mathcal{E}_s^{(\mathrm{NA})}(\phi,\chi,\tilde{\chi},\psi) = e^{-\phi} {\sum_{k\in\mathbb{Z}}}' \sum_{\ell=0}^{4|k|-1} \sum_{n\in\mathbb{Z}+\frac{\ell}{4k}} \sum_{r=0}^{\infty} C^{(\mathrm{NA})}_{r,k,\ell}(s)\, |k|^{1/2-s} e^{-4\pi|k|(\tilde{\chi}-n)^2} \\
\times H_{2r}\Big(\sqrt{8\pi|k|}(\tilde{\chi}-n)\Big) W_{-2r-\frac{1}{2},s-1}\Big(4\pi|k|e^{-2\phi}\Big) e^{8\pi i k n\chi - 4\pi i k(\psi+\chi\tilde{\chi})},
\end{aligned}
\tag{4.14}
$$

where the numerical coefficients $C^{(\mathrm{NA})}_{r,k,\ell}(s)$ will be further discussed in Section 4.6.

We shall now proceed to compute the explicit form of the Fourier expansion; that is, determine the constant term $\mathcal{E}_s^{(\mathrm{const})}$ as well as the Abelian and non-Abelian numerical Fourier coefficients $C^{(\mathrm{A})}_{\ell_1,\ell_2}(s)$ and $C^{(\mathrm{NA})}_{r,k,\ell}(s)$.

4.2 First Constant Term

The constant term[13] is defined generally as

$$
\mathcal{E}_s^{(\mathrm{const})}(\phi) = \int_0^1 d\chi \int_0^1 d\tilde{\chi} \int_0^{1/2} d\psi\ \mathcal{E}_s(\phi,\chi,\tilde{\chi},\psi),
\tag{4.15}
$$

where the integral over the NS-scalar ψ runs from 0 to 1/2 because of the extra factor of 2 in front of ψ in our parametrization of N in Eq. (2.22). Since the Cartan subgroup A appearing in the Iwasawa decomposition of $SU(2,1)$ is one-dimensional, the constant term only depends on the dilatonic scalar ϕ. Moreover, recall from the discussion in Section 2.4 that the Weyl group of $\mathfrak{su}(2,1)$ is the Weyl group of the restricted root system BC_1, which is isomorphic with $\mathbb{Z}_2$. Hence, the constant term $\mathcal{E}_s^{(\mathrm{const})}(\phi)$ consists of two contributions, $\mathcal{E}_s^{(0)}$ and $\mathcal{E}_s^{(1)}$, which are permuted by $\mathbb{Z}_2$ [64].[14]

The powers of e^{ϕ} in $\mathcal{E}_s^{(\mathrm{const})}(\phi)$ may be determined by the Laplacian condition on $\mathcal{E}_s$. In Section 3.2 we have seen that the Eisenstein series is an eigenfunction of the Laplacian $\Delta_{\mathbb{CH}^2}$ with eigenvalue $s(s-2)$. This implies that all the constant terms must individually be eigenfunctions of $\Delta_{\mathbb{CH}^2}$ with the same eigenvalue. It turns out that there is a unique solution to this, and we find that $\mathcal{E}_s^{(0)}$ must be of the form

$$
\mathcal{E}_s^{(\mathrm{const})}(\phi) = \mathcal{E}_s^{(0)} + \mathcal{E}_s^{(1)} = A(s)e^{-2s\phi} + B(s)e^{-2(2-s)\phi}.
\tag{4.16}
$$

Below we will compute the coefficients $A(s)$ and $B(s)$. The first constant term $\mathcal{E}_s^{(0)}$ corresponds to the leading order term in an expansion about the cusp $e^{\phi}\to 0$, which physically is equivalent to a weak-coupling expansion.

Our strategy for performing the Fourier expansion is to first consider the term $\omega_3 = 0$. By virtue of the constraint (4.4) we then also have $\omega_2 = 0$. The remaining sum over

[13] The terminology constant term is derived from holomorphic Eisenstein series where these terms are truly constant and independent of the scalar fields. For non-holomorphic Eisenstein series, as the one studied here, the constant terms retain a dependence on the fields corresponding to Cartan generators.

[14] We are grateful to Pierre Vanhove for helpful discussions on the constant terms.

$\omega + 1 \neq 0$ yields the first constant term $\mathcal{E}_s^{(0)}$. Then we will consider the case $\omega_3 \neq 0$ and solve the constraint (4.4) explicitly using Bézout's identity which reduces the remaining sum to that over three integers. On these we will perform Poisson resummations to uncover the second constant and the Abelian Fourier coefficients as well as the non-Abelian ones.

Therefore we start our analysis by extracting the $\omega_3 = 0$ (implying $\omega_2 = 0$) part of the sum in the Eisenstein series, $\mathcal{E}_s^{(0)}$, leaving a remainder $\mathcal{A}^{(s)}$

$$\mathcal{E}_s(\phi, \lambda, \gamma) = \mathcal{E}_s^{(0)} + \mathcal{A}^{(s)}, \tag{4.17}$$

where the leading order term is now only a sum over $\omega_1 = m_1 + im_2$

$$\mathcal{E}_s^{(0)} = e^{-2s\phi} \sum_{(m_1,m_2)\in\mathbb{Z}^2}{}' \frac{1}{(m_1^2 + m_2^2)^s} = 4\zeta_{\mathbb{Q}[i]}(s) e^{-2s\phi}\,, \tag{4.18}$$

where $\zeta_{\mathbb{Q}[i]}(s)$ is the Dedekind zeta function over the Gaussian integers

$$\zeta_{\mathbb{Q}[i]}(s) = \frac{1}{4} \sum_{\omega\in\mathbb{Z}[i]}{}' |\omega|^{-2s} = \frac{1}{4} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m^2+n^2)^s}. \tag{4.19}$$

The factor 4 is related to the center of the Gaussian integers.

The Dedekind zeta function $\zeta_{\mathbb{Q}[i]}(s)$ satisfies a functional equation that can be best described using its completed version

$$\zeta_{\mathbb{Q}[i]*}(s) := \pi^{-s}\Gamma(s)\zeta_{\mathbb{Q}[i]}(s)\,, \tag{4.20}$$

in terms of which one has

$$\zeta_{\mathbb{Q}[i]*}(1-s) = \zeta_{\mathbb{Q}[i]*}(s)\,. \tag{4.21}$$

Using results from class field theory, the Dedekind function over a quadratic number field can be written as a Dirichlet L-function times the standard Riemann zeta function. In our case this reads [65]

$$\zeta_{\mathbb{Q}[i]}(s) = \beta(s)\zeta(s), \tag{4.22}$$

where the standard Riemann zeta function is

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p\,\text{prime}} \frac{1}{1-p^{-s}} \qquad \text{for } \Re(s) > 1 \tag{4.23}$$

and $\beta(s)$ is the Dirichlet beta function,

$$\beta(s) := \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s} \qquad \text{for } \Re(s) > 0. \tag{4.24}$$

We also note that $\beta(s)$ has an Euler product representation of the form

$$\beta(s) = \prod_{p:p=1\,[4]} \frac{1}{1-p^{-s}} \prod_{p:p=3\,[4]} \frac{1}{1+p^{-s}}, \tag{4.25}$$

which together with the Euler product form of the Riemann zeta function $\zeta(s)$ above will be useful later. The functional relation for $\beta(s)$ is again best stated using its completion

$$\beta_*(s) := \left(\frac{\pi}{4}\right)^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \beta(s), \tag{4.26}$$

for which the functional relation takes the simple form

$$\beta_*(s) = \beta_*(1-s). \tag{4.27}$$

In conclusion, we have found that the first coefficient $A(s)$ in (4.16) is given by the Dedekind function $\zeta_{\mathbb{Q}[i]}(s)$ and that it is related to the term $\omega_3 = 0$ in the sum over the Gaussian integers. We will now proceed to evaluate the terms with $\omega_3 \neq 0$, contained in $\mathcal{A}^{(s)}$ of (4.17). We note that the term with $\omega_3 = 0$ and $\omega_2 \neq 0$ vanishes identically because of the quadratic constraint (4.4). Thus, $\mathcal{A}^{(s)}$ only contains terms for which $\omega_3 \neq 0$.

4.3 Solution of Constraint and Poisson Resummation

To solve the constraint (4.4) we shall make use of *Bézout's identity*, which states that for integers p_1 and p_2 the equation

$$q_1 p_2 - q_2 p_1 = d \tag{4.28}$$

has integer solutions for q_1 and q_2 if and only if d divides $\gcd(p_1, p_2)$. The most general solution can then be written as a particular solution plus appropriate integer shifts. More precisely, in the case of our constraint (4.4) we find that for $\omega_3 = p_1 + ip_2 \neq 0$ there are solutions in $\mathbb{Z}[i]^3$ if and only if

$$\frac{|\omega_2|^2}{2d} \in \mathbb{Z}\ , \qquad \text{where } d = \gcd(p_1, p_2) \tag{4.29}$$

and the most general solution for $\omega_1 = m_1 + im_2$ is then

$$\begin{aligned} m_1 &= -\frac{|\omega_2|^2}{2d} q_1 + m\frac{p_1}{d}, \\ m_2 &= -\frac{|\omega_2|^2}{2d} q_2 + m\frac{p_2}{d}. \end{aligned} \tag{4.30}$$

Here, q_1 and q_2 is any particular solution of $q_1 p_2 - q_2 p_1 = d$ and $m \in \mathbb{Z}$ is an *unconstrained* integer. Therefore, we can rewrite the constrained sum as

$$\sum_{\omega_3 \neq 0} \sum_{\substack{\omega_2 \in \mathbb{Z}[i] \\ 2d \mid |\omega_2|^2}} \sum_{m \in \mathbb{Z}} \tag{4.31}$$

where in the summand $\omega_1 = m_1 + im_2$ has to be replaced by the expression from (4.30).

Let us study this in the case of our Eisenstein series. After solving the constraint, the first term in the bracket of (4.1) becomes

$$\begin{aligned} |\omega_1 + \omega_2\lambda + \omega_3\gamma|^2 = \frac{|\omega_3|^2}{d^2}\Bigg[&\left(m - \frac{|\omega_2|^2}{2|\omega_3|^2}(q_1 p_1 + q_2 p_2) + \tilde{\ell}_1\chi + \tilde{\ell}_2\tilde{\chi} + 2d\psi\right)^2 \\ &+ \frac{1}{16d^2}\left((\tilde{\ell}_1 + 2d\tilde{\chi})^2 + (\tilde{\ell}_2 - 2d\chi)^2\right)^2\Bigg], \end{aligned} \tag{4.32}$$

where we defined

$$\tilde{\ell}_1 := \frac{d}{|\omega_3|^2}\Big[(p_1-p_2)n_1+(p_1+p_2)n_2\Big],$$
$$\tilde{\ell}_2 := \frac{d}{|\omega_3|^2}\Big[(p_1+p_2)n_1-(p_1-p_2)n_2\Big]. \tag{4.33}$$

Extracting an overall factor of $|\omega_3|^2/d^2$ the total summand may be written as follows

$$\frac{d^2}{|\omega_3|^2}\omega^\dagger\cdot\mathcal{K}\cdot\omega = \left[m-\frac{|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)+\tilde{\ell}_1\chi+\tilde{\ell}_2\tilde{\chi}+2d\psi\right]^2$$
$$+\frac{e^{-4\phi}}{d^2}\left[d^2+\frac{e^{2\phi}}{4}\Big((\tilde{\ell}_1+2d\tilde{\chi})^2+(\tilde{\ell}_2-2d\chi)^2\Big)\right]^2. \tag{4.34}$$

The integer m is now unconstrained and is amenable to Poisson resummation using the standard formula

$$\sum_{m\in\mathbb{Z}} e^{-\pi x(m+a)^2+2\pi imb} = \frac{1}{\sqrt{x}}\sum_{\tilde{m}\in\mathbb{Z}} e^{-\frac{\pi}{x}(\tilde{m}+b)^2-2\pi i(\tilde{m}+b)a}. \tag{4.35}$$

Implementing this on the remainder $\mathcal{A}^{(s)}$ defined in (4.17) yields

$$\mathcal{A}^{(s)} = \frac{\pi^s}{\Gamma(s)}e^{-2s\phi}\sum_{\tilde{m}\in\mathbb{Z}}\sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2}\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}}\frac{d}{|\omega_3|}e^{-2\pi i\tilde{m}\left(-\frac{|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)+\tilde{\ell}_1\chi+\tilde{\ell}_2\tilde{\chi}+2d\psi\right)}$$
$$\times\int_0^\infty\frac{dt}{t^{s+1/2}}e^{-\pi t\frac{d^2}{|\omega_3|^2}\tilde{m}^2-\frac{\pi}{t}\frac{|\omega_3|^2}{d^4}e^{-4\phi}\left[d^2+\frac{e^{2\phi}}{4}\left((\tilde{\ell}_1+2d\tilde{\chi})^2+(\tilde{\ell}_2-2d\chi)^2\right)\right]^2}, \tag{4.36}$$

where we have indicated explicitly the constraint from (4.29) that $2d$ must divide $|\omega_2|^2$.

