Master of science thesis in physics

M(embrane)-Theory

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Abstract

We investigate the uses of membranes in theoretical physics. Starting with the bosonic membrane and the formulation of its dynamics we then move forward in time to the introduction of supersymmetry. Matrix theory is introduced and a full proof of the continuous spectrum of the supermembrane is given. After this we deal with various concepts in M-theory (BPS-states, Matrix Theory, torodial compactifications etc.) that are of special importance when motivating the algebraic approach to M-theoretic caluclations. This approach is then dealt with by first reviewing the prototypical example of the Type IIB R^4 amplitude and then the various issues of microscopic derivations of the corresponding results through first-principle computations in M-theory. This leads us to the mathematics of automorphic forms and the main result of this thesis, a calculation of the p-adic spherical vector in a minimal representation of $SO(4,4,\mathbb{Z})$

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Preface

The thesis now in your hand has undergone several revisions before assuming this, its final form. Upon commencing work on this version, in the late summer of 2002, I set out a number of goals for myself. One of these goals was to try to write an as self-contained thesis as possible, the amount of paper in your hand at this very moment is a direct consequence of this goal. Another goal was to create a firm base of knowledge to stand on in my future research, so I traced back to the very birth of the membrane, an event taking place in an ancient era shrouded in mystery and known to some as 'the sixties'. This is where the journey taking place in this thesis begins. It then spans an interval of some forty of the lord's years, a period that saw the birth and death of many excellent attempts in physics (and in membrane theory). Roaming across this vastness of publications was a very humbling task, as I have come across many seminal ideas, but also some that make me wonder whether future generations will say about these times that"*Some things that should not have been forgotten, were lost*".

The early parts of this thesis can best be described as a collection of reviews, and though I present no new results I have tried to collect these reviews in a manner which I have found in no other publication so far. In the latter part it becomes easier to add at least some insight and new ideas to the presented material as it is both incomplete and something that I have spent a great deal of time working on myself. I have also taken the risk of including some of my own thoughts and ideas in the last chapter in hope of at best awake some interest or at least amusement.

With these words I leave the prospective reader to walk the path which I have cut through the wilderness of actions and symmetries, Godspeed.

-Tänk dig hur enkelt det var, kunde han sucka. Tänk dig 1900-talets lilla universum, en liten hemtrevlig rymd med några miljarder vintergator, några miljoner ljusår ifrån varandra. Man kunde sitta så trygg vid sitt teleskop och nästan känna hödoften och höra fågelkvittret utifrån rymderna ... - Peter Nilson

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Introduction

String theory has been said to be“21th century physics cast into the 20th century” and the same thing can undoubtedly be said about membrane theory (or M-theory). The many fundamental questions that have yet to be answered can best be summed up in the single question “What is M-theory?”. Being less general we can also ask the question, what is the membrane? The theory of membranes has been envisaged to describe a multitude of physical systems, none of which have been completely successful or adequately investigated. From electron models to bag-models onto relativistic surfaces and more recently fundamental degrees of freedom the fundamental problems essentially remains the same. This thesis is an attempt to review all of these attempts to some extent, highlighting the problems and noting the different attempts at solving them. The thread running through each and every chapter is the desire to gain understanding about the aforementioned question regarding the true nature of the membrane and it is this question that drives us through several decades of physics and a body of material so vast that no one can claim to have overlooked it all. The content of this thesis rests upon the shoulders of those who have gone before us and to which we should be ever so grateful. Are we ahead of our times? Are we presumptive in thinking that we can answer the questions that stand before us? Perhaps we are. But in science there is no way to go but forward and boldly so. It is our obligated duty as physicists, scientists, humans and inhabitants of the ever-expanding entity that is *our* universe to grab a pick-ax and hack away at the mountain of unresolved issues[1].

[1]But also to drink loads of coffee and show the world how intellect enables us to pick up beautiful women in spite of looking like road-kill ourselves.

1.1 History

The history of membranes in theoretical physics is a long and complicated one. The first real attempt to use membranes to construct a fundamental theory was [1] where Dirac considered the electron to be a charged conducting surface, "a bubble in the electromagnetic field". This theory never really became a hit and it suffers from a multitude of problems, some that are general membrane-problems that we will study here. Membranes became popular again with the rising interest in string theory during the 70's. The reason for this was quite natural, after all, if one considers extended objects of one dimension why then not consider extended object of two, three, four and generally d dimensions. The first thorough analysis of the dynamics of classical and quantum (bosonic) membranes was done by Collins and Tucker in [2]. This was before string theory was regarded as a TOE and one of the chief motivations for studying membranes was to describe the dynamics of quarks. In the above mentioned paper the authors expected to (if the theory was correct) extract the properties of quark-like constituents directly from the dynamics of the membrane. This was of course not the case, as with string theory, membrane theory can not describe strong interactions alone. The classical bosonic membrane has a continuous spectrum and this is a troublesome fact. It is related to the fact that the membrane potential is such that states can escape to infinity through thin valleys without rendering the energy infinite. Luckily this property disappears in the quantum theory. The Hamiltonian is of such a form that the quantum theory has a discrete spectrum even though the classical theory does not. A small set of Hamiltonians obey this principle and they were first studied in the paper [3] and we will later give a proof of this property in the case of the bosonic membrane. Now since there is only so much you can do with the bosonic membrane and a theory which can not incorporate fermions is only so interesting. It was natural to try to formulate a theory with supersymmetric membranes, supermembranes.

But before we dig into the historical developments concerning the supermembrane let us mention another, less successful, attempt of formulating a supersymmetric theory of membranes. Before the supermembrane was proved to exist most of the focus was on membranes with supersymmetry introduced on the worldvolume, i.e. "spinning membranes". In string theory it is possible to introduce supersymmetry in this way (it turns out to be equivalent to the target-space supersymmetric string), in membrane theory it is not. The first attempt at such an action was in [4] and many papers in the area followed this until finally the no-go theorem for spinning membranes was presented in [5].

After the paper [6] by Hughes, Liu and Polchinski in 1986, the attention of the physics community shifted from ordinary membranes to the newborn theory of supermembranes. These we previously thought not to exist since it was uncertain whether the κ-symmetry of string theory, upon which the whole supersymmetric formalism relies, could be generalized to membranes. In [6] it was shown that for 3-branes in six dimensions it could and in [7] this was extended to more general objects. A lot of work was put into investigating this new theory and the main problem became the apparent continuity of its spectrum and whether this spectrum contained massless states. In [8] it was then proved that the supermembrane indeed had a continuous spectrum and this lead to a major decline in the interest for the theory. This result was gotten through the use of Matrix theory, a brand new way of dealing with membranes through a regularization reshaping the theory into a more manageable $(0+1)$-dimensional $SU(N)$ quantum mechanical theory. That this was possible was discovered first in [9, 10] for the spherical and torodial bosonic membranes, a result subsequently extended to all topologies and also the supermembrane in [11, 12].

The discovery of the supermembrane came only a couple of years after the period often called “the first superstring revolution”, and the growing interest in string theory did not help to resolve the issues that faced physicists working on the membrane. During this time string theory went through a time of intensive development, eventually leading up to the year 1994 and “the second superstring revolution”, consisting mainly of the discovery of dualities relating the different string theories and hinting at a more fundamental theory underlying all these. This theory was then seen to be the high energy limit of 11-dimensional supergravity and to contain membranes as a dynamical object. This presented both new reasons for working on membranes as well as methods to do so. The question of the spectrum was tackled by asserting that the membrane, an integral part of M-theory, already described a ’second-quantized’ theory, whereupon the continuous spectrum is not only desired but crucial in forming the theory that we want M-theory to be. Matrix theory was seen to fit into the picture in an unprecedented way, as giving all the dynamics of M-theory in the infinite momentum frame of the light-cone gauge [13]. The perturbative picture has become ever more clear in the years that have passed and we have gained considerable insight into how the different string theories emerge as asymptotic limits of M-theory. With the discovery of M-theory the importance of non-perturbative methods became manifest and some dualities were shown to be such tools. Still there are a lot of problems that can not be solved be means of duality transformation of perturbative calculations and this leaves a large part of the M-theory picture unreachable from the roads previously laid down.

The year 1997 (modulo a few months) was an exciting one for "non-perturbative string theorists". It began with the publishing of [14] that is an extension of the important paper [15]. It was really the first effort to make use of the newly discovered dualities when doing calculations in the same way that perturbative modular invariance has been made use of. By utilization of the techniques in this paper the authors, and the authors of the subsequent papers were able to derive exact results (perturbatively as well as non-perturbatively) in Type IIB and M-theory. An intense effort was made to push these methods as far as possible, and many papers were published in the year of 1997 and early 1998 but as with many other 'booms' (and this was a small one) interest sank as complications arose. The main complication in pushing this method to the limit (which is M-theory) is that the mathematical tools that are needed are simply not developed yet, so for the few physicist that continued working on this problem this is where their work lead them, to the mathematical arena of number theory and more specifically automorphic forms.

The paper [16] was the first in which the outline of the project to perform microscopic calculations in M-theory was presented. Continuing this work in the paper [17] a crucial ingredient in the construction of the invariants giving the M-theory amplitudes was obtained. One last hurdle remains and steps have been taken [18] to overcome this as well but still the complete automorphic forms remain to be created.

1.2 Outline

This thesis is split up into three parts. The first part deals with the general theory of membranes and supermembranes which is essentially work done in "pre-M" times. We start be reviewing the theory of the bosonic membrane with special emphasis on the problems that arise in that theory and how they are resolved. Most of the work done on the bosonic membrane was done before 1986, the year in which Hughes, Liu and Polchinski published their paper [6], after which most, if not all, efforts were focused on the supermembrane. We continue by reviewing this early work on supermembranes, moving in the time period spanning from 1986 to the end of that decade. The main focus is, as in the previous section, on the actions and the problems with these actions. We start by looking at the early work on spinning membranes and then we give the proof that spelled the doom for membranes with pure local supersymmetry on the worldvolume (at least for the time being). After this we move on to membranes with target space supersymmetry as they were formulated in [6] and subsequently in [7]. We start by analyzing the dynamics of the supermembrane in a flat spacetime, we derive κ-symmetry and analyze the different parts of the action. After this we generalize our supersymmetric action to a curved background. We review the work done in [7], again with special emphasis on κ-symmetry. Next we review the connection between supermembrane theory and supersymmetric $SU(N)$ matrix theory. We use this relation to study the spectrum of this theory and also a simpler toy-model.

The second part is mostly a prelude to the third chapter because it contains material that is essential to this chapter. But it also bridges the gap between the two different eras in membrane theory. In addition to this we include some short, mostly historical, reviews that are not directly relevant but nonetheless should exist in any thesis dealing with M-theory. We begin with string theory outlining the birth and uses of it, how our view on it changed with two big revolutions and how one theory was split into five and then eventually reunited into one theory again. We treat the relation between Type IIA string theory and the membrane briefly in this section as well. The next section deals with discoveries that were made in the second superstring revolution, we discuss how dualities relate the different string theories, showing us glimpses of a larger framework, and how these dualities require the existence of dynamical objects known as D-branes. Finally we talk about the moduli space of M-theory and how dualities are transformations in this moduli space. The subsequent section is about a very important kind of states in our theories, namely BPS-states. These are also treated in a more general manner in appendix 3.3 and for the reader unfamiliar with the concept of BPS-states in supersymmetric theories it is recommended to at least briefly flip through this appendix. We talk about the different BPS-states of M-theory which will become important when we proceed to the next chapter. We then extend our considerations of dualities in the following section, discussing the groups that describe the transformations in the moduli space and the representation theory of these groups. Finally we conclude this chapter by a section dealing with the BFSS-conjecture. This section is included mostly for completeness. We will not use the material presented here in our following deliberations but this seminal idea further reveals the role of the supermembrane in M-theory and how we can proceed to analyze it.

The third part of this thesis deals with a more recent development in M-theory regarding the mathematics of automorphic forms. Our journey starts in Type IIB string theory with the paper [14], concerning the effective R^4 action, the emphasis here is on how symmetry under duality groups can help us in determining exact results (i.e. both perturbative and non-perturbative contributions). The work started in this paper is continued in [19, 20, 21], extended to other levels of compactification, other quantities and related to M-theory. The following section is dedicated to further study of these techniques within the body of M-theory. At first we motivate the attempt to carry out a first-principle computation of exact amplitudes in M-theory and after this we review the actual attempt, setting out in physics but actually ending up in a little known area of mathematics [22,23,16]. The third and last section in this chapter concerns the calculation that comes as a result of the papers reviewed in the previous section. The nature of this material [17, 24, 18] is very mathematical and here we make full use of the appendices. We also touch upon the work performed by Dr. Hegarty and myself.

The final chapter, entitled 'Conclusion', does not only contain the conclusions of this work. Therein is also collected the various ideas and thoughts that have come about in the process of working with this thesis. There is no thread running through this chapter, instead each section is related to a section in the previous three chapters. It contains various thoughts on how to approach problems and interpret results, as these are seen by a "fresh pair of eyes" (read 'beginner') in this field of physics. The first section deals with the bosonic membrane and some ideas regarding a new approach to these. This is followed by a section on the supermembrane, including spinning membranes and matrix theory. After this we deal with the use of p-adic numbers in physics and especially in the 'algebraic approach' of chapter 4. Finally the last section concerns chapter 4 of this thesis, and the project which that chapter describes.

There are a number of appendices in this thesis and a substantial part of the background material has been shifted to these sections in order to maintain some level of continuity in the previous three chapters. Some of the material concerns purely physical results, others purely mathematical results or theories. The first appendix deals with supersymmetry and supergravity. We make now claim of presenting a wholesome picture of these vast areas of physics and the appendix merely touches upon the concepts needed in the main text. The following appendix concerns p-adic analysis, a large area of mathematics that string theorists have made use of many times over the years. Finally the last appendix is about Eisenstein series which are really the main characters in this thesis. The important definitions and theorems are included here, but as with most other areas it would be impossible to give a complete review. Hopefully the reader will find enough background material here to be able to read the whole thesis since the effort has been to write an as self-contained thesis as possible.

1.3 Notation and Conventions

Since this thesis spans over many fields, a clash in notation is inevitable. I have therefore chosen to use the original notation in as many cases as possible rather then defining my own in each case. This inevitably leads a slightly more complex picture here, but helps the reader when going to original works. A list of symbols from chapter 2 is located at the very end of this thesis in order to ease the reading of that chapter.

2

The Supermembrane I: General Theory and Problems

This chapter covers the general theory of bosonic membranes and supermembranes. The work presented here was essentially done before the discovery of M-theory shed new light on the question regarding the role of membranes in fundamental theories. Special emphasis is put on the problems that are inherent in the theories that are presented here. Most of the problems that haunted the supermembrane in pre-M times remain today, but in the light of M-theory these problems are open for new interpretations. Apart from the original work referred to throughout this chapter a few previous reviews are worth mentioning, namely [25, 26, 27]

2.1 The Bosonic Membrane

We will begin by constructing and analyzing an action for a free bosonic membrane (note that the actions we present here and the analysis we perform of them could equally well have been done for a general p-brane, but we restrict our attention to the membrane in order to keep our focus), a construction that is done in almost complete analogy with the string theory case. As the membrane propagates in a D-dimensional spacetime (with $D \geq 3$) it traces out a 3-dimensional *worldvolume* on which we wish to construct a field theory. As in string theory we define scalar fields, X^μ, on this worldvolume describing the embedding of the membrane in spacetime. These fields are functions of the three variables, ξ^i, $(i = 0, 1, 2)$ parametrizing the worldvolume. The action is then simply the total volume

$$S_M = -T_3 \int d^3\xi \sqrt{-\det \partial_i X^\mu \partial_j X_\mu}, \tag{2.1}$$

where the index M on the action indicates that the space has Minkowski signature $(-, +, +, +)$, and the indices (i, j) runs from 0 to $D-1$. The constant T_3 is the tension of the membrane, in "God-given" units which has dimension (mass) $\times$ (length)$^{-2}$, and it assures us that the action is dimensionless (we will set this constant to unity in what follows). This action is immediately recognized as a generalization of the Nambu-Goto action for the string and was first proposed by Dirac in [1]. There is also a classically equivalent action that was proposed by Howe and Tucker in [4], obtained by introducing an "independent" worldvolume metric g_{ij},

$$S'_M = -\frac{1}{2} \int d^3\xi \sqrt{-g}(g^{ij} \partial_i X^\mu \partial_j X_\mu - 1), \tag{2.2}$$

where $g = \det g_{ij}$. From this action we find the equations of motion to be

$$\partial_i(\sqrt{-g} g^{ij} \partial_j X^\mu) = 0, \tag{2.3}$$

and

$$g_{ij} = \partial_i X^\mu \partial_j X_\mu, \tag{2.4}$$

i.e. the equation determining g_{ij} just says that the worldvolume metric equals the one induced by the spacetime metric. Substituting this equation for g_{ij} into the action (2.2) we recover the original action (2.1), hence the classical equivalence. If we instead substitute (2.4) into the equations of motion (2.3) we obtain the equations of motion coming from the original action. These equations will be greatly simplified if we define the canonical momenta

$$P^i_\mu(\xi) = \frac{\delta \mathcal{L}}{\delta \partial_i X^\mu(\xi)}. \tag{2.5}$$

The equations of motion then become

$$\partial_i P^i_\mu(\xi) = 0. \tag{2.6}$$

Now one might think that all we have to do is multiply this with the canonical velocity $\dot{X}^\mu$, subtract the Lagrangian density and integrate over the whole worldvolume to derive our Hamiltonian, but unfortunately things are not quite that simple.

It is a virtue of the action we have chosen that it is invariant under reparametrizations on the worldvolume

$$\xi_i \longrightarrow \xi'_i(\xi_0, \xi_1, \xi_2). \tag{2.7}$$

This will however cause problems for us in our analysis of the action. The invariance yields primary constraints on the membrane (we leave out the spacetime index here in favor of elegance)

$$\begin{aligned}\phi_1 &= (P^0)^2 - (\partial_1 X)^2(\partial_2 X)^2 + (\partial_1 X \cdot \partial_2 X)^2 \equiv 0,\\ \phi_2 &= P^0 \cdot \partial_1 X \equiv 0,\\ \phi_3 &= P^0 \cdot \partial_2 X \equiv 0,\end{aligned} \tag{2.8}$$

and an arbitrary linear combination of these should be included in the Hamiltonian of the membrane dynamics [28,29,30]. Now in actually writing down the Hamiltonian $\mathcal{H}$, we can proceed in two ways. Either we could proceed and derive a general Hamiltonian (as in [2]) or we could pick a gauge and thus rid ourselves of the restraints coming from the reparametrization invariance [25, 9]. It is a fairly simple calculation to derive the general Hamiltonian (as we will see below) and so to this end we don't really need any particular gauge. But, when we wish to quantize the theory, doing so by means of covariant quantization in a general parametrization will turn out to be overly complicated. There are essentially three main ways in which we could eliminate the constraints. We could explicitly give the "coefficient-functions" of the linear combination of constraints in $\mathcal{H}$ or we could globally specify values for ξ_i on the worldvolume. The third choice would be to further restrict the canonical variables in such a way that we fix the gauge without constraining the dynamics of the membrane. We begin though by, as promised, giving the Hamiltonian in a general gauge.

From here and on we choose ξ^0 as our time-like direction on the worldvolume. We will also refer to P^0_μ as simply P_μ when there is no possibility of confusion. It is now trivial to check that the free Hamiltonian vanishes

$$H_0 = \int d\xi^1 d\xi^2 (\dot{X}^\mu \cdot P_\mu - \mathcal{L}) = 0, \tag{2.9}$$

and hence the Hamiltonian will just become a linear combination of the constraints. So with λ, μ and ν being the coefficient functions mentioned before,

$$H = \int d\xi^1 d\xi^2 \left[\lambda(\xi)\phi_1 + \mu(\xi)\phi_2 + \nu(\xi)\phi_3\right]. \tag{2.10}$$

Choosing specific functions λ, μ and ν corresponds to specifying a particular parametrization in the action as mentioned before. Now before we move on we must make sure that these constraints are consistent. To facilitate this analysis we introduce the Poisson bracket

$$\{f, H\} = \int d\xi^1 d\xi^2 \left[\frac{\delta f}{\delta X(\xi)}\frac{\delta H}{\delta P(\xi)} - \frac{\delta f}{\delta P(\xi)}\frac{\delta H}{\delta X(\xi)}\right], \tag{2.11}$$

and we also remind ourselves of Hamilton's equation of motion which now reads

$$\dot{f} = \{f, H\} + \frac{\partial f}{\partial \xi^0}. \tag{2.12}$$

Now we begin with a dynamical system in which these constraints hold, but we must also make sure that they remain valid throughout the evolution of the system. This is our consistency condition and hence we see that

$$\dot{\phi}_i = \{\phi_i, H\} = 0, \tag{2.13}$$

must be fulfilled on the subset of configuration space where the constraints hold. Now since the free Hamiltonian, H_0, is identically zero we see that it is a sufficient condition that the constraints satisfy the following equation

$$\{\phi_i, \phi_j\} = C_{ijk}\phi_k, \tag{2.14}$$

where the C_{ijk} are some arbitrary functions of the embedding fields X^μ and the canonical momenta P_μ. This check amounts to a technically fairly simple calculation, although long and tedious, which is why we do not give the details of it here. But the result is that the constraints, and hence the equations of motion, are indeed consistent.

In accordance with Dirac's theory, to quantize this theory the scalar fields X^μ and P^ν, are now replaced by operators $\hat{X}^\mu$ and $\hat{P}^\mu$ which satisfy the commutation relations

$$\begin{aligned} [\hat{X}^\mu(\xi^0,\xi^1,\xi^2), \hat{P}^\nu(\xi^0,\xi^{1\prime},\xi^{2\prime})] &= 4\pi^2 i\hbar g^{\mu\nu}\delta(\xi^1-\xi^{1\prime})\delta(\xi^2-\xi^{2\prime}), \\ [\hat{X}^\mu(\xi^0,\xi^1,\xi^2), \hat{X}^\nu(\xi^0,\xi^{1\prime},\xi^{2\prime})] &= 0, \\ [\hat{P}^\mu(\xi^0,\xi^1,\xi^2), \hat{P}^\nu(\xi^0,\xi^{1\prime},\xi^{2\prime})] &= 0. \end{aligned} \tag{2.15}$$

Here we have assumed that the Hamiltonian and the position and momenta are Hermitian operators. This is done in order to remove any ordering ambiguities. The constraints (2.9) now also become operators

$$\begin{aligned} \hat{\phi}_1 &= \frac{1}{\hbar^2}\hat{P}^2 - \widehat{(\partial_1 X)}^2\widehat{(\partial_2 X)}^2 + (\widehat{(\partial_1 X)}\cdot\widehat{(\partial_2 X)})^2, \\ \hat{\phi}_2 &= \widehat{(\partial_1 X)}\cdot\hat{P} + \hat{P}\cdot\widehat{(\partial_1 X)}, \\ \hat{\phi}_3 &= \widehat{(\partial_2 X)}\cdot\hat{P} + \hat{P}\cdot\widehat{(\partial_2 X)}, \end{aligned} \tag{2.16}$$

and are enforced by requiring that the physical states of the theory obey (in the Heisenberg picture)

$$\hat{\phi}_i\,|p\rangle = 0. \tag{2.17}$$

To actually implement these constraints in a covariant quantization is however a complicated task, never fully completed. We will not attempt to complete it here either since it would contribute only marginally to our understanding of the membrane. Fixing a parametrization, however, will facilitate our considerations immensely, and so we turn to the question of gauge-fixing our theory to the light-cone gauge.

In string theory we can at this point pick a gauge making the equations of motion linear, which is unfortunately not possible for the membrane. This fact has a fundamental physical interpretation. Even in our theory of the "free" membrane we will always have interactions; the membrane is self-interacting and can hence change its topology at any time. Nevertheless, going to a gauge such as the light-cone gauge will reveal many facts about membrane dynamics.

To begin reformulating our theory we define the light-cone coordinates

$$X^+ = \frac{1}{2}X^0 + X^{D-1}, \tag{2.18}$$

$$X^- = X^0 - X^{D-1}, \tag{2.19}$$

$$\vec{X} = X^a \quad , \quad a = 1, 2 \ldots, D-2, \tag{2.20}$$

going to the light-cone gauge [9, 10] is then done by setting

$$X^+ = \xi_0. \tag{2.21}$$

Next we denote

$$\begin{aligned} g_{00} &= 2\dot{X}^- + \dot{\vec{X}}^2, \\ g_{0r} &= u_r = \partial_r X^- - \dot{\vec{X}}\partial_r \vec{X}, \\ g_{rs} &= \partial_r \vec{X} \partial_s \vec{X}, \end{aligned} \tag{2.22}$$

where upon the induced metric becomes

$$g_{ij} = \begin{pmatrix} 2\dot{X}^- + \dot{\vec{X}}^2 & u_1 & u_2 \\ u_1 & g_{11} & g_{12} \\ u_2 & g_{21} & g_{22} \end{pmatrix}. \tag{2.23}$$

And as before

$$g = \det g_{ij} = g'(2\dot{X}^- - \dot{\vec{X}}^2 + u_r g^{rs} u_s) = g' \cdot U, \tag{2.24}$$

where we have defined

$$g' = \det g_{rs}. \tag{2.25}$$

The Lagrangian in this gauge becomes

$$S_{lc} = -\int d^3\xi \sqrt{g'}\sqrt{U}, \tag{2.26}$$

and we can calculate the canonical momenta

$$\vec{P} = \frac{\delta\mathcal{L}}{\delta\dot{\vec{X}}} = \sqrt{\frac{g'}{U}}(\dot{\vec{X}} + \partial_r\vec{X}u_s g^{rs}), \tag{2.27}$$

$$P^- = \frac{\delta\mathcal{L}}{\delta\dot{X}^-} = -\sqrt{\frac{g'}{U}}. \tag{2.28}$$

The Hamiltonian is given by

$$\begin{aligned}\mathcal{H} &= p^-\dot{X}^- + \vec{P}\dot{\vec{X}} - \mathcal{L} = \\ &= \sqrt{\frac{g'}{U}}\{\dot{X}^- + \partial_r X^- u_s g^{rs}\},\end{aligned} \tag{2.29}$$

this Hamiltonian can then be rewritten as

$$\mathcal{H} = \frac{\vec{P}^2 + g'}{-2P^-}, \tag{2.30}$$

where we note the absence of X^-. This actually means that (by the equations of motion) P^- is just a constant. The explanation for the disapperance of X^- is that in writing down this Hamiltonian we have set all the constraint-terms to zero. After gauge-fixing we are left with the constraints

$$\Phi_r = P^-\partial_r X^- + \vec{P}\partial_r\vec{X} = 0,\ r = 1, 2, \tag{2.31}$$

then setting the constraint-terms to zero in $\mathcal{H}$ means that we further gauge-fix the theory in a way that is equivalent to setting $u_r = 0$ which reduces the symmetries of (2.31) down to only one. This residual symmetry we will study shortly but in a slightly more convenient gauge.

One could now, as we outlined for the covariant case, make operators of these fields and impose canonical commutation relations. We will however choose a different and more elaborate method of quantization, but first we must study the symmetry of (2.29).

For the next few paragraphs we drop the factor $\frac{1}{2}$ on X^+ and we make the gauge choice [31]

$$X^+ = \mathcal{P}^+ \cdot \xi^0, \tag{2.32}$$

where $\mathcal{P}^+$ is the + component of the total momentum, defined as

$$\mathcal{P}^+ = \int d\xi^1 d\xi^2 P^+. \tag{2.33}$$

The Hamiltonian becomes

$$H'_{lc} = 2\mathcal{P}^+\mathcal{P}^- = \int d\xi^1 d\xi^2 \left(P_i P^i + \sum_{i<j}(X_i X_j)^2\right), \tag{2.34}$$

with the remaining constraint

$$\partial_1 P_i \partial_2 X_i - \partial_2 P_i \partial_1 X_i = 0 \tag{2.35}$$

If we define the Poisson bracket with respect to ξ^1, ξ^2

$$\{X^i, X^j\}' = \frac{\partial X^i}{\partial \xi^1}\frac{\partial X^j}{\partial \xi^2} - \frac{\partial X^i}{\partial \xi^2}\frac{\partial X^j}{\partial \xi^1}, \tag{2.36}$$

the constraint (2.35) becomes

$$\{P^i, X^i\}' = 0. \tag{2.37}$$

It is a trivial calculation to check that the symmetry left by this constraint is

$$\begin{aligned} \xi_1 &\rightarrow \tilde{\xi}_1(\xi_1, \xi_2), \\ \xi_2 &\rightarrow \tilde{\xi}_2(\xi_1, \xi_2), \end{aligned}$$

where

$$\det \frac{\partial(\tilde{\xi}_1, \tilde{\xi}_2)}{\partial(\xi_1, \xi_2)} = 1, \tag{2.38}$$

is required for the preservation of the Poisson bracket. We also see that this is exactly the condition that the coordinate transformations preserve the area of the brane. Thus, fixing our action to light-cone gauge has reduced the full reparametrization invariance to area-preserving transformations. These transformations (and their infinitesimal counterparts) were studied in [31] as diffeomorphisms

$$\xi_1 \rightarrow \xi_1 + u_1(\xi_1, \xi_2), \tag{2.39}$$

$$\xi_2 \rightarrow \xi_2 + u_2(\xi_1, \xi_2), \tag{2.40}$$

for which the condition (2.38) becomes

$$\frac{\partial u_1}{\partial \xi_1} + \frac{\partial u_2}{\partial \xi_2} = 0. \tag{2.41}$$

The study was limited to the cases of the spherical and torodial membranes but it was shown for these that the infinitesimal transformations, corresponding to the diffeomorphisms above, form a Lie algebra, dubbed the algebra of area-preserving diffeomorphisms (or the APD algebra for short).

Now there is a highly non-trivial relation between the algebra above and matrix quantum mechanics (here we retain the factor $\frac{1}{2}$ in X^+ and the gauge $X^+ = \xi_0$). To see this we first pick a basis, Y_α, in which to expand the fields X_a and the corresponding momenta

$$X_a = \sum_\alpha X_{a\alpha} Y_\alpha, \tag{2.42}$$

$$P_a = \sum_\alpha P_{a\alpha} Y_\alpha. \tag{2.43}$$

The Hamiltonian can then be written in terms of these coefficients and

$$f_{\alpha\beta\gamma} = \int d^2\xi Y_\alpha (\partial_1 Y_\beta \partial_2 Y_\gamma - \partial_1 Y_\gamma \partial_2 Y_\beta), \tag{2.44}$$

which are the structure-constants of the infinite-dimensional APD Lie algebra. The relation between this theory and the matrix theory can now be formulated in terms of a theorem

Theorem 1 *For each membrane topology there exists a basis* $\{Y_\alpha\}_{\alpha=1}^\infty$ *of the APD algebra, and a basis* $\{T_a^{(N)}\}_{a=1}^{N^2-1}$ *of* $SU(N)$ *such that the structure constants of the latter*

$$f_{abc}^{(N)} = [T_a^{(N)}, T_b^{(N)}] T_c^{(N)}, \tag{2.45}$$

obey

$$\lim_{N\to\infty} \mathrm{tr}(-f_{abc}^{(N)}) = f_{abc} \ \forall a, b, c. \tag{2.46}$$

Thus we can now quantize this finite model

$$H_N = \sum_{a=1}^{D-1} \sum_{m=1}^{N^2-1} P_{am} P_{am} + \frac{1}{2} f_{mno}^{(N)} f_{mn'o'}^{(N)} X_{an} X_{bo} X_{an'} X_{bo'}, \tag{2.47}$$

finding ($X_a = X_{am} T_m^{(N)}$)

$$\hat{H}_N = -\Delta_{(N)} - \mathrm{tr} \sum_{a<b} [X_a, X_b]^2, \tag{2.48}$$

where the limit (2.46) assures that as $N \to \infty$ we get our quantized, discrete theory of membranes. This procedure of relating membrane theory with matrix theory is possible also for the supermembrane. We will redo this derivation in that case, making up for the slight lack of explicitness here.

Turning now to the question of the spectrum of our theory. It was long an outstanding problem whether the membrane had massless states in its spectrum or not. This question was settled once and for all by Kikkawa and Yamasaki in [32]. In this paper they calculated the Casimir energy of the membrane and on basis of this they saw that for massless states to exist the dimension of spacetime would have to differ from an integer value. Hence the bosonic membrane theory does not contain massless states. We will not address this analysis here, but instead we wish to address the question of the spectrum being continuous or discrete. The main result presented here, which concludes this treatise of the bosonic membrane, is that the quantized membrane does indeed have a discrete spectrum even though the classical theory does not. The Hamiltonian falls into a class of systems that have an infinite volume

$$\{(p, q) : p^2 + V(q) \leq E\}, \tag{2.49}$$

but finite partition function. These systems were studied extensively in [3]. Given the Golden-Thompson inequality

$$Z_Q(t) = \mathrm{tr}(e^{-tH}) \leq Z_{cl} = (2\pi)^{-\nu} \int d^\nu p d^\nu q e^{-t(p^2+V(q))}, \tag{2.50}$$

it was once believed that the opposite statement held firmly as well. This intuition was proven wrong by Simon in [3]. We gill give our proof for the Hamiltonian (2.34) and we will perform this in a four-dimensional spacetime. This will enable us to directly relate H'_{lc} to the simplest Hamiltonian in the class. Let us consider

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + x^2 y^2, \tag{2.51}$$

(now switching to the notation in [3]), which is equivalent to our membrane Hamiltonian (in this particular dimensionality). The proof (which is one of many) is now quite simple and rests upon a relation to the harmonic oscillator. For the harmonic oscillator we have the operator relation

$$-\frac{d^2}{dq^2} + \omega^2 q^2 \geq |\omega|, \tag{2.52}$$

which we can refrase, considering y to be a c-number

$$-\frac{d^2}{dx^2} + x^2 y^2 \geq |y|. \tag{2.53}$$

Is is now a simple matter to obtain the relation

$$\begin{aligned} -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2 y^2 &= \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 y^2\right) + \frac{1}{2}\left(-\frac{d^2}{dy^2} + x^2 y^2\right) - \frac{1}{2}\Delta \\ &\geq \frac{1}{2}(-\Delta + |x| + |y|) = H', \end{aligned} \tag{2.54}$$

where H' has a discrete spectrum due to

$$\begin{aligned}\mathrm{tr}(e^{-tH'}) &= [1+\mathcal{O}(1)](2\pi)^{-2}\int dxdyd^2pe^{(-tp^2-t|x|-t|y|)} \\ &= ct^{-3}[1+\mathcal{O}(1)]. \end{aligned} \tag{2.55}$$

This yields the relation

$$Z(t) = \mathrm{tr}(e^{-tH}) \leq ct^{-3}, \tag{2.56}$$

which is a crude approximation that is especially bad at high energies (small t), but nevertheless suffices. However, this proof does not work in higher dimensions, but before we discuss a proof that does hold in any given dimension we wish to give the intuition behind the fact that the classical theory has a continuous spectrum. If we look closer at the Hamiltonian (2.34) we see that the potential is such that if all but one of the scalar fields X_i are sufficiently small the other field can grow very large without rendering the potential very large. In fact if the $d-3$ scalar fields approach zero the left-over field can go off into infinity. This will look like a string attached to the membrane surface. These valleys in the membrane potential through which classical states can escape are made increasingly narrow by quantizing the theory. Because of this, wave functions fall off to zero the further out they get in the quantized scenario. In the next section we will see why this very welcome effect is lost in the supersymmetric case. There is an equivalent and more powerful method to prove a discrete spectrum. The theorem that is used for this is due to Fefferman and Phong and it concludes this section. We will not give any details here refering the reader to [3] for details. The theorem reads

Theorem 2 *Let Δ_j^λ ($\lambda > 0, j \in \mathbb{Z}^\nu$) be the cube of side $\lambda^{-1/2}$ centered at the point $\lambda^{-1/2}j$. Given $V \geq 0$ on R^ν, let $\tilde{N}(\lambda)$ be the number of cubes Δ_j^λ with $\max_{x\in\Delta_j^\lambda} V(x) \leq \lambda$. Let $N(\lambda)$ be the dimension of the spectral projection for $-\Delta + V$ on the interval $(0,\lambda)$. Then if V is a polynomial of degree d on R^ν,*

$$\tilde{N}(b\lambda) \leq N(\lambda) \leq \tilde{N}(a\lambda), \tag{2.57}$$

for all λ and suitable constants a, b, where b only depends on ν and a depends on ν and the degree d.