Recall that the Abelian terms in the Fourier expansion correspond physically to instantons with zero NS5-brane charge, independent of the NS-NS scalar ψ. We therefore split off the non-Abelian contribution with $\tilde{m}\neq 0$:

$$\mathcal{A}^{(s)} = \mathcal{D}^{(s)}+\mathcal{E}_s^{(\mathrm{NA})}, \tag{4.37}$$

where $\mathcal{E}_s^{(\mathrm{NA})}$ denotes the non-Abelian term with $\tilde{m}\neq 0$, to be considered later. From $\mathcal{D}^{(s)}$ we will be able to extract the second constant term $\mathcal{E}_s^{(1)}$ as well as the Abelian Fourier coefficients $\mathfrak{C}^{(\mathrm{A})}_{s,\ell_1,\ell_2}$. Explicitly we have

$$\mathcal{D}^{(s)} = \frac{\pi^s}{\Gamma(s)}e^{-2s\phi}\sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2}\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}}\frac{d}{|\omega_3|}\int_0^\infty\frac{dt}{t^{s+1/2}}e^{-\frac{\pi}{t}\frac{e^{-4\phi}|\omega_3|^2}{d^4}\left[d^2+\frac{e^{2\phi}}{4}\left((\tilde{\ell}_1+2d\tilde{\chi})^2+(\tilde{\ell}_2-2d\chi)^2\right)\right]^2}. \tag{4.38}$$

To get rid of the square in the exponent, we shall perform the integration over t and then choose a new integral representation of the summand. The current form of the exponent will be convenient for the evaluation of the non-Abelian terms below, but for our present purposes we shall begin to rewrite it in the following way

$$\frac{e^{-4\phi}|\omega_3|^2}{d^4}\left[d^2+\frac{e^{2\phi}}{4}\left((\tilde{\ell}_1+2d\tilde{\chi})^2+(\tilde{\ell}_2-2d\chi)^2\right)\right]^2=\frac{1}{4|\omega_3|^2}\left[|\mathcal{Y}|^2+2e^{-2\phi}|\omega_3|^2\right]^2, \quad (4.39)$$

where we defined the new variable $\mathcal{Y}=\mathcal{Y}_1+i\mathcal{Y}_2$, with

$$\begin{aligned}\mathcal{Y}_1 &:= n_1+(p_1-p_2)\tilde{\chi}-(p_1+p_2)\chi,\\ \mathcal{Y}_2 &:= n_2+(p_1+p_2)\tilde{\chi}+(p_1-p_2)\chi.\end{aligned} \quad (4.40)$$

Evaluating the integral over t then yields

$$\mathcal{D}^{(s)}=\frac{2^{2s-1}\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}e^{-2s\phi}\sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2}\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}}\frac{d}{|\omega_3|^{2-2s}}\left\{\left[|\mathcal{Y}|^2+2e^{-2\phi}|\omega_3|^2\right]\right\}^{1-2s}. \quad (4.41)$$

After replacing the term within brackets by its integral representation we obtain

$$\mathcal{D}^{(s)}=\frac{2^{2s-1}\pi^{2s-1/2}\Gamma(s-1/2)}{\Gamma(s)\Gamma(2s-1)}e^{-2s\phi}\sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2}\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}}\frac{d}{|\omega_3|^{2-2s}}\int_0^\infty\frac{dt}{t^{2s}}e^{-\frac{\pi}{t}\left[|\mathcal{Y}|^2+2e^{-2\phi}|\omega_3|^2\right]}. \quad (4.42)$$

Since all values of n_1 and n_2 are almost degenerate we shall perform a further Poisson resummation on these variables. Here we must take into account the remaining constraint that $2d$ divides $n_1^2+n_2^2$. The set of solutions to this constraint can be written as

$$n_1=n_1^0+\delta n_1\ , \qquad n_2=n_2^0+\delta n_2, \quad (4.43)$$

where $(\delta n_1,\delta n_2)$ runs over the lattice L

$$L=\{\delta(n_1,\delta n_2)\}=\left\{d(k_1+k_2,k_1-k_2)\,:\,(k_1,k_2)\in\mathbb{Z}^2\right\} \quad (4.44)$$

and (n_1^0,n_2^0) runs over all solutions of the quadratic equation $n_1^2+n_2^2=0 \mod 2d$ in a fundamental domain. We take this fundamental domain to be $0\le n_1^0<d$ and $0\le n_2^0<2d$, and it has area $2d^2$. The set of such solutions in the fundamental domains will be written as

$$\mathcal{F}(d):=\left\{n_1^0+in_2^0\,:\,n_1^2+n_2^2=0 \mod 2d,\ 0\le n_1^0<d\,,\ 0\le n_2^0<2d\right\}. \quad (4.45)$$

The cardinality of this set gives the number of solutions

$$N(d):=\sharp\mathcal{F}(d) \quad (4.46)$$

This series is multiplicative but not completely multiplicative [66] and we will give more details on it below and in Appendix B.

After inserting (4.43) into (4.42) and performing a Poisson resummation on δn_1 and δn_2, we obtain

$$\mathcal{D}^{(s)} = \frac{2^{2s-2}\pi^{2s-1/2}\Gamma(s-1/2)}{\Gamma(s)\Gamma(2s-1)} e^{-2s\phi} \sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2} \sum_{\tilde{\omega}_2\in L^*} \sum_{f\in\mathcal{F}(d)} \frac{1}{d|\omega_3|^{2-2s}} \times e^{2\pi i \Re(\tilde{\omega}_2 f)} \int_0^\infty \frac{dt}{t^{2s-1}} e^{-\pi t(\tilde{n}_1^2+\tilde{n}_2^2)-\frac{2\pi}{t}e^{-2\phi}|\omega_3|^2+2\pi i(\ell_1\chi+\ell_2\tilde{\chi})}, \tag{4.47}$$

where L^* is the lattice dual to L,

$$L^* = \left\{\tilde{\omega}_2 = \tilde{n}_1 + i\tilde{n}_2 = \frac{1}{2d}(\tilde{k}_1+\tilde{k}_2, \tilde{k}_1-\tilde{k}_2) \,:\, (\tilde{k}_1,\tilde{k}_2)\in\mathbb{Z}^2\right\}, \tag{4.48}$$

the set $\mathcal{F}(d)$ denotes the elements in the fundamental domain contributing to (4.45) for $d = \gcd(p_1,p_2)$, and we defined the new charges

$$\begin{aligned} \ell_1 &:= \tilde{n}_1(p_1+p_2) - \tilde{n}_2(p_1-p_2)\,, \\ \ell_2 &:= \tilde{n}_1(p_2-p_1) - \tilde{n}_2(p_1+p_2)\,. \end{aligned} \tag{4.49}$$

4.4 Second Constant Term

We may now extract the second constant term from the $\ell_1 = \ell_2 = 0$ part of the sum, and accordingly we split $\mathcal{D}^{(s)}$ as

$$\mathcal{D}^{(s)} = \mathcal{E}_s^{(1)} + \mathcal{E}_s^{(\mathrm{A})}, \tag{4.50}$$

where $\mathcal{E}_s^{(\mathrm{A})}$ is the Abelian term in the Fourier expansion to be considered below. The $\ell_1 = \ell_2 = 0$ part arises from the $\tilde{\omega}_2 = 0$ term which reads

$$\mathcal{E}_s^{(1)} = \frac{2^{2s-2}\pi^{2s-1/2}\Gamma(s-1/2)}{\Gamma(s)\Gamma(2s-1)} e^{-2s\phi} \sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2} \sum_{f\in\mathcal{F}(d)} \frac{1}{d|\omega_3|^{2-2s}} \int_0^\infty \frac{dt}{t^{2s-1}} e^{-\frac{2\pi}{t}e^{-2\phi}|\omega_3|^2}. \tag{4.51}$$

The sum over $f\in\mathcal{F}(d)$ is now identified with the multiplicative function $N(d)$ in (4.46) and the integral can be explicitly evaluated with the result

$$\mathcal{E}_s^{(1)} = \frac{\pi^{3/2}\Gamma(s-1/2)\Gamma(2s-2)}{\Gamma(s)\Gamma(2s-1)} e^{-2(2-s)\phi} \sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2} N(d)\frac{1}{d|\omega_3|^{2s-2}}. \tag{4.52}$$

The sum can be evaluated in terms of Riemann and Dedekind zeta functions as follows. Extract the greatest common divisor of p_1 and p_2, defining $p_1 = dp_1', p_2 = dp_2'$ with $d = \gcd(p_1,p_2)$ and $\gcd(p_1',p_2') = 1$. This yields a sum over d and coprime (p_1',p_2')

$$\sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2} N(d)d^{-1}|p|^{2-2s} = \left(\sum_{d>0} N(d)d^{1-2s}\right)\left(\sum_{(p_1',p_2')=1}\frac{1}{(p_1'^2+p_2'^2)^{s-1}}\right). \tag{4.53}$$

The second sum may be rewritten as a ratio of Riemann and Dedekind zeta functions as follows

$$\sum_{(p'_1,p'_2)=1} \frac{1}{(p_1'^2+p_2'^2)^{s-1}} = \frac{4\zeta_{\mathbb{Q}[i]}(s-1)}{\zeta(2s-2)}. \tag{4.54}$$

Let us now consider the first sum on the right hand side of (4.53). This involves the combinatorial function $N(d)$ of (4.46) (see also [66]). It gives rise to a Dirichlet series via

$$L(N,s) := \sum_{d=1}^{\infty} N(d) d^{-s} \tag{4.55}$$

that converges for $\Re(s) > 2$. We shall evaluate this Dirichlet series using its Euler product presentation. To this end we note that the multiplicative series $N(d)$ exhibits the following properties [66]

$$N(2^m) = 2^m\ , \qquad N(p^m) = \begin{cases} (m(p-1)+p)p^{m-1}\ , & p = 1[4] \\ p^{m-(m \mod 2)}\ , & p = 3[4]. \end{cases} \tag{4.56}$$

Therefore, the Dirichlet series (4.55) has an Euler product representation (see Appendix B for the derivation)

$$L(N,s) = \frac{1}{1-2^{1-s}} \prod_{p;p=1[4]} \frac{1-p^{-s}}{(1-p^{1-s})^2} \prod_{p;p=3[4]} \frac{1+p^{-s}}{(1-p^{1-s})(1+p^{1-s})}, \tag{4.57}$$

where the product runs over all primes $p > 2$. Comparing to the Euler product presentations of the Dirichlet beta function (4.25) and the Riemann zeta function (4.23), we deduce

$$L(N,s) = \frac{\beta(s-1)\zeta(s-1)}{\beta(s)}. \tag{4.58}$$

Putting everything together we then find the following expression for the constant term

$$\mathcal{E}_s^{(1)} = 4\frac{\pi^{3/2}\Gamma(s-1/2)\Gamma(2s-2)}{\Gamma(s)\Gamma(2s-1)} \frac{L(N,2s-1)}{\zeta(2s-2)} \zeta_{\mathbb{Q}[i]}(s-1) e^{-2(2-s)\phi}. \tag{4.59}$$

Referring back to the completed Dedekind zeta function (4.20) and Dirichlet beta function (4.26) we define a completed "Picard Zeta function" by

$$\mathfrak{Z}(s) := \zeta_{\mathbb{Q}[i]*}(s)\beta_*(2s-1). \tag{4.60}$$

In terms of this function the two constant terms can be neatly summarized by

$$\mathcal{E}_s^{(\mathrm{const})} = \mathcal{E}_s^{(0)} + \mathcal{E}_s^{(1)} = 4\zeta_{\mathbb{Q}[i]}(s) \left\{ e^{-2s\phi} + \frac{\mathfrak{Z}(2-s)}{\mathfrak{Z}(s)} e^{-2(2-s)\phi} \right\}. \tag{4.61}$$

4.5 Abelian Fourier Coefficients

We now turn to the Abelian Fourier coefficients, corresponding to the terms $(\tilde{n}_1, \tilde{n}_2) \neq 0$ in (4.47). The integral over t leads to a modified Bessel function,

$$\begin{aligned}\mathcal{E}_s^{(\mathrm{A})} &= \frac{2\pi^{2s-1/2}\Gamma(s-1/2)}{\Gamma(s)\Gamma(2s-1)} e^{-2\phi} \sum'_{(p_1,p_2)\in\mathbb{Z}^2} \sum'_{(\tilde{k}_1,\tilde{k}_2)\in\mathbb{Z}^2} \sum_{f\in\mathcal{F}(d)} \frac{1}{d^{2s-1}} |u|^{2s-2} \\ &\times e^{\frac{\pi i}{d}\Re[uf(1-i)]} \, K_{2s-2}\Big(2\pi e^{-\phi}|\Lambda|\Big) e^{2\pi i(\ell_1\chi+\ell_2\tilde{\chi})}, \end{aligned} \tag{4.62}$$

where we have introduced the following additional notation

$$u = \tilde{k}_1 + i\tilde{k}_2, \quad \Lambda = \ell_2 - i\ell_1 \tag{4.63}$$

for

$$\ell_1 = \frac{1}{d}(\tilde{k}_1 p_2 + \tilde{k}_2 p_1), \quad \ell_2 = \frac{1}{d}(\tilde{k}_2 p_2 - \tilde{k}_1 p_1). \tag{4.64}$$