2.2 Adding Supersymmetry

In this section we wish to formulate a supersymmetric version of the membrane theory studied so far[1]. Before doing this we must answer the

[1] See section A.1 for a short introduction to a simple supersymmetric model.

important questions of how and where to introduce this symmetry. To answer the question of where we have essentially two choices. Either we could formulate a theory with supersymmetry on the worldvolume, i.e. a spinning membrane, or as a second choice, we may introduce supersymmetry in targetspace. There is actually a third possibility, namely introducing supersymmetry both on the worldvolume and in targetspace. This formalism, called superembeddings, was pioneered in [33] but unfortunately it falls outside the scope of this thesis to review this work. Worldvolume supersymmetric membranes or spinning membranes were the target of much attention in the mid-eighties, but then a paper presenting a no-go theorem for their existence put a stop to virtually all research. Here we will review this no-go theorem and also the counterproof and try to explain how the counterproof works and why it is not interesting (see also section 3.1 for the string theory case).

The action (2.2) was introduced by Howe and Tucker in [4] precisely with the purpose of describing a spinning membrane through a supersymmetric extension. Their supersymmetric extension of the action is

$$S_{slh} = \int d^3\xi e\mathcal{L}, \tag{2.58}$$

where (we drop the spacetime index on the bosonic fields X^μ and the fermionic fields ψ^μ for the moment)

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{2}g^{ij}\partial_i X^\mu \partial_j X_\mu - \frac{1}{2}i\bar{\psi}\gamma^i D_i\psi + \frac{1}{2}i\bar{\chi}_i\gamma^j\gamma^i\partial_j X\psi - \\ & \frac{1}{16}\bar{\psi}\psi\bar{\chi}_j\gamma^i\gamma^j\chi_i - \frac{i}{8e}\epsilon^{ijk}\bar{\chi}_i\gamma_j\chi_k + \frac{1}{4}i\bar{\psi}\psi + \frac{1}{2}.\end{aligned} \tag{2.59}$$

In this Lagrangian we have introduced the dreibeins $e_i^a(\xi)$ such that

$$e_i^a\eta_{ab}e_j^b = g_{ij}, \tag{2.60}$$

$$\eta_{ab} = \mathrm{diag}(-,+,+) \quad , \quad \sqrt{-g} = \det(e_i^a) \equiv e.$$

In this action X^μ are of course our familiar bosonic coordinates, ψ^μ are our fermionic coordinates which transform as spinors. We have also introduced the vector-spinor field χ_μ, which is the Rarita-Schwinger field. The gamma matrices γ, adjoint spinor $\bar{\psi}$ and covariant derivative D_μ are given by

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad, \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad, \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma^i = e_a^i\gamma^a, \tag{2.61}$$

$$\bar{\psi} = \psi^T \gamma^0, \tag{2.62}$$

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{2}\gamma_a \omega_i^a \psi, \tag{2.63}$$

where the connection field ω_i^a is given by

$$\omega_i^c = -\frac{1}{4}\epsilon^{abc} e_i^d \left(\sum_{dab} - \sum_{abd} - \sum_{bda} \right), \tag{2.64}$$

where

$$\sum_{cab} = e_a^i e_b^j (\partial_j e_{ci} - \partial_i e_{cj} + \frac{1}{2} i \bar{\chi}_i \gamma_c \chi_j) \tag{2.65}$$

Now without going to far into the analysis of (2.58) and the result of [4] we can see that there exist additional constraints on the supergravity fields,

$$\epsilon^{ijk}(D_j \chi_k + \frac{1}{2}\gamma_j \chi_k) = 0, \tag{2.66}$$

and as a consequence of this our metric g_{ij} is no longer independent. This in turn means that we can substitute the constraint into the action (2.58) where upon it gains a term eR (R being the Ricci scalar) which renders the action inequivalent to (2.2) upon setting the fermionic parts to zero. Thus we draw the conclusion that the action(2.2) does not allow a supersymmetrization. One might ask if there are other actions that are better suited for supersymmetric extensions. As we will see below this is not the case.

Many actions were proposed in the early eighties that were thought to solve the problem above, e.g. in [34] Dolan and Tchrakian proposed the action

$$S_{DT} = -\frac{1}{2}\int d^3\xi \sqrt{-g} \left\{ \mathrm{tr}(A) - \frac{1}{2}\mathrm{tr}(A^2) + \frac{1}{2}\mathrm{tr}(A)^2 \right\}, \tag{2.67}$$

where we have defined

$$A_j^i = g^{ik} \partial_k X^\mu \partial_j X^0 \eta_{\mu 0}, \tag{2.68}$$

so that the action (2.2) becomes

$$S'_M = -\frac{1}{2}\int d^3\xi \sqrt{-g}(\mathrm{tr}(A) - 1). \tag{2.69}$$

Another action was first proposed by Inami and Yahikozawa and later by Alves and Barcelos-Neto in [35]

$$S_{IY} = -\int d^3\xi \sqrt{-g}[\mathrm{tr}(A)]^{3/2}. \tag{2.70}$$

These actions are not strictly equivalent to Dirac's action (2.1) but even if we only consider physical solutions to the equation for g_{ij} we see that local supersymmetry is still impossible. This fact is due to a theorem in [5] which states that

> If there exists any spinning membrane action there must exist a linearly realized locally supersymmetric (spinning membrane) extension of ... [(2.2)].

As we have seen above such an extension is not possible due to [4]. There are assumptions made in this theorem, a 'spinning membrane action' is defined to be an action with linearly realized local supersymmetry which reduced to Dirac's action (2.1) upon elimination of all auxiliary fields and setting the fermionic parts to zero. However, shortly after the publication of this no-go theorem a counterexample was published [36]. This counterexample relies not on a flaw in the proof but rather in the definition of what a "spinning membrane" is. It is the case that we can have actions with linearly realized supersymmetry before elimination of all auxiliary fields but after which the supersymmetry becomes non-linear. In fact these actions become extremely complicated to work with and it is because of this reason that the paper mentioned above never sparked much interest. They used the bosonic action (2.70) for their supersymmetric extension, which becomes

$$\begin{aligned} S_{SIY} &= \int d^3\xi d^2\theta E^{-1}[(\nabla^\alpha X \nabla_\alpha X)(\nabla_{\gamma\delta} X \nabla^{\gamma\delta} X)^{1/2} \\ &\quad + \frac{2}{3}(\nabla^\alpha X \nabla_\alpha^\gamma X)(\nabla^\beta X \nabla_{\beta\gamma} X)(\nabla_{\delta\lambda} X \nabla^{\delta\lambda} X)^{-1/2}], \end{aligned} \tag{2.71}$$

X now being a field in superspace. There is still no way of dealing with the quantization of an action like this, and its component form is extremely complicated.

Here we turn instead to membrane actions with targetspace supersymmetry like many physicists who worked on spinning membranes did during the late eighties. The pioneering paper in this area of research was [6], where the authors applied their previous work on partially broken global supersymmetries to study a three-brane in a six-dimensional flat space. This result was then extended by Bergshoeff, Sezgin and Townsend in [7] to any p-dimensional extended object, or p-brane, propagating in a d-dimensional curved background. We will pretty much skip the first paper and go directly to the more useful generalization (though still in flat space at first). But before we can do this we need to acquire some general feeling for the objects involved. First we introduce the mapping $Z^M(\xi)$ from the worldvolume of the membrane to a superspace

$$Z^M(\xi) = (X^\mu, \theta^\alpha), \tag{2.72}$$

which replaces the scalar fields describing the embedding in the bosonic action. Now to construct a supersymmetric generalization of the action 2.1 we need to find supersymmetric fields to use instead of the scalar fields. In our flat superspace the generators act as

$$\delta X^\mu = i\bar{\epsilon}\Gamma^\mu\theta \quad , \quad \delta\theta = \epsilon, \tag{2.73}$$

on the superspace coordinates, the ϵ is a constant anticommuting spacetime spinor. The Γ^μ are spacetime Dirac matrices. If we start from the one-forms

$$\Pi^A = dZ^M e_M^A, \tag{2.74}$$

the most general supersymmetric one-forms we can write down, e_M^A being the superspace vielbein. We now see that the pull-back to the worldvolume of these forms is exactly what we are looking for. We have

$$\Pi^A \to {}^*\Pi^A = d\xi^i\partial_i Z^M e_M^A \tag{2.75}$$

where Π^A is explicitly,

$$\Pi^\mu = dX^\mu - i\bar{\theta}\Gamma^\mu d\theta \quad , \quad \Pi^\alpha = d\theta^\alpha, \tag{2.76}$$

whereupon the pull-back explicitly becomes

$$\Pi_i^\mu = \partial_i X^\mu - i\bar{\theta}\Gamma^\mu\partial_i\theta \quad , \quad \Pi_i^\alpha = \partial_i\theta^\alpha, \tag{2.77}$$

which are easily seen to exhibit explicit invariance under the transformations (2.73). Now that we have the material to construct a supersymmetric extension of 2.1 we just substitute the scalar fields for our pull-back fields and get

$$S_{SD} = -T\int d^3\xi\sqrt{-\det(\Pi_i^\mu\Pi_i^\nu\eta_{\mu\nu})}. \tag{2.78}$$

This does however not give the full picture since we need to eliminate half of the fermionic degrees of freedom in order to match them with the bosonic ones. Thus we need to find a fermionic symmetry that allows us to gauge them away. This symmetry is known as κ-symmetry. It turns out to require additions to the action (2.78) and this addition, which will be a Wess-Zumino-type term, will put restrictions on the dimensions of our spacetime. If we were to consider general p-branes in a general d-dimensional spacetime these restrictions would give us the well known brane-scan which tells us in which dimensions the various branes can live. Here we restrict ourselves to consider only the membrane. We will actually derive the desired action in a way opposite the logical structure given above, beginning with an analysis that gives the Wess-Zumino term and then finding the κ-symmetry in order to see why this requires additions to the action.

We begin by defining an exact, Lorentz invariant and supersymmetric 4-form on superspace

$$\mathbf{h} = d\mathbf{b}, \tag{2.79}$$

the 3-form $\mathbf{b}$ is then also Lorentz invariant and supersymmetric (modulo a total derivative) and induces a 3-form on the worldvolume through the mapping $Z^M(\xi)$

$$^*\mathbf{b} = \frac{1}{6} d\xi^i d\xi^j d\xi^k b_{ijk}, \tag{2.80}$$

which retains the same properties. Then we can use this form (or rather the coefficients of this form) to construct an action invariant under the super Poincaré group

$$S_{WZ} = -\frac{T}{3}\int d^3\xi \epsilon^{ijk} b_{ijk}, \tag{2.81}$$

ϵ^{ijk} being the tensor density totally antisymmetric in its three indices. Now in order for $\mathbf{h}$ to fulfill the conditions we have set out it must be constructed from the one-forms (2.76) since these are the most general we can write down in our flat superspace. Furthermore since this term should combine with our "super-Dirac action" (2.78) it must scale as this action under constant rescalings of the coordinates. These conditions give the explicit shape of the form $\mathbf{h}$ and furthermore we get conditions on the matrices $(\Gamma_{ijk})_{\alpha\beta}$ (α, β being spinor indices) in order that $\mathbf{h}$ does not vanish. These conditions are the previously mentioned restrictions on the dimension d of the spacetime. We will not present the complete proof even for the membrane case here. It suffices to say that the membrane can exist in $4, 5, 6$ and 11 dimensions, the 11-dimensional case being the one of interest to us here. The explicit form of the pull-back of the 3-form $\mathbf{b}$ turns out to be

$$b_{ijk} = \frac{i}{2}\left(\partial_k\bar{\theta}\Gamma_{\mu\nu}\theta\right)(3\Pi_j^\nu\Pi_i^\mu - 3\Pi_j^\nu(i\partial_i\bar{\theta}\Gamma^\mu\theta) + (i\partial_j\bar{\theta}\Gamma^\nu\theta)(i\partial_i\bar{\theta}\Gamma^\mu\theta)). \tag{2.82}$$

So our complete action is of the form

$$S_{SM} = -T\int d^3\xi[\sqrt{-\det(\Pi_i^\mu\Pi_j^\nu\eta_{\mu\nu})} + \frac{1}{3}\epsilon^{ijk}b_{ijk}], \tag{2.83}$$

(we have skipped a constant in front of the Wess-Zumino term here which will be determined to be equal to 1 later because of the κ-symmetry). Since we have the explicit form of b_{ijk} in (2.82) we can now finally write down the full explicit action of our supermembrane,

$$\begin{aligned} S_{SM} &= -T\int d^3\xi[\sqrt{-\det(\Pi_i^\mu\Pi_j^\nu\eta_{\mu\nu})}+\frac{i}{2}\epsilon^{ijk}\bar\theta\Gamma_{\mu\nu}\partial_i\theta(\Pi_j^\mu\Pi_k^\nu + \\ & \qquad i\Pi_j^\mu\bar\theta\Gamma^\nu\partial_k\theta - \frac{1}{3}(\bar\theta\Gamma^\nu\partial_j\theta)(\bar\theta\Gamma^\nu\partial_k\theta))]. \end{aligned} \tag{2.84}$$

As stated earlier we have to find a symmetry that allows us to gauge away half of the fermionic degrees of freedom in order to match these with the bosonic ones. We begin by rewriting the action (2.83) in the form of Howe and Tucker,

$$S = -\frac{T}{2}\int d^3\xi[\sqrt{-g}(g^{ij}\Pi_i^\mu\Pi_j^\nu\eta_{\mu\nu}-1)+\frac{1}{3}\epsilon^{ijk}b_{ijk}] \tag{2.85}$$

Consider now the variation

$$\delta X^\mu = i\bar\theta\Gamma^\mu\delta\theta \quad , \quad \delta\theta = (1+\Gamma)\kappa(\xi), \tag{2.86}$$

where $\kappa(\xi)$ is a parameter depending on on the worldvolume coordinates and transforms as a spinor in targetspace and a scalar on the worldvolume. Γ is made up of the pull-back fields as,

$$\Gamma = \frac{1}{6\sqrt{-g}}\epsilon^{ijk}\Pi_i^\mu\Pi_j^\nu\Pi_k^\rho\Gamma_{\mu\nu\rho}. \tag{2.87}$$

Now we want to investigate this transformation and see it in action, for the pull-back fields we get that

$$\delta\Pi_i^\mu = -2i\delta\bar\theta\Gamma^\mu\partial_i\theta \quad , \quad \delta\Pi_i^\alpha = \partial_i\delta\theta^\alpha, \tag{2.88}$$

and it also follows that

$$\delta\mathbf{b} = \frac{1}{2}\Pi^\mu\Pi^\nu id\bar\theta\Gamma_\mu\Gamma_\nu\delta\theta. \tag{2.89}$$

So now we are in a position to calculate how this transformation acts on the action 2.85, we will continue to keep the transformation $\delta\theta$ implicit here since the form of it will be very intuitive. From the above we get

$$\delta S_{SM} = 2iT\int d^3\xi\delta\bar\theta[\sqrt{-g}g^{ik}\Pi_i^\mu\Gamma_\mu - \frac{1}{2}\epsilon^{ijk}\Pi_i^\mu\Pi_j^\nu\Gamma_{\mu\nu}]\partial_k\theta. \tag{2.90}$$

Now throughout this calculation we have kept g^{ij} independent, but since we do not wish to vary this field as well, we make use of the field equation $g_{ij} = \Pi_i^\mu \Pi_j^\nu \eta_{\mu\nu}$, which gives the relation

$$\epsilon^{ijk}\Pi_i^\mu \Pi_j^\nu \Gamma_{\mu\nu} = 2\sqrt{-g}g^{kl}\Pi_l^\mu \Gamma_\mu \Gamma. \tag{2.91}$$

Using this relation we can rewrite the variation of the action as

$$\delta S_{SM} = i\int d^3\xi \delta\bar{\theta}(1-\Gamma)(\sqrt{-g}g^{ij}\Pi_i^\mu \Gamma_\mu \partial_j \theta). \tag{2.92}$$

Thus we see why the particular choice of transformation for θ was made. Since the matrix Γ can be shown to satisfy $\Gamma^2 = 1$, the first factor in the variation above, and likewise in the transformation, is a projection operator. It projects θ down on a fermionic object with less degrees of freedom, actually half of the original ones. Furthermore, since the variation δS_{SM} vanishes, the transformation (2.86) turns out to be a symmetry. This is a symmetry under a transformation that removes half of the fermionic deg, thus effectively telling us that these can be gauged away without restricting the dynamics of the theory.

Since it is a necessary condition for the supermembrane that we have a matching of the bosonic and fermionic degrees of freedom, and this demands a fermionic symmetry of the above type we can also see why we get restrictions on the dimension of spacetime. If we did not have a Wess-Zumino term in (2.84) we would not get a 'projection type'-term in (2.92), so the consistency of the action requires a 3-form which in turn puts restrictions on the dimension d.

Now we wish to reformulate this theory in a curved background. This will mean increased complexity as we now have to keep spacetime coordinates and tangentspace coordinates apart, but it will also give us important insights about the supermembrane and it is of course ultimately the most interesting case. The coordinates Z^M are now those of a curved superspace, and we redefine the worldvolume fields (Π_i^M) as

$$E_i^A = \partial_i Z^M E_M^A, \tag{2.93}$$

where E_M^A are now supervielbeins in a curved space, $A = (a, \alpha)$ are tangent-space indices and $M = (\mu, \dot{\alpha})$ are spacetime indices. In the Howe-Tucker form of the action (2.85) our new action becomes

$$S'_{SM} = -\frac{T}{2}\int d^3\xi[\sqrt{-g}(g^{ij}E_i^a E_j^b \eta_{ab} - 1) - \frac{1}{3}\epsilon^{ijk}E_i^a E_j^b E_k^c B_{abc}], \tag{2.94}$$

where B_{abc} is now the curved spacetime super 3-form. This action is constructed in complete analogy with the flat spacetime case as we can see from (2.94). Now what is really interesting is to see how κ-symmetry generalizes from the flat case to the curved case, indeed this symmetry will not hold generally in a curved background. We must put additional constraints on the spacetime superform $\mathbf{H}$ (satisfying $\mathbf{H} = \mathbf{dB}$ and corresponding to $\mathbf{h}$) and the supergravity torsion T_{AB}^{C}. These constraints were found in [7], a calculation that we revisit here.

As before we require the action (2.94) to exhibit a *local* fermionic symmetry, κ-symmetry

$$\delta Z^M E_M^a = 0, \tag{2.95}$$

$$\delta Z^M E_M^\alpha = (1+\Gamma)^\alpha_\beta \kappa^\beta. \tag{2.96}$$

The variation of the action (2.94) yields

$$\delta S'_{SM} = \int d^3\xi \left[-\sqrt{-g} g^{ij} (\delta E_i^a) E_j^b \eta_{ab} - \frac{1}{3} \delta \epsilon^{ijk} E_i^a E_j^b E_k^c B_{abc} \right], \tag{2.97}$$

and by skillful manipulation of the two terms in the above variation one can in fact prove that the integrand is zero. Thus the action is κ-symmetric. However this does not hold without restrictions, and the discovery of these restrictions displayed much of the splendor that resides in the theory of the 11-dimensional supermembrane. It can be shown that the constraints that have to hold are

$$\begin{aligned} T_{\alpha\beta}^c = 2i\Gamma_{\alpha\beta}^c \quad &, \quad T_{a\beta}^c = T_{\alpha\beta}^\gamma = 0 \\ H_{\alpha\beta cd} = i(\Gamma_{cd})_{\alpha\beta} \quad &, \quad H_{\alpha abc} = H_{\alpha\beta\gamma\delta} = H_{\alpha\beta\gamma d} = 0. \end{aligned} \tag{2.98}$$

These equations follow directly from the Bianchi identities of 11-dimensional supergravity, thus linking the two theories even at this fundamental level (of course this seemingly miracoulus fact turns out to be not so miraculous once it is realized that 11-dimensional supergravity is the low-energy limit of M-theory). We also see what made the 11-dimensional supermembrane so special, because a connection between the mysterious 11-dimensional supergravity theory and some other theory was much desired.

Now we are going to return to the case of a trivial background for a moment. We wish to do for the supermembrane, essentially what we did for the bosonic membrane. We will pass to a light-cone gauge whereafter we uncover a relation between the supermembrane and supersymmetric $SU(N)$ matrix models [12, 37, 38]. This will provide astounding new insights. Some of the newer revalations that has come through this discovery are dealt with in section 3.5.

We begin by defining the light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}(X^{10} \pm X^0), \tag{2.99}$$

in accordance with the notation in [12]. Then we make the gauge choice

$$X^+ = X^+(0) + \xi_0, \tag{2.100}$$

thus reducing the number of bosonic coordinates from eleven to nine. We also define

$$\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^{10} \pm \Gamma^0), \tag{2.101}$$

and impose

$$\Gamma^+\theta = 0, \tag{2.102}$$

in order to gauge away half of the fermionic coordinates, leaving 16. Half of these 16 degrees of freedom will be regarded as momenta leaving eight to be matched up with the bosonic ones. The original Lagrangian (2.84) becomes

$$\mathcal{L} = -\sqrt{\bar{g}\Delta} + \epsilon^{rs}\partial_r X^a \bar{\theta}\Gamma^-\Gamma_a\partial_s\theta, \tag{2.103}$$

where we have defined

$$\begin{aligned} \bar{g}_{rs} &= g_{rs} = \partial_r\vec{X}\partial_s\vec{X}, \\ g_{0r} &= u_r = \partial X^- + \partial_0\vec{X}\partial_r\vec{X} + \bar{\theta}\Gamma^-\partial_r\theta, \\ g_{00} &= 2\partial_0 X^-(\partial_0\vec{X})^2 + 2\bar{\theta}\Gamma^-\partial_0\theta, \end{aligned} \tag{2.104}$$

and

$$\bar{g} = \det \bar{g}_{rs} \quad , \quad \Delta = -g_{00} + u_r\bar{g}^{rs}u_s. \tag{2.105}$$

In order to write down the Hamiltonian in this gauge we calculate the momenta which become

$$\begin{aligned} \vec{P} &= \frac{\partial\mathcal{L}}{\partial\dot{\vec{X}}} = \sqrt{\frac{\bar{g}}{\Delta}}(\dot{\vec{X}} - u_r\bar{g}^{rs}\partial_r\vec{X}), \\ P^- &= \frac{\partial\mathcal{L}}{\partial\dot{X}^-} = \sqrt{\frac{\bar{g}}{\Delta}}, \\ S &= \frac{\partial\mathcal{L}}{\partial\dot{\bar{\theta}}} = -\sqrt{\frac{\bar{g}}{\Delta}}\Gamma^-\theta, \end{aligned} \tag{2.106}$$

and give rise to the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \vec{P}\cdot\vec{X} + P^+\cdot\dot{X}^- + \bar{S}\dot{\theta} - \mathcal{L} = \\ &= \frac{\vec{P}^2 + \bar{g}}{2P^+} - \epsilon^{rs}\partial_r X^a\bar{\theta}\Gamma^-\Gamma_a\partial_s\theta. \end{aligned} \tag{2.107}$$

Before we can proceed and relate this theory with a matrix theory we must reiterate our deliberations on area-preserving diffeomorphisms, this time for the supermembrane. We use a slightly different notation here. We define the Poisson bracket of two scalar fields $A(\xi), B(\xi)$ (i.e. functions of the two parameters ξ^1, ξ^2)

$$\{A(\xi), B(\xi)\} = \frac{\epsilon^{rs}}{\sqrt{\omega(\xi)}}\partial_r A(\xi)\partial_s B(\xi), \tag{2.108}$$

where $\omega(\xi)$ is the determinant of the two-dimensional metric $\omega^{rs}(\xi)$ on the membrane (not the worldvolume). This is a Lie bracket and it can be shown that the fields $X^a(\xi)$ belong to the Cartan subalgebra of this Lie algebra. Looking at

$$\{X^a(\xi), X^b(\xi)\} = \frac{\epsilon^{rs}}{\sqrt{\omega(\xi)}}\partial_r X^a(\xi)\partial_s X^b(\xi), \tag{2.109}$$

we see that this is precisely an area element of the membrane in spacetime, and hence the residual invariance of this bracket is precisely invariance under area-preserving diffeomorphisms (APD).

We are now ready to derive a matrix theory from the supermembrane. We begin by expanding our embedding fields $X^a(\xi)$ in a complete orthonormal set of basis functions

$$\vec{X}(\xi) = \vec{X}_0 + \sum_A \vec{X}^A Y_A(\xi), \tag{2.110}$$

the space of which has the metric

$$\int d^2\xi \sqrt{\omega(\xi)} Y_A(\xi) Y_B(\xi) = \eta_{AB}. \tag{2.111}$$

We can then express our Lie bracket (2.108) in terms of this new basis

$$\{Y_A(\xi), Y_B(\xi)\} = f^C_{AB} Y_C(\xi), \tag{2.112}$$

using the completeness relation

$$\sum_A Y^A(\xi) Y_A(\xi') = \frac{1}{\sqrt{\omega(\xi)}}\delta(\xi - \xi'), \tag{2.113}$$

where the structure constants are

$$f^C_{AB} = \int d^2\xi \epsilon^{rs}\partial_r Y_A(\xi)\partial_s Y_B Y^C. \tag{2.114}$$

Stepping to a matrix theory is now done by regularizing the theory, i.e. we cut of the infinite set of modes Y_A by restricting the number of indices to a finite value Λ. This will turn our Lie algebra (2.108) into a finite Lie group, G_Λ, on which we impose the consistency condition

$$\lim_{\Lambda\to\infty} f_{AB}^{C}(G_\Lambda) = f_{AB}^{C}(APD), \tag{2.115}$$

so that as the dimension, Λ, of the Lie group approaches infinity we retrieve our original algebra of area-preserving diffeomorphisms. In [11] it was then found that the Lie group G_Λ corresponds to $SU(N)$, with $\Lambda = N^2 - 1$, for a membrane of arbitrary topology (this had previously been derived for the spherical membrane in [9] and the torodial membrane in [39, 40, 41]). The membrane Hamiltonian turns out to regularize to

$$H = \frac{1}{2}P_a^A P_{aA} + \frac{1}{4} f_{ABE} f_{CD}^{E} X_a^A X_b^B X_a^C X_b^D - \frac{1}{2} i f_{ABC} X_a^A \theta^B \gamma^a \theta^C, \tag{2.116}$$

where θ_α^A are real $SO(9)$ spinors with 16 components. This is the form of the Hamiltonian that we will use in our subsequent considerations of the supermembrane spectrum.

Before we continue let us take a closer look at the limit (2.115). It seems curious that the APD algebra for topologically inequivalent membranes are approximated by the same group $SU(N)$. This is due to the fact that the limit will be different in each case. For each new membrane topology we consider, we have to find a new basis of $N \times N$-matrices for $SU(N)$, thus getting a "new" limit. Any basis of $SU(N)$ is equivalent to all others as long as N is finite, however when $N \to \infty$ this equivalence breaks down.

This brings us to the topic of the supermembrane spectrum which was studied most extensively in [8]. We will use our newfound relation with matrix theory in order to analyze the spectrum. What we will find is that supersymmetry exactly cancels the effect that makes the quantized bosonic membrane discrete. First we will show how this cancellation works in a simple supersymmetric extension of the theory (2.51) since we in this case know for a fact that the bosonic spectrum is discrete. This example is important since it contains all the essential features of the supermembrane case. It is equally important to keep this example in mind in order to not loose track among all the technicalities of the full supermembrane proof which we will perform directly after the example.

We begin by studying the supersymmetric model with Hamiltonian

$$H_{toy} = \{Q, Q^\dagger\} = \begin{pmatrix} -\Delta + x^2y^2 & x + iy \\ x - iy & -\Delta + x^2y^2 \end{pmatrix}, \tag{2.117}$$

where

$$Q = Q^\dagger = \begin{pmatrix} -xy & i\partial_x + \partial_y \\ i\partial_x - \partial_y & xy \end{pmatrix}. \tag{2.118}$$

In the bosonic case (2.51) we saw that the Hamiltonian was bounded from below (2.54) by the groundstate energy (regarding either x or y as a constant). In the supersymmetric case above, it turns out that there is no groundstate energy. It vanishes when supersymmetry is turned on because the off-diagonal terms in (2.117) now affects the system. Thus the bound H_{toy} becomes trivial

$$H_{toy} \geq 0, \tag{2.119}$$

and we can construct states which can escape to infinity without having to add an infinite amount of energy. To show this we want to create a wave packet that can be shown explicitly to behave as stated above. The off-diagonal terms in (2.117) are the ones that make a negative contribution to the energy and so consequently we want to write down a function that maximizes their effect. Also, we want to see explicitly how to push the function to infinity. We make the ansatz

$$\psi_t(x, y) = \chi(x - t)\psi_0(x, y)\xi_F, \tag{2.120}$$

where

$$\xi_F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{2.121}$$

precisely maximizes the negative contribution since

$$\xi_F^T H_{toy} \xi_F = H - x, \tag{2.122}$$

(where H is the Hamiltonian (2.51)). We have the parameter t in (2.120) and $\chi(x)$ is a smooth function with compact support such that χ becomes zero unless x is of order t. This means that as t increases $\psi_t(x, y)$ is pushed in the x-direction and moves to infinity along $y = 0$ as $t \to \infty$. If we look at the fermionic contribution to the energy expectation value of ψ_t we see that it is $-t\mathcal{O}(1)$ (as t grows and χ becomes dominant), which is just what is needed to cancel the energy of the groundstate in a y-harmonic oscillator. Therefore we choose $\psi_0(x, y)$ to be the groundstate wave function of such an oscillator

$$\psi_0(x, y) = \left(\frac{|x|}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}|x|y^2}. \tag{2.123}$$

Now calculating the expectation value as $t \to \infty$ we get

$$\lim_{t\to\infty} (\psi_t, H_{toy}^{\nu} \psi_t) = \int dx \chi(x)^* (-\partial_x^2)^{\nu} \chi(x), \tag{2.124}$$

which is finite as expected and also tells us that the spectrum of H_{toy} is the whole of the positive real line. If we then pick χ such that $\|\chi\| = 1$ and

$$\|(-\partial_x^2 - E)\chi\|^2 < \frac{\epsilon}{2}, \tag{2.125}$$

for any given energy eigenvalue $E \geq 0$ and arbitrary $\epsilon > 0$, we see that as t gets very large and χ becomes dominant in ψ_t we can write

$$\|\psi_t\| = 1 \quad , \quad \|(H_{toy} - E)\psi_t\|^2 < \epsilon. \tag{2.126}$$

Thus any given E is an energy eigenvalue of H_{toy}, and as the values of E extend from zero to infinity we see that we have indeed proven that the spectrum is continuous.

As we have mentioned before, in the membrane picture this is due to the fact that the membrane can grow infinitely thin spikes. In the matrix picture, on the other hand, we see that it is because the potential is similar to that of (2.117) and wave funtions can escape through the

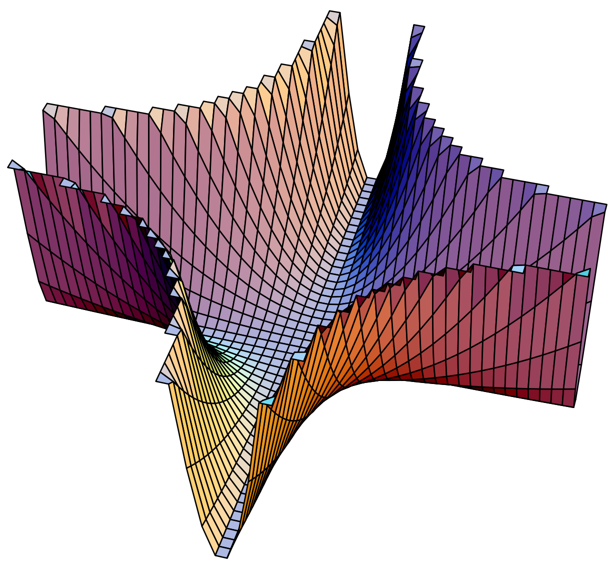

Figure 2.1: The potential x^2y^2

valleys with only a finite energy contribution.

In the full supermembrane case we will find a similar scenario. Now it will be possible for the wave functions to escape to infinity along the directions that correspond to the generators of the Cartan subalgebra belonging to the algebra of the group $SU(N)$. This fact is expressed in the following theorem [8].

Theorem 3 *Let G be a compact Lie group (of finite dimensionality) and H the associated Hamiltonian operator (2.116). Then for any energy value $E \in [0, \infty[$ and any $\epsilon > 0$, there exists a G-invariant wave funtion ψ such that*

$$\|\psi\| = 1 \quad \text{and} \quad \|(H - E)\psi\|^2 < \epsilon. \tag{2.127}$$

In particular, the spectrum of H is continuous and equal to the interval $[0, \infty[$.

It is this theorem that we will review the proof of during the rest of this section (for the group $SU(N)$).

The first thing we wish to do is give a brief outline of the proof. The most crucial step in the proof is gauge fixing. The idea is that one of the matrices, e.g. X_9, can always be diagonalized. This will turn the wave functions on the Hilbert space, $\mathcal{H}$, into reduced wave functions. Furthermore, the Hamiltonian turns into the reduced Hamiltonian. The essential feature of this gauge is that the reduced Hamiltonian splits into four terms that are separately invariant under the Cartan subgroup K. We now make an ansatz for a wave function, as in the previous toy-model example, after which we can study the groundstate for each of the terms in $\mathcal{H}$ separately. On the basis of this analysis we can then conclude the proof.

Looking at the Hamiltonian (2.116) we will in the following regard this Hamiltonian as regularized in accordance with (2.115). In order to quantize the theory we impose the canonical commutations relations

$$[P_a^A, X_b^B] = -\delta_{ab}\delta_{AB}, \tag{2.128}$$
$$\{\theta_\alpha^A, \theta_\beta^B\} = \delta_{\alpha\beta}\delta_{AB}, \tag{2.129}$$

where the Clifford algebra (2.129) of the fermionic coordinates will play a very important role in a little while.