These charges are manifestly integral since d divides p_1 and p_2. This last relation can also be written as

$$\Lambda = \frac{u\omega_3}{d} = u\omega_3', \tag{4.65}$$

where $\omega_3' = \omega_3/d$ is a Gaussian number with coprime real and imaginary part. To extract the Abelian Fourier coefficients $\mathfrak{C}^{(\mathrm{A})}_{s\ell_1,\ell_2}(\phi)$ we therefore replace the sum over ω_3 and u by a sum over d, Λ and ω_3' where the primitive Gaussian integer ω_3' has to be a Gaussian divisor of Λ, to wit

$$\begin{aligned}\mathcal{E}_s^{(\mathrm{A})} &= C_s^{(\mathrm{A})} e^{-2\phi} \sum'_{\Lambda\in\mathbb{Z}[i]} \left\{ \sum_{\omega_3'|\Lambda} \left|\frac{\Lambda}{\omega_3'}\right|^{2s-2} \left(\sum_{d>0} \frac{1}{d^{2s-1}} \sum_{f\in\mathcal{F}(d)} e^{\frac{\pi i}{d}\Re\left[\frac{\Lambda}{\omega_3'}f(1-i)\right]} \right) \right\} \\ &\times K_{2s-2}\Big(2\pi e^{-\phi}|\Lambda|\Big) e^{2\pi i(\ell_1\chi+\ell_2\tilde{\chi})}, \end{aligned} \tag{4.66}$$

where the coefficient is given by

$$C_s^{(\mathrm{A})} = \frac{2\pi^{2s-1/2}\Gamma(s-1/2)}{\Gamma(s)\Gamma(2s-1)} = \frac{8\zeta_{\mathbb{Q}[i]}(s)\beta(2s-1)}{\mathfrak{Z}(s)}. \tag{4.67}$$

To make contact with the general discussion in Section 4.1 above, we write this result as a sum over the real variables ℓ_1 and ℓ_2:

$$\mathcal{E}_s^{(\mathrm{A})} = 2\zeta_{\mathbb{Q}[i]}(s)\frac{e^{-2\phi}}{\mathfrak{Z}(s)} \sum'_{(\ell_1,\ell_2)\in\mathbb{Z}^2} \mu_s(\ell_1,\ell_2)\left[\ell_1^2+\ell_2^2\right]^{s-1} K_{2s-2}\Big(2\pi e^{-\phi}\sqrt{\ell_1^2+\ell_2^2}\Big) e^{2\pi i(\ell_1\chi+\ell_2\tilde{\chi})}, \tag{4.68}$$

where we defined the summation measure

$$\mu_s(\ell_1,\ell_2) := 4\beta(2s-1)\sum_{\omega_3'|\Lambda} |\omega_3'|^{2-2s} \left(\sum_{d>0} d^{1-2s} \sum_{f\in\mathcal{F}(d)} e^{\frac{\pi i}{d}\Re\left[\frac{\Lambda}{\omega_3'}f(1-i)\right]}\right) \tag{4.69}$$

containing the sum over primitive Gaussian divisors of $\Lambda = \ell_2 - i\ell_1$. The sum over d in the parentheses may be carried out for fixed Λ and ω_3' to give the Gaussian divisor function (see Appendix B for the derivation)

$$\sum_{d>0} d^{1-2s} \sum_{f\in\mathcal{F}(d)} e^{\frac{\pi i}{d}\Re\left[\frac{\Lambda}{\omega_3'}f(1-i)\right]} = \frac{1}{4\beta(2s-1)} \sum_{z|\frac{\Lambda}{\omega_3'}} |z|^{4-4s}\,, \tag{4.70}$$

whence the instanton measure (4.69) simplifies to

$$\mu_s(\ell_1,\ell_2) = \sum_{\omega_3'|\Lambda} |\omega_3'|^{2-2s} \sum_{z|\frac{\Lambda}{\omega_3'}} |z|^{4-4s}\,. \tag{4.71}$$

Thus, the abelian summation measure (4.71) involves both a sum over primitive divisors of Λ and a sum over all divisors of Λ/ω_3'. By comparing (4.68) to (4.13) we may now extract the numerical abelian Fourier coefficients:

$$C^{(\mathrm{A})}_{\ell_1,\ell_2}(s) = \frac{2\zeta_{\mathbb{Q}[i]}(s)}{\mathfrak{Z}(s)}\mu_s(\ell_1,\ell_2)\left[\ell_1^2+\ell_2^2\right]^{s-1}. \tag{4.72}$$

4.6 Non-Abelian Fourier Coefficients

Finally we consider the non-Abelian term $\mathcal{E}_s^{(\mathrm{NA})}$ in (4.37). This term reads

$$\begin{aligned}\mathcal{E}_s^{(\mathrm{NA})} &= \frac{\pi^s}{\Gamma(s)}e^{-2s\phi}\sum_{\tilde{m}\in\mathbb{Z}}{}' \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}} \sum_{(p_1,p_2)\in\mathbb{Z}^2}{}' \frac{d}{|\omega_3|}e^{-2\pi i\tilde{m}\left(-\frac{|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)+\tilde{\ell}_1\chi+\tilde{\ell}_2\tilde{\chi}+2d\psi\right)}\\ &\times\int_0^\infty \frac{dt}{t^{s+1/2}}e^{-\pi t\frac{d^2}{|\omega_3|^2}\tilde{m}^2-\frac{\pi}{t}\frac{|\omega_3|^2}{d^4}e^{-4\phi}\left[d^2+\frac{e^{2\phi}}{4}\left((\tilde{\ell}_1+2d\tilde{\chi})^2+(\tilde{\ell}_2-2d\chi)^2\right)\right]^2}.\end{aligned} \tag{4.73}$$

The integral is of Bessel type and yields

$$\begin{aligned}\mathcal{E}_s^{(\mathrm{NA})} &= \frac{2\pi^s}{\Gamma(s)}e^{-2s\phi}\sum_{\tilde{m}\in\mathbb{Z}}{}' \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}} \sum_{(p_1,p_2)\in\mathbb{Z}^2}{}' \left[\frac{d}{|\omega_3|}\right]^{s+1/2}\left[\frac{|\tilde{m}|^2}{\Re(S_{\ell_1,\ell_2,k})}\right]^{s-1/2}\\ &\times K_{s-1/2}\Big(2\pi\Re(S_{\ell_1,\ell_2,k})\Big)e^{-2\pi i\Im(S_{\ell_1,\ell_2,k})}e^{-\frac{\pi i}{2kd}(\ell_1^2+\ell_2^2)(q_1p_1+q_2p_2)},\end{aligned} \tag{4.74}$$

where the real and imaginary parts of $S_{\ell_1,\ell_2,k}$ are given by

$$\begin{aligned}\Re(S_{\ell_1,\ell_2,k}) &= |k|e^{-2\phi}+\frac{1}{4|k|}\Big[(\ell_1+2k\tilde{\chi})^2+(\ell_2-2k\chi)^2\Big],\\ \Im(S_{\ell_1,\ell_2,k}) &= \ell_1\chi+\ell_2\tilde{\chi}+2k\psi,\end{aligned} \tag{4.75}$$

and we also defined[15]

$$\begin{aligned}k &:= \tilde{m}d,\\ \ell_1 &:= \tilde{m}\tilde{\ell}_1 = \frac{k}{|\omega_3|^2}\Big[(p_1-p_2)n_1+(p_1+p_2)n_2\Big],\\ \ell_2 &:= \tilde{m}\tilde{\ell}_2 = \frac{k}{|\omega_3|^2}\Big[(p_1+p_2)n_1-(p_1-p_2)n_2\Big].\end{aligned} \tag{4.76}$$

[15]The non-Abelian charges ℓ_i defined in (4.76) should not be confused with the Abelian charges ℓ_i in (4.49).

In Gaussian notation, $\Lambda = \ell_2 - i\ell_1$, the last two relations amount to

$$\Lambda = \frac{(1-i)k\bar{\omega}_2}{\bar{\omega}_3}. \tag{4.77}$$

By comparing the expression (4.74) with the general form of the non-Abelian term (4.14) it is clear that they don't match. More precisely, the non-Abelian term (4.74) is currently written as a sum of Gaussian wavefunctions in the $(\chi, \tilde{\chi})$ plane, while the general form (4.14) is written in terms of an basis of invariant wavefunctions on the twisted torus parametrized by $(\chi, \tilde{\chi}, \psi)$. These are eigenmodes of ∂_χ and $\partial_{\psi+\chi\tilde{\chi}}$ and the associated charges kn and k are naturally quantized. However, it is clear from (4.74) that in the Gaussian basis all the charges are not quantized since (ℓ_1, ℓ_2) are not integral. To extract the correct non-Abelian Fourier coefficients $C^{(\mathrm{NA})}_{k,\ell}(s)$ we must therefore transform (4.74) into the correct basis. This can be achived through Fourier transform on the variable χ (or $\tilde{\chi}$ in the other polarization).

To perform the Fourier transform we go back to the integral representation in (4.73). The integrand is quartic in χ and therefore not immediately amenable for Fourier transform. To remedy this we make the following change of integration variables:

$$t \longrightarrow \frac{t|\omega_3|^2 A}{k^2}, \tag{4.78}$$

where

$$A(y,\chi,\tilde{\chi}) = k\Big[y + \Big(\chi + \frac{\ell_1}{2k}\Big)^2 + \Big(\tilde{\chi} - \frac{\ell_2}{2k}\Big)^2\Big], \tag{4.79}$$

and we recall that $y = e^{-2\phi}$. Implementing this in (4.73) we obtain

$$\begin{aligned}\mathcal{E}^{(\mathrm{NA})}_s &= \frac{\pi^s}{\Gamma(s)} y^s \sideset{}{'}\sum_{\tilde{m}\in\mathbb{Z}} \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}} \sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2} \frac{d|k|^{2s-1}}{|\omega_3|^{2s}}\\ &\times e^{-2\pi i\tilde{m}\left(-\frac{|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)+\tilde{\ell}_1\chi+\tilde{\ell}_2\tilde{\chi}+2d\psi\right)} \int_0^\infty \frac{dt}{t^{s+1/2}} A^{1/2-s} e^{-\pi\left(t+\frac{1}{t}\right)A},\end{aligned} \tag{4.80}$$

which is now Gaussian in both χ and $\tilde{\chi}$. We proceed to rewrite this expression as follows

$$\begin{aligned}\mathcal{E}^{(\mathrm{NA})}_s &= \frac{\pi^s}{\Gamma(s)} y^s \sideset{}{'}\sum_{\tilde{m}\in\mathbb{Z}} \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}} \sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2} \frac{d|k|^{2s-1}}{|\omega_3|^{2s}} e^{\frac{\pi i\tilde{m}|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)}\\ &\times e^{-4\pi ik\psi-2\pi i\ell_2\tilde{\chi}} \int_0^\infty \frac{dt}{t^{s+1/2}} e^{-\pi k\left(t+\frac{1}{t}\right)\left[y+\left(\tilde{\chi}+\frac{\ell_1}{2k}\right)^2\right]} f(y,\chi,\tilde{\chi};t),\end{aligned} \tag{4.81}$$

where all the χ-dependence is contained in the function

$$f(y,\chi,\tilde{\chi};t) = A^{1/2-s} e^{-\pi k\left(t+\frac{1}{t}\right)\left(\chi-\frac{\ell_2}{2k}\right)^2-2\pi i\ell_1\chi}. \tag{4.82}$$

The Fourier transform over χ is implemented as follows

$$f(y,\chi,\tilde{\chi};t) = e^{-4\pi ik\chi\tilde{\chi}}\int dn\hat{f}(n)e^{8\pi ikn\chi}, \tag{4.83}$$

with

$$\hat{f}(n) = 4|k|\int d\xi\ e^{-8\pi ikn\xi+4\pi ik\xi\tilde{\chi}}f(y,\xi,\tilde{\chi};t). \tag{4.84}$$

To perform the integral over ξ we shall utilize the following formula (Eq. 3.462.3 in [67])

$$\int_{-\infty}^{\infty} dx\ (ix)^{\nu}e^{-\beta^2x^2-iqx} = \sqrt{\pi}2^{-\nu/2}\beta^{-\nu-1}e^{-\frac{q^2}{8\beta^2}}D_{\nu}\Big(\frac{q}{\sqrt{2}\beta}\Big), \tag{4.85}$$

where D_ν is the parabolic cylinder function. In order to make use of this formula we must further manipulate the expression in (4.82) such that we separate the variables in the prefactor $A^{1/2-s}$. This can be done using the binomial expansion which is justified at the cusp $y\to\infty$ and yields

$$A^{1/2-s}(y,\xi,\tilde{\chi}) = \sum_{q=0}^{\infty}\frac{\Gamma(s-\frac{1}{2}+q)}{\Gamma(q+1)\Gamma(s-\frac{1}{2})}\Big[y+\Big(\tilde{\chi}+\frac{\ell_1}{2k}\Big)^2\Big]^{1-2s-2q}\Big[i\big(\xi-\frac{\ell_2}{2k}\big)\Big]^{2q}. \tag{4.86}$$

Inserting this into (4.81) and Fourier transforming over χ we obtain

$$\begin{aligned}\mathcal{E}_s^{(\mathrm{NA})} &= \frac{4\pi^s}{\Gamma(s)}y^s\sideset{}{'}\sum_{\tilde{m}\in\mathbb{Z}}\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}}\sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2}\sum_{q=0}^{\infty}\frac{d||k|^{2s}}{|\omega_3|^{2s}}e^{\frac{\pi i\tilde{m}|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)}\frac{\Gamma(s-\frac{1}{2}+q)}{\Gamma(q+1)\Gamma(s-\frac{1}{2})}\\
&\quad\Big[y+\Big(\tilde{\chi}+\frac{\ell_1}{2k}\Big)^2\Big]^{1-2s-2q}\int_0^{\infty}\frac{dt}{t^{s+1/2}}e^{-\pi k\left(t+\frac{1}{t}\right)\left[y+\left(\tilde{\chi}+\frac{\ell_1}{2k}\right)^2\right]}\int dn\ e^{-\frac{\pi i}{k}\ell_2(\ell_1-4kn)}\\
&\quad\times\Big\{\int d\xi\Big[i\big(\xi-\frac{\ell_2}{2k}\big)\Big]^{2q}e^{-\pi k\left(t+\frac{1}{t}\right)\left(\xi-\frac{\ell_2}{2k}\right)^2}e^{-i(2\pi\ell_1-4\pi k\tilde{\chi}+8\pi kn)\left(\xi-\frac{\ell_2}{2k}\right)}\Big\}\\
&\quad\times e^{8\pi ikn\chi-4\pi ik(\psi+\chi\tilde{\chi})}.\end{aligned} \tag{4.87}$$

After evaluating the integral over ξ using Eq. (4.85) the non-Abelian term becomes

$$\begin{aligned}\mathcal{E}_s^{(\mathrm{NA})} &= \frac{4\pi^s}{\Gamma(s)}y^s\sideset{}{'}\sum_{\tilde{m}\in\mathbb{Z}}\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}}\sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2}\sum_{q=0}^{\infty}\frac{d|k|^{2s-q-1/2}}{(4\pi)^q|\omega_3|^{2s}}e^{\frac{\pi i\tilde{m}|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)}\\
&\quad\times\frac{\Gamma(s-\frac{1}{2}+q)}{\Gamma(q+1)\Gamma(s-\frac{1}{2})}\Big[y+\Big(\tilde{\chi}+\frac{\ell_1}{2k}\Big)^2\Big]^{1-2s-2q}\int_0^{\infty}\frac{dt}{t^{s+1/2}}e^{-\pi k\left(t+\frac{1}{t}\right)\left[y+\left(\tilde{\chi}+\frac{\ell_1}{2k}\right)^2\right]}\\
&\quad\times\Big(t+\frac{1}{t}\Big)^{-q-1/2}\int dn\ e^{-\frac{\pi i}{k}\ell_2(\ell_1-4kn)}e^{-\frac{\pi}{k}\frac{(\ell_1-2k\tilde{\chi}+4kn)^2}{(t+1/t)}}H_{2q}\left(\sqrt{\frac{\pi}{k}}\frac{\ell_1-2k\tilde{\chi}+4kn}{\sqrt{t+1/t}}\right)\\
&\quad\times e^{8\pi ikn\chi-4\pi ik(\psi+\chi\tilde{\chi})},\end{aligned} \tag{4.88}$$

where we have also used the fact that for integral index the parabolic cylinder function $D_m(x)$ can be written in terms of Hermite polynomials as follows

$$D_\nu(x) = 2^{-\nu/2} e^{-x^2/4} H_\nu\left(\frac{x}{\sqrt{2}}\right). \tag{4.89}$$

This has the advantage of explicitly extracting the leading order behaviour of $D_\nu(x)$ in the limit of large argument

$$D_\nu(x) \sim x^\nu e^{-x^2/4}. \tag{4.90}$$

Let us now comment on the structure of Eq. (4.88). After Fourier transforming we see that the non-Abelian term indeed corresponds to an expansion in terms of the invariant wavefunctions on the twisted torus as in the general form of Eq. (4.14). However, we have unfortunately not been able to further manipulate Eq. (4.88) into the form displayed in (4.14) and therefore we cannot extract the numerical Fourier coefficients $C_{k,\ell}^{(\mathrm{NA})}(s)$ in as compact a form as the Abelian coefficients (4.72). Nevertheless, as a consistency check of our analysis we shall show that the leading order exponential behaviour of (4.88) near the cusp $y \to \infty$ coincides with that of Eq. (4.14). To this end we may take the saddle point approximation for the integral over t in (4.88) for which the saddle point is located at $t = 1$. We thus find that the leading exponential dependence of (4.88) at the saddle point is given by e^{-S} with

$$S = 2\pi|k|\Big[y + \Big(\tilde{\chi} + \frac{\ell_1}{2|k|}\Big)\Big] + \frac{\pi}{2|k|}\big(\ell_1 - 2|k|\tilde{\chi} + 4|k|n\big)^2. \tag{4.91}$$

Rearranging terms, this can be written as

$$S = 2\pi|k|y + 4\pi|k|\big(\tilde{\chi} - n\big)^2 + 4\pi|k|\Big(n + \frac{\ell_1}{2|k|}\Big)^2. \tag{4.92}$$

Using the asymptotic behaviour of the Whittaker function $W_{k,m}(x) \sim e^{-x/2}$ one may indeed verify that the first two terms in (4.92) exactly coincide with the leading behaviour of the general expression (4.14) in the limit $y \to \infty$. The last term in (4.92) is moduli-independent and should presumably be absorbed into the measure after summing over ℓ_1 and ℓ_2. We further expect that the summation over ℓ_1 and ℓ_2 (or, more precisely, over ω_2 and ω_3) will restrict the integral over n such that it localizes on the points in $\mathbb{Z} + \ell/(4|k|)$ as is expected from the general expression (4.14). We stress that the result (4.92) is valid in the polarization (4.8) we have chosen. There is an analogous result for the other polarization.

4.7 Functional Relation for the Poincaré series

The expression (4.61) for the two constant terms suggests a functional relation for the $SU(2,1;\mathbb{Z}[i])$ Poincaré series defined in (3.11). As a consequence of (3.14), the constant terms of the Poincaré series of orders s and $2-s$ satisfy the relation

$$\mathfrak{Z}(s)\mathcal{P}_s^{(\mathrm{const})} = \mathfrak{Z}(2-s)\mathcal{P}_{2-s}^{(\mathrm{const})}. \tag{4.93}$$

Eq. (4.61) can be viewed as an extension of Langlands's constant term formula [64] for Eisenstein series for special linear groups to the case of the unitary group $SU(2,1)$. The completed Picard zeta function $\mathfrak{Z}(s)$ plays the same role as the completed Riemann zeta function $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ in Langlands's formula. The completed Riemann zeta function has the functional relation $\xi(s) = \xi(1-s)$; there is no analogous relation for our completed Picard zeta function $\mathfrak{Z}(s)$. Note also that (4.61) is consistent with the fact the constant terms are permuted by the restricted Weyl group of $SU(2,1)$, which acts by $s \leftrightarrow 2-s$.

On general grounds, the functional relation for the constant terms should extend to the full Eisenstein series [64]. This is manifest for the Abelian terms (4.68) when the instanton measure (4.71) is combined with the factor $|\Lambda|^{2(s-1)} = [\ell_1^2+\ell_2^2]^{s-1}$ and recalling the symmetry of the Bessel function $K_{2s-2}(x) = K_{2-2s}(x)$. By these arguments, the functional relation satisfied by the Poincaré series for the Picard modular group $SU(2,1;\mathbb{Z}[i])$ reads

$$\mathfrak{Z}(s)\mathcal{P}_s = \mathfrak{Z}(2-s)\mathcal{P}_{2-s}\,. \tag{4.94}$$

This functional relation must also hold on the non-Abelian terms but due to their unwieldy form this is not manifest.

5. Instanton Corrections to the Universal Hypermultiplet

In this section we use the Eisenstein series for the Picard modular group constructed in Sections 3 and 4 to conjecture the exact form of the D2 and NS5 instanton corrections to the universal hypermultiplet metric $\mathcal{M}_{\mathrm{UH}}$. We start by recalling some aspects of quantum corrections to hypermultiplet moduli spaces in type II Calabi-Yau compactifications, with particular emphasis on recent developments involving twistor theory.

5.1 Quantum Corrected Hypermultiplet Moduli Spaces in IIA and IIB

Perturbative corrections to hypermultiplet moduli spaces are well understood, and it has been established that the metric on $\mathcal{M}_{\mathrm{H}}$ receives only tree-level and one-loop corrections, but no perturbative corrections beyond one loop [18–21,68,69]. For the universal hypermultiplet this was rigorously proven in [68]. The general form of the perturbative corrections can be inferred from compactifications of higher derivative couplings in ten dimensions [18], or via an explicit string theory calculations in $D = 4$ [21]. In the string frame, the tree-level correction is of the form $\zeta(3)\chi_X g_s^{-2}$, where χ_X is the Euler number of the Calabi-Yau threefold X. while the one-loop correction to the metric on $\mathcal{M}_{\mathrm{H}}$ is of the form $\zeta(2)\chi_X$. The complete perturbatively corrected metric can be found in [69, 28].

As discussed in Section 1, it is also known since [16] that $\mathcal{M}_{\mathrm{H}}$ should receive non-perturbative corrections due to D2-brane and NS5-brane instantons. Computing these contributions has remained a long standing problem, mainly due to the intricacies of quaternionic-Kähler geometry, and and to the lack of instanton calculus techniques in string theory. Nevertheless, in a series of recent papers [28–32, 70], a subset of the non-perturbative corrections to $\mathcal{M}_{\mathrm{H}}$ have been understood. The key idea, following advances in the mathematics [22–24] and physics literature [25–27] on quaternionic-Kähler spaces, is that linear deformations of the hypermultiplet moduli space $\mathcal{M}_{\mathrm{H}}$ can be lifted to the twistor space $\mathcal{Z}_{\mathcal{M}_{\mathrm{H}}}$, a $\mathbb{C}P^1$ bundle over $\mathcal{M}_{\mathrm{H}}$. In contrast to the latter, $\mathcal{Z}_{\mathcal{M}_{\mathrm{H}}}$ admits a

Kähler-Einstein metric, and quantum corrections to $\mathcal{M}_{\rm H}$ can in principle be encoded in the Kähler potential, with suitable conditions to ensure that the Kähler metric is Einstein. These conditions can be solved in terms of holomorphic data, by using the canonical complex contact structure on $\mathcal{Z}_{\mathcal{M}_{\rm H}}$ [29] (see, e.g., [71] for an introduction to contact geometry). By Darboux's theorem, the complex contact structure can be specified by providing complex contact transformations between local complex Darboux coordinate systems $\big(\xi^{\Lambda}_{[i]}, \tilde{\xi}^{[i]}_{\Lambda}, \alpha_{[i]}\big)$ $(\Lambda = 1, \dots, h_{2,1}+1)$. These complex contact transformations are defined on the overlap $U_i \cap U_j$ of two coordinate systems, and generated by holomorphic functions $S_{ij}\big(\xi^{\Lambda}_{[i]}, \tilde{\xi}^{[j]}_{\Lambda}, \alpha_{[j]}\big)$, subject to co-cycle conditions, local contact transformations and reality constraints. The geometry of the hypermultiplet moduli space can then be extracted by determining the contact twistor lines, i.e. expressing the generic coordinates $\big(\xi_{\Lambda}, \tilde{\xi}^{\Lambda}, \alpha\big)$, in some patch U, in terms of the coordinates $x^{\mu} \in \mathcal{M}_{\rm H}$ on the base manifold, and the complex coordinate $z \in \mathbb{C}P^1$ on the fiber. Deformations of the contact transformations S_{ij} then determine the corrections to the contact twistor lines, from which the corrected geometry of $\mathcal{M}_{\rm H}$ may be extracted in terms of the contact potential $e^{\Phi_{[i]}(x^{\mu},z)}$, which determines the Kähler potential on $\mathcal{Z}_{\mathcal{M}_{\rm H}}$ through

$$\mathcal{K}^{[i]}_{\mathcal{Z}_{\mathcal{M}_{\rm H}}} = \log \frac{1+z\bar{z}}{|z|} + \Re\big[\Phi_{[i]}(x^{\mu}, z)\big]. \tag{5.1}$$

For details on this construction we refer the reader to [28,29]. In Section 5.2 we will discuss in more detail the twistor space of the universal hypermultiplet.

5.2 Twistorial Interpretation of the Eisenstein Series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$

As was briefly discussed in Section 5.1, the quantum corrected geometry of $\mathcal{M}_{\rm H}$ is preferably not described directly by its quaternionic-Kähler metric, but rather through certain properties of its twistor space $\mathcal{Z}_{\mathcal{M}_{\rm H}}$, namely the contact twistor lines $(\xi_{\Lambda}, \tilde{\xi}^{\Lambda}, \alpha)$, in some local patch $U \subset \mathcal{Z}_{\mathcal{M}_{\rm H}}$, and the contact potential $e^{\Phi(x^{\mu},z)}$ which determines the Kähler potential on $\mathcal{Z}_{\mathcal{M}_{\rm H}}$. In this section we will take some initial steps in finding the exact deformed geometry of the twistor space $\mathcal{Z}_{\mathcal{M}_{\rm H}}$ of the universal hypermultiplet in the presence of D2-brane and NS5-brane instantons. More precisely, we shall propose a non-perturbative completion of the contact potential $e^{\Phi(x^{\mu},z)}$ on the north pole $z=0$ of the twistor space. To find the global form of the deformed geometry, one must also provide a completion of the twistor lines themselves, but this is beyond the scope of the present work. Before we proceed, let us recall some of the salient features of the twistorial description of the universal hypermultiplet.