The Hamiltonian (2.116) operates in a Hilbert space, $\mathcal{H}$, which consists of wave functions, $\psi(X_1, \ldots, X_9)$, taking values in a fermionic Fock space $\mathcal{H}_F$. This space carries a representation of the (Clifford) algebra (2.129), as well as a unitary representation of the gauge group $SU(N)$. The Hilbert space, $\mathcal{H}$, has a scalar product

$$(\psi, \phi) = \int \prod_{a,A} dX_a^A (\psi(X), \varphi(X))_F, \tag{2.130}$$

where $(\cdot,\cdot)_F$ is the unique scalar on $\mathcal{H}_F$, such that the spoinors θ^A_α are hermitian operators. The wave functions ψ in $\mathcal{H}$, the space of all physical states, must then be of finite norm and obey

$$\psi(UXU^{-1}) = V_F(U)\psi(X)\forall U \in SU(N), \tag{2.131}$$

i.e. invariance under the gauge group. V_F is some unitary representation of $SU(N)$ in $\mathcal{H}_F$.

We now set out to fix a gauge in such a way that the original system does not gain any additional constraint. We introduce the set p of all matrices, Z, on the form

$$Z = i\begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & \lambda_N \end{pmatrix}, \lambda_n \in \mathbb{R}, \tag{2.132}$$

with

$$\mathrm{tr}Z = 0, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N, \tag{2.133}$$

which satisfies

$$X = UZU^{-1}, \forall X \in su(N), \tag{2.134}$$

U being some element in the group $SU(N)$ and Z a unique element in p. Also if X is a regular element (has pairwise different eigenvalues) then the element U is unique up to a multiplication by an element in K. This means that we have a parametrization of $su(N)$ in terms of elements $Z \in p$ and $U \in G/K$. From this it can be shown that any integrable, complex valued and $SU(N)$-invariant function, $f(X)$, satisfies

$$\int \prod_A dX^A f(X) = C \int_p \prod_i dZ^i \det z f(Z), \tag{2.135}$$

where Z_{IJ} is a real, anti-symmetric matrix defined by

$$z_{IJ} = Z^k f_{kIJ}, \tag{2.136}$$

which, for $SU(N)$, has determinant

$$\det z = \prod_{m<n}^{N} (\lambda_m - \lambda_n)^2. \tag{2.137}$$

We are thus ready to fix a gauge; to any $\psi \in \mathcal{H}$ we define

$$\hat{\psi}(X_1, \dots, X_9, Z) = (C \det z)^{1/2} \psi(X)\big|_{X_9 = Z}, \tag{2.138}$$

the associated, reduced wave function. These functions contain no less information then the full wave functions and we can use these to define the Hilbert space $\hat{\mathcal{H}}$. This space consists of all functions $\hat{\psi}$ (still taking values in $\mathcal{H}_F$) such that they can be normalized with the scalar product

$$(\hat{\psi}, \hat{\varphi}) = \int \prod_{a,A} dX_a^A \int_p \prod_i dZ^i (\hat{\psi}(X,Z), \hat{\varphi}(X,Z))_F, \tag{2.139}$$

and remain invariant under K

$$\hat{\psi}(UXU^{-1}, Z) = V_F(U)\hat{\psi}(X,Z), U \in K. \tag{2.140}$$

To each element in $\hat{\mathcal{H}}$ there exists a unique element in $\mathcal{H}$, so when we set out to create a wave packet later on we only have to check that the reduced wave function has the desired properties.

The final step, before proposing a wave function that can be pushed to infinity as desired, is to evaluate the reduced Hamiltonian, $\hat{H}$, defined by

$$\hat{H}\hat{\psi} = \widehat{H\psi}. \tag{2.141}$$

To write down this Hamiltonian explicitly we have to calculate the terms coming from the $(P_9^A)^2$ part. Using the generator corresponding to X_9^A and the symmetry of the light-cone action, it can be shown that the reduced $(P_9^A)^2$ term becomes

$$(w^T w)_{IJ} \hat{L}_I \hat{L}_J - \sqrt{\det z} \left(\frac{\partial}{\partial Z^k} - w_{IJ} f_{IJk} \right) \frac{\partial}{\partial Z^k} \frac{1}{\sqrt{\det z}}, \tag{2.142}$$

where we have defined

$$w_{IJ} z_{JK} = \delta_{IK}, \tag{2.143}$$

$$\hat{L}_I = f_{IJK}(X_a^J P_a^K - \frac{i}{2}\theta_\alpha^J \theta_\alpha^K) + i f_{IJK} X^J \frac{\partial}{\partial X^K}. \tag{2.144}$$

The Hamiltonian $\hat{H}$, upon decomposing the coordinates

$$X_a^A \to (Z_a^i, Y_a^I) \quad , \quad a = 1, \ldots, 8, \tag{2.145}$$
$$Z_a^i \equiv X_a^i \quad , \quad Y_a^I \equiv X_a^i, \tag{2.146}$$

splits into four terms

$$\hat{H} = H_1 + H_2 + H_3 + H_4. \tag{2.147}$$

These terms are explicitly

$$
\begin{aligned}
H_1 &= -\frac{1}{2}\left(\frac{\partial}{\partial Z^k}\right)^2 - \frac{1}{2}\left(\frac{\partial}{\partial Z_a^k}\right)^2, & (2.148)\\
H_2 &= -\frac{1}{2}\left(\frac{\partial}{\partial Y_a^I}\right)^2 + \frac{1}{2}(z^T z)_{IJ} Y_a^I Y_a^J, & (2.149)\\
H_3 &= -\frac{1}{2} i\theta^I (z_{IJ}\gamma_9 + z_{IJ}^a \gamma_a)\theta^J, & (2.150)\\
H_4 &= \frac{1}{4} f_{AIJ} f_{AKL} Y_a^I Y_b^J Y_a^K Y_b^L + f_{AiJ} f_{AKL} Z_a^i Y_b^J Y_a^K Y_b^L + \\
&\quad + \frac{1}{2} f_{AiJ} f_{AkL} Z_a^i Y_b^J (Z_a^k Y_b^L - Z_b^k Y_a^L) + \\
&\quad + \frac{1}{2}(w^T w)_{IJ} \hat{L}_I \hat{L}_J - \frac{1}{2} i f_{IAB} Y_a^I \theta^A Y_a^B, & (2.151)
\end{aligned}
$$

with

$$z_{IJ}^a = Z_a^k f_{kIJ}. \tag{2.152}$$

Now we are ready to propose a construction of the appropriate wave packet. There are no fundamental differences here from the simple example earlier in the section and the reader will find the choice of wave function completely analogous to that example.

We begin by defining a smooth function $\chi(Z, Z_a), Z, Z_a \in k$, with compact support. As in (2.120) the function depends on a parameter t and Z as

$$\chi(Z - tV, Z_a), \tag{2.153}$$

$$V = \begin{pmatrix} s-1 & 0 & 0 & \dots & 0 \\ 0 & s-2 & 0 & \dots & 0 \\ 0 & 0 & s-3 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & s-N \end{pmatrix}, s = \frac{(N+1)}{2}, \tag{2.154}$$

such that when $t \to \infty$, the wave packet gets pushed to infinity along the direction specified by V.

We now make the ansatz

$$\hat{\psi}_t(Z, Z_a, Y_a^I) = \xi(Z - tV, Z_a)\varphi_0(Z, Y_a^I)\xi_F(Z, Z_a), \tag{2.155}$$

for the wave packet. In what follows we will see that φ_0 and ξ_F, which we choose to be the ground-state wave functions for H_2 and H_3 respectively, are separately invariant under K. Consequently, since χ depends only on $Z, V \in k$, the wave function $\hat{\psi}_t$ is in the Hilbert space $\hat{\mathcal{H}}$.

If we define

$$\Omega = \sqrt{z^T z}, \tag{2.156}$$

we easily see that (2.149) actually describes an $8(N^2 - N)$-harmonic oscillator. On the basis of this we can immediately write down the wave function describing the ground-state of H_2

$$\varphi_0(Z, Y_a^I) = \pi^{2(N-N^2)}(\det\Omega)^2 e^{-\frac{1}{2}\Omega_{IJ}Y_a^I Y_a^J}. \tag{2.157}$$

Here we introduce the inner product

$$(\varphi, \psi)_Y \equiv \int \prod_{a,I} Y_a^I \varphi^* \psi, \tag{2.158}$$

and note the normalization

$$(\varphi_0, \varphi_0)_Y = 1. \tag{2.159}$$

We can also give the energy in this ground-state

$$H_2\varphi_0 = 4\mathrm{tr}\Omega\varphi_0. \tag{2.160}$$

Arrival at the ground-state wave function ξ_F for H_3 requires a lengthier calculation which we will try to compactify here.

The matrices z_{IJ}, z_{IJ}^a can be diagonalized and their eigenvectors, E_{mn}^I, define an orthonormal basis in which the θ^I's can be expanded

$$\theta^I = \sum_{m\neq n} \theta^{mn} E_{mn}^I. \tag{2.161}$$

A general argument will then show that the ground-state energy of H_3 looks like

$$H_3\xi_F = -8\sum_{m<n}^{N} \omega_{mn}\xi_F, \tag{2.162}$$

with

$$\omega_{mn} = \sqrt{(\lambda_m - \lambda_n)^2 + (\lambda_m^a - \lambda_n^a)^2}, \tag{2.163}$$

λ_m and λ_m^a being the eigenvalues of Z and Z_a respectively. The spinor coefficients θ^{mn} we then redefine as

$$\tilde{\theta}^{mn} = \frac{1}{\sqrt{2\omega_{mn}}}\left(\sqrt{\omega_{mn} + \lambda_m - \lambda_n} - \frac{(\lambda_m^a - \lambda_n^a)\gamma_a\gamma_9}{\sqrt{\omega_{mn} + \lambda_m - \lambda_n}}\right)\theta^{mn}, \tag{2.164}$$

and yet again to chiral spinors by the projection

$$\theta_\pm = \frac{1 \pm \gamma_9}{2}\theta, \tag{2.165}$$

so that the Hamiltonian H_3 becomes

$$H_3 = \sum_{m<n}^{N} \omega_{mn} (\tilde{\theta}_+^{mn\dagger} \tilde{\theta}_+^{mn} + \tilde{\theta}_-^{mn} \tilde{\theta}_-^{mn\dagger} - 8). \tag{2.166}$$

The ground-state wave function can consequently be written as

$$\xi_F = \left(\prod_{m<n}^{N} \prod_{\alpha'=1}^{8} (\tilde{\theta}_-^{mn\dagger})_{\alpha'} \right) \xi_0, \tag{2.167}$$

where ξ_0 satisfying

$$\omega^{mn} \xi_0 = 0 \;\forall m < n, \tag{2.168}$$

is independent of λ and normalized to 1. Furthermore, ξ_F can be shown to be K-invariant and satisfy

$$\|\mathcal{D}\xi_F\|_F \leq \frac{C}{t}, \tag{2.169}$$

for any number of diffrentiations, $\mathcal{D}$, with respect to λ_m and λ_m^a, and for a large value on t. This result is in analogy with one for φ_0 saying that

$$|(\mathcal{D}_1\varphi_0, \mathcal{D}_2\varphi_0)_Y| \leq C t^{\dim \mathcal{D}_1 + \dim \mathcal{D}_2}, \tag{2.170}$$

for some choice of operators $\mathcal{D}_1, \mathcal{D}_2$, possibly containing derivatives with respect to Y_a^I and λ_m, and for suitable C and large t.

From the equality

$$(H_2 + H_3)\hat{\psi}_t = 8 \sum_{m<n}^{N} (\lambda_m - \lambda_n - \omega_{mn})\hat{\psi}_t, \tag{2.171}$$

we see that the energy approaches zero as $\lambda_m - \lambda_n \to \infty$. Since φ_0 and ξ_F are both normalized to 1, the normalization of $\hat{\psi}_t$ depends only on the normalization of χ. We have

$$(\hat{\psi}_t, \hat{\psi}_t) = \langle \chi, \chi \rangle, \tag{2.172}$$

where

$$\langle \chi_1, \chi_2 \rangle = \int \prod_i Z^i \prod_{k,a} dZ_a^k \chi_1^* \chi_2, \tag{2.173}$$

form this and the fact that χ has compact support we can draw the conclusion that

$$\lim_{t\to\infty} \|(H_2 + H_3)\hat{\psi}_t\| = 0. \tag{2.174}$$

Also for H_4 we can use the inequalities (2.169) and (2.170), since all terms in (2.151) have dimension $\leq -\frac{1}{2}$ we conclude that

$$\lim_{t\to\infty} \|H_4\hat{\psi}_t\| = 0. \tag{2.175}$$

Thus we only have to consider H_1 and from (2.148) we see that when it acts on φ_0 and ξ_F we can use the relations (2.169) and (2.170) again to set the contributions to some term of order t^{-1}. It follows then that

$$\lim_{t\to\infty}(\psi_t, H^\nu \psi_t) = \lim_{t\to\infty}(\hat{\psi}_t, H_1^\nu \hat{\psi}_t) = \langle \chi, H_1^\nu \chi\rangle \,, \nu = 0, 1, 2, \tag{2.176}$$

Thus if we pick an arbitrary energy eigenvalue E and an $\epsilon > 0$, we can choose χ such that

$$\langle \chi, \chi\rangle \quad , \quad \langle (H_1 - E)\chi, (H_1 - E)\chi\rangle < \frac{\epsilon}{2}, \tag{2.177}$$

and from the properties of the wave function ψ_t and its associated reduced function $\hat{\psi}_t$ it follows that

$$|((H - E)\psi_t, (H - E)\psi_t) - \langle (H_1 - E)\chi, (H_1 - E)\chi\rangle \,| < \frac{\epsilon}{2}. \tag{2.178}$$

Thus for a sufficiently large t we get

$$\|\psi_t\| = 1 \quad , \quad \|(H - E)\psi_t\|^2 < \epsilon, \tag{2.179}$$

which finally proves the theorem stated earlier in this section. It should be emphasized that, even though the review of the proof here contains many details that one generally does not find in a review, it still leaves out some details. The reader who whishes to study the proof in full detail should refer to [8]. In that paper the authors noted that for the case of the supermembrane “our result does not bode well for its future”, and the interest in membrane theory dropped drastically after they published their result. However many years later the continuous spectrum of the membrane was turned into a virtue in a new and bold interpretation. This new view of the supermembrane should be reviewed in the light of M-theory, which is why we have postponed the discussion until section 3.5.

3

The Supermembrane II: M-theory

This section bridges the gap between the material presented in the previous chapter, mainly developed before 1990, and the following chapter which contains physics that have only very recently been developed. Apart from studying the role of the membrane in M-theory we will treat the different topics that are needed as background material in the forthcoming chapter and also include some subjects needed in order to maintain some level of completeness. Due to the staggering amount of open problems in M-theory, many of them very fundamental, the field is in a constant state of diversification. This means that a complete and stringent review of what has been done is virtually inconstructible. This is of course also the case for the following chapter and therefore we make no attempt at such a review settling instead for a chapter that is the mere handmaiden of chapter 4 and also lacks a clear logical structure. Sometimes concepts will be used before being properly introduced but in these cases the proper pointers and references are included.

3.1 Five String Theories

String theory first arose as a theory of strong interaction [42,43,44] in the late sixties. This attempt ultimately failed and the theory was replaced by quantum chromodynamics. In the beginning of the seventies it was then conjectured [45,46,47] that string theory should be elevated from a theory of strong interactions to a theory incorporating all the forces in nature, including gravity. The work on string theory during the period from 74 to 84 was largely done by a small group of people, mostly because the problems that plagued the theory seemed insurmountable at the time. Then came the period which we now call the first superstring revolution. The explosion of interest that was set off was mainly due to the discovery of the conditions for anomaly cancellation in various theories. In [48] it was shown that as for chiral theories the only anomaly free theory in ten dimensions was Type IIB supergravity (which we know to be the low-energy limit of IIB superstring theory). The next anomaly cancellation was revealed in [49, 50, 51] for Type I strings, that is open strings which have gauge degrees of freedom added to their endpoints. It can be shown by considering the field theoretic low-energy effective action that anomaly cancellation occurs for the gauge groups $SO(32)$ and $E_8 \times E_8$ [52]. This led to the discovery of the two heterotic string theories [53].

Thus the search for a single unified theory of all interactions and the attempt to unify gravity and quantum mechanics, had led to five(!) different string theories.

The resolution to this big mystery came after about ten years (notice that we are skipping a long and important period of theoretical high-energy physics) in the so called second superstring revolution. The main themes here were the discoveries of dualities between the different string theories and the role of D-branes. This hinted toward the existence of a single theory containing all the others as limits, a master theory, mysteriously named M-theory, related somehow to eleven dimensions.

Before considering some specific properties of Type II string theory and moving on to M-theory, let us briefly review the most basic construction of string theory and its features. The action (2.1) for the bosonic string is defined by the area of the worldsheet that the string traces out as is propagates

$$S = \int d^2\sigma \det \sqrt{\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}}, \qquad (3.1)$$
$$i, j = 1, 2, \ \mu, \nu = 0, 1, \ldots, D-1,$$

which is the Nambu-Goto action of the relativistic string. We also have a $(p = 1)$ Polyakov type action

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}, \qquad (3.2)$$

where g^{ij} is a two-dimensional auxiliary metric. Note the absence of a cosmological term; this will turn out to be very fortunate. Apart from the ordinary reparametrization invariance this action is Weyl invariant (unlike the Nambu-Goto action), i.e., symmetric under rescalings of the metric

$$g_{ij}(\sigma) \longrightarrow e^{2\omega(\sigma)} g_{ij}(\sigma), \qquad (3.3)$$

and these three symmetries (two reparametrization invariances and one Weyl) are exactly what we need to gauge-fix away the unphysical degrees of freedom in g_{ij}. When we chose a gauge analogous to the light-cone gauge in chapter 2 this will eliminate any unphysical degrees of freedom, and so we can proceed to quantize the theory by imposing equal-time commutator-relations in the standard way. One problem however, remains in the classical light-cone gauge-fixed theory; Lorentz invariance is hidden but must still remain a symmetry of the theory and in the the quantization procedure an anomaly arises. We get conditions for the preservation of Lorentz invariance. Amon others $D = 26$, i.e., the dimension of spacetime must be 26, which is true for both the open and closed string. The lightest state in bosonic string theory is a tachyon (i.e. it has negative mass squared) meaning that the vacuum state (the state with no string) is actually unstable. Whether or not the bosonic string theory has some stable vacuum is still a matter for research.

Unlike the case in membrane theory, it is possible to formulate a string theory with worldsheet supersymmetry. Here we wish to construct such a theory by extending the conformal symmetry of (3.3) to a superconformal symmetry. We begin by introducing a worldsheet spinor field ψ^μ into the bosonic action (in a conformal gauge), extending it to

$$-\frac{1}{2\pi}\int d^2\sigma(\partial_i X^\mu \partial^i X_\mu - i\bar{\psi}^\mu \rho^j \partial_j \psi_\mu), \tag{3.4}$$

where ψ is a two-component spinor

$$\psi^\mu = \begin{pmatrix} \psi_1^\mu \\ \psi_2^\mu \end{pmatrix}, \tag{3.5}$$

with

$$\bar{\psi}^\mu = \psi^\mu \rho^0. \tag{3.6}$$

The matrices ρ^i are given by

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{3.7}$$

satisfying

$$\{\rho^i, \rho^j\} = -2\eta^{ij}, \tag{3.8}$$

(η^{ij} being a two-dimensional Minkowski metric). This action has a conserved supercurrent

$$J_i = \frac{1}{2}\rho^j \rho_i \psi^\mu \partial_j X_\mu, \tag{3.9}$$

coming from the supersymmetry transformations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \quad , \quad \delta\psi^\mu = -i\rho^i \partial_i X^\mu \epsilon. \tag{3.10}$$

An alternative formulation of this theory with a more manifest version of supersymmetry would come from introducing a superspace. This is done by defining two new coordinates θ^a, $a = 1, 2$ which are anti-commuting and form a Majorana spinor together. A general function on this space, a superfield, is given by the power series

$$Y^\mu(\sigma, \theta) = X^\mu(\theta) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma), \tag{3.11}$$

which is exact due to the anti-commutation relations for θ. $B^\mu(\theta)$ is an auxiliary field which allows us to demonstrate that the supersymmetry algebra closes without the use of field equations. A supersymmetry transformation in the formerly used notation is now equivalent to a transformation of the superspace coordinates

$$\delta\theta^a = [\bar{\epsilon}Q, \theta^a] = \epsilon^a, \tag{3.12}$$

$$\delta\sigma^i = [\bar{\epsilon}Q, \sigma^i] = i\bar{\epsilon}\rho^i\theta, \tag{3.13}$$

generated by

$$Q_a = \frac{\partial}{\partial\bar{\theta}^a} + i(\rho^i\theta)_a\partial_i, \tag{3.14}$$

with ϵ^a being some infinitesimal parameter. For the superfield Y^μ we then get

$$\delta Y^\mu = [\bar{\epsilon}Q, Y^\mu] = \bar{\epsilon}QY^\mu, \tag{3.15}$$

and expanding this according to (3.11) we get back the transformations (3.10). The next step in order to create supersymmetric field theories within this formalism is to introduce a superspace covariant derivative

$$D = \frac{\partial}{\partial\bar{\theta}} - i\rho^\alpha\theta\partial_\alpha, \tag{3.16}$$

which produces superfields from superfields under derivation. Using integration over all of superspace we are now ready to write down an action

$$S = \frac{i}{4\pi}\int d^2\sigma d^2\theta \bar{D}Y^\mu DY_\mu, \tag{3.17}$$

where we have

$$DY^\mu = \psi^\mu + \theta B^\mu - i\rho^\alpha\theta\partial_\alpha X^\mu + \frac{i}{2}\bar{\theta}\theta\rho^\alpha\partial_\alpha\psi^\mu, \tag{3.18}$$

$$\bar{D}Y^\mu = \bar{\psi}^\mu + B^\mu\bar{\theta} + i\partial_\alpha X^\mu\bar{\theta}\rho^\alpha - \frac{i}{2}\bar{\theta}\theta\partial_\alpha\bar{\psi}^\mu\rho^\alpha. \tag{3.19}$$

Due to the measure of the fermionic integration only terms quadratic in θ will survive the integration, thus the action can be written as

$$-\frac{1}{2\pi}\int d^2\sigma(\partial_\alpha X^\mu\partial^\alpha X_\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu - B^\mu B_\mu), \tag{3.20}$$

varying this action gives us the field equation for B

$$B_\mu = 0, \tag{3.21}$$

so we can just forget about the term $B^\mu B_\mu$ in (3.20) and thus we retain the action (3.4).

The fields living on the string worldsheet naturally have to satisfy a periodicity condition, i.e. invariance under

$$\sigma_2 \rightarrow \sigma_2 + 2\pi. \tag{3.22}$$

For our worldsheet spinor fields $\psi^\mu(\sigma_1, \sigma_2)$ this leaves the freedom to choose sign, i.e.

$$\psi^\mu(\sigma_1, \sigma_2 + 2\pi) = +\psi^\mu(\sigma_1, \sigma_2), \tag{3.23}$$

$$\psi^\mu(\sigma_1, \sigma_2 + 2\pi) = -\psi^\mu(\sigma_1, \sigma_2), \tag{3.24}$$

are equally viable fields. This splits all ψ^μ into two different sets, the first called Ramond fields (which we will denote by an R) and the second Neveu-Schwarz fields (NS) so that in the expansion of the different fields the sum runs over integers in the R case and half-integers in the NS case. By enforcing R and NS boundary condition differently on the left- and right-moving modes of ψ^μ and $\bar{\psi}^\mu$ respectively we get four different sector in the theory, R-R, NS-NS, NS-R and R-NS. One must now enforce a number of consistency conditions in order to arrive at a sound theory. These conditions pick out subsets of these sectors containing the states that form the spectrum of the theory. Since our interest is in membranes mainly we will not mention much about the Type I and Heterotic string theories, but the relation between membranes and Type II string theories is something that we will rely heavily on in the following sections and therefore a few words about these are in order.

The two Type II theories are tachyon free theories of closed superstrings in 10 dimensions, Type IIA and IIB corresponding to non-chiral and chiral theories, respectively. Type IIB string theory and supergravity have been dealt with in appendix A at the level of detail which is necessary in this thesis. Regarding the Type IIA theory we will use the strong coupling limit relation to M-theory frequently in what follows so we will now review the dimensional reduction that yields the IIA supergravity from the 11D supergravity in order to obtain the relations between the string theory coupling and the eleventh direction in M-theory. We will only consider the techniques in the bosonic parts of the two theories since this is all we need for our further considerations.

The bosonic part of 11-dimensional supergravity is described by

$$S_{11} = \frac{1}{l_p^9} \int d^{11}x \sqrt{-g} \left(R - \frac{l_p^9}{48} (dC)^2 \right) + \frac{\sqrt{2}}{2^7 \cdot 3^2} \int C \wedge dC \wedge dC, \tag{3.25}$$

where l_p is the 11-dimensional Planck length, g the metric, R the corresponding Ricci scalar and C is the 3-form coupling to the membrane. Taking the 11-dimensional metric

$$ds_{11}^2 = R_s^2(dx^s + A_\mu dx^\mu)^2 + ds_{10}^2, \tag{3.26}$$

and substituting it into (3.25) we obtain

$$S_{10} = \frac{1}{l_p^9}\int d^{10}xR_s\sqrt{-g}\left[R + \left(\frac{\partial R_s}{R_s}\right)^2 + R_s^2(dA)^2 + l_p^6(dC)^2 + \frac{l_p^6}{R_s^2}(dB)^2\right] + \int B\wedge dC\wedge dC. \tag{3.27}$$

Here, R_s is the radius of the eleventh direction x^s and A_μ is the $U(1)$ gauge field that arises in order to maintain consistency under the isometry in the compact direction. The 3-form $C_{\mu\nu\rho}$ is split into a 2-form $B_{\mu\nu} = C_{\mu\nu s}$ and a 3-form $C_{\mu\nu\rho}$. Comparing this to the ordinary Type IIA supergravity action

$$S_{IIA} = \frac{1}{l_s^8}\int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(R + 4(\partial\phi)^2 - \frac{l_s^4}{12}(dB)^2\right) - \frac{l_s^2}{4}(dA)^2 - \frac{l_s^6}{48}(dC)^2\right] + \int B\wedge dC\wedge dC, \tag{3.28}$$

we make the identifications

$$R_s = l_s g_s, \qquad \frac{R_s}{l_p^3} = \frac{1}{l_s^2}, \tag{3.29}$$

where upon we see that the limit $R_s \to \infty$ taking us from 10 to 11 dimensions indeed corresponds to a strong coupling limit. Thus we have seen that from the 11-dimensional supergravity action, by an appropriate dimensional reduction, we can obtain the Type IIA supergravity action describing the dynamics of the massless NS-NS fields $g_{\mu\nu}$ (metric), $B_{\mu\nu}$ (anti-symmetric tensor) and ϕ (dilaton) as well as the R-R fields A_μ and $C_{\mu\nu\rho}$ (gauge fields). The string can also be obtained from the membrane in this manner, by a *double dimensional reduction* [54]. By considering the bosonic part of (2.94) and identifying the direction ξ^3 on the membrane worldvolume with the eleventh direction x^s in spacetime and then considering the background fields independent of x^s we retain the superstring action. The intuitive picture here is that the membrane completely wraps the direction x^s and when this is compactified on a circle of small radius the membrane becomes a string.

This short text is by no means meant to act as a source on string theory. The issues we have discussed here (and not in the appendices) are merely included in order to achieve a slightly higher level of completeness; for material on string theory we refer the reader to [55, 56].

3.2 Dualities, D-branes and Moduli

Even before the second superstring revolution the notion of duality was familiar. In string theory it was discovered as early as 1984 that T-duality transformations related different string theories [57,58]. T-duality is most easily described for a theory compactified on a circle, for example Type II string theory, when compactified on a circle of radius R gives Kaluza-Klein momenta of the form

$$p_{KK} = \frac{n}{R}, n \in \mathbb{Z}. \tag{3.30}$$

I addition to these Kaluza-Klein modes the string can wrap, m times, around the compact direction, and taking also this into account we obtain the momenta

$$p = \frac{n}{2R} \pm mR, \tag{3.31}$$

for left- and right-moving modes respectively. It is evident from this that the mass-squared $\mathcal{M}^2$ is invariant under $R \leftrightarrow \frac{1}{2R}$ upon also interchanging $n \leftrightarrow m$. This is T-duality, where the inversion of the radius corresponds to the exchange of windings against Kaluza-Klein momenta. It can also be shown that the transformation above reverses the chirality of the left-going (or right-going, depending on definition) modes of the string, turning a chiral theory into a non-chiral theory and vice versa. Therefore we conclude that T-duality relates IIA and IIB and that IIA $\leftrightarrow$ IIB under $R \leftrightarrow \frac{1}{2R}$ in our particular case. In a similar manner T-duality relates the heterotic string theories $SO(32)$ and $E_8 \times E_8$.

The duality we are about to study next led to the profound insights about string theory that we already encountered in the previous section. S-duality relates strong and weak coupling, and even before its advent it had long been known that 11-dimensional supergravity reduced to 10-dimensional non-chiral supergravity (IIA) upon compactifying on a circle and discarding the Kaluza-Klein modes. But when keeping these modes and also keeping the radius of compactification finite it was noticed [59, 60] that the relationship between the string coupling constant and the radius R_s of the eleventh direction hinted toward an 11-dimensional theory corresponding to the strong coupling limit of Type IIA string theory. This we saw already in the previous section by an analysis of

Type IIA supergravity and 11-dimensional supergravity, the theory to which this new mysterious theory, namely M-theory, has to reduce in the low-energy limit. Furthermore, in addition to reducing to Type IIA string theory when compactified on S^1 it can be shown that M-theory reduces to heterotic $E_8 \times E_8$ when compactified on the orbifold $S^1/\mathbb{Z}^2$.

The Type IIB theory turns out to be invariant under this duality. We will not expand much further on this here, in section A.4, where this symmetry of Type IIB is used, the formalism needed to check this is presented.

Finally, for completeness, we state the fact that the Type I string theory is S-dual to the heterotic $SO(32)$ theory. Thus we have completed the information needed in order to understand the following popular picture.

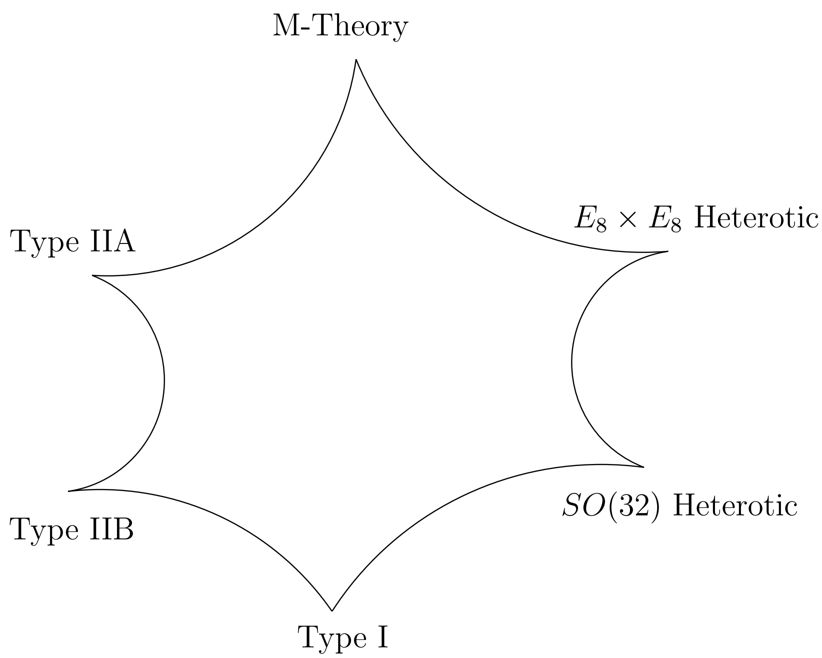

Figure 3.1: The moduli space of M-theory

Let us summarized:

- T-duality: A perturbative duality that relates different compactifications and reveals the relations IIA $\leftrightarrow$ IIB and $SO(32) \leftrightarrow E_8 \times E_8$.
- S-duality: A non-perturbative duality relating a strongly coupled region to a weakly coupled one. Relates IIA $\leftrightarrow$ M-theory, $E_8 \times E_8$ $\leftrightarrow$ M-theory and Type I $\leftrightarrow$ $SO(32)$.
- U-duality: A non-perturbative duality that incorporates both S- and T-duality [61]. We will examine it more closely in section 3.4.

We will end this section by considering the inside of the M-theory star in figure 3.1, but before we do this we need to mention something about a type of object that will appear often in what follows, D-branes.

We have already stated that IIB string theory is self-dual under S-duality, furthermore the fundamental string in Type IIB is charged under the NS-NS field $B^{(1)}$ (see section A.4) and not under the R-R field $B^{(2)}$. But S-duality mixes these two charges and consequently if IIB is supposed to be self-dual under S-duality then it must contain objects charged under this R-R field. It does, D-strings, furthermore it contains a whole menagerie of these D-objects or D-branes. They are BPS-states and non-perturbative in nature which explains why their relevance was not fully understood before the second (non-perturbative) superstring revolution. The study of D-branes in various contexts virtually exploded after the paper [62] and we will only state a few facts here that are of general or direct relevance in the forthcoming sections.

D-branes were in fact studied much further back in time, originally introduced as dynamic hyper-surfaces in spacetime on which open strings could end [63, 64], but the paper [62] very much revolutionized the way people thought about them and also revealed a number of new features.

The D-string or D1-brane lives in Type IIB string theory, more generally we have only Dp-branes with odd p in IIB and with even p in IIA. They are solitonic objects, the forces in between them cancel which is why we can compose D-branes into stacks of arbitrary size in order to obtain states of arbitrary charge. This is, as we will see in section 3.5, of paramount importance in matrix theory. Further facts about D-branes can be found in sections 3.5 (D0-branes) and A.3 (D-instantons or D(-1)-branes), as well as spread out through the text.

We have seen how dualities enter into M-theory and relate the different string theories. We have also talked about how this introduces D-branes in a new way. The string theories are actually asymptotic limits in the moduli space of M-theory (this is why we picture them as cusps in the M-theory star), dualities transform between these different $g_s \to 0$ limits in the moduli space but still only allows us to investigate a set of moduli with zero measure. The BPS-states that we are about to study in the next section are states that behave nicely in limits opposite to the $g_s \to 0$ limit and can, because of this, be studied at points away from the string theoretical points of the moduli space. M-theory is in some sense a theory in the complete moduli space, but it is believed that there should be some sort of potential (or procedure) that fixes the moduli completely.

3.3 BPS States

In section A.2 of the appendices we give a brief introduction to supermultiplets and BPS-states (the reader unfamiliar with the concept of BPS-states is well advised to read this section prior to reading this and indeed the rest of this thesis), while here we wish to extend these considerations into the realm of M-theory. Much of the chapter following this one is devoted to the attempt to use the microscopic degrees of freedom in M-theory in order to do calculations reproducing results previously known from string theory. But very little is known about these microscopic degrees of freedom. One of the few ways of studying these is through 11-dimensional supergravity. The central principle in 11-dimensional supergravity, $N = 1$ supersymmetry, is independent of energy-scale and should thus be valid even in M-theory, and the states of M-theory should therefore organize into multiplets of the super-Pioncaré algebra. This algebra contains central charges Z^{IJ} and Z^{IJKLM} (upper-case letter shall often denote 11-dimensional indices) which are interpreted as the electric and magnetic charges of various extended M-theory objects. These objects are the:

- Membrane (M2-brane): The main character of this thesis. It is charged under the 3-form C_{IJK}.
- 5-brane (M5-brane): Another important object that will come into play in the amplitudes we consider in the next chapter. It is charged under the 6-form dual to the 3-form C_{ijk}.
- KK6-brane: The Kaluza-Klein 6-brane (or monopole) will not be considered here since it is of less importance for our purposes.
- 9-brane: A non-dynamical object that is not charged and which will not appear in this thesis.

BPS-states are shared between the two energy regimes of M-theory and 11-dimensional supergravity. Therefore they give us the opportunity to study M-theoretical objects as solitonic solutions in 11-dimensional supergravity. These states are also as we know solutions that satisfy non-renormalization theorems, i.e. are protected from the perils of quantization. They are also stable which means that they survive strong-coupling limits (or equivalently they remain in any compactification of M-theory). There are three standard solutions that describe extended objects which satisfy $\frac{1}{2}$-BPS conditions, the membrane, 5-brane and KK6-brane. The dynamics of membranes we have already studied extensively, the 5-brane dynamics are governed by a six-dimensional tensor theory whereas the KK6-brane is described by a gauged sigma model.

We have already mentioned how the membrane reduces to the Type IIA string in a double-dimensional reduction of M-theory and table 3.2 completes the picture to the extent that is needed here. It displays the relevant BPS-states in M-theory and how these reduce depending on whether they wrap or do not wrap the compact direction. As mentioned

M-Theory	Type IIA
Supergraviton ($P^{11} = 1/R$)	$D0$-brane
Wrapped $M2$-brane	String
Unwrapped $M2$-brane	$D2$-brane
Wrapped $M5$-brane	$D4$-brane
Unwrapped $M5$-brane	NS 5-brane

Figure 3.2: Object correspondence between Type IIA and M-theory

before these states behave very nicely in the different limits through which we can enter M-theory, and are therefore ideal for the purpose of checking dualities and the relations that these infer on the objects in various string theories. BPS-states in relation to duality and the different compactifications of M-theory is the subject of the next section.

3.4 Representation Theory of Duality Groups

When we studied dualities in section 3.2 we took a fairly simplistic view of them, T-duality inverting the radius of compactification in a theory compactified on S^1 and S-duality simply inverting the coupling constant. In truth dualities act on all of the moduli of a theory thus constituting a transformation in the full moduli space. These transformations are described by the action of various groups, duality groups, which we will study here. For example S-duality acts by matrix multiplication on the matrix composed of the scalars of Type IIB. We use this and the symmetry of Type IIB in section A.4 to create an $Sl(2, \mathbb{Z})$ multiplet of strings. In this section we will be concerned with T-duality to some extent but mostly U-duality, a symmetry of M-theory that we aim to use in order to derive amplitudes in the following chapter.

We have previously said that T-duality relates different compactifications, more precisely it is a perturbative "symmetry" of the compactified M-theory, meaning that it hold in every order of the perturbation theory. In section A.4 we saw how the scalar fields ϕ, $C^{(0)}$ could be written in a matrix to realize the symmetry under $Sl(2,\mathbb{Z})$. In the same manner the NS-NS fields g_{ij} and B of Type IIB can be written as

$$M = \begin{pmatrix} g^{-1} & g^{-1}B \\ -Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \tag{3.32}$$

displaying the fact that these fields parametrizes the coset

$$\frac{SO(d,d,\mathbb{R})}{SO(d) \times SO(d)}. \tag{3.33}$$

The form of this matrix stems from a fundamental mathematical fact that also has physical implications. We can decompose our T-duality group in what is called the Iwasawa decomposition

$$G = K \cdot A \cdot N, \tag{3.34}$$

where K, A and N are maximal compact, Abelian and nilpotent subgroups respectively . The matrix M can then be written

$$M = v^T v, \tag{3.35}$$

where v is the vielbein on the group manifold $SO(d,d,\mathbb{R})$ chosen from the Iwasawa decomposition as $v = 1 \cdot a \cdot n$ on the basis of fixing the invariance under K in such a was that we arrive at the correct coset. When quantized this symmetry must be constrained to the discrete subgroup $SO(d,d,\mathbb{Z})$, the T-duality group of a torodially compactified string theory.

Instead of approaching U-duality as the symmetry of M-theory which contains both the T-duality and S-duality groups as subgroups we wish to see how it arises from 11-dimensional supergravity. We have previously stated that supersymmetry is a principle that is expected to hold in the high energy limit of 11-dimensional supergravity. The super-Poincaré group decomposes when we compactify on T^d, leaving the symmetry $Sl(d+1,\mathbb{R})$ as the 11-dimensional unbroken part and gaining an additional symmetry; R-symmetry. This symmetry turns out to be a maximal compact subgroup of the global symmetry group G_{d+1}, the Cremmer-Julia symmetry group, which contains the symmetries, additional to the super-Poincaré invariance, of the supergravity theory. These groups and their maximal compact subgroups are given in table 3.3. We will often write G_{d+1} as $E_{d+1(d+1)}$. These symmetries can not persist to be continuous in the quantized limit as the charges under the gauge potentials of the theory become discrete. Thus they have to reduce to discrete subgroups of G_{d+1}; the U-duality groups (table 3.4). They can be seen to contain both the S-duality and T-duality groups as subgroups.

$d+1$	G_{d+1}	H_{d+1}
1	$\mathbb{R}^+$	1
2	$Sl(2,\mathbb{R}) \times \mathbb{R}^+$	$U(1)$
3	$Sl(2,\mathbb{R}) \times Sl(3,\mathbb{R})$	$SO(3) \times U(1)$
4	$Sl(5,\mathbb{R})$	$SO(5)$
5	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$
6	$E_{6(6)}(\mathbb{R})$	$USp(8)$
7	$E_{7(7)}(\mathbb{R})$	$SU(8)$
8	$E_{8(8)}(\mathbb{R})$	$SO(16)$

Figure 3.3: The Cremmer-Julia groups and their maximal compact subgroups

$d+1$	U-duality
1	1
2	$Sl(2,\mathbb{Z})$ $(E_{2(2)}(\mathbb{Z}))$
3	$Sl(2,\mathbb{Z}) \times Sl(3,\mathbb{Z})$ $(E_{3(3)}(\mathbb{Z}))$
4	$Sl(5,\mathbb{Z})$ $(E_{4(4)}(\mathbb{Z}))$
5	$SO(5,5,\mathbb{Z})$ $(E_{5(5)}(\mathbb{Z}))$
6	$E_{6(6)}(\mathbb{Z})$
7	$E_{7(7)}(\mathbb{Z})$
8	$E_{8(8)}(\mathbb{Z})$

Figure 3.4: The U-duality groups

Turning now to the representation theory of these groups, U-duality is the only symmetry that is thought to be valid throughout the high energy limit of 11-dimensional supergravity. Thus in dividing the spectrum into multiplets, irreducible representation, of these symmetry groups we can expect the content of these to remain the same when passing from 11-dimensional supergravity to M-theory. We will limit our discussion to BPS-states here since these are the only ones over which we have some control in high energy and strong coupling limits. Actually , so far, when we say multiplets of the U-duality group we mean only the irreducible representations of a much smaller group, the Weyl group, and after creating these we will see how they can be extended to the whole group. The Weyl group has a set of generating elements corresponding to

$$S_I : \quad R_I \leftrightarrow R_{I+1} \tag{3.36}$$

$$T : \quad R_1 \to \frac{l_p^3}{R_2 R_3}, l_p^3 \to \frac{l_p^6}{R_1 R_2 R_3}, \tag{3.37}$$

which act on the vectors in the weight space as Weyl reflections. We can now derive the fundamental weights dual to the simple roots and these in turn make up the Dynkin basis of our weight space. From these weights we can pick out the corresponding irreducible representations, the fundamental representations. By representing the vectors in the weight space by monomials in the directions R_I, i.e. tensions of various objects, we can directly realize the action of the elements (3.36) and (3.37). The orbits[1] of each of these weights yields the irreducible representation in which the fundamental weight is the highest weight. The multiplets which we will use later on are the string multiplet and the membrane multiplet, corresponding to the weights $\mathcal{T}_1 = \frac{R_1}{l_p^3}$ and $\mathcal{T}_2 = \frac{1}{l_p^3}$ respectively. However acting with the Weyl generators alone does not generate the full multiplet of the U-duality group since we are only studying its Weyl group in that case. In order to extend these multiplets we need to act on the highest weight in each multiplet with Borel generators as well (simply put those generators that are not Weyl generators). In our case these turn out to be: a generator that interchanges the eleventh direction with one of the others and generators shifting the gauge potentials of M-theory by a constant. Proceeding in this manner we see that the string multiplet consists of not just membranes wrapped around one compact dimension ($\frac{R_I}{l_p^3}$) but also 5-branes wrapped around four dimensions as well as the higher-dimensional objects of M-theory wrapped in different ways depending on in which dimensionality we are, i.e. which U-duality group we are representing. The same argument can be used for the membrane multiplet (and indeed all the other multiplets) thus yielding the wanted split of the spectrum. [22] [65] [22]

3.5 The BFSS-Conjecture

Previously (section 2.1 and 2.2) we have seen how matrix theory can be defined through a regularization of the membrane and supermembrane [9, 10,12]. We saw that matrix theory became a very powerful tool in dealing with the theory of supermembranes, but this new found tool did however also help us to prove once and for all that the supermembrane really does have a continuous spectrum [8]. A continuous spectrum is a devastating attribute for a classical theory that needs to be first quantized, but a necessity for a theory that is already a second quantized theory in the

[1]The group orbit of an element v in the representation space M is defined as $\mathcal{O}_v = \{w \in M : \exists \gamma \in G : w = (R(\gamma))(v)\}$, i.e. given an element v in the representation space the orbit is the set of elements which can be reached from v by the action of the representation of some element γ in G

sense that it describes multiparticle states. In the paper [13] it was conjectured that matrix theory, in the large N limit, in fact describes all of M-theory (in the infinite momentum frame). Here we refer to this result as the BFSS-conjecture after the authors. We will spend the next few pages reviewing this original work, the motivations for the conjecture, and some of the work that has been done since.

But before we can do this we need to state some preliminary facts about D-branes (section 3.2), relations of these to supersymmetric Yang-Mills theories and dimensional reductions thereof. By considering the open string in a fixed D-brane background it can be shown that [64, 66] the low-energy action for the Dp-brane becomes $U(1)$ SuperYang-Mills (SYM), dimensionally reduced to $p+1$ dimensions from ten dimensional $U(1)$, $\mathcal{N}=1$ SYM. The action of this theory is given by[2]

$$S = \frac{1}{g_{YM}^2}\int d^{10}\xi \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\psi}\Gamma^\mu\partial_\mu\psi\right), \tag{3.38}$$

where $F^{\mu\nu}$ is the field strength and ψ a 16 component Majorana-Weyl spinor of $SO(1,9)$. This fact can be generalized to the case of N parallel D-branes (of equal dimension p). In this case we have fields A^μ_{ij} corresponding to oriented strings stretching between the ith and the jth D-brane, with μ being the 10-dimensional spacetime index. The mass of this field is proportional to the length of the string and thus as the D-branes gets closer to one another this field becomes massless. It has then been shown [66] that these fields (in a static gauge on a flat background) obey the dynamics of a 10-dimensional $U(N)$ SYM, dimensionally reduced to $p+1$ dimensions, in a low energy limit. This theory is (with the fields properly rescaled described by the action

$$S = \frac{1}{g_{YM}^2}\int d^{10}\xi(-\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + 2i\mathrm{tr}\bar{\psi}\Gamma^\mu D_\mu\psi), \tag{3.39}$$

$$D_\mu = \partial_\mu - iA_\mu, \tag{3.40}$$

which we dimensionally reduce by considering all of the fields as independent of the coordinates $p+1,\ldots,9$. The field A_μ now becomes decomposed into $9-p$ adjoint scalar fields, X^a, and a $(p+1)$-dimensional gauge field A_α, the action for which has the bosonic part

$$S = \frac{1}{g_{YM}^2}\int d^{p+1}\xi\,\mathrm{tr}(-F_{\alpha\beta}F^{\alpha\beta} - 2(D_\alpha X^a)^2 + [X^a, X^b]^2). \tag{3.41}$$

[2]In this discussion of D-branes we set $2\pi\alpha' = 1$.

The fields X^a are interpreted as describing fluctuations in the directions transverse to the D-brane and A_α is a gauge field living on the worldvolume. This action is then indeed the action describing the low-energy dynamics of our branes in a static gauge (i.e. the $p+1$ dimensions of the worldvolume are directly identified with directions in spacetime) and a flat background. By considering a state of this system in which the fermionic fields as well as the field strenghts $F_{\alpha\beta}$ become zero, and furthermore the scalar fields are constant and commuting, we see that X^a can be diagonalized to yield

$$X^a = \begin{pmatrix} x_1^a & 0 & 0 & \cdots & 0 \\ 0 & x_2^a & 0 & \cdots & 0 \\ 0 & 0 & x_3^a & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & x_N^a \end{pmatrix}. \tag{3.42}$$

In this way we see that x_n^a can be associated with the position of the nth D-brane in the ath transverse direction. The main motivation behind [13] concerns precisely such a "stack" of $D0$-branes to which we now devote some space. The low-energy action of this system is that of $\mathcal{N}=1$, $D=10$, SYM reduced to one dimension, and the vector field A_μ decomposes into X^a, $a=1,\ldots,9$ and A_0 which we gauge-fix to zero in the following. The full Lagrangian of this theory in terms of the nine Hermitian $N\times N$-matrices, X^a and the 16-component $SO(9)$[3] spinors θ now becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2g\sqrt{\alpha'}}\mathrm{tr}\left[\dot{X}^a\dot{X}_a + \frac{1}{(2\pi\alpha')^2}\sum_{a<b}[X^a,X^b]^2 + \frac{1}{2\pi\alpha'}\theta^T i\dot{\theta} - \right. \\ &- \left.\frac{1}{(2\pi\alpha')^2}\theta^T\Gamma_a[X^a,\theta]\right]. \end{aligned} \tag{3.43}$$

As before the matrices X^a can be diagonalized and their elements interpreted as the positions of the $D0$-branes hence giving us the configuration space $(\mathbb{R}^9)^N$ modulo permutations of the branes (since they are identical).

Previously (section 3.1) we have reviewed T-duality in the context of string theory, now we wish to do the same for SYM in the case of N $D0$-branes. We choose the simple example when the branes move in $\mathbb{R}^8\times S^1$, that is, we compactify the X^9 direction on a circle. The general technique to study such a system of D-branes is by considering the orbifold $\mathbb{R}^9/\Gamma$, where Γ is a discrete group, and then constraining the theory in such a way that it remains invariant under the action of Γ. Intuitively we

[3]The Clifford algebra $SO(9)$ is generated by the 16×16-matrices Γ^a

see that the strings connecting the N different $D0$-branes can now wind around the compact direction X^9, since $S^1 = \mathbb{R}/\mathbb{Z}$ this is equivalent to considering an infinite number of "copies" of the $D0$-branes. This can be seen by imagining that the compact direction is broken up, strings connecting one brane to another by winding n times in the compact direction are now equivalent to strings connecting the brane to the nth copy of the other brane. As such the transverse components X^a of the fields, A^{μ}_{ij} describing the connection between the ith and the jth brane, turn into $X^a_{mi,nj}$, describing the connection between the mth copy of the ith $D0$-brane and the mth copy of the jth brane. The indices m, n lies in the infinite set $\mathbb{Z}$ and it follows that we gain an infinite number of degrees of freedom. The resulting theory is essentially a quantum field theory in a compact space, or equivalently an $U(\infty)$ quantum mechanics theory. Then enforcing the constraints of Γ gives the form of the infinite block matrices $X^a_{mi,nj}$ and shows that in the X^9 direction we can interpret this as a matrix representation of the operator

$$X^9 = i\hat{\partial} + A(\hat{x}), \tag{3.44}$$

which is a gauge covariant derivative that operates on a Fourier expansion

$$\phi(\hat{x}) = \sum_n \hat{\phi}_n e^{in\hat{x}/\hat{R}_9}, \tag{3.45}$$

$$\hat{R}_9 = \frac{\alpha'}{R_9} = \frac{1}{2\pi R_9}, \tag{3.46}$$

of the function $\phi(\hat{x})$. For the other transverse directions as well as the fermionic fields we have that they obey the same relation but without the homogeneous term $A(\hat{x})$. Thus we now see that we can encode the $U(\infty)$ matrices in these derivatives, the "$N \times N \times \infty \times \infty$" infinity disappears into the operator and leaves a $1 + 1$ (adds $\hat{x}$ as a degree of freedom) SYM theory of $N \times N$ $U(N)$ matrices on the dual circle. Therefore we now know the exact relation between the two theories, the one degree of freedom in the SYM theory on $\mathbb{R}^8 \times S^1$ corresponds to the $1 + 1$ degrees of freedom in the SYM on the dual circle of radius $\hat{R}_9$ exactly through this encoding of the winding modes into Fourier modes of S^1. Subsequently it is possible to show that the Lagrangians of the two theories are equivalent under this relation. This result is beautiful in its simplicity and upon compactifying further we get equally beautiful results until we eventually run into problems. Matrix theory compactified on a torus works fine, as well as compactifications on 3-tori and 4-tori, but as for compactification on a 5-torus and higher dimensional tori we can not resolve the problems in the resulting theory (see [67] for a review of this). Now we are ready to leave the $D0$-branes preliminaries and "get down to business" with the BFSS-conjecture, for those who wish to go further into the details of the D-branes and matrix theory there are a number of good references [68, 69].

From section 3.2 we know that M-theory compactified on a circle S^1 becomes Type IIA string theory, consequently there exists a number of relations between objects in M-theory and objects in Type IIA. The most important of these, at least for our present considerations, it the roles of the "uncompactified" partner of the Type IIA $D0$-brane in M-theory. The perturbative string states in IIA does not carry a RR-charge and consequently no momentum in the compactified direction. The $D0$-brane does however couple to the RR-gauge field A_μ in IIA (see table 3.3) which in turn corresponds to the Kaluza-Klein photon of $g_{\mu 11}$ ($g_{\mu\nu}$ being the 11-dimensional metric). Subsequently, the $D0$-brane corresponds to the supergraviton with momentum in the compactified direction. The momentum P^{11} is given by

$$P^{11} = \frac{1}{R}, \tag{3.47}$$

R being the radius of the compactified direction, there also exists supergravitons with larger momentum, $P^{11} = N/R$ and these can be shown to correspond to the stacks of N $D0$-branes that we dealt with earlier. The exact formulation of the conjecture in [13],

> M-theory formulated in the infinite momentum frame is exactly equivalent to the $N \to \infty$ limit of the supersymmetric quantum mechanics described by (3.43).

Let us therefore review some of the basics of M-theory in the infinite momentum frame (IMF).

To pass to the IMF we pick a direction, X^{11}, which we call the longitudal direction, then we disregard any physical systems except those where the momenta P^{11} are larger than any other relevant quantity in the system. The two main features of the IMF is now that states with negative or vanishing momenta P^{11} decouples from the theory and the Lorentz invariance turns into a Super Galilean symmetry in which P^{11} plays the role of mass. In our case the choice of longitudal direction X^{11} will coincide with the direction which we choose to compactify. Thus the longitudal momentum becomes naturally quantized in units of $\frac{1}{R}$, we get $P^{11}\frac{N}{R}$ and begin to see how an argument for the validity of the conjecture might look. Taking M-theory (here thought of as the strong coupling limit of Type IIA) to the IMF is seen to be equivalent to taking the limit $\frac{N}{R} \to \infty$ at the same time as $R \to \infty$. Now we can formulate our first concrete evidence in favor of the BFSS-conjecture, for it can be shown that the matrix theory has exactly the same invariance under the Super Galilean group as M-theory (or even general Lorentz covariant 11-dimensional theories) in the IMF. Furthermore we have in the large N limit that we retain the 11-dimensional Lorentz invariance. Also we see that states in M-theory with large P^{11} are made up almost entirely of (strong coupling partners of) $D0$-branes, so taking the limit where P^{11} becomes infinite seems to support the idea of the $D0$-branes as the only dynamical degrees of freedom. The existence of a membrane in matrix theory is another strong evidence in favor of BFSS but the most important one, which was pointed out in [13] is that classical interactions in 11-dimensional supergravity are exactly reproduced by loop effects in matrix theory. Their example regarded the equivalence of interaction between a pair of gravitons in the linearized theory of gravity and a 1-loop calculation for a pair of $D0$-branes. This evidence has been extended to other interactions in supergravity but above the level of linear interaction there are several complications. Mainly the result in the linear case rests upon usage of non-renormalization theorems for the loop amplitudes, there is evidence that such theorems does not exist for higher levels.

To discuss these calculations in any kind of detail would be to go to far into this area. But we have nevertheless presented a large portion of the evidence in the case for the BFSS-conjecture. Others have been discovered in the years since [13] for a fairly recent and very comprehensive review see [70]. Among the problems with matrix theory the issue of compactification and the failure to reproduce higher-order non-linear effects in supergravity are certainly the greatest. Others exist as well, and the future for the BFFS-conjecture looks a tad more bleak now than it did at the time of its birth.

4

The Supermembrane III: An Algebraic Approach

The modular invariance on the string worldsheet proved to be an extremely powerful tool in perturbative string theory. With the discovery of dualities, exact symmetries of M-theory, the question naturally arises if these can be used in calculations in the same way as modular invariance? In this chapter we will examine the efforts pursued along this line of thought in M-theory, starting in string theory and ending up in a little explored area of mathematics. The later parts of this chapter requires the use of mathematical tools that are not heard of very often within the body of physics It would be impossible to go through all the background mathematics, and therefore some of it has been transferred to appendices. It is recommended to at least briefly look through these prior to reading this chapter.

4.1 Using Exact Symmetries in M-Theory

We will begin by briefly reviewing the paper [15] upon which the actual result to be presented in this section is based. In this paper the authors calculated the corrections to the gravitational classical equations of motion that are expected since string/M-theory is in a sense an extension of Einstein's theory of gravity. Starting from the standard free Lagrangian $\mathcal{L}_0$ of string theory, a new effective action can be constructed by calculating tree-level scattering amplitudes and adding the contribution of these to $\mathcal{L}_0$. In Type II one does not get any contributions from the amplitudes below the four-point amplitude, these are the same as we would find in supergravity. Thus the equations of motion are those of Einstein.

However, in the four-point graviton scattering amplitude we will find additional interactions due to the exchange of massive string states, and these contributions will have to be added to the effective action. The four-point graviton tree amplitude in Type II (both A and B [71]) is given by

$$A_4 = \frac{\kappa^2}{128} \frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(1+\frac{1}{8}s)\Gamma(1+\frac{1}{8}t)\Gamma(1+\frac{1}{8}u))} K(\varepsilon^{(i)}, k^{(j)}) \tilde{K}(\varepsilon^{(i)}, k^{(j)}), \quad (4.1)$$

where $k_\mu^{(i)}$ are the momenta and $\varepsilon_{\mu\nu}^{(i)} = \varepsilon_\mu^{(i)} \varepsilon_\nu^{(i)}$, the polarization tensor. κ is the gravitational coupling constant and $K, \tilde{K}$, kinematic factors (that arise from an integration over the fermionic zero modes). The gamma function expression can be expanded to yield

$$\frac{\Gamma(-\frac{1}{8}s)\Gamma(-\frac{1}{8}t)\Gamma(-\frac{1}{8}u)}{\Gamma(1+\frac{1}{8}s)\Gamma(1+\frac{1}{8}t)\Gamma(1+\frac{1}{8}u))} = -\frac{2^9}{stu} - 2\zeta(3) + \ldots, \quad (4.2)$$

the first term in this expression gives the scattering amplitude in the supergravity theory. The leading term in the string theory correction becomes

$$\Delta A = -\frac{1}{128} \kappa^2 2\zeta(3) K(\varepsilon^{(i)}, k^{(j)}) \tilde{K}(\varepsilon^{(i)}, k^{(j)}), \quad (4.3)$$

where the factor $K\tilde{K}$ can be interpreted as coming from a term, Y, in the effective action which can be written as an integral over fermionic zero modes

$$Y = \int d\psi_L^\alpha d\psi_R^\beta \exp[\bar{\psi}_L^\alpha \Gamma_{\alpha\beta}^{\mu\nu} \psi_L^\beta \bar{\psi}_R^{\alpha'} \Gamma_{\alpha'\beta'}^{\sigma\tau} \psi_R^{\beta'} R_{\mu\nu\sigma\tau}], \quad (4.4)$$

or equivalently by defining $t^{\mu_1\mu_2\ldots\mu_8}$ satisfying

$$\sqrt{\det \Gamma^{\mu\nu} F_{\mu\nu}} = t^{\mu_1\mu_2\ldots\mu_8} F_{\mu_1\mu_2} \ldots F_{\mu_7\mu_8}, \quad (4.5)$$

it can be written as

$$Y = t^{\mu_1\mu_2\ldots\mu_8} t^{\nu_1\nu_2\ldots\nu_8} R_{\mu_1\mu_2\nu_1\nu_2} \ldots R_{\mu_7\mu_8\nu_7\nu_8}. \quad (4.6)$$

When this quantity is added to the ordinary Einstein-Hilbert action

$$I = \int \sqrt{g}(R + Y), \quad (4.7)$$

it yields an effective action that correctly describes scattering amplitudes in string theory including terms of order R^4. It is evident that this new term will alter the gravitational equations of motion (since generally,

$\sqrt{g}Y$, does not vanish) which was the goal of [15]. For our considerations the importance of this result comes second to the calculation itself since we will now study a similar calculation, more explicitly, done in the background of a D-instanton.

The basics of IIB supergravity and the D-instanton solution is briefly covered in section A.2. In the current section we assume familiarity with the material presented in the appendix and move right on to the calculation [14].

From our previous considerations we know that in calculating amplitudes, effects of fermionic zero modes has to be taken into consideration, the integration over these will produce kinematic factors which in turn contribute to the effective action. The bosonic zero modes in the supergravity instanton background is just the position of the D-instanton. The fermionic ones can be obtained by general supersymmetry methods. In string theory, our primary area of interest, an instanton amplitude as the one we are currently considering corresponds to a disk worldsheet (at least to first order), satisfying Dirichlet boundary conditions, with open string fermionic states, corresponding to the fermionic zero modes, attached to the boundary, as well as a closed string state, attached to the interior of the disk.

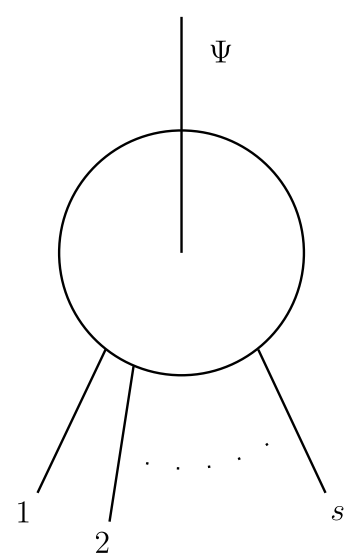

Figure 4.1: Tadpole diagram with closed string stated and fermionic zero modes coupling to a open string worldsheet.

In this way we can construct any scattering amplitude by adding the appropriate number of fermionic vertices to the disk diagram (what we add depends on the closed string state coupling to the interior).

Our goal now is to study scattering of four external gravitons in this background, corresponding to a diagram with four disconnected disks each coupling to an external graviton vertex and four fermionic states.

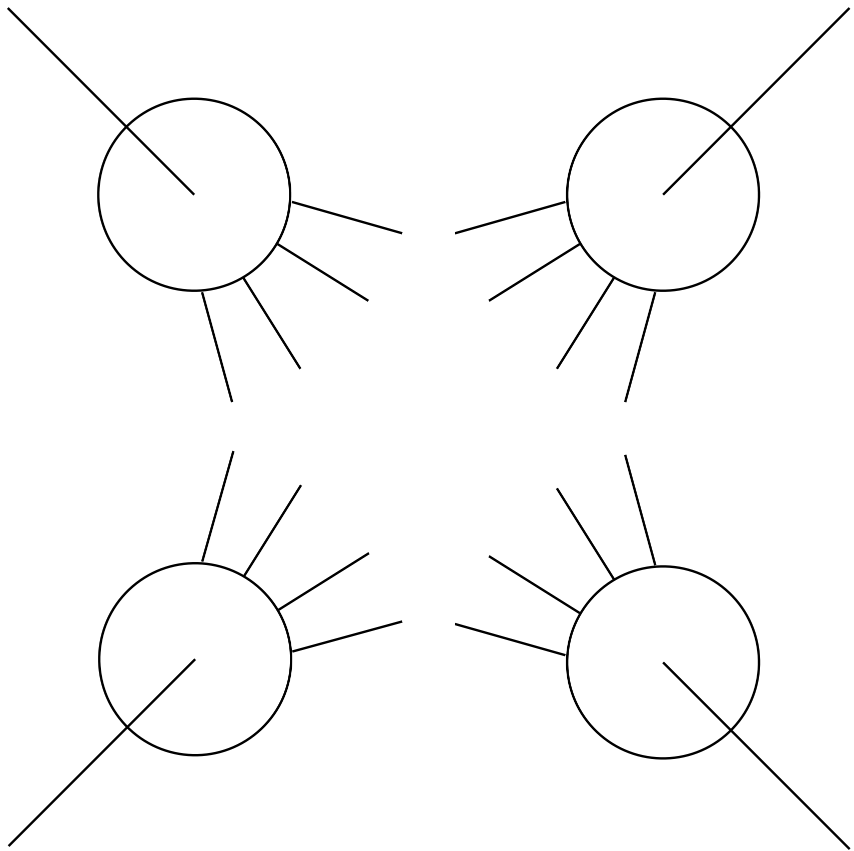

Figure 4.2: R^4 tadpole diagram

The amplitude becomes

$$A_4 = C \int d^{10}y d^{16}\epsilon_0 \prod_{r=1}^{4} \left(\bar{\epsilon_0}\gamma^{\mu_r\sigma_r\rho}\epsilon_0\bar{\epsilon_0}\gamma^{\nu_r\tau_r\rho}\epsilon_0\zeta^{\mu_r\nu_r}k_r^{\sigma_r}k_r^{\tau_r}e^{i\sqrt{\kappa}k_r\cdot y} \right), \quad (4.8)$$

where y is the position of the instanton, parametrizing the bosonic zero modes as stated earlier. The Grassmann variable ϵ_0 corresponds to the fermionic zero modes of the spin-$1/2$ complex Weyl-fermion λ in Type IIB supergravity. The polarization tensor is $\zeta^{\mu_r\nu_r}$ and $k_r^{\sigma_r}$ are the momenta of the gravitons, overall factors, as well as the coupling constant κ are contained in C. Choosing a simple frame this integral can be evaluated to yield

$$A_4 = Ce^{2\pi i\tau_0}\int d^{10}y e^{i\sum_r k_r\cdot y} \tag{4.9}$$
$$\left(\hat{t}^{i_1j_2\ldots i_4j_4}\hat{t}_{m_1n_2\ldots m_4n_4} - \frac{1}{4}\epsilon^{i_1j_2\ldots i_4j_4}\epsilon_{m_1n_2\ldots m_4n_4}\right) R^{m_1n_1}_{i_1j_1}R^{m_2n_2}_{i_2j_2}R^{m_3n_3}_{i_3j_3}R^{m_4n_4}_{i_4j_4},$$

with

$$\epsilon_{a_1a_2\ldots a_8}\gamma^{i_1j_1}_{a_1a_2}\cdots\gamma^{i_4j_4}_{a_7a_8} = t^{i_1j_1\ldots i_4j_4} = \hat{t}^{i_1j_1\ldots i_4j_4} + \frac{1}{2}\epsilon^{i_1j_1\ldots i_4j_4}, \tag{4.10}$$
$$\epsilon_{\dot{a}_1\dot{a}_2\ldots\dot{a}_8}\gamma^{i_1j_1}_{\dot{a}_1\dot{a}_2}\cdots\gamma^{i_4j_4}_{\dot{a}_7\dot{a}_8} = t^{i_1j_1\ldots i_4j_4} = \hat{t}^{i_1j_1\ldots i_4j_4} - \frac{1}{2}\epsilon^{i_1j_1\ldots i_4j_4}, \tag{4.11}$$

and the factor $e^{2\pi i\tau}$ is due to boundaries on the worldsheet of the string.(the spinor ϵ_0 here has been expressed in $SU(8)$ spinor components and the supergravity field τ is set to τ_0 since it is constant in the instanton background, see section A.2). The term bilinear in the Levi-Cevita tensor vanishes when the integration is performed, and what remains is the term bilinear in $\hat{t}$ and this term is precisely of the same form as in (4.6) and also turns out to be of the same form as the 1-loop 4-graviton amplitude in the zero instanton sector. These three contributions are collected to a complete effective R^4 action (in the Einstein frame)

$$S_{R^4} = (\alpha')^{-1}(a\zeta(3)\tau_2^{3/2} + b\tau_2^{-1/2} + ce^{2\pi i\tau} + \ldots)R^4 = (\alpha')^{-1}f(\tau,\bar{\tau})R^4, \tag{4.12}$$

(R^4 here includes the contractions with $\hat{t}$ and '...' is a reservation for possible corrections). The coefficients a, b, c in front of the tree-level, one-loop and instanton terms respectively depend on the normalization and are not of great importance for our considerations at the moment, therefore we do not determine them.

This is where symmetry under $Sl(2,\mathbb{Z})$ (S-duality) comes in. The amplitude and therefore the function $f(\tau,\bar{\tau})$ must be invariant under $Sl(2,\mathbb{Z})$. Based on what we know of this function other than its symmetry properties led the authors of [14] to conjecture that the function should be given by

$$f(\tau,\bar{\tau}) = \sum_{(p,q)\neq(0,0)} \frac{\tau_2^{3/2}}{|p+q\tau|^3}. \tag{4.13}$$

Expanding this function with the help of a Poisson resummation formula etc. leads to the expression

$$
\begin{aligned}
f(\tau,\bar{\tau}) &= 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3}\tau_2^{-1/2} + \\
&+4\pi^{3/2}\sum_{m,n\geq 1}\left(\frac{m}{n^3}\right)^{1/2}\left(e^{2\pi i m n\tau}e^{-2\pi i m n\bar{\tau}}\right)\times \\
&\left(1+\sum_{k=1}^{\infty}(4\pi m n\tau_2)^{-k}\frac{\Gamma(k-1/2)}{\Gamma(-k-1/2)}\right), \qquad (4.14)
\end{aligned}
$$

where we recognize the first two terms as the tree-level and one-loop term. The third term is a sum over all multiply charged instantons and anti-instantons, which is as expected since $Sl(2,\mathbb{Z})$ invariance requires that we sum over all instanton states. Only (multiply charged) single instanton configurations arise since multi-instanton configurations require more disks to couple to the extra fermionic zero modes, and will thus only contribute in higher order functions. The function f in (4.13) is in fact a well known one. It is an Eisenstein series, a non-holomorphic modular form. The fact that we use such a form and consider it to render an exact result constitutes the basis for our considerations in the following section. For now it is sufficient to keep in mind that f is such an object, we will return to its properties later on.