On the Twistor Space of the Universal Hypermultiplet

The twistor space $\mathcal{Z}_{\mathcal{M}_{\rm UH}}$ of the classical moduli space $\mathcal{M}_{\rm UH}$ can be nicely described group-theoretically as follows. Viewing the $\mathbb{C}P^1$ twistor fiber as $S^2 = SU(2)/U(1)$, the fibration of $SU(2)/U(1)$ over $\mathcal{M}_{\rm UH}$ is such that the $SU(2)$ cancels: [72,53]:

$$\mathcal{Z}_{\mathcal{M}_{\rm UH}} = \frac{SU(2)}{U(1)} \ltimes \frac{SU(2,1)}{SU(2)\times U(1)} = \frac{SU(2,1)}{U(1)\times U(1)}. \tag{5.2}$$

The twistor space $\mathcal{Z}_{\mathcal{M}_{\rm UH}}$ is a complex 3-dimensional contact manifold, with local coordinates $(\xi, \tilde{\xi}, \alpha)$. These coordinates parametrize the complexified Heisenberg group $N_{\mathbb{C}}$, or, equivalently, coordinates on the complex coset space $P_{\mathbb{C}}\backslash SL(3,\mathbb{C})$, where $P_{\mathbb{C}}$ is the complexification of the parabolic subgroup $P \subset SU(2,1)$ discussed in Appendix A and $SL(3,\mathbb{C})$ is the complexification of $SU(2,1)$. In terms of the coordinates $(\xi, \tilde{\xi}, \alpha)$ on $P_{\mathbb{C}}\backslash SL(3,\mathbb{C})$ the Kähler potential of $\mathcal{Z}_{\mathcal{M}_{\rm UH}}$ takes the following form [53]

$$K_{\mathcal{Z}_{\mathcal{M}_{\rm UH}}} = \frac{1}{2}\log\left[\Big((\xi-\bar{\xi})^2+(\tilde{\xi}-\bar{\tilde{\xi}})^2\Big)^2+4\Big(\alpha-\bar{\alpha}+\bar{\xi}\tilde{\xi}-\xi\bar{\tilde{\xi}}\Big)^2\right]. \tag{5.3}$$

The contact twistor lines for the unperturbed twistor space correspond to the change of variables that relate the coordinates $(\xi, \tilde{\xi}, \alpha)$ on $\mathcal{Z}_{\mathcal{M}_{\rm UH}}$ to the coordinates $x^\mu = \{e^\phi, \chi, \tilde{\chi}, \psi\}$ on the base $\mathcal{M}_{\rm UH}$ and the coordinate z on the fiber $\mathbb{C}P^1 = SU(2)/U(1)$. These twistor lines were obtained in [53]. In our notations they read

$$\begin{aligned}
\xi &= -\sqrt{2}\chi + \frac{1}{\sqrt{2}}e^{-\phi}\big(z - z^{-1}\big),\\
\tilde{\xi} &= -\sqrt{2}\tilde{\chi} - \frac{i}{\sqrt{2}}e^{-\phi}\big(z + z^{-1}\big),\\
\alpha &= \quad 2\psi - e^{-\phi}\Big[z(\tilde{\chi}+i\chi) - z^{-1}(\tilde{\chi}-i\chi)\Big].
\end{aligned} \tag{5.4}$$

Plugging these into (5.3) we find that the Kähler potential for the twistor space of the classical universal hypermultiplet, in the coordinates $x^\mu \in \mathcal{M}_{\rm UH}$ and $z \in \mathbb{C}P^1$, reads

$$K_{\mathcal{Z}_{\mathcal{M}_{\rm UH}}} = \log\frac{1+z\bar{z}}{|z|} - 2\phi, \tag{5.5}$$

which indeed agrees with the general form of the Kähler potential in Eq. (5.1) upon identifying the classical contact potential as follows

$$e^{\Phi_{\rm classical}(x^\mu, z)} = e^{-2\phi}. \tag{5.6}$$

Deformations of the twistor space may now be encoded in deformations of the Kähler potential $K_{\mathcal{Z}_{\mathcal{M}_{\rm UH}}}$, or, equivalently, in deformations of the classical contact potential $e^{\Phi_{\rm classical}}$.

Non-Perturbative Completion of the Contact Potential

The purpose of the present work is to propose a non-perturbative completion of the classical contact potential $e^{\Phi_{\rm classical}}$ which includes the contributions from D2-brane and NS5-brane instantons. Assuming $SU(2,1;\mathbb{Z}[i])$-invariance of the exact effective action, we thus wish to complete the contact potential with the Eisenstein series $\mathcal{E}_s(\phi,\chi,\tilde{\chi},\psi)$. We first observe that the usual tree-level α'^3 correction $\chi_X\zeta(3)$ to the type IIA hypermultiplet moduli space is not expected to exist for compactification on a rigid Calabi-Yau. Indeed, it usually comes from a degenerate contribution in the Gromov-Witten sum in type IIB on the mirror Calabi-Yau, but, as we have stressed in Section 1, rigid Calabi-Yau threefolds do not have mirrors. Another way to realize that the tree-level correction is absent for the universal sector is that the corresponding contribution to the prepotential $i\zeta(3)X_0^2$ can be reabsorbed into the classical prepotential $iX_0^2/2$ of the universal hypermultiplet by a field redefinition. This is of course not so for generic (non-rigid) Calabi-Yau manifolds.

Perturbative Contributions

We further note that by construction the Eisenstein series $\mathcal{E}_s(\phi, \chi, \tilde{\chi}, \psi)$ is independent of the fiber coordinate z of the twistor space and may thus only be considered as a function on the north pole $z = 0$ of $\mathcal{Z}_{\mathcal{M}_{\mathrm{UH}}}$. Our main conjecture is then that the exact contact potential is given by[16]

$$e^{\Phi_{\mathrm{exact}}(x^\mu, z=0)} = e^{\phi}\mathcal{P}_{3/2}(\phi, \chi, \tilde{\chi}, \psi), \tag{5.7}$$

where $\mathcal{P}_s$ is the Poincaré series constructed in Section 3.2. The choice of order $s = 3/2$ in (5.7) is fixed by demanding that the constant terms in the Fourier expansion of $\mathcal{P}_s$ reproduces the correct classical and one-loop contributions. By comparing (5.8) with (5.6) we see that the perturbative terms indeed exhibit the correct dilaton powers to match the expected classical and one-loop terms. From the results of the previous section on the general Fourier expansion of $\mathcal{E}_s = 4\zeta_{\mathbb{Q}[i]}\mathcal{P}_s$, we then find that the perturbative contributions to (5.7) are

$$e^{\Phi_{\mathrm{exact}}} = e^{-2\phi} + C_1(3/2) + \cdots \tag{5.8}$$

where C_1 is the one-loop coefficient

$$C_1(3/2) = \frac{\mathfrak{Z}(1/2)}{\mathfrak{Z}(3/2)} = \pi^2 \frac{\beta(0)\beta(1/2)\zeta(1/2)}{\beta(3/2)\beta(2)\zeta(3/2)} \approx -2.32607 \tag{5.9}$$

where we made use of (4.60), (4.22). The numerical value does not directly correspond to the known coefficient $2\zeta(3)$ for the one-loop coefficient, a puzzle which we presently do not know how to resolve.

Let us also emphasize that the exact contact potential $e^{\Phi_{\mathrm{exact}}}$ is *not* invariant under $SU(2,1;\mathbb{Z}[i])$ but transforms as a modular form:

$$e^{\Phi_{\mathrm{exact}}} \;\longmapsto\; |C + D\mathcal{Z}|\; e^{\Phi_{\mathrm{exact}}}, \tag{5.10}$$

where the overall weight is due to the transformation of the prefactor e^{ϕ} (see Section 2.2). This ensures that the Kähler potential $K_{\mathcal{Z}_{\mathcal{M}_{\mathrm{UH}}}}$ transforms by a Kähler transformation, and hence that the deformed metric on $\mathcal{M}_{\mathrm{UH}}$ remains $SU(2,1;\mathbb{Z}[i])$-invariant.

D2-Brane Instantons

Now let us look closer into the non-perturbative contributions. From the Fourier expansion of $\mathcal{E}_s$ presented in Section 4 we find that $e^{\Phi_{\mathrm{exact}}}$ takes the general form

$$e^{\Phi_{\mathrm{exact}}} = e^{\Phi_{\mathrm{pert}}} + e^{\Phi_{\mathrm{D2}}} + e^{\Phi_{\mathrm{D2/NS5}}}, \tag{5.11}$$

where $e^{\Phi_{\mathrm{D2}}}$ corresponds to the Abelian term containing the contributions from D2-brane instantons, and $e^{\Phi_{\mathrm{D2/NS5}}}$ corresponds to the non-Abelian term containing the combined effects from D2- and NS5-brane instantons.

[16] A similar proposal for the $SL(3,\mathbb{Z})$-completion of the contact potential in type IIB on generic Calabi-Yau threefolds was put forward in [35].

In the weak-coupling limit $g_s = e^{\phi} \to 0$ we may utilize the asymptotic expansion of the modified Bessel function

$$K_t(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{n\geq 0} \frac{\Gamma\left(t+n+\frac{1}{2}\right)}{\Gamma(n+1)\Gamma\left(t-n+\frac{1}{2}\right)}(2x)^{-n} = \sqrt{\frac{\pi}{2x}} e^{-x}\Big[1+\mathcal{O}(1/x)\Big], \tag{5.12}$$

and from Eq. (4.68) we find that $e^{\Phi_{\mathrm{D2}}}$ is of the form

$$e^{\Phi_{\mathrm{D2}}} = \frac{1}{4\mathfrak{Z}(3/2)} e^{-\phi/2} \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2}{}' \mu_{3/2}(\ell_1,\ell_2)(\ell_1^2+\ell_2^2)^{1/4} e^{-2\pi S_{\ell_1,\ell_2}}\Big[1+\mathcal{O}(g_s)\Big]. \tag{5.13}$$

We thus find that $e^{\Phi_{\mathrm{D2}}}$ exhibits exponentially suppressed corrections in the limit $g_s \to 0$, weighted by the instanton action

$$S_{\ell_1,\ell_2} = e^{-\phi}\sqrt{\ell_1^2+\ell_2^2} + i\ell_1\chi + i\ell_2\tilde{\chi}. \tag{5.14}$$

This is recognized as the action for Euclidean D2-branes wrapping special Lagrangian 3-cycles in the homology class $\ell_1\mathcal{A}+\ell_2\mathcal{B} \in H_3(\mathcal{X})$, where $\mathcal{A}$ and $\mathcal{B}$ are the universal 3-cycles of the rigid Calabi-Yau threefold $\mathcal{X}$. The infinite series within the brackets in (5.13) should in the spirit of [5] be interpreted as perturbative excitations in the instanton background.

Let us now comment on the summation measure $\mu_{3/2}(\ell_1,\ell_2)$ in (5.13). The general form of $\mu_s(\ell_1,\ell_2)$ was derived in Section 4.5 (see also Appendix B) and for $s=3/2$ it takes the form

$$\mu_{3/2}(\ell_1,\ell_2) = \sum_{\omega_3'|\Lambda} |\omega_3'|^{-1} \sum_{z|\frac{\Lambda}{\omega_3'}} |z|^{-2} \tag{5.15}$$

where we recall that $\Lambda = \ell_2 - i\ell_1$ is a complex combination of the electric and magnetic charges (ℓ_1,ℓ_2). The instanton measure $\mu_{3/2}(\ell_1,\ell_2)$ thus "counts" the degeneracy of Euclidean D2-branes on the universal 3-cycles of the rigid Calabi-Yau manifold $\mathcal{X}$. For the special case of only A-type instantons with $\ell_2 = 0$, the measure reduces to

$$\mu_{3/2}(\ell_1,0) = \sum_{\omega_3'|\ell_1} |\omega_3'|^{-1} \sum_{z|\frac{\ell_1}{\omega_3'}} |z|^{-2} \tag{5.16}$$

If ℓ_1 is a product of inert primes only (those of the form $p = 4n+3$, see Appendix B), the first sum collapses and the instanton measure is identical to that from the study of D(-1) instantons in IIB string theory (see, e.g., [5]) where this measure is related to ways of assembling a given instanton from ones of smaller charge. In the more general case there are additional contributions from factoring an integer ℓ_1 over the Gaussian numbers, e.g. $2 = -i(1+i)^2$ or $5 = (2+i)(2-i)$. This indicates a qualitatively different behaviour of pure type A branes depending on their charge that can sometimes be composed of dyonic instantons carrying both electric and magnetic charge.

Therefore we may conclude that $SU(2,1;\mathbb{Z}[i])$-invariance predicts additional contributions to the purely electric D2-brane instantons compared to the results of [29] restricted to the special case of universal A-cycles. In [29] the instanton measure for electric D2-branes on rigid 3-cycles was also dictated by $SL(2,\mathbb{Z})$-invariance through mirror symmetry from the type IIB side. This corresponds to the standard sum over divisors of the electric charge and hence does not display the additional Gaussian divisors since it does not capture dyonic instantons. We interpret the extra contribution as a generic feature of rigid Calabi-Yau compactifications where $SL(2,\mathbb{Z})$ is not a natural duality group, but should be replaced by $SU(2,1;\mathbb{Z}[i])$. Under the c-map, the measure (5.15) should also reproduce the degeneracies of BPS black holes in type IIB arising from D3-branes wrapping special Lagrangian 3-cycles. It would be interesting to investigate these issues further.