The important question is how to relate this result to M-theory. In [19] an exact expression for the R^4 term in the effective action of M-theory compactified on a torus T^2 was suggested. The motivation relies on, and is very similar to, that in [14]. Let us very briefly review how the argument goes. The action suggested is (R^4 still denotes contractions with t)

$$S_{R^4} = \frac{1}{3(4\pi)^7 l_{11}}\int d^9x\sqrt{-G^{(9)}}\left(V_2^{-1}f(\Omega,\bar{\Omega}) + \frac{2\pi}{3}V_2\right)R^4, \qquad (4.15)$$

where $4\pi^2 l_{11}^2 V_2$ is the volume of the internal torus (l_{11} being the 11-dimensional Planck length), $G^{(9)}$ is the nine-dimensional metric and $f(\Omega,\bar{\Omega})$ is the same Eisenstein series as we encountered earlier, although this time a function of a different variable, to which we will come back shortly. By compactifying the Type II string theory on a circle, S^1, or radius r, to nine dimensions, the previously obtained result can be written as (now using the normalization of [19])

$$S_{R^4} = \frac{1}{3\cdot 2^8\kappa_{10}^2}\int d^9x\sqrt{-g^{(9)}}r\left[2\zeta(3)\tau_2^2 + \frac{2\pi^2}{3}\left(1+\frac{1}{r^2}\right)+\dots\right]R^4, \qquad (4.16)$$

which holds equally well in Type IIA as in Type IIB, now related by

$$r_B = \frac{1}{r_A}, \quad e^{-\phi_B} = r_A e^{-\phi_A}, \quad C^{(0)} = C^{(1)} (= C^{(1)}_{10}). \tag{4.17}$$

This can be written in a more condensed manner as

$$\tau^B = \tau^A, \tag{4.18}$$

by defining

$$\tau^A = \tau_1^A + i\tau_2^A = C^{(1)} + i r_A e^{-\phi_A}. \tag{4.19}$$

Taking into account the contribution from D-instanton sectors we arrive at the familiar result (previously motivated) that all the terms can be summed up in an Eisenstein series. An interpretation of this result in M-theory is obtained by noting that the modulus, τ^B, of nine-dimensional Type IIB should be equal to the modular parameter, Ω, of the internal torus of M-theory compactified on T^2 [72]. This implies that the symmetry under $Sl(2,\mathbb{Z})$ in Type IIB corresponds to a geometrical symmetry in nine-dimensional M-theory. To realize this we note that in the M-theory (i.e. 11-dimensional supergravity) frame the metric can be parametrized as

$$ds^2 = G^{(10)}_m n dx^m dx^n + R_{11}^2 (dx^{11} - C^{(1)}_m dx^m)^2, \tag{4.20}$$

with $G^{(10)}_{mn} = R_{11}^{-1} g^A_{mn}$ and R_{11} the radius of the eleventh direction. Compactifying further on a circle of radius R_{10} leads to

$$g^A_{1010} = R_{10}^2 R_{11} = G_{1010} R_{11} \quad , \quad \tau_2^A = R_{11}^{-3/2}, \tag{4.21}$$

$$\tau_2^B = \frac{R_1 0}{R_1 1} \quad , \quad r_B = \frac{1}{R_{10}\sqrt{R_{11}}}. \tag{4.22}$$

This leads us to believe that the 11-dimensional metric can be written such that

$$\sqrt{-G^{(11)}} = \sqrt{G^T}\sqrt{-G^{(9)}}, \tag{4.23}$$

with

$$G^T = \frac{1}{l_{11}^2} \begin{pmatrix} R_{10}^2 + R_{11}^2 (C^{(1)})^2 & -R_{11}^2 C^{(1)} \\ -R_{11}^2 C^{(1)} & R_{11}^2 \end{pmatrix}, \tag{4.24}$$

the metric on the internal two-torus. The complex structure, Ω, of this torus can now, using the equivalences (4.22), be equated as

$$\Omega = \Omega_1 + i\Omega_2 = C^{(1)} + i\frac{R_{10}}{R_{11}} = C^{(0)} + i e^{-\phi_B} = \tau_B. \tag{4.25}$$

Finally before arguing for the exact term (4.15) we rewrite the term S_{R^4}, (4.16) in the quantities best suited for Type IIA, IIB and M-theory respectively

$$\frac{1}{3\cdot 2^8\kappa_{10}^2}\int d^9x\sqrt{-g^{A(9)}}r_A\left[2\zeta(3)(\tau_2^A)^2+\frac{2\pi^2}{3}\left(1+\frac{1}{r_A^2}\right)+\ldots\right]R^4$$

$$\frac{1}{3\cdot 2^8\kappa_{10}^2}\int d^9x\sqrt{-g^{B(9)}}r_B\left[2\zeta(3)(\tau_2^B)^2+\frac{2\pi^2}{3}\left(1+\frac{1}{r_B^2}\right)+\ldots\right]R^4 \tag{4.26}$$

$$\frac{l_{11}^6}{3\cdot 2^8\kappa_{11}^2}\int d^9x\sqrt{-G^{(9)}}2\pi R_{11}R_{10}\left[2\zeta(3)\frac{l_{11}^3}{R_{11}^3}+\frac{2\pi^2}{3}+\frac{2\pi^2}{3}\frac{l_{11}^3}{R_{10}^2R_{11}}+\ldots\right]R^4,$$

where we have used[1]

$$\kappa_{10}^2=\frac{k_{11}^2}{2\pi R_{11}(\lambda^A)^2}=2^6\pi^7\alpha'^4, \tag{4.27}$$

λ^A being the Type IIA coupling constant. The ellipsis on each line in equation (4.26) denote non-perturbative contributions expressed as a power series in $e^{2\pi i\tau^A}$, $e^{2\pi i\tau^B}$ and $e^{2\pi i\Omega}$ respectively (as well as the exponential of the conjugate moduli). T-duality equates the first two types of non-perturbative corrections ($D0$-branes in the Type IIA case and the familiar D-instanton in IIB) and (4.25) equates these with the third kind. The last line in (4.26) can in turn be rewritten as

$$S_{R^4}=\frac{1}{3(4\pi)^7l_{11}}\int d^9x\sqrt{-G^{(9)}}\times$$
$$\left\{V_2^{-1/2}\left[2\zeta(3)(\Omega_2)^{3/2}+\frac{2\pi^2}{3}(\Omega_2)^{-1/2}+\ldots\right]+\frac{2\pi^2}{3}V_2\right\}, \tag{4.28}$$

where we have used

$$\Omega_2=\frac{R_{10}}{R_{11}}\quad,\quad V_2=\frac{R_{10}R_{11}}{l_{11}^2}. \tag{4.29}$$

This expression looks like an expansion of a modular function of Ω (for large Ω_2) and by comparison with the expression (4.15) one can find strong evidence for the exactness of (4.15). First of all we only find two perturbative terms in (4.28) just as expected from our previous considerations, adding support to the conjecture that no perturbative corrections

[1] Although we have set $\alpha'=1$ as usual in these expressions

arise above one-loop level (this conjecture was proved in [73] for IIB compactified on T^2). Upon taking the limits $r_A \to \infty$ and $r_B \to \infty$ one retains the correct perturbative terms in IIA and IIB respectively and these are also related by T-duality in the proper way. Also, upon taking $V_2 \to \infty$, that is, decompactifying the internal torus, the only term that survives in (4.28) is the last one, forming

$$S_{R^4} = \frac{1}{18 \cdot (4\pi)^7 l_{11}^3} \int d^{11}x \sqrt{-G^{(11)}} R^4, \tag{4.30}$$

which is finite (for further details on the rather lengthy motivations we refer the reader to [14, 19]).

Thus we now possess strong evidence that (4.15) is indeed the exact R^4 contribution to the effective action. In [19, 20] this is further studied in compactifications on T^3 which strengthens the arguments. However, the modular function f, which we now know to be an Eisenstein series, $\mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s}$, is uniquely determined by the fact that it satisfies the eigenvalue equation

$$\Delta\varepsilon_s = s(s-1)\varepsilon_s \quad , \quad \Delta = 4\tau_2^2 \partial_\tau \partial_{\bar{\tau}}, \tag{4.31}$$

and there is another type of solution to this equation, namely cusp forms. Cusp form solutions satisfy the equation with $s \in 1/2 + i\mathbb{R}$. The contribution of these forms will not alter the desired properties and thus one has to find some principle that rules out the existence of these kinds of terms in (4.15). In [19] it was conjectured that supersymmetry (and U-duality) were such principles and this conjecture was proved in [74] using the formalism due to Berkovits. Thus we can now conclude that if we trust the evidence that speaks for the exactness of (4.15) then this expression must also be uniquely given by the Eisenstein series $\mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s=3/2}$.

In [19, 20, 21] these matters are studies further in compactifications down to seven dimensions (in M-theory and IIB). Upon compactifying further like this a richer structure appears, in addition to non-perturbative D-instanton corrections (which remain in all kinds of torodial compactifications due to the point-like behavior of the D-instanton) the effective IIB action receives contribution due to solitonic IIB objects whose world-volumes wrap around the compact dimensions. These contributions arise for the first time when we compactify on T^2, in the form of (p, q)-strings[2] (wrapped D-strings). The contribution of (p, q)-strings can be calculated from the $(1, 0)$-string (the fundamental IIB string) [21] using the symmetry under $Sl(2, \mathbb{Z})$. The main principle in [21], which generalizes that of earlier papers which we have reviewed in this section is that the full

[2]See section A.4 in the appendices.

result should be invariant under U-duality. On the basis of this principle the authors calculate an effective R^4 action of IIB string theory on $R^8 \times T^2$ which is in agreement with that calculated in [20]. Now keeping the $\epsilon_8\epsilon_8 R^4$ term (which we have discarded in previous calculations) and inferring a parity violating term, $\epsilon_8 t_8 R^4$ (that contribute in IIA) we write the effective action in the eight-dimensional Einstein frame and with the normalization of [21] as

$$S_8^{R^4} = \mathcal{N}_8 \int d^8x \sqrt{-g_E^{(8)}} \left[\Delta_{tt} t_8 t_8 + \frac{1}{4}\Delta_{\epsilon\epsilon}\epsilon_8\epsilon_8 + \Theta\epsilon_8 t_8 \right] R^4. \tag{4.32}$$

The coefficients Δ_{tt}, $\Delta_{\epsilon\epsilon}$ and Θ in this equation should, upon decompactification of the internal torus, satisfy

$$\lim_{V_2\to\infty} \frac{\Delta_{tt}}{V_2} = \lim_{V_2\to\infty} \frac{\Delta_{\epsilon\epsilon}}{V_2} = \sqrt{\tau_2} f(\tau,\bar{\tau}) \quad , \quad \lim_{V_2\to\infty} \frac{\Theta}{V_2} = 0, \tag{4.33}$$

where $f(\tau,\bar{\tau})$ is the previously obtained function. The reason for the dissimilarities in the structure of (4.33) and the actions we have previously seen is due to the fact that $\Delta_{\epsilon\epsilon}$ now depends on the moduli and the term $\Delta_{\epsilon\epsilon}\epsilon_8\epsilon_8 R^4$ is therefore no longer a total derivative and can not be discarded. Also since our previous concern has been only Type IIB string theory where no parity violating term exists the need for Θ has only just now arisen. Let us briefly touch upon the calculation and form of the coefficients Δ_{tt}, $\Delta_{\epsilon\epsilon}$ and Θ.

Upon compactifying on a two-torus we arrive at an action that is invariant under $Sl(2,\mathbb{Z})$ acting on both the complex structure, U, of the internal torus, the usual scalar moduli, τ, as well as the T-modulus

$$T = B_N + iV_2. \tag{4.34}$$

The perturbative tree-level result is obtained in a straightforward manner by compactifying the ten-dimensional result. The one-loop correction is calculated by evaluating the amplitude yet again, but we will not go into the details of this calculation here. The resulting perturbative corrections become

$$\Delta_{tt}^{\text{pert}} = 2\zeta(3)V_2\tau_2^2 - 2\pi\log\left(V_2|\eta(T)|^4\right) - 2\pi\log\left(U_2|\eta(U)|^4\right), \tag{4.35}$$
$$\Delta_{\epsilon\epsilon}^{\text{pert}} = 2\zeta(3)V_2\tau_2^2 - 2\pi\log\left(V_2|\eta(T)|^4\right) + 2\pi\log\left(U_2|\eta(U)|^4\right), \tag{4.36}$$
$$\Theta = 4\pi\text{Im}\left[\log\eta(U)^4\right], \tag{4.37}$$

expressed in the field of the moduli previously defined. The D-instanton contribution remains the same, as motivated above, and the (p,q)-string instanton contribution can be found by imposing invariance under $Sl(2,\mathbb{Z})$

while extending the $(0,1)$-string instanton contribution to the general (p,q)-case. The contribution from all (p,q) D-strings to the coefficients Δ_{tt} and $\Delta_{\epsilon\epsilon}$ becomes

$$I_{p,q} = -8\pi \sum_{(p,q)=1} \mathrm{Re} \log \left[\prod_{n=1}^{\infty} \left(1 - e^{2\pi i n T^{p,q}}\right) \right], \qquad (4.38)$$

with

$$T_{p,q} = (qB_R - pB_N) + i|p + q\tau|V. \qquad (4.39)$$

No contribution beyond perturbation theory is expected to the parity violating term $\Theta\epsilon_8 t_8 R^4$. Thus we now have all the constituents of the exact eight-dimensional R^4 term, we write these together as

$$\Delta_{tt} = V_2\sqrt{\tau_2} f(\tau,\bar\tau) - 2\pi \log V_2 - 2\pi \log\left(U_2|\eta(U)|^4\right) + I_{p,q}, (4.40)$$

$$\Delta_{\epsilon\epsilon} = V_2\sqrt{\tau_2} f(\tau,\bar\tau) + 2\pi \log V_2 - 2\pi \log\left(U_2|\eta(U)|^4\right) + I_{p,q}. (4.41)$$

The U-duality group in eight dimensions is $Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$, where $Sl(2,\mathbb{Z})$ acts on the complex modulus, U, of the internal torus. Looking at the expressions (4.40) and (4.41) we see that these are manifestly invariant under $Sl(2,\mathbb{Z})$ since the log is the order-1 Eisenstein series of $Sl(2,\mathbb{Z})$ and the other terms are independent of U. The invariance under $Sl(3,\mathbb{Z})$ is not seen so easily, but it can be shown [21] that upon expanding the Eisenstein series $\mathcal{E}^{Sl(3,\mathbb{Z})}_{\mathbf{2};s=3/2}$ in the proper way, all the constituents (except the order-1 $Sl(2,\mathbb{Z})$ Eisenstein series) are obtained.

Upon compactifying even further, to seven dimensions, one obtains the result as an order-5/2 Eisenstein series of $Sl(5,\mathbb{Z})$, the U-duality group in seven dimensions.

Given this body of evidence it seems reasonable to conjecture that at any level of compactification the exact R^4 amplitude should be given by an Eisenstein series of the U-duality group. The reason for the growing complexity of these amplitudes at higher level of compactification is directly mirrored in this conjecture, as we increase the number of compactified dimensions the number of moduli, upon which the perturbative and non-perturbative parts can depend, grows as well. With this, the U-duality group also grows thus rendering the Eisenstein series more and more complex.

R^4 amplitudes are but one example of quantities that can be handled in this way, more generally any quantity saturating a BPS bound behaves in the same manner This is due to the fact that they receive contribution from at most one order in perturbation theory (which should also only depend on BPS-states), in the R^4 case, one-loop. Also non-perturbative corrections must keep an appropriate number of supersymmetries unbroken.

Now that we have treated the IIB case in some detail (only once making real contact with M-theory) we wish to see how far this method can be pushed. If we could understand precisely how the methods we have used here manifest themselves in M-theory then perhaps they could be extended further, allowing us to calculate other exact quantities that at our current level of understanding are unreachable. This is the subject of the next section in this chapter.

4.2 Membrane Amplitudes as Automorphic Forms

In the previous section we have studied exact BPS-saturated amplitudes as automorphic forms in compactifications down to seven dimensions. We have seen that these amplitudes can be determined on the basis of U-duality and supersymmetry as Eisenstein series of the corresponding U-duality group $E_{d+1(d+1)}(\mathbb{Z})$. In table 3.4 we have listed the U-duality groups in various dimensions.

These amplitudes should depend on the scalars in the symmetric space $K\backslash E_{d+1(d+1)}(\mathbb{Z})$ (see section C.1). So far we have not been in need of Eisenstein series of other groups then $Sl(d,\mathbb{Z})$, but as we compactify further table 3.4 clearly shows that this need arises. In order to construct Eisenstein series of the above groups that is a viable amplitude we have to specify a representation of $E_{d+1(d+1)}(\mathbb{Z})$ which contains the perturbative results. In section C.1 we have seen that the one-loop result could be written as ($d \neq 1, 2$)

$$I_d = \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbf{S};1}, \tag{4.42}$$

and furthermore the tree-level term can be written

$$\mathcal{E}^{G(\mathbb{Z})}_{\mathbf{1};s} = 2\zeta(2s), \tag{4.43}$$

as an Eisenstein series for any group G in the singlet representation. Our goal is to find a representation of $E_{d+1(d+1)}(\mathbb{Z})$ which unifies both these representations of $SO(d,d,\mathbb{Z})$. We have in fact already encountered such a representation in section 3.3, the string multiplet of the U-duality group. This representation is described by the charges corresponding to wrappings of membranes, 5-branes and KK6-branes around one, four and five compact dimensions respectively. It would be impossible to give a thorough introduction to the representation theory of the U-duality group here (we refer the reader to [22, 23, 75, 76] for further details) and therefore the statement above is granted the status of postulate in order to allow us to proceed with the motivation. With this information at hand we could proceed and conjecture that the exact non-perturbative amplitude in any number of dimensions is given by the Eisenstein series $\mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{string};3/2}$ but this would not be valid in the $d = 1, 2$ case since the one-loop amplitude also contains the Eisenstein series in the conjugate spinor representation of $SO(d, d, \mathbb{Z})$ there. So we have to find a representation of $E_{d+1(d+1)}(\mathbb{Z})$ that can be decomposed into (among other) the conjugate spinor representation of $SO(d, d, \mathbb{Z})$. This turns out to be the membrane multiplet described by charges corresponding to membranes, 5-branes and KK6-branes around zero, three and four compact dimensions respectively. Therefore it seems plausible that the full R^4 amplitude in any dimension should be written as

$$f_{R^4} = \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{string};3/2} + \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\mathbf{membrane};1}, \tag{4.44}$$

where we have discarded any factor in front of the two terms. For $d > 2$ it can be shown (proved for $d = 3, 4$ and strongly implicated for $d > 4$) that the two series in (4.44) are actually equal to each other, which fits nicely with the fact that in those dimensionality's the spinor and conjugate spinor representations are equally viable for expressing the one-loop result. Expanding this amplitude at weak coupling reveals the perturbative and instanton terms as we have seen in previous cases. Since we aim at interpreting this result as due to objects in M-theory it is useful to study the amplitude (4.44) at large volume. Doing this we obtain [16]

$$\begin{aligned} f_{R^4} &= \frac{\pi^2 l_{11}^6}{3} + \sum_{m^i \in \mathbb{Z}^{d+1}} \frac{l_{11}^9}{[(m^i)^2]^{3/2}} + \pi \sum_{m^3 \neq 0} \frac{l_{11}^9}{\sqrt{(m^3)^2}} + \\ &+ \pi l_{11}^6 \sum_{m^3 \neq 0} \left[\frac{l_{11}^6}{(m^3)^2}\right]^{1/2} \mu(m^3) \exp\left(-\frac{2\pi}{l_{11}^3}\sqrt{(m^3)^2} + 2\pi i m^3 C_3\right) \times \\ &\times \left(1 + \mathcal{O}\left(\frac{1}{l_{11}^3}\right)\right), \end{aligned} \tag{4.45}$$

where $m^3 = m^{ijk}$, $C_3 = C_{ijk}$ (the three-form gauge field) and the instanton summation measure is given by

$$\mu(m^3) = \sum_{n | m^{ijk}} n. \tag{4.46}$$

In this expansion we see the first three terms as perturbative terms (in the sense that they are not of the same form as the instanton terms) and the following sum as a sum over membrane instantons, where the membrane worldvolume wraps the three-torus $T^3 \subset T^{d+1}$. The last correction could possibly incorporate instanton contribution from higher dimensional objects in M-theory (5-branes etc.). Such contribution should occur for $d+1 \geq 6$ but this discussion disregards from such contributions and this from such compactifications.

There is really very little we can say about (4.45) since we lack a great deal of understanding about the contributing objects. On the other hand, if we could derive the coupling (4.44) from first principles in M-theory then we would know exactly what objects contribute to the different parts in (4.45) and also how these objects contribute. Before we attempt this we return to the Type IIB one-loop amplitude in order to get a better understanding of this in a number theoretical sense.

We have already seen that the one-loop amplitude can be obtained as an integral of the partition function, $Z_{d,d}(g;B;\tau)$, over the fundamental domain of $Sl(2,\mathbb{Z})$

$$I_d = 2\pi \int_{\mathcal{F}(Sl(2,\mathbb{Z}))} \frac{d^2\tau}{\tau_2^2} Z_{d,d}(g;B;\tau). \tag{4.47}$$

The result of this integration is an $SO(d,d,\mathbb{Z})$ Eisenstein series, an automorphic form. This is true also for higher point functions, the only change is that now we have to insert an $Sl(2,\mathbb{Z})$ invariant function, $\Phi(\tau,\bar{\tau})$, a modular form, that incorporates the effects of the extra legs on the loop. Integrating this modular form against the partition function

$$I'_d = \int_{\mathcal{F}(Sl(2,\mathbb{Z}))} \frac{d\tau d\bar{\tau}}{\tau_2^2} Z_{d,d}(g;B;\tau)\Phi(\tau,\bar{\tau}), \tag{4.48}$$

constitutes a *theta lift* from from $Sl(2,\mathbb{Z})$ modular forms to $SO(d,d,\mathbb{Z})$ automorphic forms. With this we get a completely new interpretation of the partition function $Z_{d,d}$; it is now a *theta correspondence*, invariant under $Sl(2,\mathbb{Z}) \times SO(d,d,\mathbb{Z})$ and generating lifts from $Sl(2,\mathbb{Z})$ forms to $SO(d,d,\mathbb{Z})$ forms. It is with this view on amplitude calculations that we move on to make conjectures about the M-theory case. We wish to present and motivate the proposal [16] that the coupling (4.44) should be given by

$$\int_{\mathcal{F}} \Xi_{d+1} = \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\textbf{string};3/2} + \mathcal{E}^{E_{d+1(d+1)}(\mathbb{Z})}_{\textbf{membrane};1}, \tag{4.49}$$

where the analog of the partition function $Z_{d,d}$ is Ξ_{d+1}. In this case, for reasons that will become apparent in a little while we conjecture that this function should depend on γ_{ab}, g_{ij} and C_{ijk} and be invariant under $Sl(3,\mathbb{Z}) \times E_{d+1()d+1}(\mathbb{Z})$. Furthermore the integration should be over the fundamental domain $\mathbb{F}$ of $Sl(3,\mathbb{Z})$.

Based on the expression (4.45) a few assumptions can be justified. The first is that the membrane gives the "fundamental" degrees of freedom in M-theory. This strong claim is in part justified from the fact that the membrane gives the fundamental degrees of freedom in string theory upon double dimensional reduction. Certainly the membrane gives the relevant degrees of freedom in the expression (4.45) albeit modulo the fact that we have disregarded from the cases where the 5-brane contributes. Also we are assuming that the only contributing topology is that of the torus T^3. This also seems to be justified from the fact that the membrane instantons in (4.45) wrap subtori T^3 of T^{d+1}. As a consequence of this there should be an $Sl(3,\mathbb{Z})$ modular invariance that restricts the integration to the fundamental domain of $Sl(3,\mathbb{Z})$. Based on these assumptions (as well as the, most likely, faulty assumption that only the zero modes contribute, in analogy with the string theory case) the authors of [16] proceeded to calculate the amplitude. Based on the work in [77] they formed the amplitude

$$\mathcal{A}_4 = \mathrm{STr}(V^1\Delta V^2\Delta V^3\Delta V^4\Delta), \tag{4.50}$$

where the V^i are the vertex operators that govern the emission of massless particles from the supermembrane. These vertex operators should of course also govern the emission of gravitons and the derivation of these is done in (the light-cone gauge) analogy with the string theory case and the case of the 11-dimensional superparticle. They incorporate the vertex operators in both these cases, reducing to the correct superstring vertex operator upon double dimensional reduction. The propagator $\Delta = \int_0^\infty dt\exp[-tH]$ where the Hamiltonian H splits into

$$H = H_{\mathrm{class}} + H_0 + H_{\mathrm{int}}, \tag{4.51}$$

a classical part corresponding to the bosonic zero modes, a superharmonic oscillator part H_0 and an interaction part H_{int}. The amplitude can be argued to factorize as

$$\begin{aligned}\mathcal{A}_4 &= \int_0^\infty \int d^{11}x\,\langle x|e^{-tH_{\mathrm{class}}}|x\rangle \sum_{\mathcal{N}} \langle\mathcal{N}|(-)^F V^1|_{\theta_0^4}V^2|_{\theta_0^4}V^3|_{\theta_0^4}V^4|_{\theta_0^4}|\mathcal{N}\rangle \\ &\times \sum_m \langle\langle m||(-)^F e^{-t(H_0+H_{\mathrm{int}})}||m\rangle\rangle\,,\end{aligned} \tag{4.52}$$

where $|x\rangle$ and $|\mathcal{N}\rangle$ are the bosonic and fermionic zero modes respectively, $||m\rangle\rangle$ denotes the discrete eigenstates of $H_0 + H_{\mathrm{int}}$. The second factor in (4.52), the trace over fermionic zero modes, yields the correct tensor structure, just like in the previous cases (see section 4.1), and the last factor can be argued to vanish due to the non-contribution of the non-zero modes. Thus what remains to be calculated is the classical part (the first factor). In [16] this is done using $Sl(3,\mathbb{Z})$ modular invariance techniques. However, parts of the integration remain unconstrained by the $Sl(3,\mathbb{Z})$ invariance and behaves very badly. We will not dwell on this calculation here since in the end it yields the summation measure

$$\mu'(m^3) = \sum_{n|m^{ijk}} \sum p|(m^{ijk}/n)np^2, \tag{4.53}$$

which obviously is in conflict with the measure predicted by U-duality, indeed this expression is not even invariant under U-duality. As a consequence of this not even the "perturbative terms" in (4.45) are correctly reproduced in this amplitude since they are closely related to the summation measure.

The reasons for the shortcomings of this approach are not fully known, although it is thought that the assumption of exclusive contribution from zero modes is one flaw, the apparent lack of manifest (and perhaps also hidden) invariance under U-duality in (4.52) another. The remedy for both these flaws regards the nature of the partition function Ξ_{d+1}.

To understand what this function should be let us return to the type IIB case yet again. The partition function $Z_{d,d}$ in this case is a restriction of a function carrying a larger symmetry, namely the symplectic theta series

$$\theta_{Sp(g)}(\Omega_{AB}) = \sum_{m^A \in \mathbb{Z}^g} e^{-\pi m^A \Omega_{AB} m^B}, \tag{4.54}$$

this series being invariant under the symplectic group $Sp(g,\mathbb{Z})$. Our partition function turns out to be given by

$$Z_{d,d}(g;B;\tau) = \theta_{Sp(g)}(\tau \otimes (g+B)), \tag{4.55}$$

where the tensor product '$\otimes$' gives an embedding $Sl(2) \times SO(d,d) \subset Sp(2d)$. This is known as a dual pair in the mathematics literature and defined by the fact that the centralizer[3] of each subgroup is equal to the centralizer of the other subgroup. Therefore is seems reasonable to assume that the partition function Ξ_{d+1} which we seek is also a restriction of some generalization of this theta series. In fact the relevant dual pairs have already been classified as shown in table 4.3 (compare with table 3.4). Strictly speaking the last dual pair in this table should not be included since it is related to a level of compactification where instantons

[3]The centralizer of a subgroup is the set of elements in the larger group which commute with all elements in the subgroup.

$d+1=2$	$\mathbb{R}^+ \times Sl(3) \times Sl(2) \subset Sl(5)$
$d+1=3$	$\mathbb{R}^+ \times Sl(3) \times Sl(2) \times Sl(3) \subset E_{6(6)}$
$d+1=4$	$\mathbb{R}^+ \times Sl(3) \times Sl(5) \subset Sl(8)$
$d+1=5$	$\mathbb{R}^+ \times Sl(3) \times SO(5,5) \subset E_{8(8)}$
$d+1=6$	$Sl(3) \times E_{6(6)} \subset E_{8(8)}$

Figure 4.3: Dual pairs related to various level of compactification.

coming from wrapped 5-branes contribute (it is however interesting to note the absence of $\mathbb{R}^+$ in this case, see section 5.4 for an extended discussion).

Our next objective must therefore be to find theta series of the groups that embed the dual pairs in table 4.3. As we search the mathematics literature however, we will find that they do not exist! At least not explicitly constructed, even though having such objects readily at hand would be useful not only to physicists but mathematicians as well. If we could find (or after constructing them ourselves) these theta series, the process of which is the subject of the following section, we would have to constrain them to reproduce the correct invariance as is done in (4.55). Then we would presumably gain a great deal of knowledge about the nature of the (BPS) membrane and its role in M-theory.

4.3 Calculation of Automorphic Forms

We are about to set out on a journey that will take us deep into the misty mountains of number theory, on the roads not traveled before by physicists. Like Gandalf to Frodo a few words of encouragement are in order for the brave few who intend to carry this ring to the bitter end.

This section concerns a particular method of constructing theta series. There are essentially three ingredients in this stew, and after introducing the method by means of a few examples we will examine each part in excruciating detail. For those who have not come in contact with p-adic numbers (and analysis on these) before, appendix B is mandatory reading since the latter parts of this section rely heavily on p-adic analysis.

Before we begin let us be clear on the point that there are still a large number of unanswered questions in this area (both mathematical and physical). We will stumble upon them as we go, not paying particular attention to them until the conclusions 5.4.

The example we chose for introducing the machinery used hereafter is the simplest possible one, the Jacobi theta series

$$\theta(\tau) = \tau_2^{1/4} \sum_{m\in\mathbb{Z}} e^{i\pi\tau m^2} = \tau_2^{1/4} \sum_{m\in\mathbb{Z}} f_\tau(m),\ \tau \in \mathcal{H}. \tag{4.56}$$

This series is manifestly invariant under shifts $\tau \to \tau + 2$ (the factor $\tau_2^{1/4}$ is inserted to cancel the modular weight) and the behavior under inversion $\tau \to -1/\tau$ follows after Poisson resumming

$$\tau_2^{1/4} \sum_{m\in\mathbb{Z}} f_\tau(m) = \tau_2^{1/4} \sum_{p\in\mathbb{Z}} \tilde{f}_\tau(p), \tag{4.57}$$

where

$$\tilde{f}(p) = \int dx f(x) e^{2\pi i p x}, \tag{4.58}$$

yielding

$$\theta(-\frac{1}{\tau}) = \sqrt{i}\theta(\tau). \tag{4.59}$$

We see now that this series is an $Sl(2,\mathbb{Z})$ (holomorphic) modular form, but is there a better way of seeing this than testing explicitly. We would like to find a construction of this series that displays the invariance properties and is general enough to be applicable in the construction of other invariants. We will essentially use the results presented in [78, 79, 17] although discarding much of the mathematical rigor. First we aim to show that the theta series (4.56) can be written

$$\theta(\tau) = \langle \delta, \rho(g_\tau) \cdot f \rangle, \tag{4.60}$$

where

$$\delta(x) = \sum_{m\in\mathbb{Z}} \delta(x-m), \qquad g_\tau = \frac{1}{\sqrt{\tau_2}} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau^2 \end{pmatrix}, \tag{4.61}$$

$$f(x) = e^{-\pi x^2}. \tag{4.62}$$

The inner product $\langle \cdot, \cdot \rangle$ is just an integration $\int dx$ and $\rho(g_\tau) \cdot f$ is a representation element (acting on f) from the metaplectic representation[4] (of $Sl(2,\mathbb{R})$)

$$\rho \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \phi(x) \to e^{i\pi t x^2} \phi(x), \tag{4.63}$$

$$\rho \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} : \phi(x) \to e^{t/2} \phi(e^t x), \tag{4.64}$$

$$\rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \phi(x) \to e^{i\pi/4} \tilde{\phi}(-x). \tag{4.65}$$

[4] For more material on the metaplectic representation of $Sl(2,\mathbb{Z})$ see [80]. For our purposes it is sufficient to know that it fulfills the criterions needed to be called minimal.

This representation acts on the Schwartz space of functions[5] of which our function f is an element. Setting $t_1 = \tau_1$ and $e^{t_2} = \sqrt{\tau_2}$ we can multiply the first two matrices to obtain

$$\begin{pmatrix} e^{-t_2} & 0 \\ 0 & e^{t_2} \end{pmatrix} \begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-t_2} & t_1 e^{-t_2} \\ 0 & e^{t_2} \end{pmatrix} = \frac{1}{\sqrt{\tau_2}} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau^2 \end{pmatrix}, \quad (4.66)$$

corresponding to the action

$$\phi(x) \to \tau_2^{1/4} \phi(\sqrt{\tau_2} x) \to \tau_2^{1/4} e^{i\pi\tau_1 x^2} \phi(\sqrt{\tau_2} x), \quad (4.67)$$

which applied to the function f reads

$$f(x) = e^{-\pi x^2} \to \tau_2^{1/4} e^{i\pi\tau x^2} = f_\tau(x). \quad (4.68)$$

Thus, inserting the distribution δ,

$$\langle \delta, \rho(g_\tau) \cdot f \rangle = \int dx \sum_{m\in\mathbb{Z}} \delta(x-m) \tau_2^{1/4} e^{i\pi\tau x^2} = \tau_2^{1/4} \sum_{m\in\mathbb{Z}} e^{i\pi\tau m^2}. \quad (4.69)$$

Hence we have shown that the construction (4.60) applies to this case. Before proceeding let us take a closer look at the metaplectic representation given in this example.

From this representation we can obtain a representation of the Lie algebra $Sl(2,\mathbb{R})$ by linearizing the first two elements, (4.63) and (4.64), and then by Weyl reflecting the generator E_+ corresponding to the positive root in order to obtain the generator corresponding to the negative root

$$E_+ = i\pi x^2, \quad (4.70)$$

$$E_- = \frac{i}{4\pi} \partial_x^2, \quad (4.71)$$

$$H = \frac{1}{2}(x\partial_x + \partial_x x). \quad (4.72)$$

In this case there is only one compact generator (the maximal compact subgroup is $U(1)$) namely $E_+ - E_-$ and we see that our function f satisfies the eigenvalue equation

$$(E_+ - E_-) f = \frac{i}{2} f, \quad (4.73)$$

with lowest possible eigenvalue $i/2$. This is no coincidence as we will see shortly. Before proceeding to the next example we shall introduce the

[5] A function, $f : \mathbb{R} \to \mathbb{R}$, is called a Schwartz function if it, and all its derivatives, goes to zero, as $|x| \to \infty$, faster than any inverse power of x.

terminology (for a general Lie group G now) that will be used throughout the rest of this chapter, and also relate these concepts to the previous example.

The representation in the previous case was the metaplectic representation; but in general it should be a so called minimal representation of G. For our purposes it is enough to consider such a representation to be one with the smallest possible representation space (with respect to dimensionality)[6]. This representation is not unique in our case, nor is it unique for any A_n, but for the other cases we will study here the minimal representation is the only one of its kind. Most often when we work with the representation to display something explicitly it is the representation of the algebra that we will be dealing with. The construction of this representation is actually quite similar to the one we have seen in the previous example.

The function f is called a *spherical vector*, defined by the property of being invariant under the action of the maximal compact subgroup K of the group G. This means that the function f given by (4.62) is not strictly speaking a spherical vector since then it should be annihilated by the compact generator $E_+ - E_-$. However f is the lowest state admitted by the representation and in this specific example it functions as a spherical vector. The invariance of the spherical vector under K means that the full theta series depends only on variables lying in $K\backslash G$ (in the previous example up to a phase though).

The distribution δ introduces the invariance property of the theta series and its relation to the spherical vector is hidden in the previous example when written in the form (4.61). If we rewrite it instead as an infinite product over all primes[7]

$$\delta = \sum_{m\in\mathbb{Z}} \delta(x-m) = \sum_{m\in\mathbb{Q}} \delta(x-m) \prod_p \gamma_p(x), \tag{4.74}$$

we see a clear relation. In the following the letter 'p' will always denote a prime number. The theta series that we will examine are (unlike the Jacobi theta series that has the trivial summation measure $\mu(x) = 1$) weighted sums, and we will write the summation as $\sum_x \mu(x)$, where $\mu(x)$ is the summation measure. Skipping the delta function

$$\sum_{m\in\mathbb{Z}} 1 = \sum_{m\in\mathbb{Q}} \prod_p \gamma_p(x), \tag{4.75}$$

[6]This is of course not the strict mathematical definition of a minimal representation. It is often defined by the property that its Gelfand-Kirillov dimension is minimal. See [79] for further details.

[7]$\gamma_p(x)$ is 1 on the p-adic integers and 0 elsewhere. See appendix B for further material on p-adic analysis.

we see that this is just an elaborate method of writing '1', but it displays the connection between the distribution and the spherical vector since $\gamma_p(x)$ is invariant under the p-adic Fourier transform, and hence may be thought of as the p-adic counterpart of the Gaussian. The defining property of the spherical vector (4.62) is that it is invariant under Fourier transformation and thus $\gamma_p(x)$ is its p-adic counterpart, the p-adic spherical vector. So, given any (real) spherical vector we should be able to calculate the distribution (or really the summation measure) as an adelic product over the p-adic spherical vector. In the following example we will see this principle at work in a more complex case.

The next example is a familiar one, the Eisenstein series of $Sl(2,\mathbb{Z})$ (see section C.1)

$$\mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s}(\tau) = \sum_{(m,n)\neq(0,0)} \left(\frac{\tau_2}{|m+n\tau|^2}\right)^s, \tag{4.76}$$

a non-holomorphic modular form[8] on the symmetric space $U(1)\backslash Sl(2,\mathbb{R})$. We are familiar with the action of $Sl(2,\mathbb{R})$ on this space being written as $\tau \to (a\tau+b)/(c\tau+d)$. We consider the representation

$$\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \phi(x,y) \to \phi(ax+by, cx+dy), \tag{4.77}$$

corresponding to

$$E_+ = x\partial_y, \quad E_- = y\partial_x, \quad H = x\partial_x - y\partial_y. \tag{4.78}$$

However, this representation is not irreducible and any function of $(x^2 + y^2)$ is invariant under the action of K in this case. An irreducible representation in one variable can be found by Poisson resumming the Eisenstein series (4.76) to yield[9]

$$\begin{aligned}\mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s}(\tau) = 2\zeta(2s)\tau_2^s + \frac{2\sqrt{\pi}\tau_2^{1-s}\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)} + \\ + \frac{2\pi^s\sqrt{\tau_2}}{\Gamma(s)} \sum_{N\in\mathbb{Z}} \mu_s(N) N^{s-1/2} K_{s-1/2}(2\pi\tau_2 N) e^{2\pi i \tau_1 N},\end{aligned} \tag{4.79}$$

where we have set $N = nm$ and

$$\mu_s(N) = \sum_{n|N} n^{-2s+1}. \tag{4.80}$$

[8]A function, $f(\tau)$, $\tau \in \mathcal{H}$, which (among other things) satisfies $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^k f(\tau)$, where the integers a,b,c,d satisfy $ab - cd = 1$.

[9]This was also done in section 4.1 although there the Bessel function K was also expanded.

From this expression we can (through a few calculations) read off the correct representation

$$\tilde{\rho}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \phi(x) \to e^{-itx}\phi(x), \tag{4.81}$$

$$\tilde{\rho}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} : \phi(x) \to e^{-2(s-1)t}\phi(e^{2t}x), \tag{4.82}$$

corresponding to

$$E_+ = ix, \quad E_- = i(x\partial_x + 2 - 2s)\partial_x, \quad H = 2x\partial_x + 2 - 2s, \tag{4.83}$$

and acting on the spherical vector

$$f_s(x) = x^{s-1/2}K_{s-1/2}(x), \tag{4.84}$$

where $K_{s-1/2}(x)$ is a modified Bessel function[10]. The representation element g_τ acts upon $f_s(x)$ as

$$f_s(x) = x^{s-1/2}K_{s-1/2}(x) \to \sqrt{\tau_2}x^{s-1/2}K_{s-1/2}(\tau_2 x)e^{-i\tau_1 x}, \tag{4.85}$$

(where we have rescaled by a factor 2π and dropped the factor in front of the sum in (4.79)), with the distribution

$$\delta_s(x) = \sum_{N\in\mathbb{Z}} \mu_s(N)\delta(x-N). \tag{4.86}$$

We can retrieve the series (4.79) by writing

$$\begin{aligned} &\langle \delta_s(x), \rho(g_\tau)\cdot f_s(x)\rangle = \\ &= \int dx \sum_{N\in\mathbb{Z}} \mu_s(N)\delta(x-N)\sqrt{\tau_2}x^{s-1/2}K_{s-1/2}(\tau_2 x)e^{-i\tau_1 x} \\ &= \sqrt{\tau_2}\sum_{N\in\mathbb{Z}} \mu_s(N)N^{s-1/2}K_{s-1/2}(\tau_2 N)e^{-i\tau_1 N}, \end{aligned} \tag{4.87}$$

albeit rescaled by a factor 2π and modulo the first two (degenerate) terms. This method of obtaining the constituents of (4.60) in order to form a theta series lies close at hand for physicists since the resummation (4.79) corresponds to a weak coupling expansion of the ten-dimensional Type IIB R^4 coupling[11]. However in the following we are going to look at theta series whose form is not known at all, in which case this method

[10] Note that in this representation we have discarded the first two terms in (4.79).

[11] See section C.2 for a discussion of this theta series in a different, less physics related, representation.

is obviously quite useless and we are forced to obtain general principles to construct (4.60). To this we return after studying the summation measure (4.80) in greater detail.

In the case of the Jacobi theta series we saw that the (trivial) summation measure could be written as an adelic product over p-adic spherical vectors. In this case we do not have a trivial summation measure though, but we still expect the relation between the summation measure and the real spherical vector to hold. The question becomes, what is the p-adic analogue of modified Bessel functions. It can be shown that (see section C.2 for a simple proof)

$$\sum_{N\in\mathbb{Z}} \mu_s(N) = \sum_{N\in\mathbb{Q}} \prod_p f_{p;s}(N), \tag{4.88}$$

where

$$f_{p;s}(x) = \gamma_p(x) \frac{1-p^{-2s+1}\left|x\right|_p^{2s-1}}{1-p^{-2s+1}}, \tag{4.89}$$

is the p-adic spherical vector[12]. The question now becomes, how can we derive this from the real spherical vector (4.84). The answer is that up until now we could not. This is so since no strict definition of the p-adic counterpart of the modified Bessel function $K_s(x)$ exists. We define it to be of the form (4.89) though we do not claim any relation to Bessel functions other then through its properties as a spherical vector.

The construction that we have just reviewed can easily be generalized to the $Sl(n,\mathbb{Z})$ case by using the fundamental representation of section C.1 and Poisson resumming only one direction. The representation (of the algebra) as well as the real and p-adic spherical vectors are of an completely analogous form.

In what follows we will mix general considerations with more complex examples in order to acquire a good understanding for the construction without resorting to the classical mathematical definition-theorem structure. We shall begin by studying the minimal representation followed by the spherical vector and lastly its p-adic counterpart.

The construction of the minimal representation as we have seen it relies fully on the existence of a unique 5-grading of all simple Lie algebras (modulo the choice of Cartan subalgebra and system of simple roots

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2. \tag{4.90}$$

Here, $G_{\pm 2}$ consists of the generators corresponding to the highest and lowest root $E_{\pm\omega}$, $G_{\pm 1}$ contains some of the generators corresponding to

[12] In [17] this is called a p-adic Bessel function which is misleading. p-adic Bessel functions certainly exist [81], but the function (4.89) is not related to them unless we define it to be the modified Bessel function. This seems highly questionable however.

positive and negative roots respectively (the condition will follow) and G_0 contains the generators of the remaining roots as well as the Cartan generators. The representation now follows from the fact that the subspace $G_1 \oplus G_2$ closes as an algebra, to be more specific a Heisenberg subalgebra

$$[E_{\alpha_1}, E_{\alpha_2}] = (\alpha_1, \alpha_2)E_\omega \qquad E_{\alpha_1}, E_{\alpha_2} \in G_1, \tag{4.91}$$

where $(\cdot, \cdot)$ is a symplectic form. We choose a *polarization* by picking the simple root, β_0, to which the affine root connects in the extended Dynkin diagram (this choice is not unique in the $Sl(n)$ case, we choose the root α_1 at the left-most end of the Dynkin diagram), this root lies in G_1. The positive roots in G_1 then split into three different sets depending on their inner product with β_0 We denote these roots as follows

$$\beta_i : \langle \beta_i, \beta_0 \rangle = 1, \qquad i > 0 \tag{4.92}$$

$$\gamma_i : \langle \gamma_i, \beta_0 \rangle = -1 \qquad (\gamma_i = \omega - \beta_i), \tag{4.93}$$

$$\gamma_0 : \langle \gamma_0, \beta_0 \rangle = -1 \qquad (\gamma_0 = \omega - \beta_0), \tag{4.94}$$

$$\beta_0 : \langle \beta_0, \beta_0 \rangle = 2, \tag{4.95}$$

and choose a representation of the corresponding generators as

$$E_\omega = iy, \qquad E_{\gamma_i} = ix_i, \tag{4.96}$$

$$E_{\beta_i} = y\partial_i, \tag{4.97}$$

the dimension of this space will be determined when we turn to examples[13]. Thus we have picked a representation for the first two spaces in

$$\begin{aligned} G_2 &= \{E_\omega\}, \\ G_1 &= \{E_{\beta_i}, E_{\gamma_i}\}, \\ G_0 &= \{E_{-\alpha_j}, H_{\alpha_k}, E_{\alpha_j}\}, \\ G_{-1} &= \{-E_{\beta_i}, E_{-\gamma_i}\}, \\ G_{-2} &= \{E_{-\omega}\}, \end{aligned} \tag{4.98}$$

and before we turn to the realization of the other generators we display this choice of polarization for a specific example, $SO(4,4)$.

$SO(4,4)$ has the Dynkin diagram invariant under permutations of

[13]We will not prove that this is a minimal representation here, but the argument relies on a branching of G into $Sl(2) \times H$ where one can pick a (minimal) orbit such that the quantization of this orbit gives the minimal representation. This corresponds to the choice of polarization.

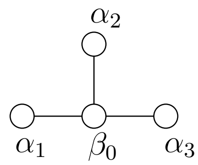

Figure 4.4: Dynkin diagram of D_4

$\alpha_1, \alpha_2, \alpha_3$ (triality), the simple roots are given by

$$\alpha_1 = (\ 1 \quad -1 \quad 0 \quad 0\ ), \tag{4.99}$$
$$\alpha_2 = (\ 0 \quad 0 \quad 1 \quad 1\ ), \tag{4.100}$$
$$\alpha_3 = (\ 0 \quad 0 \quad 1 \quad -1\ ), \tag{4.101}$$
$$\beta_0 = (\ 0 \quad 1 \quad -1 \quad 0\ ). \tag{4.102}$$

And we can explicitly calculate

$$\beta_0 = (\ 0 \quad 1 \quad -1 \quad 0\ ), \qquad \gamma_0 = (\ 1 \quad 0 \quad 1 \quad 0\ ), \tag{4.103}$$
$$\beta_1 = (\ 1 \quad 0 \quad -1 \quad 0\ ), \qquad \gamma_1 = (\ 0 \quad 1 \quad 1 \quad 0\ ), \tag{4.104}$$
$$\beta_2 = (\ 0 \quad 1 \quad 0 \quad 1\ ), \qquad \gamma_2 = (\ 1 \quad 0 \quad 0 \quad -1\ ), \tag{4.105}$$
$$\beta_3 = (\ 0 \quad 1 \quad 0 \quad -1\ ), \qquad \gamma_3 = (\ 1 \quad 0 \quad 0 \quad 1\ ), \tag{4.106}$$

$$\omega = (\ 1 \quad 1 \quad 0 \quad 0\ ), \tag{4.107}$$

and write down the elements of the representation

$$E_{\beta_0} = y\partial_0, \qquad E_{\gamma_0} = ix_0, \tag{4.108}$$
$$E_{\beta_1} = y\partial_1, \qquad E_{\gamma_1} = ix_1, \tag{4.109}$$
$$E_{\beta_2} = y\partial_2, \qquad E_{\gamma_2} = ix_2, \tag{4.110}$$
$$E_{\beta_3} = y\partial_3, \qquad E_{\gamma_3} = ix_3, \tag{4.111}$$

$$E_\omega = iy. \tag{4.112}$$

This calculation is as seen very simple and straightforward. It is merely a question of determining the simple roots and calculating the inner product to make the split according to the polarization.

Our next objective is to extend this representation to the full algebra, which is done by Weyl reflections. We need two transformations in order to generate the full representation. The first, S, maps β_i to γ_i, that in this representation this maps momenta to positions, i.e. it is a Fourier transformation

$$(Sf)(y, x_0, \ldots, x_{d-1}) = \int \frac{\prod_{i=0}^{d-1} p_i}{(2\pi y)^{d/2}} f(y, p_0, \ldots, p_{d-1}) e^{\frac{i}{y}\sum_{i=0}^{d-1} p_i x_i}. \tag{4.113}$$

This transformation, which acts by conjugation, can also be checked to map α_i to α_{-i} and clearly leaves ω invariant. The other generator, A,

G	I_3	H_0
$Sl(d)$	0	$Sl(n-3)$
$SO(d,d)$	$x_1 \sum_i x_{2i}x_{2i+1}$	$SO(n-3,n-3)$
E_6	det	$Sl(3)\times Sl(3)$
E_7	Pf	$Sl(6)$
E_8	$27^{\otimes_s 3}\|_1$	E_6

Figure 4.5: Groups and their corresponding cubic form I_3 as well as subgroup H_0.

maps $\beta_0 \to -\beta_0$, $\gamma_0 \to \omega$ and $\beta_i \to \alpha_j$ (see equation (4.98)), and is given by

$$(Af)(y,x_0,\ldots,x_{d-1}) = e^{\frac{-iI_3}{yx_0}} f(-x_0,y,\ldots,x_{d-1}), \tag{4.114}$$

where I_3 is a cubic form given for each relevant group in table 4.5.

All other generators are obtained by commutation of the previously determined ones. One additional fact is important to mention in relation to the generator A; there is a set of generators that are invariant under the action of A and they make up a linearly realized subalgebra H_0. This subalgebra is generated by the generators corresponding to the simple roots not connected to β_0 (or the opposite root in the $Sl(n)$ case) in the Dynkin diagram, the choice of polarization being invariant under this algebra.

Let us apply this procedure to our $SO(4,4)$ example, in this case all simple roots attach to β_0 in the Dynkin diagram, and the cubic form is given by $I_3 = x_1x_2x_3$. By acting with A we get

$$AE_{\beta_1}A^{-1} = e^{\frac{-ix_1x_2x_3}{yx_0}}(-x_0\partial_1)e^{\frac{ix_1x_2x_3}{yx_0}} = -x_0\partial_1 - \frac{ix_2x_3}{y} = E_{\alpha_1}, \tag{4.115}$$

$$AE_{\beta_2}A^{-1} = e^{\frac{-ix_1x_2x_3}{yx_0}}(-x_0\partial_2)e^{\frac{ix_1x_2x_3}{yx_0}} = -x_0\partial_2 - \frac{ix_1x_3}{y} = E_{\alpha_2}, \tag{4.116}$$

$$AE_{\beta_3}A^{-1} = e^{\frac{-ix_1x_2x_3}{yx_0}}(-x_0\partial_3)e^{\frac{ix_1x_2x_3}{yx_0}} = -x_0\partial_3 - \frac{ix_1x_2}{y} = E_{\alpha_3}, \tag{4.117}$$

and then with S, yielding

$$SE_{\alpha_1}S^{-1} = x_1\partial_0 + iy\partial_2\partial_3 = E_{-\alpha_1}, \tag{4.118}$$

$$SE_{\alpha_2}S^{-1} = x_2\partial_0 + iy\partial_1\partial_3 = E_{-\alpha_2}, \tag{4.119}$$

$$SE_{\alpha_3}S^{-1} = x_3\partial_0 + iy\partial_1\partial_2 = E_{-\alpha_3}. \tag{4.120}$$

By acting with A on E_{β_0} and $E_{-\alpha_i}$ we produce

$$AE_{\beta_0}A^{-1} = -x_0\partial + \frac{ix_1x_2x_3}{y^2} = E_{-\beta_0}, \quad (4.121)$$

$$AE_{-\alpha_1}A^{-1} = x_1\partial + \frac{x_1}{y}(1 + x_2\partial_2 + x_3\partial_3) - ix_0\partial_2\partial_3 = E_{-\beta_1}, \quad (4.122)$$

$$AE_{-\alpha_2}A^{-1} = x_2\partial + \frac{x_2}{y}(1 + x_1\partial_1 + x_3\partial_3) - ix_0\partial_3\partial_1 = E_{-\beta_2}, \quad (4.123)$$

$$AE_{-\alpha_3}A^{-1} = x_3\partial + \frac{x_3}{y}(1 + x_1\partial_1 + x_2\partial_2) - ix_0\partial_1\partial_2 = E_{-\beta_3}, \quad (4.124)$$

and acting with S gives

$$\begin{aligned} SE_{-\beta_0}S^{-1} &= 3i\partial_0 + iy\partial\partial_0 - y\partial_1\partial_2\partial_3 + \\ &+ i(x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\partial_0 = E_{-\gamma_0}, \quad (4.125) \end{aligned}$$

$$SE_{-\beta_1}S^{-1} = iy\partial_1\partial + i(2 + x_0\partial_0 + x_1\partial_1)\partial_1 - \frac{x_2x_3}{y}\partial_0 = E_{-\gamma_1}, \quad (4.126)$$

$$SE_{-\beta_2}S^{-1} = iy\partial_2\partial + i(2 + x_0\partial_0 + x_2\partial_2)\partial_2 - \frac{x_1x_3}{y}\partial_0 = E_{-\gamma_2}, \quad (4.127)$$

$$SE_{-\beta_3}S^{-1} = iy\partial_3\partial + i(2 + x_0\partial_0 + x_3\partial_3)\partial_3 - \frac{x_1x_2}{y}\partial_0 = E_{-\gamma_3}, \quad (4.128)$$

finally the generator corresponding to the lowest root, E_ω, is created by acting with A on $E_{-\gamma_0}$

$$\begin{aligned} AE_{-\gamma_0}A^{-1} = 3i\partial + iy\partial^2 + \frac{i}{y} + ix_0\partial_0\partial \; + \frac{x_1x_2x_3}{y^2}\partial_0 \\ + \frac{i}{y}(x_1x_2\partial_1\partial_2 + x_3x_1\partial_3\partial_1 + x_2x_3\partial_2\partial_3) + \\ i(x_1\partial_1 + x_2\partial_2x_3\partial_3)(\partial + \frac{1}{y}) + x_0\partial_1\partial_2\partial_3. \quad (4.129) \end{aligned}$$

The remaining generators we have to fix by commutation

$$\begin{aligned} [E_{\beta_0}, E_{-\beta_0}] &= -y\partial + x_0\partial_0 = H_{\beta_0}, && (4.130) \\ [E_{\beta_1}, E_{-\beta_1}] &= -1 - x_0\partial_0 + x_1\partial_1 - x_2\partial_2 - x_3\partial_3 = H_{\beta_1}, && (4.131) \\ [E_{\beta_2}, E_{-\beta_2}] &= -1 - x_0\partial_0 - x_1\partial_1 + x_2\partial_2 - x_3\partial_3 = H_{\beta_2}, && (4.132) \\ [E_{\beta_3}, E_{-\beta_3}] &= -1 - x_0\partial_0 - x_1\partial_1 - x_2\partial_2 + x_3\partial_3 = H_{\beta_3}. && (4.133) \end{aligned}$$

Thus we have fully worked out the standard minimal representation of $SO(4,4)$ in every detail. The complexity that creeps in to some of the calculations here grows rapidly with more complex algebras, so brute force is not the ideal way of going about calculations in this representation as we will see shortly.

Now we turn our attention to the spherical vector, there is really very little left to say here about this object, at least in general terms. We have defined it as a representation element invariant under the maximal compact subgroup K. In our representation this means that $f(y, x_0, \ldots, x_{d-1})$ should solve

$$(E_\alpha \pm E_{-\alpha}) f(y, x_0, \ldots, x_{d-1}) = 0, \tag{4.134}$$

(where the sign is chosen such that $E_\alpha \pm E_{-\alpha}$ is a compact generator). We do not have to solve these equations for all roots α, it is enough to solve them for simple roots since the other equations can be obtained by commuting the generators corresponding to simple roots. As stated above, brute force calculations is not preferable in this representation, even in the $SO(4,4)$ case we have a system of four rather dull partial differential equations to solve. Instead of solving these we will use our previously acquired knowledge from Eisenstein series and $\frac{1}{2}$-BPS amplitudes to find the spherical vector in a representation identical to the one arising in string theory. This spherical vector will then be mapped to the representation that we have now become familiar with.

We already know of the embedding $SO(d,d,\mathbb{R}) \times Sl(2,\mathbb{R}) \subset Sp(2d,\mathbb{R})$, the minimal representation of $Sp(2d,\mathbb{R})$ is an irreducible representation of $SO(d,d,\mathbb{R})$ but we can reduce this to a minimal representation by considering only functions that are invariant under $Sl(2,\mathbb{R})$. This we already knew, since enforcing this invariance on

$$\frac{SO(d,d,\mathbb{R})}{SO(d) \times SO(d)} \times \frac{Sl(2,\mathbb{R})}{U(1)}, \tag{4.135}$$

corresponds to integrating over the last factor, i.e., calculating a $\frac{1}{2}$-BPS one-loop amplitude. Taking the partition function

$$\theta_{Sp(2d)}(g; B; \tau) = V_d \sum_{m^i, n^i} e^{\frac{-\pi(m^i + n^i\tau) g_{ij} (m^j + n^j \bar{\tau})}{\tau_2} + 2i\pi m^i B_{ij} m^j}, \tag{4.136}$$

and performing the integration

$$\theta_{SO(d,d)}(g; B) = 2\pi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \theta_{Sp(2d)}(g; B; \tau), \tag{4.137}$$

produces

$$\begin{aligned} \theta_{SO(d,d)}(g; B) = \frac{2\pi^2}{3} V_d + 2V_d \sum_{m^i \neq 0} \frac{1}{m^i g_{ij} m^j} \\ +4\pi V_d \sum_{(m^i, n^i)/Sl(2)} \frac{e^{-2\pi\sqrt{(m^{ij})^2} + 2\pi i m^{ij} B_{ij}}}{\sqrt{(m^{ij})^2}}. \end{aligned} \tag{4.138}$$

The last sum in this expression (the instanton contribution) can be written as a sum over rank 2 ($r(m^{ij}) = 2$) anti-symmetric matrices m^{ij}

$$\sum_{r(m^{ij})=2} \mu(m^{ij}) \frac{e^{-2\pi\sqrt{(m^{ij})^2}+2\pi i m^{ij} B_{ij}}}{\sqrt{(m^{ij})^2}}, \tag{4.139}$$

which constitute a $2n - 3$ dimensional representation (only two rows in the matrices are non-zero and then we have to remove the diagonal elements and one element more is fixed by anti-symmetry); this is the same dimensionality as the minimal representation of $SO(d, d)$. The spherical vector of this representation can actually be read off from the above expression by going to the origin of the moduli space[14] (i.e. by setting $g_{ij} = \delta_{ij}$ and $B_{ij} = 0$)

$$\tilde{f}_{SO(d,d)} = \frac{e^{-2\pi\sqrt{(m^{ij})^2}}}{\sqrt{(m^{ij})^2}}. \tag{4.140}$$

The summation measure in the above sum is the familiar

$$\mu(m^{ij}) = \sum_{n|m^{ij}} n, \tag{4.141}$$

which we will mention more about later.

Now we want to translate this spherical vector into the standard minimal representation. We will do this in the $SO(4, 4)$ case in order to be able to express ourselves as explicitly as possible. The trick we use to do this is to identify an Abelian subalgebra that is generated by shifting B_{ij} by a constant in (4.139). An equivalent algebra can also be found in the standard minimal representation and by using the explicit form of the generators in this representation we can find a common eigenstate. Choosing the generators $\{E_{\alpha_3}, E_{\beta_3}, E_{\gamma_0}, E_{\gamma_1}, E_{\gamma_2}, E_\omega\}$ (on the basis of them having coefficient 1 in front of α_3 in the basis of simple roots), we label the eigenvalues of these simultaneously diagonalizable generators $i(m^{43}, m^{24}, m^{14}, m^{23}, m^{13}, m^{12})$. Their form in the standard minimal representation hints at the common eigenstate

$$\psi_{m^{ij}} = \delta(y - m^{12})\delta(x_0 - m^{13})\delta(x_1 - m^{14})\delta(x_2 - m^{23})e^{\frac{im^{24}x_3}{m^{12}}}, \tag{4.142}$$

provided that the following condition holds

$$m^{43} = -\frac{m^{14}m^{23}}{m^{12}} - \frac{m^{13}m^{24}}{m^{12}}, \tag{4.143}$$

[14] Remember that the moduli lie in a symmetric space $K\backslash G$, consequently by going to the origin we retrieve the function which does not change when we “transform” this space in the larger space G.

but this is just the $r(m^{ij}) = 2$ condition in the $d = 4$ case. This state displays an intertwiner between the two representations (the integration over all y, x_i of the state $\psi_{m^{ij}}$ becomes a Fourier transformation in x_3)

$$\begin{aligned}\tilde{f}(m^{ij}) &= \int dy dx_0 d^3x_i \psi_{m^{ij}} f(y, x_0, x_i) \\ &= \int dx_3 e^{\frac{im^{24}x_3}{m^{12}}} f(m^{12}, m^{13}, m^{14}, m^{23}, x_3),\end{aligned} \tag{4.144}$$

and more importantly

$$f(y, x_0, x_i) = \int \frac{dm^{24}}{y} e^{\frac{-2\pi i m^{24} x_3}{y}} \tilde{f}\left(y, x_0, x_1, x_2, m^{24}, \frac{x_1 x_2 + x_0 m^{24}}{y}\right). \tag{4.145}$$

The computation of this integral (the details of which we refer the reader to [17]) yields the spherical vector in the standard minimal representation

$$f_{SO(4,4)} = \frac{4\pi}{\sqrt{y^2 + x_0^2}} K_0(S_1') e^{-i\frac{x_0 x_1 x_2 x_3}{y(y^2+x_0^2)}}, \tag{4.146}$$

where

$$S_1' = \frac{\sqrt{(y^2 + x_0^2 + x_1^2)(y^2 + x_0^2 + x_2^2)(y^2 + x_0^2 + x_3^2)}}{y^2 + x_0^2}. \tag{4.147}$$

This function could actually be obtained as a solution to the differential equation that the system of equations reduces to when introducing the variable S_1 which is essentially the method that is used to retrieve the spherical vector in the exceptional cases in [17].

In the following cases we will be brief since the method has already been outlined and the long and tedious calculations do nothing to enrich the discussion. The data and some of the calculations can be found in [17]. The method used in the $SO(4, 4)$ case can be generalized to $SO(d, d)$ in a straightforward manner, yielding

$$f_{SO(d,d)} = \left(\frac{y^2 + x_0^2 + x_1^2}{(y^2 + x_0^2)^2 + (y^2 + x_0^2)P + Q^2}\right)^{\frac{d-4}{2}} \frac{K_{\frac{d-4}{2}}(S_1) e^{-iS_2}}{\sqrt{y^2 + x_0^2}}, \tag{4.148}$$

where we have introduced the forms

$$I_2 = x_1^2 + P, \quad I_3 = x_1 Q, \quad I_4 = x_1^4 + P^2 - 2Q^2, \tag{4.149}$$

with

$$P = \sum_{j=2}^{2d-5} x_j^2, \qquad Q = \sum_{i=1}^{d-3} (-)^{i+1} x_{2i} x_{2i+1}, \tag{4.150}$$

such that

$$S_1 = \frac{\sqrt{(y^2+x_0^2)^3 + (y^2+x_0^2)I_2 + (y^2+x_0^2)(I_2^2 - I_4)/2 + I_3^2}}{y^2+x_0^2}, \quad (4.151)$$

$$S_2 = \frac{x_0 I_3}{y(y^2+x_0^2)}. \quad (4.152)$$

This spherical vector can be written in a more convenient manner

$$f_{SO(d,d)} = \frac{1}{R}\left(\frac{||(y,x_0)||}{R}\right)^{d-4} \mathcal{K}_{\frac{d-4}{2}}\left(||X, \nabla_X\left(\frac{I_3}{R}\right)||\right) e^{-i\frac{x_0 I_3}{yR^2}}, \quad (4.153)$$

where $\mathcal{K}_d = x^{-d}K_d(x)$, $R = ||(y,x_0)||$, $X = (y, x_0, \ldots, x_{2d-5})$ and ∇_X is the gradient with respect to the coordinates X.

We will now briefly review the simplest of the exceptional cases, E_6, and then end this discussion of the spherical vectors by quoting the results for E_7 and E_8.

The method in this and indeed all exceptional cases relies on two observations, the first is that the compact generator corresponding to β_0 performs a rotation in (y, x_0) and thus restricts the spherical vector to depend on y and x_0 through $y^2+x_0^2$ only. The second observation is that the spherical vector must be invariant under K_0, the maximal compact subgroup of the linearly realized H_0 (not G). This allows us to write down the invariants

$$I_2 = \mathrm{tr}(Z^T Z), \quad I_3 = -\det(Z), \quad I_4 = \mathrm{tr}(Z^T Z Z^T Z)), \quad (4.154)$$

where

$$Z = \begin{pmatrix} x_1 & x_3 & x_6 \\ x_2 & x_5 & x_9 \\ x_4 & x_7 & x_8 \end{pmatrix}, \quad (4.155)$$

and make an ansatz for the spherical vector expressed in these invariants and $y^2+x_0^2$. Putting this ansatz to the test with some of the compact generators further reveals the structure of the solution and allows us to extend its invariance properties to all of K. We will not go into more details other than what is stated above. The calculation amounts to solving differential equations, and the resulting function becomes

$$f_{E_6} = \frac{K_{1/2}(S_1) e^{-i\frac{x_0 I_3}{y(y^2+x_0^2)}}}{(y^2+x_0^2)\sqrt{S_1}}, \quad (4.156)$$

where S_1 is given by (4.151) albeit with our new forms I_2, I_3 and I_4.

The E_7 and E_8 cases are solved in complete analogy with the E_6 case. For E_7 we define

$$Z = \begin{pmatrix} 0 & -x_1 & x_2 & -x_4 & -x_6 & -x_9 \\ & 0 & x_3 & -x_5 & -x_8 & x_{12} \\ & & 0 & x_7 & x_{11} & -x_{15} \\ & & & 0 & -x_{14} & x_{13} \\ & & & & 0 & x_{10} \\ & & & & & 0 \end{pmatrix}, \tag{4.157}$$

and write down the invariants

$$I_2 = -\frac{1}{2}\mathrm{tr}(Z^2), \quad I_3 = -\mathrm{Pf}Z, \quad I_4 = \frac{1}{2}\mathrm{tr}(Z^4), \tag{4.158}$$

the spherical vector becomes

$$f_{E_7} = \frac{K_1(S_1)}{(y^2+x_0^2)^{3/2}S_1} e^{-i\frac{x_0 I_3}{y(y^2+x_0^2)}}, \tag{4.159}$$

with S_1 as previously defined. For E_8 we can define

$$Z = \begin{pmatrix} 0 & x_5 & x_8 & x_{10} & x_{12} & x_{15} \\ & 0 & x_9 & x_{11} & x_{14} & x_{17} \\ & & 0 & x_{13} & x_{16} & x_{20} \\ & & & 0 & x_{19} & x_{23} \\ & & & & 0 & x_{26} \\ & & & & & 0 \end{pmatrix}, \tag{4.160}$$

$$Y_1 = \begin{pmatrix} x_7 \\ -x_6 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} x_{18} \\ x_{21} \\ x_{24} \\ -x_{27} \\ x_{25} \\ -x_{22} \end{pmatrix}, \tag{4.161}$$

giving the invariants

$$I_2 = -\mathrm{tr}(Z^2)/2 + \mathrm{tr}(Y_i Y_i^T), \tag{4.162}$$

$$I_3 = \mathrm{Pf}Z + \mathrm{tr}(Y_1^T Z Y_2), \tag{4.163}$$

$$\begin{aligned} I_4 = &\ \frac{1}{2}\mathrm{tr}(Z^4) + \mathrm{tr}((Y_i Y_i^T)^2) 2\mathrm{tr}(Y_i^T Z^2 Y_i) \\ & -(\mathrm{tr}Z)^2(\mathrm{tr}Y_i Y_i^T) + \frac{1}{2}\epsilon^{ijklmn} Z_{ij} Z_{kl} Y_{1m} Y_{2n}, \end{aligned} \tag{4.164}$$

which we then use to derive the spherical vector

$$f_{E_8} = \frac{K_2(S_1)}{(y^2 + x_0^2)^{5/2} S_1^2} e^{-i \frac{x_0 I_3}{y(y^2+x_0^2)}}. \tag{4.165}$$

Before proceeding with a discussion of the summation measure let us conclude our deliberations on the (real) spherical vector with a short summary.