NS5-Brane Instantons

As mentioned above, the NS5-brane instanton contribution to the contact potential is encoded in the non-Abelian term $\mathcal{E}^{(\mathrm{NA})}_{3/2}$ of the Fourier expansion in Section 4. Although we have not been able to derive the summation measure in this case, we can still extract the instanton action by taking the semiclassical limit. This corresponds to the asymptotic behaviour of (4.14) in the limit $y \to \infty$, or, equivalently, to the saddle point approximation of the t-integral in (4.88) as analyzed in Section 4.6. Expanding the Whittaker function around ∞ yields

$$\begin{aligned} W_{k,m}(x) &= e^{-x/2}x^k \sum_{n\geq 0} \frac{\Gamma\left(m-k+n+\frac{1}{2}\right)\Gamma\left(m+k+\frac{1}{2}\right)}{\Gamma(n+1)\Gamma\left(m-k+\frac{1}{2}\right)\Gamma\left(m+k+n+\frac{1}{2}\right)} x^{-n} \\ &= e^{-x/2}x^k\Big[1+\mathcal{O}\big(1/x\big)\Big]. \end{aligned} \tag{5.17}$$

Implementing this in (4.14) and extracting the leading $r=0$ term, we deduce that the leading order contribution to $e^{\Phi_{\mathrm{D2/NS5}}}$ is given by

$$e^{\Phi_{\mathrm{D2/NS5}}} \sim e^{\phi} {\sum_{k\in\mathbb{Z}}}' \sum_{\ell=0}^{4|k|-1} \sum_{n\in\mathbb{Z}+\frac{\ell}{4|k|}} C_{k,\ell}|k|^{-s} e^{-2\pi S_{k,q}}\Big[1+\mathcal{O}\big(g_s^4\big)\Big], \tag{5.18}$$

where we have defined

$$S_{k,q} = |k|e^{-2\phi} + 2|k|\big(\tilde{\chi}-n\big)^2 - 4iq\chi + 2ik(\psi+\chi\tilde{\chi}). \tag{5.19}$$

This is interpreted as the Euclidean action of combined D2/NS5-brane instantons with D2-brane charge $q \equiv nk$ and NS5-brane charge k. Note that even in the absence of D2-brane instanton contributions, $q=0$, the real part of the action receives a contribution from the background Ramond-Ramond flux $\tilde{\chi}$. This is similar to the D2/NS5-instanton action found previously in [70].

We emphasize that the result (5.18) only displays the contribution from *A-type* D2-brane instantons. The B-type D2-branes are exposed by going to the alternative polarization displayed in (4.11), but then the A-type D2-brane effects are not visible. Hence, $SU(2,1;\mathbb{Z}[i])$-invariance predicts that when the NS5-brane charge k is non-zero, is not possible to exhibit instanton contributions with both types of D2-brane charges turned on simultaneously. This is in contrast to the analysis of [35] in which case $SL(3,\mathbb{Z})$-invariance was used to exhibit D(-1), D5 and NS5-brane effects in type IIB on Calabi-Yau threefolds. Although in this case one must also choose a polarization for the diagonalization of the Heisenberg group of $SL(3,\mathbb{Z})$, it turned out that the non-Abelian Fourier coefficients contained an additional phase factor which encodes the effects of the B-type D2-brane instantons after transforming to the type IIA language using mirror symmetry. There is in fact a representation-theoretic explanation for this discrepancy. The semiclassical limit of the spherical vector for the principal series of $SL(3,\mathbb{R})$ exhibits a cubic phase factor [73] which precisely accounts for the extra D2-brane contribution to the instanton action [35]. However, the corresponding spherical vector for $SU(2,1)$ does not exhibit such a phase factor [53], thus explaining the absence of B-type D2-brane effects in (5.18). This observation reinforces our point of view that the case of rigid Calabi-Yau compactifications should be analyzed separately from generic compactifications, and our results should therefore not be directly compared to previous results on the type IIA side which relied on mirror symmetry from type IIB [31, 29].

Note further that in the special case when the D2-brane instantons are turned off, $q = 0$, and with zero flux $\tilde{\chi} = 0$, the action (5.19) reproduces the well-known pure NS5-brane instanton action of [16]:

$$S_k = |k|e^{-2\phi} + 2ik\psi. \tag{5.20}$$

We finally mention the interesting observation that through the expansion of the Whittaker function, $SU(2,1;\mathbb{Z}[i])$-invariance predicts an infinite series of perturbative excitations around the NS5-brane instanton background. This is in marked contrast to the case of type IIA Euclidean NS5-branes wrapping $K3\times T^2$, in which case the instanton background only receives perturbative corrections up to one loop [74].

A. Spherical Vector and p-Adic Eisenstein Series

There exists a general method for constructing automorphic forms on a coset space G/K, as developed in [50–52]. In this Appendix, we explain this method and show that it may be used to construct the Eisenstein series $\mathcal{E}_s(\phi,\lambda,\gamma)$ for the Picard modular group. This alternative approach also sheds light on the relation between the quadratic constraint (3.3) and the representation theoretic structure of the Eisenstein series. This Appendix may be viewed as an extension of the analysis of [53] to the automorphic setting.

A.1 Formal Construction

In general, to construct an automorphic form Ψ on G/K, invariant under a discrete subgroup $G(\mathbb{Z})\subset G$, we require three ingredients: (1) a K-invariant *spherical vector* $f_K\in\mathcal{H}$ ($\mathcal{H}$ being a Hilbert space of square integrable functions), (2) a linear representation ρ of G acting on $\mathcal{H}$, and (3) a $G(\mathbb{Z})$-invariant *distribution* $f_{\mathbb{Z}}\in\mathcal{H}^\star$ in the dual space of $\mathcal{H}$. Using the natural pairing $\langle\ ,\ \rangle$ between $\mathcal{H}$ and $\mathcal{H}^\star$, the automorphic form Ψ can then be defined formally as

$$\Psi(g) := \langle f_{\mathbb{Z}}, \rho(g)\cdot f_K\rangle\,, \tag{A.1}$$

with $g\in G$. By virtue of the Iwasawa decomposition,

$$G = NAK, \tag{A.2}$$

an arbitrary group element $g \in G$ splits as $g = nak := \mathcal{V}k$, and, since f_K is K-invariant, Ψ simplifies to

$$\Psi(\mathcal{V}) = \langle f_{\mathbb{Z}}, \rho(\mathcal{V}) \cdot f_K \rangle \,. \tag{A.3}$$

The coset representative $\mathcal{V} \in G/K$ transforms by $k^{-1} \in K$ from the right and $\gamma \in G(\mathbb{Z})$ from the left,

$$\mathcal{V} \longmapsto \gamma \mathcal{V} k^{-1}. \tag{A.4}$$

On Ψ the right action by k^{-1} becomes a left action on f_K, which is invariant by definition, and the left action of γ becomes a right action on $f_{\mathbb{Z}}$, which is also invariant. Hence, $\Psi(\mathcal{V})$ is by construction a function on the double quotient $G(\mathbb{Z})\backslash G/K$ as desired.

Although very appealing, in practice this method is complicated by the fact that the distribution $f_{\mathbb{Z}}$ is in general difficult to obtain. There is however a powerful mathematical technique, developed in [50–52], to compute $f_{\mathbb{Z}}$ using p-adic number theory. In this approach, the distribution $f_{\mathbb{Z}}$ is reinterpreted in terms of a *p-adic spherical vector* f_p that can be straightforwardly constructed from its real counterpart f_K in a way that will be explained shortly. The automorphic form Ψ can then be rewritten in the following way [50–52]

$$\Psi(\mathcal{V}) = \sum_{\vec{x} \in \mathbb{Q}^n} \Big[\prod_{p<\infty} f_p(\vec{x}) \Big] \rho(\mathcal{V}) \cdot f_K(\vec{x}), \tag{A.5}$$

where $\vec{x}$ is a vector of rational numbers in $\mathbb{Q}^n$, and the product is over all prime numbers p. We shall now see that the Eisenstein series $\mathcal{E}_s(\phi, \lambda, \gamma)$, constructed in Section 3.1, has a natural interpretation in terms of Eq. (A.5).

A.2 Real and p-Adic Spherical Vector

To apply this method to obtain an Eisenstein series for the Picard modular group we shall begin by explicitly constructing a spherical vector f_K, invariant under $SU(2) \times U(1)$, which belongs to the *principal continuous series* of $SU(2,1)$-representations. This means that f_K belongs to the Hilbert space $\mathcal{H} = L^2(P\backslash SU(2,1))$ of real-valued, square-integrable functions on the coset space $P\backslash SU(2,1)$, where P is the parabolic subgroup of $SU(2,1)$ corresponding to the Lie algebra

$$\mathfrak{p} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \subset \mathfrak{su}(2,1), \tag{A.6}$$

associated with the 5-grading of $\mathfrak{su}(2,1)$ defined in (2.8). The parabolic group P thus corresponds to the subgroup of lower-triangular matrices,

$$P = \left\{ \begin{pmatrix} t_1 & & \\ * & t_2 & \\ * & * & t_3 \end{pmatrix} \in SU(2,1) \;:\; t_1 t_2 t_3 = 1 \right\}. \tag{A.7}$$

The coset space $P\backslash SU(2,1)$ can be identified with the Heisenberg group N, which is parameterized as follows

$$n = e^{xX_1 + \tilde{x}\tilde{X}_1 + yX_2} = \begin{pmatrix} 1 & i\bar{C}_2 & C_1 \\ & 1 & C_2 \\ & & 1 \end{pmatrix} \equiv \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{pmatrix} \in N, \tag{A.8}$$

where

$$C_1 := 2y + \frac{i}{2}|C_2|^2\,, \quad C_2 := x + \tilde{x} + i(\tilde{x} - x) \tag{A.9}$$

and the last equality in (A.8) defines the row vectors of the Heisenberg group element. It is important to note that the two complex variables are not independent, but obey the relation

$$|C_2|^2 - 2\Im(C_1) = 0\,. \tag{A.10}$$

This relation is of course obvious in the present context, but we will see that this is exactly what gives rise to the quadratic constraint in the lattice construction of Section 3.1.

Any function $f \in L^2(P\backslash SU(2,1))$ obeys

$$\rho(g)\cdot f(n) = f(ng) = f(pn') = \chi(p)f(n'), \qquad p \in P,\ n' \in P\backslash SU(2,1), \tag{A.11}$$

where $\chi(p)$ is an infinitesimal character. For ρ in the principal continuous series we may choose the character:

$$\chi_s(p) := t_1^{-2s}, \qquad p = \begin{pmatrix} t_1 & & \\ * & t_2 & \\ * & * & t_3 \end{pmatrix} \in P, \tag{A.12}$$

where we added a subscript on the character to indicate that the principal series depends on a single real parameter s. We are therefore interested in functions $f(x,\tilde{x},y) \in L^2(P\backslash SU(2,1))$ of three real variables which transform by the overall character t_1^{-2s}.

As was apparent from Eq. (A.11), a general group element $g \in SU(2,1)$ acts on $n \in P\backslash SU(2,1)$ from the right, and since this will destroy the upper triangular structure, a compensating transformation of $p \in P$ from the left is needed to restore the upper triangular form of n. This left-action of P on the second and third rows, $\vec{r}_2$ and $\vec{r}_3$, of n is quite complicated, while the action on the first row $\vec{r}_1$ is very simple: $p \in P$ simply modifies $\vec{r}_1$ by an overall factor of t_1. Moreover, the action of $k \in SU(2)\times U(1)$ leaves invariant the (complex) norms of the rows $\vec{r}_i$. We may therefore construct a spherical vector $f_K \in L^2(P\backslash SU(2,1))$ as the norm of the first row $\vec{r}_1$ of n, raised to the appropriate power of s [53][17]:

$$f_K(x,\tilde{x},y) := |\vec{r}_1|^{-2s} = \left(1+|C_1|^2+|C_2|^2\right)^{-s} = \Big(1+2(x^2+\tilde{x}^2)+4y^2+(x^2+\tilde{x}^2)^2\Big)^{-s}.$$

This object is indeed invariant under $SU(2)\times U(1)$, since the right action of k on n is a "rotation" that preserves the norm, while the compensating left action of p merely modifies f_K by an overall factor t_1^{2s}, which in turn is canceled against the character $\chi_s(p) = t_1^{-2s}$ which is present since f_K is in the principal series. We have thus found our desired spherical vector.

The next step is to compute the action of $\rho(\mathcal{V})$ on f_K. Following the prescription above, this can be done by first computing $n\cdot\mathcal{V} = p_0\cdot n'$, with

$$p_0 = \begin{pmatrix} e^{-\phi} & & \\ & 1 & \\ & & e^{\phi} \end{pmatrix} \in P, \qquad n' = \begin{pmatrix} 1 & ie^{\phi}(\bar{\lambda}+\bar{C}_2) & e^{2\phi}(\gamma + i\bar{C}_2\lambda + C_1) \\ & 1 & e^{\phi}(\lambda + C_2) \\ & & 1 \end{pmatrix} \in P\backslash SU(2,1). \tag{A.13}$$

[17]See also [73] for a similar construction in the context of $SL(3,\mathbb{R})$.