The spherical vector is defined as an object invariant under the maximal compact subgroup K of G, and it ensures that the theta series only depends on variables lying in the symmetric space $K\backslash G$. In the standard minimal representation it can be obtained by solving a system of partial differential equations corresponding to the action of the simple roots. The task of deriving the spherical vector in the $SO(d,d)$ case could be completed by considering a $\frac{1}{2}$-BPS one-loop string theory amplitude, reading off the spherical vector and then transforming it to the standard minimal representation via an intertwiner. For the exceptional cases we really have to solve the differential equations. However, the fact that there is a linearly realized subgroup H_0 of G and that this subgroup in turn has a maximal compact subgroup K_0 under which the spherical vector must be invariant, facilitates this task greatly. Here we only sketched how the solutions were obtained and proceeded to quote them from [17].

We can now rewrite the solutions in a form well suited for the continuation of this chapter, in analogy with (4.153) we can write

$$f_{E_d} = \frac{1}{R^{s+1}} \mathcal{K}_{s/2} \left(||X, \nabla_X \left(\frac{I_3}{R} \right) || \right) \exp \left(-i \frac{x_0 I_3}{y R^2} \right), \tag{4.166}$$

with $s = 1, 2, 4$ for $d = 6, 7, 8$ respectively and the constituents defined in analogy with the ones appearing in (4.153).

This, the last, part of this section concerns the last object we have left to study, the distribution δ. Actually we will reduce our studies, first to the summation measure μ since the correct delta function and sum are rather easily determined, and then down to the p-adic counterpart of the previously acquired spherical vectors. This is because we have seen strong evidence picking out the adelic product over all primes of the p-adic spherical vector as the correct summation measure. As explained in the beginning of this section there are still many unanswered questions surrounding this area, many of them reside in the particular branch we are about to study now. We will begin by tracking back to the beginning of this section to review how the p-adic spherical vector arises there. Then we will discuss some of the issues in [17] and end by quoting some results and technicalities from the calculation [24] and the notoriously complicated paper [18].

The first example of theta series we used was the Jacobi theta series, here the summation measure was quite trivial, just '1'. We were able to rewrite this summation measure as a product over all primes of the p-adic Gaussian, $\gamma_p(x)$, the real spherical vector being the real Gaussian. In the example following this we studied Eisenstein series and saw that the summation measure in that case could be written as

$$\prod_p \frac{1 - p^{-2s+1} |x|_p^{2s-1}}{1 - p^{-2s+1}}. \tag{4.167}$$

The real spherical vector in this case was given by a modified Bessel function, and this was the first time we saw its p-adic analog (with respect to K-invariance). Furthermore we have studied an equivalent representation in D.2 related to the above one by a Fourier transformation and seen that the p-adic spherical vectors are also related to one another by p-adic Fourier transformations. The following example is a more complicated one, the $SO(4,4)$ theta series. We studied this (or generally $SO(d,d)$ theta series) in a 'stringy' representation and saw that the summation measure in this representation was very similar to the summation measure in the previously studied Eisenstein series. In fact it can be written as an adelic product over the p-adic spherical vector

$$\tilde{f}_p = \gamma_p(m^{ij}) \frac{1 - p\,|m^{ij}|_p}{1 - p}, \tag{4.168}$$

and since the 'stringy' representation is intertwined with the standard minimal representation by a Fourier transform the p-adic spherical vector should be obtained by performing a p-adic Fourier transformation. The expression that should be calculated is a p-adic integral over $\mathbb{Q}_p$, the integrand is a product of gp's and a quotient containing a max function as we have seen in a previous example. Finally there is an additive character $\psi\left(\frac{x_3 m^{24}}{y}\right)$ that makes this integral a Fourier transformation. Both (one) γ_p and the max function depend on the combination $\frac{x_1 x_2 + x_0 m^{24}}{y}$, where m^{24} is the dummy variable in the integral. The technique used in [24] was then to simply split the integral into several different cases depending on the result of the max function, ordering among the p-adic norms is enforced by multiplying with γ_p's of quotients of the variables. For example

$$\gamma_p\left(\frac{m^{24}}{x_1}\right) = \begin{cases} 1, & |m^{24}|_p \leq |x_1|_p\,, \\ 0, & |x_1|_p > |m^{24}|_p\,. \end{cases} \tag{4.169}$$

Thus by multiplying the integrand with such a factor we restrict the integration region to the disk $|m^{24}|_p \leq |x_1|_p$, so with this method the whole task of integrating becomes reduced to keeping track of integration regions and calculating integrals within these. After splitting into cases depending on the result of max we split yet again, this time the $\frac{x_1 x_2 + x_0 m^{24}}{y}$ contribution, all expressions with γ_p's or p-adic norms, depending on more complex combinations than pure quotients of p-adic variables can be split up into several cases and then integrated. By proceeding in this manner we can calculate this integral fully by standard p-adic techniques in the different regions, the resulting expression has a high level of complexity and in [24] no apparent simplification was spotted that would render the expression simple.

A much more elegant (and most likely equivalent) method was used in [18] to obtain the spherical vector

$$f_p = \begin{cases} \psi\left(-\frac{x_1 x_2 x_3}{x_0}\right) |x_0|_p^{-1} \left[1 + v\left(x, \text{grad}\left(\frac{x_1 x_2 x_3}{x_0}\right) x\right)\right], & v(x) \geq 0, \\ 0, & v(x) < 0. \end{cases} \tag{4.170}$$

Here, ψ is the additive character on $\mathbb{Q}_1$ (see appendix B) and $v(x)$ is the valuation in $\mathbb{Q}_p$ ($|x|_p = p^{-v(x)}$, see the definition of $|x|_p$ in appendix B), x represents all the variables $x_0, x_1, \ldots$. The valuation of multiple variables means the valuation of the variable resulting from $\max(|x_0|_p, \ldots)$ (in the $v(x)$ case), and 'grad' is the ordinary real gradient, performed as if the variables were real, then we simply state that the variables and thus the result is p-adic, this is a 'real glitch' in an otherwise p-adic calculation. There are a number of lemmas underlying the calculations in this paper that we will not go into here. The confident reader may find them in [18].

This result can be extended to $SO(d, d)$, $d \geq 5$

$$f_p = \begin{cases} \psi\left(-\frac{I_3}{y x_0}\right) |x_0|_p^{-1} \max\left(1, \left|\frac{x_1}{x_0}\right|_p\right)^{d-4} \frac{p^{d-4} K_p(x)^{-d-4} - 1}{p^{d-4} - 1}, & K_p(x) \leq 1 \\ 0, & K_p(x) > 1, \end{cases} \tag{4.171}$$

where

$$K_p(x) = \left| x, \text{grad}\left(\frac{I_3}{x_0}\right) x \right|_p . \tag{4.172}$$

Here I_3 is the previously defined cubic form and the $|(\cdot, \cdot, \cdots)|_p$ is shorthand for the max of the p-adic norms of the elements in this 'vector'.

Last but not least we can quote and understand the p-adic spherical vectors of the main characters in this section, the exceptional groups.

These functions takes the form

$$f_p = \begin{cases} \psi\left(-\frac{I_3}{yx_0}\right)|x_0|_p^{-s-1}\,\frac{p^s K_p(x)^{-s}-1}{p^s-1}, & K_p(x) \leq 1 \\ 0, & K_p(x) > 1, \end{cases} \tag{4.173}$$

where as before $s = 1, 2, 4$ for $d = 6, 7, 8$ respectively.

Ending this section and chapter here might seem abrupt but the discussion that takes place after these results are more appropriately placed in the next chapter, Conclusions, where we without further ado point the reader (section 5.4).

5

Conclusion

In this chapter I have collected some conclusions, speculations and ideas that have come up during the writing of this work. Some of these conclusions are entirely my own and the same goes for the (wild) speculations and (far-fetched) ideas. In the cases where my present or future judges determine me to be not completely off track, I take full credit. In the cases where the 'track' is nowhere to be seen and long forgotten, I blame ignorance of youth.

5.1 The Bosonic Membrane

The biggest problem that the membrane suffers from is probably our interpretation of it. What, at the first stages of formulation, looks like a nice theory, just an integral over a manifold, turns out to be a hideously beautiful theory. Not only do we discover interactions between the embedding-fields X^μ, but we also discover that the spectrum becomes continuous due to the fact that we can add (a finite number of) spikes to the membrane that go off to infinity in one point. This does not only affect the spectrum of the theory but also contradicts the most basic assumptions of all, that the membrane should be a manifold.

One interesting proposal for how to study the membrane self-interactions is the mathematics of cobordisms. Two n-dimensional manifolds are said to be cobordant if their disjoint union is a manifold, i.e. if the two manifolds are borders of another $n+1$-dimensional manifold. The strict mathematical definition does put greater restraints on what kind of manifolds we can deal with, but the technicalities are not of great importance here. Let us assume for the moment that the membrane really is a nice-looking manifold, and that we have picked a gauge so that the timelike direction is identifiable. Then the cobordism between the two manifolds that is the membrane at two fixed times τ_1, τ_2, is really just the worldvolume of the propagation during this time. The topology of this three-dimensional manifold is not explicitly given, as in the case of membrane self-interaction it can have an arbitrary number of holes. Using methods like this it might be possible to study the interactions and determine for example how our choice of gauge restrains the topology of the membrane etc. There has been work along similar lines in gravity [82] where it holds importance for determining if the universe can go through topology-transitions other than the Big Bang (such as dynamical creation of Einstein-Rosen bridges).

But, before this formalism could be applied to our microscopic membrane we would have to deal with the question of what happens at the singular points of our "manifold". The mathematics of cobordism is quite complex and there has yet to come any real attempt along these lines in high-energy physics.

5.2 Membranes, Supersymmetry and Matrices

With the exception of a small attempt at perturbative calculations recently, the interest in bosonic membranes has pretty much died away. Apart from not containing fermions or being able to describe multi-particle states (due to the discrete spectrum of the quantized theory) there has been some indications that the theory may have anomalies. This would render the quantized theory inconsistent and finalize the "hammering in" of the last nail in its coffin. Similarly, the so-called spinning membrane was dealt a major blow several years ago with the appearance of a no-go theorem concerning its existence. The interesting thing though, is that after the publication of this theorem a couple of Lagrangians were presented that evaded its implications. There actually is a way to get around this no-go theorem, but the models that exemplified this procedure were exceedingly complicated. One might ask oneself if our inability to formulate spinning membrane models is due to the fact that no such models exist or if it is due to our lack of understanding for those kinds of supersymmetric theories.

Another fascinating area to ask questions within is matrix theory. Matrix theory has become a wonderful looking-glass through which we have been able to sneak a peak inside M-theory. But there still remains some concern and questions about the nature of this membrane regularization. Particularly, since we have such limited knowledge about how the membrane really behaves, can we be sure that we do not loose information when cutting off the basis at a finite number of elements. The membrane is an object that behaves very strangely. Can we be sure that all that strangeness is translated over into the matrix theory or are we simply getting to see the well behaved cousin of membrane theory. To be more specific, can we really approximate a geometric object that could be extremely singular with a finite basis? No matter how large an N we take it still has to be finite whereupon we loose the ability to map the effects of these singularities in matrix theory.

The limit $N \to \infty$, "des pudels kern" or matrix theory, is another source for concern. This is a mathematical technicality that we should have a good understanding of, since the physical predictions relying on it are immense. For example, it has been proved a long time ago that the matrix regularization works for the spherical and torodial membrane. These proofs have since then been extended to any compact Kähler manifold. But a membrane need not be compact! Indeed with spikes going off to infinity it will be exceedingly non-compact. What actually happens in the limit turning a rather well-behaved Yang-Mills theory into a theory of membranes would be very nice to know. Perhaps it would also tell us more about how the more unpleasant details of the membrane survive in the matrix regularization.

5.3 p-adic Numbers in Physics

This is not the first time that the mathematical area of p-adic analysis enters the arena of modern physics It has been used many times before in different areas of physics. What separates the sometimes less successful attempts from this one is the motivation. Let us briefly review some of the uses p-adic numbers have found in physics prior to this occasion.

In the paper [83] Freund and Olson first considered the possibility of non-archimedean (p-adic) strings. Their conjecture was then out to the test in [84] where they calculated some amplitudes in this theory. A key feature in their work is that, in the end, all quantities are real or complex, the complex parameter, z, which is integrated over in the ordinary amplitude is turned into a parameter residing in a quadratic extension of the p-adic field $\mathbb{Q}_p$ (quadratic extensions have not been dealt with in appendix B so we refer the reader to e.g. [85] for further details). They start from the complex (tree-level) 4-point amplitude

$$A_4^{(\infty)}(k_1k_4, k_2k_4) = \kappa^2 \int d^2|z_4|^{k_1k_4/2}|1 - z_4|^{k_2k_4/2}, \tag{5.1}$$

where $k_1, \ldots, k_4$ are momenta and κ the string coupling constant, z_4

is, as stated above, a complex variable. The transition to a quadratic extension, $\mathbb{Q}_p(\sqrt{\tau})$, of $\mathbb{Q}_p$ (the details of which we do not dwell upon here) then produces the p-adic string amplitude

$$A_{4,\tau}^{(p)}(k_1k_4, k_2k_4) = \kappa^2 \int_{\mathbb{Q}_p(\sqrt{\tau})} dt\rho_a^{(p)}(t)\,|t\bar{t}|_p^{-1}\,\rho_b^{(p)}(1-t)\,|(1-t)(1-\bar{t})|_p^{-1}, \tag{5.2}$$

which can be exactly calculated to yield

$$A_{4,\tau}^{(p)} = \lambda^2 \prod_{x=s,t,u} \frac{1-p^{-(x+16)/8}}{1-p^{(x+8)/8}}. \tag{5.3}$$

In this expression we have put τ (the number around which the quadratic extension is formed) equal to p since it turns out to be the interesting case, λ is the coupling constant times a numerical factor. This result can now be generalized to N-point functions in a straightforward manner. The crucial feature here, which makes p-adic strings interesting, is that these amplitudes can be calculated exactly. A connection with ordinary string amplitudes can be made by factorizing these [86] into an infinite product of p-adic amplitudes, thus these constitute an adelic integral. The motivations behind this branch of string theory is quite simply, simplicity. The prospect of feasible calculations is simply too good to reject, and although the interest in p-adic strings has cooled substantially in recent years, the connections to ordinary string theory makes it impossible to completely reduce the status of this theory to that of a toy-model. The reader is referred to [84, 86, 87, 88, 85] for further material on p-adic string theory.

Another interesting use for p-adic numbers in physics is that presented in [89, 90]. Here the resulting amplitudes are no longer real/complex but p-adic, and the motivations is not merely about calculability but stems from (slightly circumstantial) facts about the spacetime itself. Volovich argues that since the archimedean principle seems to break down at the Planck-scale perhaps we should consider the possibility of constructing a spacetime over a non-archimedean field. The main candidate of fields, he argues, is the field of p-adic numbers $\mathbb{Q}_p$. Volovich takes the ordinary Veneziano amplitude

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}, \tag{5.4}$$

and replaces it with a p-adic amplitude by replacing the momentum vectors making up the arguments s and t by vectors in a p-adic space $\mathbb{Q}_p^D$ and the gamma functions by the Morita gamma functions Γ_p, this getting

$$A_p(s,t) = \frac{\Gamma_p(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma_p(-\alpha(s)-\alpha(t))}. \tag{5.5}$$

But the considerations of Volovich does not stop with string theory, indeed he suggests a transfer of much of the modern day formalism to a p-adic spacetime and since their first considerations in the late eighties this project has gained some support.

So, whereas the p-adic string of Freund et al. is due to a relations between real and p-adic numbers the conjecture of Volovich et al. is due to a theoretical observation of the geometry of space and time. Which of these two approaches (if any) is then the right one? one might ask oneself. The string with the p-adic worldsheet merely presents a tool for making calculations that are insurmountable in the real case, there is no actual claim of the strings being observable phenomena. Volovich on the other hand makes a very strong claim about the structure of spacetime itself, and although the foundation upon which he bases is argument seems stable enough (smoothness breaking down below the Planck length) it is questionable if one can draw the conclusions that he draws (that the correct approach is to exchange the base field to a p-adic one).

The way that p-adic numbers surface in the material covered in chapter 4 is very much similar to the way that it appears in Freund's p-adic strings. We calculate the instanton measure by means of a p-adic method in much the same way that we can express a string amplitude in terms of p-adic string amplitudes. In both p-adic string theory and the algebraic approach to M-theory, p-adic analysis is nothing more than a mathematical tool, the end result is always real(or complex). We can use this tool (in any area of physics) on the merits of the theorem due to Ostrowski saying that there exists no other norms other than the p-adic and real one. What separates the case presented in this thesis from that of p-adic string theory is that we have a very poor understanding of what the corresponding 'real' technique should be for calculating the instanton measure, so the use of a p-adic method is not only warranted but necessary.

5.4 The Algebraic Approach

The algebraic approach described in chapter 4 is far from a "closed chapter". Many avenues still remain to be explored. Focusing for the moment on the project outlined in [16] and disregarding from the possible spin-offs, one notes that it remains yet to be completed. The representation needed has been known of for many years. The spherical vectors were calculated in [17] and the p-adic spherical vectors in [18]. All the building blocks are at hand and what remains is to assemble them into an object carrying the desired properties of invariance. There are several important questions that one might ask oneself before commencing the completion of this construction. How should we proceed? What can we, both physically and mathematically, expect? How would we proceed once we have this object at hand and how can we learn anything from it?

Presumably one should, to attempt an answer to the first question, start by taking the p-adic functions from [18] and turn them into summation measures, thus ridding the subsequent calculations from any p-adic elements. After this one should probably attempt to perform the integrations which requires the distribution, thus spurring its creation. After this we would hopefully have our much desired theta series. Although as is well known , the unwelcome arrival of 'problems' seldom fail to occur.

What then, can we expect from these theta series. Mathematicaly it is somewhat unclear, and physically it is completely unknown. Our lack of understanding for the M-theoretical degrees of freedom coming into play here is directly mirrored in our lack of predictions regarding the physical contents of the theta series. As to how we should proceed once we have these objects at hand we can rely on our old friend number theory. Resumming and expanding should hopefully enable us to recognize different parts of the expressions as related to the objects we do know to exist in M-theory (membranes, 5-branes, KK6-branes, etc.). Ideally we will also be able to spot both conjectured and unexpected relationships between these; perhaps forcing us to reconsider our view on these M-theoretical objects. Very recently there appeared a note [91] on a subject best described as a spin-off. We have mentioned very little about the cubic forms in table 4.5 that were utilized in the construction of the minimal representation. In the note [91] these are used to create cubic free field theories, a subject and an approach that holds some great promises for future work.

What would also be interesting is to consider more closely from a physical point of view; the construction. We have already talked about the use of p-adic numbers, but we might also benefit from examining the physical intuition (or lack thereof) lurking behind the scenes. We are making a conjecture based mainly on a highly non-trivial mathematical structure, and then we proceed to prove this by, in a construction, making use of representation theory. This is a long way from Galileo rolling marbles down a slide.

We conclude this section by expanding some more on the exceptional dual pair as promised in the text.

In table 4.3 we included the $d+1=6$ case and observed that the relevant dual pair in that case did not include a factor $\mathbb{R}^+$. This means that we get rid of the troublesome integration over the volume-factor. This yield a manifestly finite integral and thus the need for a cut-off disappears. It is interesting that this case coincides with the level of compactification where 5-brane instantons come into play. With our limited understanding of the dynamics behind the different terms in (4.45) it is difficult to say anything about whether or not there is any 5-brane contribution there. It would be great if the theta series corresponding to the $d+1=6$ case would actually reproduce the full amplitude, in which case we could perhaps see explicitly the relation between the membrane and 5-brane or the possible containment of the 5-brane dynamics within the (BPS) membrane degrees of freedom, hinting at the conjectured "fundamentality" of these.

Closer speculation and hopefully more concrete results are left for future work.

A

Brief Introduction to Supersymmetry and Supergravity

This appendix is a very brief introduction to supersymmetry and supergravity. We only barely scratch the surface of this vast area of modern physics here, for more extensive reviews see [92, 93, 94, 95, 96] (supersymmetry), [96] (supergravity).

A.1 The Wess-Zumino Model

Supersymmetry (SUSY) is an extension of special relativity that introduces a symmetry between bosonic and fermionic states. The idea of such a symmetry first arose in the early 70's and has since then developed into one of the main interests of modern theoretical physics to a stage where we now take it for granted. Nevertheless experiments have during this time also reached a stage where it is now possible to experimentally verify SUSY (though not at the energy scale in which it appears in string/M-theory).

We will introduce the concept of supersymmetry by means of an example, the Wess-Zumino model in a 4-dimensional spacetime. This is a theory of two real scalar (actually scalar and pseudoscalar respectively) fields A and B, a spinor (which we take to be Dirac for now) ψ and two other real scalar (and pseudoscalar respectively) fields F, G which we will find to be auxiliary later on. We form the Lagrangian

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu A \partial^\mu A - \frac{1}{2}\partial_\mu B \partial^\mu B - \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi + \frac{1}{2}F^2 + \frac{1}{2}G^2, \qquad \text{(A.1)}$$

where γ^μ are Dirac gamma matrices (in the chiral representation)

$$\gamma^\mu = i \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \mu = 0, 1, 2, 3 \tag{A.2}$$

$$\begin{cases} \sigma^\mu = (\mathbb{1}, \sigma^i) \\ \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i) \end{cases}, i = 1, 2, 3, \tag{A.3}$$

and $\bar{\psi}$ is the Dirac conjugate of ψ, defined by

$$\bar{\psi} = \psi^\dagger \gamma^0. \tag{A.4}$$

Now we introduce a transformation that interchanges the bosonic fields with the fermionic ones (and vice versa), we choose

$$\delta A = i\bar{\varepsilon}^a \psi_a, \tag{A.5}$$

and

$$\delta B = i\bar{\varepsilon}\gamma^5 \psi, \tag{A.6}$$

where

$$\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3, \tag{A.7}$$

is inserted because B should be a pseudo-scalar. The transformation of the spinor becomes

$$\delta\psi_a = -\partial_\mu (A + i\gamma^5 B)_a^b (\gamma^\mu \varepsilon)_b + ((F + i\gamma^5 G)\varepsilon)_a \tag{A.8}$$

and consequently

$$\delta\bar{\psi} = \bar{\varepsilon}\gamma^\mu \partial_\mu (A + i\gamma^5 B) + \bar{\varepsilon}(F + i\gamma^5 G). \tag{A.9}$$

The transformations of the fields F and G become

$$\delta F = -i\bar{\varepsilon}\gamma^\mu \partial_\mu \psi, \tag{A.10}$$
$$\delta G = \bar{\varepsilon}\gamma^5 \gamma^\mu \partial_\mu \psi, \tag{A.11}$$

and this along with the field equations

$$\Box A = 0 \quad , \quad \Box B = 0, \tag{A.12}$$

$$\gamma^\mu \partial_\mu \psi = 0, \tag{A.13}$$

$$F = 0 \quad , \quad G = 0, \tag{A.14}$$

implies that F and G are auxiliary fields that take the theory off-shell when we consider them to be non-zero. However, without the auxiliary fields the supersymmetry algebra

$$[\delta_1, \delta_2]A = -2i(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)\partial_\mu A = 2\varepsilon^\mu P_\mu, \tag{A.15}$$

$$\varepsilon^\mu = \bar{\varepsilon}_2\gamma^\mu\varepsilon_1 \quad , \quad P_\mu = -i\partial_\mu, \tag{A.16}$$

would not close without the use of the equations of motion (and also the off-shell degrees of freedom would not match). Checking the invariance of the action under the supersymmetry transformations one arrives at the conclusion that δ is a symmetry if and only if we have

$$\bar{\varepsilon}\psi = \bar{\psi}\varepsilon, \tag{A.17}$$

which is equivalent to saying that the spinor ψ is a Majorana spinor and not a Dirac spinor. Hence we have to redefine ψ in light of this new fact, this is a direct consequence of supersymmetry. Equation (A.15) leads us to define

$$\delta = \bar{\varepsilon}^a Q_a, \tag{A.18}$$

where Q_a are the supersymmetry generators, generating supersymmetry transformations in the same way that P_μ generates translations and $M^{\mu\nu}$ Lorentz rotations. The supersymmetry algebra can now be written as

$$\{Q_a, \bar{Q}_b\} = 2i(\gamma^\mu)_{ab}\partial_\mu. \tag{A.19}$$

Interaction- and mass-terms can now be created, but no general method exists to do this in the formalism we have presented here, one has to rely on a trial-and-error approach. Because of this and the fact that we want to find a way to better depict supersymmetry, we introduce the superspace formalism.

We already know how to work with coordinate representations of P_μ and $M^{\mu\nu}$

$$P_\mu = -i\partial_\mu, \tag{A.20}$$

$$M_{\mu\nu} = -\frac{1}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu), \tag{A.21}$$

and now we wish to realize the generator Q_a in the same way. But since Q_a satisfies anti-commutation relations we need anti-commuting coordinates in order to make this realization work. So we define a superspace made up by the coordinates (x^μ, θ^a), satisfying

$$\theta^a\theta^b = -\theta^b\theta^a, \tag{A.22}$$

$$\theta^a\theta^a = 0, \tag{A.23}$$

where θ^a is a four-component Majorana spinor (due to the fact that Q_a is a Majorana spinor). In the following we will write the anti-commuting coordinates θ^a in terms of its component (two-component) Weyl spinors $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$, so the superspace will be made up by the directions $z^M = (x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. The supersymmetry generator

$$Q_a = \begin{pmatrix} Q_\alpha \\ \bar{Q}_{\dot{\alpha}} \end{pmatrix}, \tag{A.24}$$

can now be realized as

$$\begin{aligned} Q_\alpha &= \partial_\alpha - i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, && \text{(A.25)} \\ \bar{Q}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu, && \text{(A.26)} \end{aligned}$$

so we see that just as Lorentz rotations and translations, supersymmetry is now a geometrical transformation in superspace. By switching sign on the second term in (A.25) and (A.26) we get a different operator that is still supersymmetric, this is the supersymmetry-covariant derivative

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, && \text{(A.27)} \\ \bar{D}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu. && \text{(A.28)} \end{aligned}$$

This operator will play an important role when we define superfields below, and also when defining extended actions (with mass-terms and interactions etc.). Now we wish to write down a general field in this space, a superfield. This can be done by expanding in the θ-coordinate, such a power-series terminates quickly due to the (Grassmann) properties of θ. Generally we have

$$\begin{aligned} \Phi(z^M) &= a(x) + \theta^\alpha\lambda_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\bar{\lambda}'^{\dot{\alpha}}(x) + \frac{1}{2}\theta^\alpha\theta^\beta m_{\alpha\beta}(x) + \\ &+ \frac{1}{2}\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}n^{\dot{\alpha}\dot{\beta}}(x) + \frac{1}{2}\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}v_\mu(x) + \frac{1}{2}\theta^2\bar{\theta}_{\dot{\alpha}}\bar{\chi}'^{\dot{\alpha}}(x) + \\ &+ \frac{1}{2}\bar{\theta}^2\theta^\alpha\chi_\alpha(x) + \frac{1}{4}\theta^2\bar{\theta}^2 d(x), \end{aligned} \tag{A.29}$$

(with $\theta^2 = \theta^\alpha\theta_\alpha$). It turns out that the complex scalar field $\Phi(z^M)$ does not correspond to any irreducible representation of the supersymmetry algebra and consequantly we need to find some condition that projects the field Φ into such a representation. It can be shown that there are two different conditions that we could impose, either we could demand that the superfield be real, whereupon it is called a vector-superfield. We will not use this condition here, instead we will demand that our superfield be chiral, which is the same as saying that it should satisfy

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{A.30}$$

This constraint is trivially consistent under supersymmetry since D anti-commutes with Q. The constraining turns the superfield (A.29) into

$$\begin{aligned}\tilde{\Phi}(z^M) &= \phi(x) + \theta\lambda(x) + i\theta\sigma^\mu\theta\partial\mu\phi(x) + \frac{1}{2}\theta^2 F(x) + \\ & i(\theta\sigma^\mu\bar{\theta})(\theta\partial_\mu\lambda(x)) + \frac{1}{2}\theta^2\bar{\theta}^2\Box\theta(x), \qquad \text{(A.31)}\end{aligned}$$

this can be seen either by expanding the field and imposing the constraint or by preforming a similarity trnasformation, S, on the field and then imposing the constraint which now reads

$$\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}(S\Phi(x^M)) = 0. \qquad \text{(A.32)}$$

In order to create superspace actions we have to clear up what it means to integrate over the anti-commuting coordinates. The measure is defined by

$$\int (d\theta)^a\theta_b = \delta^a_b, \qquad \text{(A.33)}$$

$$\int (d\bar{\theta})^{\dot{a}}\theta_{\dot{b}} = \delta^{\dot{a}}_{\dot{b}}, \qquad \text{(A.34)}$$

$$\int d\theta = 0, \qquad \text{(A.35)}$$

normalized as

$$\int d^2\theta\theta^2 = 1, \qquad \text{(A.36)}$$

$$\int d^2\bar{\theta}\bar{\theta}^2 = 1, \qquad \text{(A.37)}$$

thus only terms quadratic in both θ and $\bar{\theta}$ survive the $d^4xd^2\theta d^2\bar{\theta}$ integration.

With this in mind we try to write down a superspace action for the Wess-Zumino model, forming the product $\tilde{\Phi}\tilde{\Phi}^*$ reveals

$$\begin{aligned}S &= -\int d^4xd^2\theta d^2\bar{\theta}\tilde{\Phi}\tilde{\Phi}^* = \\ &= \int d^4x(-\partial\phi\partial\phi^* + i\lambda\sigma^\mu\partial_\mu\lambda - FF^*), \qquad \text{(A.38)}\end{aligned}$$

after integrating out the θ-dependence. Thus defining

$$\phi = \frac{1}{\sqrt{2}}(A + iB), \qquad \text{(A.39)}$$

$$F = \frac{1}{\sqrt{2}}(F + iG), \qquad \text{(A.40)}$$

$$\frac{\lambda_\alpha}{\lambda^{\dot{\alpha}}} = \psi, \qquad \text{(A.41)}$$

we retrive our old component-field action.

As we now have constructed the free Lagrangian in superspace the other possible terms may be written down. we have a mass-term

$$S_m = m \int d^4x (d^2\theta \tilde{\Psi}^2 + \text{h.c.}), \tag{A.42}$$

and an interaction term

$$S_g = \frac{2g}{3!} \int d^4x (d^2\theta \tilde{\Phi}^3 + \text{h.c.}), \tag{A.43}$$

both of which can readily be put in component form with our previous definitions of the fields ϕ, F and λ. Also with superspace techniques the equations of motion for both the free and the coupled theory can be retrieved. This concludes our little treatise of this simple model and we refer the reader to the references given in the beginning of this chapter for further material on supersymmetry.

A.2 Supersymmetry Multiplets and BPS States

The multiplets (irreducible representations of the supersymmetry algebra) of a theory, with N supersymmetry

$$\{Q^A_\alpha, \bar{Q}^B_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{AB}, \tag{A.44}$$
$$\{Q^A_\alpha, Q^B_\beta\} = \{\bar{Q}^A_{\dot{\alpha}}, \bar{Q}^B_{\dot{\beta}}\} = 0, \tag{A.45}$$

is created by letting the fermionic creation and annihilation operators, made up by the supercharges, act on the (Clifford) vacuum. Let us take the example of massive one-particle states in the rest frame, $P = (-M, 0, 0, 0)$, of a four-dimensional theory. The supersymmetry algebra takes the form

$$\{Q^A_\alpha, \bar{Q}^B_{\dot{\alpha}}\} = 2M\delta_{\alpha\dot{\alpha}}\delta^{AB}, \tag{A.46}$$
$$\{Q^A_\alpha, Q^B_\beta\} = \{\bar{Q}^A_{\dot{\alpha}}, \bar{Q}^B_{\dot{\beta}}\} = 0, \tag{A.47}$$

(we have set the central charges to zero here). We can define creation and annihilation operators by rescaling according to

$$a^A_\alpha = \frac{1}{\sqrt{2M}} Q^A_\alpha \quad , \quad (a^A_\alpha)^\dagger = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot{\alpha}A}. \tag{A.48}$$

These operators satisfy the algebra

$$\{a^A_\alpha, (a^B_\beta)^\dagger\} = \delta^\beta_\alpha \delta^A_B, \tag{A.49}$$
$$\{a^A_\alpha, a^B_\beta\} = \{(a^A_\alpha)^\dagger, (a^B_\beta)^\dagger\} = 0. \tag{A.50}$$

We build states by acting on the vacuum, $|0\rangle$. A general state is given by

$$|n\rangle^{\alpha_1\ldots\alpha_n}_{A_1\ldots A_n} = \frac{1}{\sqrt{n}}(a^{A_1}_{\alpha_1})^\dagger \ldots (a^{A_n}_{\alpha_n})^\dagger |0\rangle \,. \tag{A.51}$$

Each pair of indices on this state takes $2N$ values (A_n runs from 1 to N and α_n denotes which one of the two possible Weyl component spinors we are refering to) and consequently n must be less than or equal to $2N$. For any n we can choose the indices in $\binom{2N}{n}$ different ways, so from this we see that the size of the multiplet (or dimension of the representation) becomes

$$\sum_{n=0}^{2N} \binom{2N}{n} = 2^{2N}. \tag{A.52}$$

We call this multiplet the fundamental matter multiplet (granted that $|0\rangle$ is a unique vacuum), and it consists of 2^{2N-1} fermionic states and 2^{2N-1} bosonic states.

Looking instead at the massless multiplet ($P^2 = 0$) we choose the frame, $P = (-E, 0, 0, E)$, where the supersymmetry algebra reduces to

$$\{Q^A_\alpha, \bar{Q}^B_{\dot{\alpha}}\} = 2\begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}\delta^{AB}, \tag{A.53}$$

$$\{Q^A_\alpha, Q^B_\beta\} = \{\bar{Q}^A_{\dot{\alpha}}, \bar{Q}^B_{\dot{\beta}}\} = 0. \tag{A.54}$$

Since we see from this that Q^A_2 and $\bar{Q}^A_{\dot{2}}$ totally anti-commute we set them to zero and attain only N creation and annihilation operators, given by

$$a^A = \frac{1}{\sqrt{2E}}Q^A_1 \quad , \quad (a^A)^\dagger = \frac{1}{\sqrt{2E}}\bar{Q}_{\dot{1}A}, \tag{A.55}$$

and satisfying the algebra

$$\{a^A, (a^B)^\dagger\} = \delta^A_B, \tag{A.56}$$

$$\{a^A, a^B\} = \{(a^A)^\dagger, (a^B)^\dagger\} = 0. \tag{A.57}$$

A general state is given by

$$|n\rangle_{A_1\ldots A_n} = (a^{A_1})^\dagger \ldots (a^{A_n})^\dagger |0\rangle \,, \tag{A.58}$$

and the dimension of this representation (or size of the multiplet),

$$\sum_{n=0}^{N} \binom{N}{n} = 2^N, \tag{A.59}$$

becomes substantially smaller than in the massive case.