Applying this to the spherical vector $f_K(x,\tilde{x},y) = f_K(n)$ yields

$$\rho(\mathcal{V}) \cdot f_K(n) = f_K(n\mathcal{V}) = f_K(p_0 n') = \chi_s(p_0) f_K(n') = e^{2s\phi}|\vec{r}_1{}'|^{-2s}, \tag{A.14}$$

which may be written explicitly in the form

$$\rho(\mathcal{V}) \cdot f_K(C_1, C_2) = e^{-2s\phi}\Big(|\bar{C}_1 - iC_2\bar{\lambda} + \bar{\gamma}|^2 + e^{-2\phi}|C_2 + \lambda|^2 + e^{-4\phi}\Big)^{-s}. \tag{A.15}$$

The p-adic spherical vector $f_p(C_1, C_2)$ can now be found by the following method [50–52]: replace the complex norm $|\cdot|$ in the real spherical vector f_K by the p-adic counterpart $|\cdot|_p^{\mathbb{Q}[i]}$. Actually, this differs slighthly from the analysis in [50–52] (which dealt with the real Euclidean norm $||\cdot||$) in the sense that we must here consider the p-adic norm associated with the quadratic extension $\mathbb{Q}[i]$ of the rational numbers $\mathbb{Q}$. In this case the complex p-adic norm is defined as follows [75]:

$$|z|_p^{\mathbb{Q}[i]} := \sqrt{\big|z\bar{z}\big|_p}, \qquad z \in \mathbb{Q}[i], \tag{A.16}$$

where the right hand side is evaluated using the standard p-adic norm $|\cdot|_p$ since $z\bar{z} \in \mathbb{Q}$. For more information on p-adic numbers we refer the reader to [75]. The p-adic spherical vector is then given by

$$f_p(C_1, C_2) := \Big[|v_1|_p^{\mathbb{Q}[i]}\Big]^{-2s} = \max\Big(1, \sqrt{\big|C_1\bar{C}_1\big|_p}, \sqrt{\big|C_2\bar{C}_2\big|_p}\Big)^{-2s}. \tag{A.17}$$

A.3 Product over Primes

The automorphic form $\Psi(\mathcal{V})$ in this representation now reads

$$\Psi(\mathcal{V}) = \sum_{(x,\tilde{x},y)\in\mathbb{Q}^3} \Big[\prod_{p<\infty} \max\Big(1, \sqrt{\big|C_1\bar{C}_1\big|_p}, \sqrt{\big|C_2\bar{C}_2\big|_p}\Big)^{-2s}\Big] \rho(\mathcal{V}) \cdot f_K(x,\tilde{x},y). \tag{A.18}$$

We can also write the summation over the complex rational variables $(C_1, C_2) \in \mathbb{Q}[i]^2$ instead of the rational variables $(x,\tilde{x},y) \in \mathbb{Q}^3$, if we incorporate the constraint from Eq. (A.10) as follows

$$\Psi(\mathcal{V}) = \sum_{(C_1,C_2)\in\mathbb{Q}[i]^2} \delta\Big(|C_2|^2 - \Im(C_1)\Big)\Big[\prod_{p<\infty} f_p(C_1, C_2)\Big] \rho(\mathcal{V}) \cdot f_K(C_1, C_2). \tag{A.19}$$

Next we must evaluate the infinite product over prime numbers. To this end we split the rational variables C_1 and C_2 in the following way:[18]

$$C_1 = \frac{\omega_1}{\omega_3}, \quad C_2 = \frac{i\bar{\omega}_2}{\bar{\omega}_3}, \tag{A.20}$$

with $\omega_j \in \mathbb{Z}[i]$, for $j = 1,2,3$ and $\gcd(\omega_1, \omega_2) = 1$. We can now evaluate the infinite product over primes with the simple result

$$\prod_{p<\infty} \max\Big(1, \sqrt{\Big|\frac{\omega_1\bar{\omega}_1}{\omega_3\bar{\omega}_3}\Big|_p}, \sqrt{\Big|\frac{\omega_2\bar{\omega}_2}{\omega_3\bar{\omega}_3}\Big|_p}\Big)^{-2s} = |\omega_3|^{-2s}. \tag{A.21}$$

[18] We note that the greatest common divisor in $\mathbb{Q}[i]$ is defined up to Gaussian units which are a subgroup of order 4 in the Gaussian integers $\mathbb{Z}[i]$.

We also multiply the constraint $|C_2|^2 - \Im(C_1)$ by a factor of $|\omega_3|^2$ which yields

$$|\omega_3|^2\Big(|C_2|^2 - \Im(C_1)\Big) = |\omega_2|^2 - 2\Im(\omega_1\bar{\omega}_3) = \vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = 0. \tag{A.22}$$

Combining Eqs. (A.19), (A.21) and (A.22) then yields the final form of $\Psi(\mathcal{V})$:

$$\Psi(\mathcal{V}) = \sum_{\substack{\vec{\omega}\in\mathbb{Z}[i]^3 \\ \omega_3\neq 0,\ \gcd(\omega_1,\omega_2)=1}} \delta(\vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega})e^{-2s\phi}\Big[|\bar{\omega}_1+\bar{\omega}_2\bar{\lambda}+\bar{\omega}_3\bar{\gamma}|^2+e^{-2\phi}|\bar{\omega}_2-i\bar{\omega}_3\lambda|^2+e^{-4\phi}|\omega_3|^2\Big]^{-s}, \tag{A.23}$$

which we recognize as the Eisenstein series $\mathcal{E}_s(\phi,\lambda,\gamma)$ constructed in Section 3.1. More correctly, this result is equal to $\mathcal{E}_s(\phi,\lambda,\gamma)$ modulo the term in the sum with $\omega_3 = 0$, since this term is not allowed in Eq. (A.23) by virtue of Eq. (A.20). The same phenomenon also happens when constructing non-holomorphic $SL(2,\mathbb{Z})$-invariant Eisenstein series using the p-adic approach [50, 52].

B. Gaussian Integers, Dirichlet series and the Abelian measure

In this appendix, we collect for the reader's convenience some standard facts about Dirichlet series, Gaussian integers and analyse the norm constraint (4.29) entering at various places in the Fourier expansion in detail.

B.1 Euler products and Dirichlet series

A series $a(n)$ for $n \in \mathbb{N}$ is called multiplicative iff $a(n_1n_2) = a(n_1)a(n_2)$ whenever n_1 and n_2 are coprime [76]. The associated Dirichlet series

$$L(a,s) = \sum_{n>0} a(n)n^{-s} \tag{B.1}$$

constructed from a multiplicative $a(n)$ can be recast as an Euler product over the primes $(p > 1)$

$$L(a,s) = \prod_{p\text{ prime}} P(p,s)\,, \tag{B.2}$$

where

$$P(p,s) = \sum_{k\geq 0} a(p^k)p^{-ks}\,. \tag{B.3}$$

As an example consider the multiplicative series (4.46). One finds

$$P(2,s) = \sum_{k\geq 0}(2^{1-s})^k = \frac{1}{1-2^{1-s}} \tag{B.4}$$

and for Pythagorean primes $p = 1 \mod 4$

$$P(p,s) = \sum_{k\geq 0}(p^{1-s})^k + \frac{p-1}{1-s}\partial_p\sum_{k\geq 0}(p^{1-s})^k = \frac{1-p^{-s}}{(1-p^{1-s})^2}\,. \tag{B.5}$$

For primes of the form $p = 3 \mod 4$ one has

$$P(p,s) = \sum_{k\geq 0}(p^{2-2s})^k + p^{-s}\sum_{k\geq 0}(p^{2-2s})^k = \frac{1+p^{-s}}{(1-p^{1-s})(1+p^{1-s})}\,, \tag{B.6}$$

whence one recovers (4.57).

B.2 Inert, split and ramified structure of Gaussian primes

The Gaussian integers $\mathbb{Z}[i]$ are a principal ideal domain [77, 78]. There are four units given by ± 1 and $\pm i$; prime factorization over $\mathbb{Z}[i]$ is uniquely defined up to unit migration. The prime factors entering can be of three types. We will use the notation g for Gaussian primes and p for standard primes:

(i) The ramified case corresponds to $g = 1 + i$, arising from the standard prime $p = 2$ by taking a square root.

(ii) Inert primes $g = p = 4n + 3$ (for some $n \in \mathbb{N}$), coming from such prime numbers over the usual integers.

(iii) Split primes coming in complex conjugate pairs g and $\bar{g}$, arising from usual prime numbers of the form $p = 4n + 1$. These are called Pythagorean primes since they are of the form $p = a^2 + b^2$ and then $g = a + ib$. By Fermat's theorem on the sums of squares these are exactly the primes in $\mathbb{N}$ admitting a factorization of this type.

The structure of Gaussian primes over the usual primes is recorded by the Dirichlet beta function.

B.3 Analysis of norm constraint

The norm constraint (4.29) requires to find, for a fixed integer d, all Gaussian integers with norm squared divisible by $2d$. In this appendix we write this constraint as

$$|\alpha|^2 \equiv 0 \mod 2d. \tag{B.7}$$

B.3.1 Multiplicative structure

The solutions to the norm constraint possess a multiplicative structure. Let d_1 and d_2 be coprime integers and let α_1 and α_2 be Gaussian integers such that $2d_i$ divides $|\alpha_i|^2$. Then clearly $\alpha_1\alpha_2$ satisfies the norm constraint for $d_1 d_2$. Due to prime factorization of Gaussian integers we know that this describes all solutions and it is therefore sufficient to study the solutions to the norm constraint for powers of primes $d = p^k$. There are three qualitatively different cases.

(i) $p = 2$, whence $d = 2^k$. The structure of the set of solutions looks different for k even and odd. For k even one has that

$$\alpha = 2^{k/2}(n_1 + n_2) \qquad \text{for } n_1 + n_2 \in 2\mathbb{Z} \tag{B.8}$$

solves the constraint, whereas for k odd

$$\alpha = 2^{(k+1)/2}(n_1 + n_2) \tag{B.9}$$

solves the constraint without restriction on the integers n_1 and n_2.

(ii) $p = 4n+3$. Again one has to distinguish k even and k odd in solving (B.7) for $d = p^k$. For k even one has

$$\alpha = p^{k/2}(n_1 + n_2) \qquad \text{for } n_1 + n_2 \in 2\mathbb{Z} \tag{B.10}$$

solves the constraint, whereas for k odd

$$\alpha = p^{(k+1)/2}(n_1 + n_2) \qquad \text{for } n_1 + n_2 \in 2\mathbb{Z} \tag{B.11}$$

solves the constraint. Note that there are restrictions on the integers n_1 and n_2 in both cases.

(iii) $p = 4n+1$. This case is the most complicated one. Any such prime can be written as $p = a^2 + b^2$ for some integers a and b and we assume $a > b$ without loss of generality. To describe the set of solutions to (B.7) for $d = p^k$ we again distinguish even and odd k. An important auxiliary definition is furnished by

$$e_k = (a - ib)^k(1 + i) \qquad \Rightarrow \qquad |e_k|^2 = 2p^k\,, \tag{B.12}$$

providing an elementary solution of the constraint. With the help of the Gaussian integer e_k one can define the following pairs of lattices for k odd and $j = 0, \ldots, \frac{k-1}{2}$

$$\begin{aligned} \Lambda_{j+1} &= \left\{p^j(k_1 e_{k-2j} + k_2 i e_{k-2j}) \,:\, k_1, k_2 \in \mathbb{Z}\right\}\,, \\ \bar{\Lambda}_{j+1} &= \left\{p^j(k_1 \bar{e}_{k-2j} + k_2 i \bar{e}_{k-2j}) \,:\, k_1, k_2 \in \mathbb{Z}\right\}\,. \end{aligned} \tag{B.13}$$

The set of all solutions for k odd is then given by

$$\bigcup_{j=0}^{(k-1)/2} (\Lambda_{j+1} \cup \bar{\Lambda}_{j+1})\,. \tag{B.14}$$

For k even one also requires the lattice

$$\Lambda_{k+1} = \left\{p^{k/2}(k_1 + ik_2) \,:\, k_1, k_2 \in \mathbb{Z} \text{ and } k_1 + k_2 \in 2\mathbb{Z}\right\} \tag{B.15}$$

and then all solutions are given by

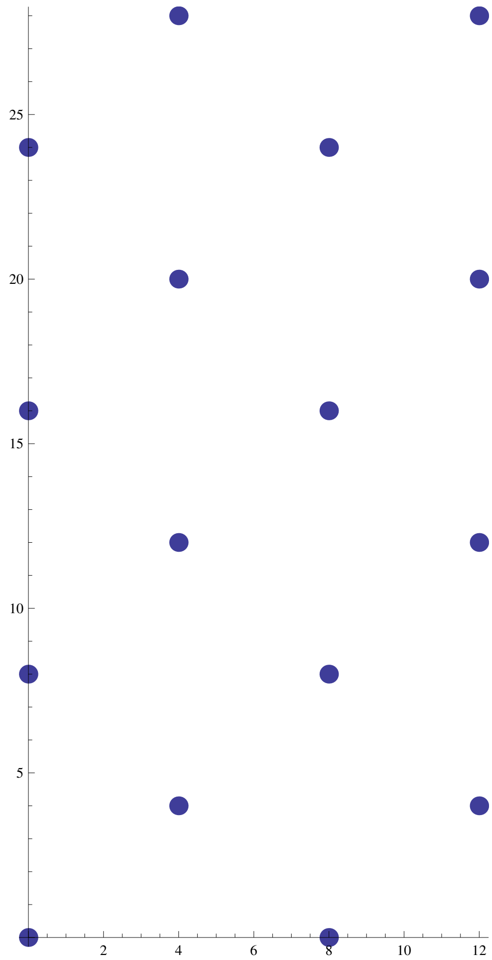
$$\bigcup_{j=0}^{k/2-1} (\Lambda_{j+1} \cup \bar{\Lambda}_{j+1}) \cup \Lambda_{k+1}\,. \tag{B.16}$$

Pictures of the three kinds of solution sets will be given momentarily when discussing the restriction to a fundamental domain under the action of a translation group.

B.3.2 Restriction to fundamental domain

In the abelian measure we made use of writing the solution to the constraint in terms of solutions in a fundamental domain in (4.43). We denote by

$$\mathcal{F}(d) = \left\{\alpha \in \mathbb{Z}[i] \,:\, |\alpha|^2 \equiv 0 \mod 2d \text{ and } 0 \le \Re(\alpha) < d,\, 0 \le \Im(\alpha) < 2d\right\} \tag{B.17}$$



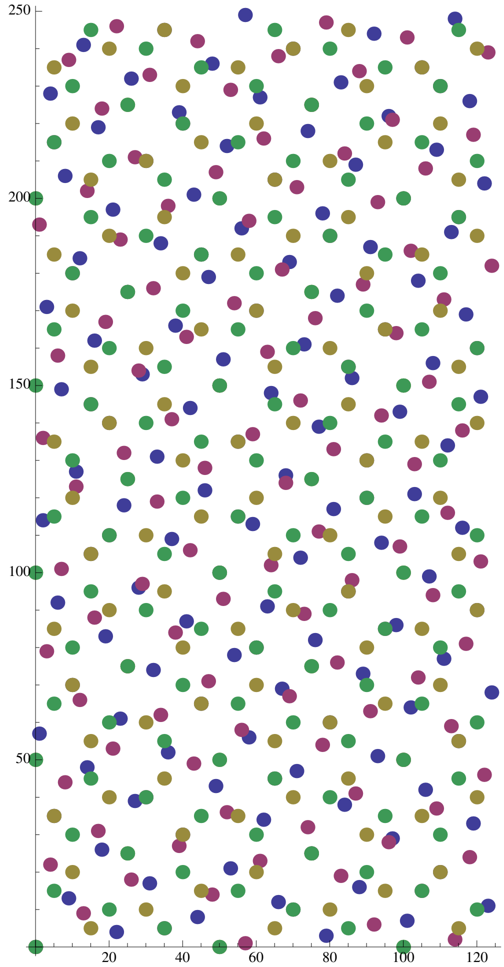


Figure 1: Left: *The set* $\mathcal{F}(2^4)$ *is a* $\pi/4$ *rotated and rescaled square lattice.* Right: *The set* $\mathcal{F}(5^3)$ *as the intersection of four lattices with common points.*

the set of solutions to (B.7) in the fundamental domain. From the analysis above we know that for d_1 and d_2 coprime the following holds

$$\mathcal{F}(d_1 d_2) \cong \mathcal{F}(d_1) \times \mathcal{F}(d_2) \tag{B.18}$$

where the solutions are of the form $d_2 f_1 + d_1 f_2$ for $f_i \in \mathcal{F}(d_i)$ up to translation by the lattice L of (4.44) defining the fundamental domain. Therefore it is sufficient to restrict to $d = p^k$ being a power of a prime. For describing (B.17) more explicitly we have to make recourse to the results of the preceding section and distinguish three cases.

(i) $d = 2^k$. Here one simply restricts the integers n_1 and n_2 in (B.8) and (B.9). The number of points in the fundamental domain is

$$\sharp\, \mathcal{F}(2^k) = 2^k = N(2^k) \tag{B.19}$$

in agreement with (4.56). The easiest way of doing the counting is by computing the sizes of the fundamental cell of the lattices and comparing to the total area of the fundamental domain $2d^2$. A typical lattice is depicted on the left of fig. 1. For $k=1$ there are two points in the fundamental domain.

(ii) $d=p^k$ for $p=4n+3$. The counting works similar but one has to take into account the additional constraint in (B.11) leading to

$$\sharp\,\mathcal{F}(p^k) = p^{k-(k \mod 2)} = N(p^k)\,. \tag{B.20}$$

The form of the lattice is identical to that of the left part of fig. 1. For $k=1$ there is only a single point $\alpha=0$ in the fundamental domain.

(iii) $d=p^k$ for $p=4n+1$. The counting of points in the fundamental domain is now more involved since the individual lattices in (B.14), or (B.16), have common points that should not be overcounted. Each lattice has p^k points but, for example, there are p^{2j} common points for the lattices Λ_{j+1} and $\bar{\Lambda}_{j+1}$ forming the square lattice

$$\Lambda_{j+1}\cap\bar{\Lambda}_{j+1} = \Big\{p^{k-j}(k_1+ik_2)\,:\,k_1+k_2\in 2\mathbb{Z}\Big\}\,, \tag{B.21}$$

implying

$$\sharp\,\big((\Lambda_{j+1}\cup\bar{\Lambda}_{j+1})\cap\mathcal{F}(d)\big) = 2p^k - p^{2j}\,. \tag{B.22}$$

One can also show that

$$\sharp\,\big((\Lambda_{j+1}\cup\bar{\Lambda}_{j+1})\cap(\Lambda_{j+2}\cup\bar{\Lambda}_{j+2})\cap\mathcal{F}(d)\big) = 2p^{k-1} - p^{2j} \tag{B.23}$$

and that all common points between pairs of lattices whose indices are farther apart are already contained in the intersection above. Putting everything together one arrives at the following count of points in the fundamental domain for k odd

$$\sharp\,\mathcal{F}(p^k) = 2p^k - p^0 + \sum_{j=1}^{(k-1)/2}\big(2p^k - p^{2j} - (2p^{k-1}-p^{2j-2})\big) = (k+1)p^k - kp^{k-1} = N(p^k)\,. \tag{B.24}$$

For k even the analysis is similar. An example of lattices with intersection points can be found on the right of fig. 1.

B.4 Rewriting the abelian measure

We now turn to deriving (4.71) from the abelian measure (4.69). This involves mainly demonstrating the equality (4.70). To this end we introduce an additional function on the Gaussian integers

$$\begin{aligned}\nu_s(q) &= |q|^{2s-2}\beta(2s-1)\left(\sum_{d>0} d^{1-2s}\sum_{f\in\mathcal{F}(d)} e^{\frac{\pi i}{d}\Re[qf(1-i)]}\right)\\ &= |q|^{2s-2}\,\beta(2s-1)\sum_{d>0} d^{1-2s}a_q(d)\,.\end{aligned} \tag{B.25}$$

The function $\nu_s(q)$ is related to the l.h.s. of (4.70) in an obvious way.

B.4.1 Evaluation of the auxiliary functions $a_q(d)$ and $\nu_s(q)$

The first observation is that the series $a_q(d)$ is multiplicative in d for fixed q (but not in q). This follows from (B.18) and a simple rewriting of the exponent. Therefore it is sufficient to determine $a_q(d)$ for $d = p^k$. This is where the description of the sets $\mathcal{F}(p^k)$ enters. The series $\nu_s(q)$ can be shown to be multiplicative so we only require $a_q(d)$ for $q = g^m$ as a power of a Gaussian prime and $d = p^k$ the power of standard prime.

The next observation is that

$$a_{g^m}(p^k) = \sum_{f \in \mathcal{F}(p^k)} e^{\frac{\pi i}{p^k} \Re[g^m f(1-i)]} = \sum_{f \in g^m \mathcal{F}(p^k)} e^{\frac{\pi i}{p^k} \Re[f(1-i)]} \tag{B.26}$$

by rotating (and rescaling) the set of fundamental solutions. If g does not divide p then the rotated set is an equivalently good fundamental set of solutions. Hence

$$a_{g^m}(p^k) = a_1(p^k) \quad \text{if } g \text{ does not divide } p. \tag{B.27}$$

For this reason we will first evaluate $a_1(p^k)$ and treat the case when g divides p afterwards.

It turns out that it suffices to count the number of times the lattice containing only the point $\alpha = 0$ in $\mathcal{F}(p^k)$ appears in the sum over $\mathcal{F}(p^k)$. For all other lattices the sum over phases is zero. Hence one finds immediately

$$a_1(2^k) = 0 \qquad \text{for } k > 0 \tag{B.28}$$

and for $p = 4n + 3$ that

$$a_1(p^k) = \begin{cases} 1 \ , \ k = 1 \\ 0 \ , \ k > 1 \end{cases} . \tag{B.29}$$

For $p = 4n + 1$ one has to count more carefully due to the intersection points. For $k = 1$ the origin is the only common point in Λ_1 and $\bar{\Lambda}_1$ and hence is overcounted once leading to $a_1(p) = -1$. For $k > 1$ this is offset by the intersection with $\Lambda_2 \cup \bar{\Lambda}_2$, making $a_1(p^k)$ vanish. In total one has therefore for the Pythagorean primes

$$a_1(p^k) = \begin{cases} -1 \ , \ k = 1 \\ \ \ 0 \ \ , \ k > 1 \end{cases} . \tag{B.30}$$

Constructing the Dirichlet series in (B.25) via its Euler product therefore leads to, after referring back to (4.25)

$$\sum_{d>0} d^{1-2s} a_1(d) = \frac{1}{\beta(2s-1)} \qquad \Rightarrow \qquad \nu_s(1) = 1 \, . \tag{B.31}$$

For $q = g^m$ one can perform a scaling of the lattices involved. Starting with the case of $g = 1 + i$ the value $p = 2$ is important and one has

$$(1+i)^m \mathcal{F}(2^k) \cong [\mathcal{F}(2^{k-m})]^{2^m} \tag{B.32}$$

defining the right hand side to consist only of the origin for $k \leq m$. Counting now the number of times the origin appears leads to

$$a_{(1+i)^m}(2^k) = \begin{cases} 0 & \text{for } k > m \\ 2^k = N(2^k) & \text{for } k \leq m \end{cases} . \tag{B.33}$$

The corresponding auxiliary series (B.25) is then

$$\begin{aligned} \nu_s((1+i)^m) &= 2^{m(s-1)}\beta(2s-1)\left(1+\sum_{k=1}^{m} 2^{k(2-2s)}\right)\sum_{d>0} d^{1-2s}a_1(d) \\ &= \frac{1}{4}2^{m(1-s)}\sum_{z|(1+i)^m} |z|^{4s-4} \end{aligned} \tag{B.34}$$

and so is the usual divisor function multiplied by the right power to make it symmetric under $s \leftrightarrow 2-s$.

A similar analysis can be carried out for inert primes leading to

$$a_{p^m}(p^k) = \begin{cases} 0 & \text{for } k \geq 2(m+1) \\ p^{k-k \mod 2} = N(p^k) & \text{for } k < 2(m+1) \end{cases} . \tag{B.35}$$

Therefore the full result for (B.25) for inert primes is

$$\begin{aligned} \nu_s(p^m) &= p^{2m(s-1)}\beta(2s-1)\left(1+\sum_{k=1}^{2m+1} p^{k-(k \mod 2)+k(1-2s)}\right)\frac{\sum_{d>0} d^{1-2s}a_1(d)}{1+p^{1-2s}} \\ &= \frac{1}{4}p^{2m(1-s)}\sum_{z|p^m} |z|^{4s-4}. \end{aligned} \tag{B.36}$$

For split primes one finds

$$a_{g^m}(p^k) = \begin{cases} 0 & \text{for } k > m+1 \\ -p^m & \text{for } k = m+1 \\ p^{k-1}(p-1) & \text{for } k < m+1 \end{cases} . \tag{B.37}$$

Therefore the full result for split primes is

$$\begin{aligned} \nu_s(g^m) &= \frac{p^{m(s-1)}}{1-p^{1-2s}}\left(1+\sum_{k=1}^{m}(p-1)p^{k-1+k(1-2s)} - p^{m+(m+1)(1-2s)}\right) \\ &= \frac{1}{4}|g|^{2m(1-s)}\sum_{z|g^m} |z|^{4s-4}. \end{aligned} \tag{B.38}$$

In summary the function $\nu_s(q)$ defined in (B.25) takes the value

$$\nu_s(q) = \frac{1}{4}|q|^{2-2s}\sum_{z|q} |z|^{4s-4}, \tag{B.39}$$

for any Gaussian integer $q \neq 0$ and therefore is a Gaussian divisor function in disguise.

B.4.2 The abelian instanton measure

The abelian instanton measure of (4.69) is thus given by a sum over primitive divisors of $\Lambda = \ell_2 - i\ell_1$

$$\mu_s(\ell_1, \ell_2) = 4 \sum_{\omega_3'|\Lambda} |\Lambda|^{2-2s} \nu_s \left(\frac{\Lambda}{\omega_3'}\right) , \tag{B.40}$$

which, together with (B.39), leads to (4.71).

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