Finally we come to the case with non-zero central charge which we again study in the rest frame. The algebra becomes (setting $P^2 = -M^2$)

$$\begin{aligned} \{Q_\alpha^A, \bar{Q}_{\dot{\alpha}}^B\} &= 2M\delta_{\alpha\dot{\alpha}}\delta^{AB}, && \text{(A.60)} \\ \{Q_\alpha^A, Q_\beta^B\} &= \varepsilon_{\alpha\beta} Z^{AB}, && \text{(A.61)} \\ \{\bar{Q}_{\dot{\alpha}}^A, \bar{Q}_{\dot{\beta}}^B\} &= \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}^{AB}. && \text{(A.62)} \end{aligned}$$

The central charges Z^{AB} (anti-symmetric in the indices A, B) commutes with all the other generators and can thus be brought into block diagonal form by a $U(N)$ transformation U_B^A

$$\tilde{Z}^{AB} = U_C^A U_D^B Z^{CD}, \tag{A.63}$$

consequently (for the case where N is even)

$$\tilde{Z} = \varepsilon \otimes D, \tag{A.64}$$

where D is a diagonal $\frac{N}{2} \times \frac{N}{2}$-matrix with (real) eigenvalues Z_m and ε is the two-dimensional Levi-Cevita symbol. If we split the indices $A = (a, m)$, $B = (b, n)$ with $a, b = 1, 2$, $m, n = 1, \ldots, \frac{N}{2}$, and perform a $U(N)$ transformation on the supercharges as well, $\tilde{Q}_\alpha^A = U_B^A Q_\alpha^B$, we can rewrite the algebra as

$$\begin{aligned} \{\tilde{Q}_\alpha^{am}, \tilde{\bar{Q}}_{\dot{\alpha}}^{bn}\} &= 2M\delta_{\alpha\dot{\alpha}}\delta^{ab}\delta^{mn}, && \text{(A.65)} \\ \{\tilde{Q}_\alpha^{am}, \tilde{Q}_\beta^{bn}\} &= \varepsilon_{\alpha\beta}\varepsilon^{ab}\delta^{mn} Z_n, && \text{(A.66)} \\ \{\tilde{\bar{Q}}_{\dot{\alpha}}^{am}, \tilde{\bar{Q}}_{\dot{\beta}}^{bn}\} &= \varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{ab}\delta^{mn} Z_n. && \text{(A.67)} \end{aligned}$$

We can construct fermionic creation and annihilation operators as linear combinations of these supersymmetry charges

$$\begin{aligned} a_\alpha^m &= \frac{1}{\sqrt{2}} \left(\tilde{Q}_\alpha^{1m} + \varepsilon\alpha\beta \tilde{Q}_\beta^{2m} \right), && \text{(A.68)} \\ b_\alpha^m &= \frac{1}{\sqrt{2}} \left(\tilde{Q}_\alpha^{1m} - \varepsilon\alpha\beta \tilde{Q}_\beta^{2m} \right), && \text{(A.69)} \end{aligned}$$

and similarly for the conjugates $(a_\alpha^m)^\dagger$, $(b_\alpha^m)^\dagger$, these operators satisfy the algebra

$$\begin{aligned} \{a_\alpha^m, a_\beta^n\} = \{a_\alpha^m, b_\beta^n\} &= \{b_\alpha^m, b_\beta^n\}, && \text{(A.70)} \\ \{a_\alpha^m, (a_\beta^n)^\dagger\} &= \delta_{\alpha\beta}\delta^{mn}(2M + Z_n), && \text{(A.71)} \\ \{b_\alpha^m, (b_\beta^n)^\dagger\} &= \delta_{\alpha\beta}\delta^{mn}(2M - Z_n). && \text{(A.72)} \end{aligned}$$

The unitarity of our theory now requires the right-hand side of these equations to be non-negative which enforces

$$M \geq \max \left\{ \frac{Z_n}{2} \right\}, \tag{A.73}$$

this is a so called Bogomolnyi (or BPS) bound. If this equivalence is saturated ($M = 2Z$) for some number of eigenvalues Z_n then the corresponding creation and annihilation operators vanish. Thus if r eigenvalues saturated the bound (with $0 \leq r \leq N/2$) then $2r$ of the b-operators vanish and $2N - 2r$ creation and annihilation operators remain. In the maximal case when all of the b-operators vanish ($r = N/2$) we are left with a multiplet of the same size as the massless multiplet (but still with mass) as can easily be seen by setting $Z = 2M$ in the above algebra.

The analysis done in this thesis does however concern amplitudes or terms in the effective action and not particluar states. Amplitudes (we interchange the words 'term' and 'amplitude' freely here) are said to be BPS-saturated if they recieve contributions exclusively from BPS states, in section 4.1 we see explicitly that this is the case for R^4 terms in the effective IIB action. It falls outside the scope of this thesis to discuss the general details of BPS-saturated amplitudes and the non-renormalization theorems that these obey, some of these concepts are dealt with in chapter 3 and 4, here we merely state some facts about these amplitudes that are relevant to the considerations of this thesis.

BPS-saturated amplitudes:

- Obtain perturbative corrections exclusively from BPS-states.
- Obtain perturbative corrections at only one order in the perturbation theory.
- Obtain non-perturbative corrections from instantons preserving the same amount of supersymmetry as the perturbative ones.

A.3 Instanton Solutions in Type IIB Supergravity

We will not pay any attention to the full IIB supergravity theory here, since the solution we seek [97] is one where only two scalar fields, the dilaton, ϕ, and and the Ramon-Ramond scalar, a, as well as the metric, are non-trivial. The Lagrangian for these fields can be written as

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial a)^2, \tag{A.74}$$

(with a metric of signature $(-++\ldots+)$) or by defining

$$F_{\mu_1\mu_2\ldots\mu_9} = e^{2\phi}\epsilon^{\mu}_{\mu_1\mu_2\ldots\mu_9}\partial_\mu a, \tag{A.75}$$

as

$$\hat{\mathcal{L}} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(9!)}e^{-2\phi}F_{\mu_1\mu_2\ldots\mu_9}F^{\mu_1\mu_2\ldots\mu_9}. \tag{A.76}$$

We are interested in the equations of motion in a space with Euclidian signature. The equations that we obtain from the Euclidian version of (A.74) are equivalent to those obtained from the Lagrangian

$$\tilde{\mathcal{L}} = R - \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}e^{2\phi}(\partial\alpha)^2 \tag{A.77}$$

(with $\alpha = ia$), furthermore the action derived from this Lagrangian is equal to the action derived from (A.76) modulo unimportant surface terms (also note that $\hat{\mathcal{L}}$ does not change form under a Wick rotation). The Euclidean equations of motion are

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}(\partial_\mu\phi\partial_\nu\phi - e^{2\phi}\partial_\mu\alpha\partial_\nu\alpha) = 0,\\ &\nabla_\mu(e^{2\phi}\partial^\mu\alpha) = 0,\\ &\nabla^2\phi - e^{2\phi}(\partial\alpha)^2 = 0. \end{aligned} \tag{A.78}$$

We now require our solution to be BPS, i.e. preserve half of the supersymmetries. The condition for this, in the background we have chosen, turns out to be

$$d\alpha = \pm e^{-\phi}d\phi \quad , \quad g_{\mu\nu} = \eta_{\mu\nu}, \tag{A.79}$$

which, together with (A.78), implies

$$\partial^2(e^\phi) = 0. \tag{A.80}$$

This equation has a solution

$$e^\phi = (e^{\phi(r=\infty)} + \frac{c}{r^8}), \tag{A.81}$$

which is spherically symmetric and constitutes an instanton. The constant c is related to the R-R electric charge of the instanton

$$Q^{(-1)} = 2\pi n, n \in \mathbb{Z}, \tag{A.82}$$

by

$$c = \frac{3|n|}{\pi^{3/2}}. \tag{A.83}$$

This solution is now transformed from the Einstein frame (with $ds_E^2 = dx^2$) to the string frame (with the metric depending on the dilaton), here the metric is given by

$$ds^2 = e^{\phi/2} ds_E^2 = (e^{(r=\infty)} + \frac{c}{r^8})^{1/2}(dr^2 + r^2 d\Omega_9^2), \tag{A.84}$$

where $d\Omega_9^2$ is the $SO(9)$-invariant line element on S^9. This metric can be seen to be invariant under

$$r \longrightarrow (ce^{-\phi(r=\infty)})^{1/4}\frac{1}{r}, \tag{A.85}$$

and the configuration can thus be interpreted as an Einstein-Rosen bridge (wormhole) in spacetime. The charge $Q^{(-1)}$ of the instanton is interpreted as the charge flowing down the throat of the wormhole. This constitutes a violation of the conservation of charge in related physical processes. We shall not comment further on the interpretation of the D-instanton as a wormhole, instead refering the reader to [97] and references therein.

It is also important to note that a is the non-constant part of the field $C^{(0)}$ which in our solution can be written as

$$C^{(0)} = \chi + ia(r), \tag{A.86}$$

with χ constant. This field is the antisymmetric tensor field of Type IIB supergravity, it always appears in combination with the dilaton field as

$$\tau \equiv \tau_1 + i\tau_2 = C^{(0)} + ie^{-\phi}. \tag{A.87}$$

We will use this notation often in section 4.1.

A.4 Type IIB and (p,q)-strings

We make repeated use of, what we call, (p,q)-strings in chapter 3, these strings constitue an infinite family of solutions in Type IIB supergravity, a multiplet of $Sl(2,\mathbb{Z})$ [72].

As in the preceeding section we do not work with the full IIB supergravity action here, since we wish to study solutions that carry a particular charge we can throw away any of the fields that correspond to charges not carried by our solution. Subsequently, since our solution is a string and we know that p-branes carry charges that correspond to $(p+2)$-form field strengths, only charges generated by 3-forms will be of interest. The non-zero fields we write as the covariant action

$$S = \frac{1}{2\pi\kappa_{10}^2}\int d^{10}x\sqrt{-g_B^{(10)}}(R + \frac{1}{4}\mathrm{tr}(\partial\mathcal{M}\partial\mathcal{M}^{-1}) - \frac{1}{12}H^T\mathcal{M}H), \tag{A.88}$$

where

$$\mathcal{M} = e^{\phi} \begin{pmatrix} |\tau|^2 & C^{(0)} \\ C^{(0)} & 1 \end{pmatrix}, \tag{A.89}$$

and

$$H = \begin{pmatrix} H^{(1)} \\ H^{(2)} \end{pmatrix} \tag{A.90}$$

, is a vector made up by the 3-form field strengths ($H = dB$) $H^{(1)}$ and $H^{(2)}$ carrying NS-NS and R-R charge respectively. This classical action is invariant under $Sl(2,\mathbb{R})$ acting as

$$\mathcal{M} \to \Lambda\mathcal{M}\Lambda^T \quad , \quad B \to (\Lambda^T)^{-1}B, \tag{A.91}$$

and since the transformation $(\Lambda^T)^{-1}$ mixes the NS-NS field $B^{(1)}$ with the R-R field $B^{(2)}$ we see that it becomes meaningless to talk about solutions carrying charge under only one of the fields. Thus we study solutions that carry charges (p,q) where p and q are integers (due to a Dirac quantization condition) and relatively prime (due to the fact that it can be shown that integers that are not relatively prime leads to solutions that are not stable, they will decompose into multiple strings, the number of which is given by $\gcd(p,q)$). Of the integers p and q at least one should also be positive since $(-p,-q)$ can be obtained from (p,q) by a reversal of the orientation $x^1 \to -x^1$.

The solution we seek can be obtained by extending a macroscopic string solution [98] of the equations of motion belonging to the theory

$$S_0 = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g^{(10)}}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-\phi}H^2\right), \tag{A.92}$$

which is the action consisting of the fields that all supergravity theories share in ten dimensions ($H = dB$ is a NS-NS field strength, we identify it with $H^{(1)}$).

The solution is given by

$$ds^2 = A^{-3/4}\left[-dt^2 + (dx^1)^2\right] + A^{1/4}d\mathbf{x}\cdot\mathbf{x}, \tag{A.93}$$

$$B_{01} = e^{2\phi} = A^{-1}, \tag{A.94}$$

with

$$A = 1 + \frac{Q}{3r^6}. \tag{A.95}$$

This is a static string oriented in the x^1-direction, we have denoted $\mathbf{x} = (x^2, x^3, \ldots, x^9)$, $r = \mathbf{x}\cdot\mathbf{x}$ and Q is the electric charge that our solution carries under B. The charge (p,q) in the solution we wish to find is measured in units of this Q.

The (p,q)-string solution is now obtained by making the ansatz

$$A_{p,q} = 1 + \frac{\alpha_{p,q}}{3r^6}, \tag{A.96}$$

where $\alpha_{p,q} = \Delta_{p,q}^{1/2} Q$ and the new tension of the string becomes $T_{p,q} = \Delta_{p,q}^{1/2} T$, where T is the tension of the former solution. By applying an $Sl(2,\mathbb{R})$ transformation and imposing the quantization condition we arrive at the solution

$$\begin{aligned}
ds^2 &= A_{p,q}^{-3/4}\left[-dt^2 + (dx^1)^2\right] + A_{p,q}^{1/2} d\mathbf{x}\cdot\mathbf{x}, && \text{(A.97)}\\
B_{01}^{(1)} &= e^{\phi_0}(p - qC_0^{(0)})\Delta_{p,q}^{-1/2} A_{p,q}^{-1}, && \text{(A.98)}\\
B_{01}^{(2)} &= e^{\phi_0}(q|\tau_0|^2 - pC_0^{(0)})\Delta_{p,q}^{-1/2} A_{p,q}^{-1}, && \text{(A.99)}\\
\tau &= \frac{pC_0^{(0)} - q|\tau_0|^2 + ipe^{-\phi_0} A_{p,q}^{1/2}}{p - qC_0^{(0)} + iqe^{-\phi_0} A_{p,q}^{1/2}}, && \text{(A.100)}
\end{aligned}$$

with

$$\Delta_{p,q} = e^{\phi_0}(qC_0^{(0)} - p)^2 + e^{-phi_0} q^2. \tag{A.101}$$

The τ_0 appearing in these expressions is

$$\lim_{r\to\infty} \tau = \tau_0 = C_0^{(0)} + ie^{-\phi_0}, \tag{A.102}$$

the asymptotic value of τ in a region very far from the string. We have to specify a τ_0 to get a complete solution since τ_0 is the vacuum in which the string lives, it can be shown that there are degeneracies for certain vacua. This fact and the issue of compactification (which leads to the important result that the scalar moduli of nine-dimensional Type IIB should be equated with the complex structure of the internal torus of M-theory on T^2) will not be dealt with here. We refer the reader to [72] for further details.

B

The Field of p-adic Numbers

To define the field of real numbers $\mathbb{R}$ (from the rationals $\mathbb{Q}$) we form Cauchy sequences or rational numbers and consider the elements of $\mathbb{R}$ to be equivalence classes of these sequences (see for example [99]). To this end we need a *norm* $||\ ||$on $\mathbb{Q}$.

Definition 1 *A norm is a map*

$$\|\ \| : \mathbb{K} \to \mathbb{R}_+, \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$$

such that $\forall\ x, y \in \mathbb{K}$

$$\|x\| = 0 \Leftrightarrow x = 0\ ,\ \|xy\| = \|x\| \cdot \|y\|\ ,\ \|x + y\| \leq \|x\| + \|y\|$$

And with the help of this norm we define Cauchy sequences as

Definition 2 *A sequence* $(x_n) \in \mathbb{K}$ *is said to be a Cauchy sequence with respect to a norm* $\|\ \|$ *if*

$$\forall \epsilon > 0 \exists M\ :\ m, n \geq M \Rightarrow \|x_m - x_n\| < \epsilon \tag{B.1}$$

In the above definitions $\mathbb{K}$ is any field but in our considerations it will be the rationals $\mathbb{Q}$. When *Cauchy-completing* $\mathbb{Q}$ to construct the reals $\mathbb{R}$ the norm is obviously the ordinary absolute value, but it is also possible to complete the set $\mathbb{Q}$ with another norm and thus end up with another field. Such a norm is the *p-adic norm*, $|\ |_p$, which can be defined for any prime p, and the field is the p-adic numberfield, $\mathbb{Q}_p$. To define the p-adic norm we first make the observation that given a prime p and an $x \in \mathbb{Q}$, there exists unique integers a, b, c, such that

$$x = \frac{a}{b} p^n\ ,\ (a, b) = (a, p) = (b, p) = 1\ b > 0.$$

We then define the p-adic norm as

Definition 3

$$|x|_p : \mathbb{Q} \to \mathbb{R}_+ : |x|_p = \begin{cases} \frac{1}{p^n} & , \textit{if } x \neq 0 \\ 0 & , \textit{if } x = 0 \end{cases}$$

In fact, this norm and our ordinary absolute value (in this context often denoted by $| \, |_\infty$) are the only non-trivial norms that exist on $\mathbb{Q}$ due to a theorem by Ostrowski. The next step is to define the set C_p of all Cauchy sequences with respect to our p-adic norm (the prime p is fixed) and then on this set define an addition and multiplication

Definition 4 *On the set*

$$C_p = \{(x_n) : (x_n) \textit{ is a Cauchy sequence with respect to } | \, |_p\} \tag{B.2}$$

we define a multiplication "·" and an addition "+" as

$$(x_n) \cdot (y_n) = (x_n \cdot y_n)$$
$$(x_n) + (y_n) = (x_n + y_n)$$

It is trivial to show that the sum and product of two Cauchy sequences is also a Cauchy sequence (the set C_p and the algebraic structure defined by the two operations above constitutes a commutative ring with unity). The final step in the construction of $\mathbb{Q}_p$ is to define an equivalence relation on C_p and identify the classes of equivalent sequences with the elements of $\mathbb{Q}_p$

Definition 5 *Two Cauchy sequences (x_n) and (y_n) are said to be equivalent ($(x_n) \sim (y_n)$) iff*

$$|x_n - y_n|_p \to 0 \textit{ as } n \to \infty$$

The set of all such equivalence classes with the above defined operations constitute a field denoted $\mathbb{Q}_p$ and called the p-adic numbers

There is an equivalent (and more elegant) way to define $\mathbb{Q}_p$ albeit less intuitive. The way to do this is to note that the set off all null-sequences, N_p (i.e. the sequences whose elements tend to zero with respect to $| \, |_p$), is a maximal ideal of the ring C_p, the quotient C_p/N_p is thus a field, which is exactly $\mathbb{Q}_p$. Another useful set to have is the set $\mathbb{Z}_p$ of p-adic integers defined as

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \tag{B.3}$$

which turns out to be the maximal compact subring of $\mathbb{Q}_p$ with an inclusion $\mathbb{Z} \to \mathbb{Z}_p$ (i.e. $\mathbb{Z}$ is a subring of $\mathbb{Z}_p$) which also generalizes to the whole field $\mathbb{Q}_p$ which has an inclusion $\mathbb{Q} \to \mathbb{Q}_p$ (the elements of $\mathbb{Q}$ are the constant sequences, i.e. all entries equal). We can represent a p-adic number x as a series

$$x = \sum_{n=n_{\min}}^{\infty} a_n p^n \ , \ a_n \in \{0, 1, 2, \ldots, p-1\} \ , \ n_{\min} \in \mathbb{Z}$$

This way to think of p-adic numbers also explains why this field of numbers was introduced in the first place. Writing functions as a series in irreducible (or "prime") polynomials (Laurent series) is very useful in complex analysis, but when we try to do this with rational numbers in number theory (the irreducible polynomials now corresponding to prime numbers) we find that the series does not converge with respect to the ordinary norm. It does however converge with respect to the p-adic norm!

What we want to do next is to perform analysis on $\mathbb{Q}_p$ and for this we need a topology, so we take the norm $| \ |_p$ and form *neighbourhoods*

Definition 6 *A neighbourhood of a point $a \in \mathbb{Q}_p$ is defined as*

$$D_a(r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\} \ , \ r \in \mathbb{R}_+.$$

These are the basis for open sets in $\mathbb{Q}_p$

(the reader not familiar with fundamental analysis and point-set topology should note that this is not the most general way of constructing a topology on a given set, one can do without the norm) It turns out that the neighbourhoods (and thus all open sets) are also closed, which means that our topology is *totally disconnected* meaning that every subset of $\mathbb{Q}_p$ with more than two points is disconnected (can be partitioned into two open sets not sharing any points). Now we have all the tools we need to construct an analysis, but before we do this we introduce yet another algebraic structure, the ring of *adeles*

Definition 7 *We define an adele x as an infinite sequence*

$$x = (x_\infty, x_2, x_3, \ldots, x_p, \ldots) \ , \ x_\infty \in \mathbb{Q}_\infty \ , \ x_p \in \mathbb{Q}_p$$

where at most a finite number of x_p's are not in $\mathbb{Z}_p$

The set $\mathbb{A}$ of all adeles form a ring under componentwise addition and multiplication (i.e. the same as for $\mathbb{Q}_p$) and the subset of elements in $\mathbb{A}$ with a multiplicative inverse is called the *ideles*.

What we want to do now is to construct an integration on the field of p-adic numbers (as well as the adeles as we will see later). The first thing to note is that $\mathbb{Q}_p$ has a, up to normalization, unique Haar measure (i.e. a translation invariant measure) that we normalize to that it takes the value 1 on $\mathbb{Z}_p$

$$\int_{\mathbb{Z}_p} dx = 1$$

Now the mathematical theory of integration on locally compact fields(in mathematical literature often called local fields) is rather large and complicated, I will therefore refrain from giving even the briefest introduction to the subject, instead only mentioning the concepts that are useful to us. But before we really get down to integration we have to define the concept of *character* and write down some of the characters that are interesting to us.

Definition 8 *An additive character ψ on $\mathbb{Q}_p$ is a mapping*

$$\psi : \mathbb{Q}_p \to \mathbb{C}^* \textit{ such that } \forall x, y \in \mathbb{Q}_p \ \psi(x+y) = \psi(x)\psi(y)$$

where $\mathbb{C}^*$ is the multiplicative group of nonvanishing complex numbers

We can now write down the most general additive character we have in $\mathbb{Q}_p$ which is

$$\psi_u(x) = e^{i2\pi[ux]} \ , \ u \in \mathbb{Q}_p$$

where (remembering our way to represent p-adic numbers explained above) we have that $[x]$ is the "fractional part"

$$[x] = \sum_{n=n_{\min}}^{-1} a_n p^n$$

of our series expansion

$$x = \sum_{n_{\min}}^{-1} a_n p^n + \sum_{n=0}^{\infty} a_n p^n \ , \ a_n \in \{0, 1, 2, \ldots, p-1\}$$

The next type of character we introduce is

Definition 9 *A multiplicative character ξ on $\mathbb{Q}_p$ is a mapping*

$$\xi : \mathbb{Q}_p^* \to \mathbb{C}^* \textit{ such that } \forall x, y \in \mathbb{Q}_p \xi(xy) = \xi(x)\xi(y)$$

Again we are interested in writing down the most general multiplicative character which takes the form

$$\xi_s(x) = |x|_p^s \theta(x)$$

where $\theta(x)$ is a character on the set $U_{\mathbb{Z}_p}$ of all non-zero elements in $\mathbb{Z}_p$ with inverses in $\mathbb{Z}_p$ (the set of *units*). Since we always work with explicit characters further explanation of these general forms of characters is not useful. More precisely the definitions of the characters above are from the additive and multiplicative groups of $\mathbb{Q}_p$ to the multiplicative group of all non-vanishing complex numbers.

We now turn to integration with the goal of illustrating the method as well as giving explicit examples of integrals. We have all the ingredients we need to write down an integral and so (without proof or definition of the integral since it is analgous to the real Riemann-integral case) we write down an integral and illustrate the techniques by examples.

Example 1 We start with a trivial example of integration of an additive character over the p-adic integers just to point out a result that is important to keep in mind

$$\int_{\mathbb{Z}_p} \psi_1(x)dx = \int_{\mathbb{Z}_p} e^{i2\pi[x]}dx = \int_{\mathbb{Z}_p} (1 + 2\pi i[x] + (2\pi i)^2 \frac{[x]^2}{2!} + \ldots)dx$$

but since x is a p-adic integer and hence can be written as $x = a_0 + a_1 p + a_2 p + \ldots$ it follows that $[x] = 0$ which gives the result

$$\int_{\mathbb{Z}_p} e^{2\pi i[x]}dx = 1$$

Example 2 Our next example illustrates the technique we use to compute integrals of multiplicative characters. We start by rewriting the intgral as

$$\int_{\mathbb{Z}_p} |x|_p^7 dx = \sum_{k=0}^{p-1} \int_{C_k} |x|_p^7 dx \tag{B.4}$$

where $C_k = \{x = a_0 + a_1 p + a_2 p^2 + \ldots \in \mathbb{Z}_p \ : a_0 = k\}$. Now C_k can be translated into C_m and so with the translation invariance of the measure we get that

$$\int_{C_k} 1 \cdot dx = \int_{C_m} 1 \cdot dx \ \forall k, m \in \{0, 1, \ldots, p-1\}.$$

This, together with the fact that $\mathbb{Z}_p = \bigcup_{k=0}^{p-1} C_k$ gives us that

$$1 = \int_{\mathbb{Z}_p} dx = \sum_{k=0}^{p-1} \int_{C_k} dx = p \cdot \int_{C_k} dx$$

and so

$$\int_{C_k} 1 \cdot dx = p^{-1}$$

Now one forms subsets of each C_k such that $C_k = \bigcup_{k'=0}^{p-1} C_{k,k'}$ and $C_{k,k'}$ is the subset of all $x \in \mathbb{Z}_p$ such that $a_0 = k$ and $a_1 = k'$. In analogy with the above result we get

$$\int_{C_{k,k'}} 1 \cdot dx = p^{-2}$$

And finally

$$\int_{C_{k_1,k_2,\ldots,k_r}} 1 \cdot dx = p^{-r}$$

which is all we need to calculate the integral of any well-behaved function. Applying this together with the fact that for all $x \in C_k$ with $k \neq 0$ we jave $|x|_p = 1$ and thereby get

$$\int_{\mathbb{Z}_p} dx|x|_p^7 = \sum_{k=0}^{p-1} \int_{C_k} dx|x|_p^7 = \frac{p-1}{p} + \int_{C_0} dx|x|_p^7.$$

Next we (as above) do the same for C_0 and get (for $x = C_{0,k}$, $k \neq 0$, $|x|_p = p^{-1}$)

$$\int_{C_0} dx|x|_p^7 = \sum_{k=0}^{p-1} \int_{C_{0,k}} dx|x|_p^7 = \frac{p-1}{p^2} p^{-7} + \int_{C_{0,0}} dx|x|_p^7$$

And for $C_{0,0}$

$$\int_{C_{0,0}} dx|x|_p^7 = \sum_{k=0}^{p-1} \int_{C_{0,0,k}} dx|x|_p^7 = \frac{p-1}{p^3} p^{-14} + \int_{C_{0,0,0}} dx|x|_p^7$$

Clearly the remaining integral goes to zero as the integration region shrinks and we are left with a geometric series

$$\int_{\mathbb{Z}_p} dx|x|_p^7 = \frac{p-1}{p} + \frac{p-1}{p^2} p^{-7} + \frac{p-1}{p^3} p^{-14} + \ldots$$

Which gives us the final result

$$\int_{\mathbb{Z}_p} dx|x|_p^7 = \frac{p-1}{p} \frac{1}{1-p^{-8}}$$

The examples above have been restricted to the integration region $\mathbb{Z}_p$ which is not adequate for our purpose. Our next objective is therefore to extend these methods of integration to the whole of $\mathbb{Q}_p$. We now denote the set of all p-adic numbers of the form $x = a_{-m}p^{-m} + a_{-m+1}p^{-m+1} + \ldots + a_0 + a_1 p \ldots$ by $K^{(-m)}$ and by a similar argument to that in the case of C_k, K_k^{-m} now corresponding to C_k, we find that

$$\int_{K^{-m}} dx \cdot 1 = p^m$$

We now see that $\mathbb{Q}_p = \bigcup_{m=-\infty}^{\infty} K^{-m}$ which from our result in Example 2 yields that

$$\int_{\mathbb{Q}_p} dx|x|_p^7 = \sum_{m=-\infty}^{\infty} \int_{K^{-m}} dx|x|_p^7 = \sum_{m=-\infty}^{\infty} \frac{p-1}{p} \frac{p^{m+7m}}{1-p^{-8}}$$

Which diverges just as in the case of real integration since the volume of $\mathbb{Q}_p$ is infinite just as the volume of $\mathbb{R}$.

Finally in this chapter we give some integrals that are examples of the p-adic Fourier transform. These were extremely important in the calculations that we performed in [24]. The proofs are left to the reader as an exercise

Example 3

$$\int_{|x|_p \leq |R|_p} dx\psi_u(x) = |R|_p \, \gamma_p(Ru), \tag{B.5}$$

Example 4

$$\int_{|x|_p = |R|_p} dx\, |x|_p \, \psi_u(x) = |R|_p \, \gamma_p(Ru) - \frac{1}{p} \, |R|_p \, \gamma_p(Rup), \tag{B.6}$$

Example 5

$$\int_{|x|_p < |R|_p} dx\, |x|_p \, \psi_u(x) = \frac{1}{p} \, |R|_p \, \gamma_p(Rup), \tag{B.7}$$

Example 6

$$I = \int_{|x|_p \leq |R|_p} |x|_p \, \psi(ux), \tag{B.8}$$

For readers wanting do go further than I have done here in the world of p-adic analysis I recommend [100], [85] and [101]

C

A Touch of Number Theory

The latter parts of this thesis carries with it a strong smell of number theory. In this appendix we review some of the background material that may come in handy while reading.

C.1 Eisenstein Series

Eisenstein series are the main characters of this thesis, this section contains some definitions and results that are, in general, related to physics and especially to our area of consideration. Let us turn to the general definition[1] definition of an Eisenstein series. The goal is to create an object that carries some invariance properties, so first we have to specify under what action our series should be invariant, generally we choose a group $G(\mathbb{Z})$ defining a symmetric space $K\backslash G(\mathbb{R})$, where K is the maximal compact subgroup of G. Denoting $\mathcal{R}$ some representation of G we can now write our Eisenstein series as

$$\mathcal{E}^{G(\mathbb{Z})}_{\mathcal{R};s}(g) = \sum_{m \in \Lambda_{\mathcal{R}} \backslash \{0\}} \delta(m \wedge m) \left[m \cdot \mathcal{R}^t \mathcal{R}(g) \cdot m \right]^{-s}, \tag{C.1}$$

where g is some element in $K\backslash G(\mathbb{R})$ and m is a vector in the integer lattice $\Lambda_{\mathcal{R}}$ transforming under $\mathcal{R}$. Physical amplitudes, which is what are trying to encode in these series, depend on scalar fields taking values in symmetric spaces, like $g \in K\backslash G(\mathbb{R})$. Furthermore the vectors m of the lattice $\Lambda_{\mathcal{R}}$ labels the BPS-states and $m \cdot \mathcal{R}^t \mathcal{R}(g) \cdot m$ gives their tension

[1] Actually in the mathematics literature a more general definition is used but we will not address it here since the definition (C.1) has a much clearer meaning in this context.

(or mass squared). The product $\wedge$ is integer valued and has the property of projecting the symmetric tensor product of the representation $\mathcal{R}$ with itself onto the highest irreducible component. We will not dwell upon the definition of this product here, it is sufficient to know that $m \wedge m = 0$ imposes the 1/2-BPS condition, the details become much clearer in the examples to follow.

Let us turn to the first example of Eisenstein series encountered in chapter 3, the S-symmetric R^4 coupling in ten-dimensional Type IIB

$$\mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};3/2} = \frac{\zeta(3)}{l_P^8} \sum_{(p,q)=1} \frac{1}{\mathcal{T}^3_{p,q}}, \tag{C.2}$$

with

$$\mathcal{T}_{p,q} = \frac{|p+q\tau|}{l_s^2}, \tag{C.3}$$

for a general s this series takes the form

$$\mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s} = \sum_{(m,n)\neq 0} \left[\frac{\tau_2}{|m+n\tau|^2}\right]^s . \tag{C.4}$$

Since this is uncompactified Type IIB the only existing moduli is the scalar moduli $\tau \in U(1)\backslash Sl(2,\mathbb{R})$ upon which the amplitude depends. $\mathbf{2}$ denotes the two-dimensional vector representation of $Sl(2)$ and (m,n) are vectors in the lattice transforming under $Sl(2)$. Looking at the expression (C.2) we can now identify all the constituents from the definition (C.1), the (p,q)-string solution of Type IIB is stable (BPS) if p and q are relatively prime integers so the sum over $(p,q)=1$ enforces the BPS-condition.

For general s, the Eisenstein series is determined by the fact that it satisfies the eigenvalue equation

$$\Delta_{U(1)\backslash Sl(2)}\ \mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s} = \frac{s(s-1}{2}\ \mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s}, \tag{C.5}$$

where

$$\Delta_{U(1)\backslash Sl(2)} = \frac{1}{2}\tau_2^2(\partial^2_{\tau_1} + \partial^2_{\tau_2}), \tag{C.6}$$

however the Eisenstein series does not give the full spectrum of this Laplacian, cusp forms are also possible solutions. In the Type IIB case that we are considering at the moment, contribution from cusp forms is ruled out by proving that s is uniquely determined to be 3/2 (this fixes the solution of the eigenvalue equation).

The extension of this result to $Sl(d,\mathbb{Z})$ Eisenstein series is relevant to compactifications of general diffeomorphism invariant theories on T^d.

When compactified the theory acquires an $Sl(d,\mathbb{Z})$ symmetry with respect to the metric under which all amplitudes are invariant. This leads us to define an Eisenstein series

$$\mathcal{E}^{Sl(d,\mathbb{Z})}_{\mathbf{d};s} = \hat{\sum}_{m^i} \left[m^i g_{ij} m^j\right]^{-s}, \tag{C.7}$$

where g_{ij} ($g = e^t e$, for the veilbein e) is the metric on the internal torus, $\mathbf{d}$ is the fundamental representation of $Sl(d,\mathbb{Z})$ (the hat on the sum denotes the exclusion of $m^i = 0$). The metric is only defined up to orthogonal rotations $SO(2,\mathbb{R})$ so the moduli should live in $SO(d,\mathbb{R})\backslash Sl(d,\mathbb{R})$[2] and our Eisenstein series turns out to satisfy

$$\Delta_{Sl(d)}\ \mathcal{E}^{Sl(d,\mathbb{Z})}_{\mathbf{d};s} = \frac{s(d-1)(2s-d)}{2d}\ \mathcal{E}^{Sl(d,\mathbb{Z})}_{\mathbf{d};s}, \tag{C.8}$$

where

$$\Delta_{Sl(d)} = \frac{1}{4} g_{ik} g_{jl} \frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} - \frac{1}{4d}\left(g_{ij}\frac{\partial}{\partial g_{ij}}\right)^2 + \frac{d+1}{4} g_{ij} \frac{\partial}{\partial g_{ij}}. \tag{C.9}$$

This result can be generalized to other, higher dimensional representations, we will not pursue this line here, nor will we dwell upon the properties of (C.7) regarding analyticity and large volume behavior, referring the reader to [23] for further details.

Instead we move on to the case of $SO(d,d,\mathbb{Z})$ Eisenstein series which arise for example in Type II string theories where we call it T-duality. In stringtheory however we have objects that transform as spinor representations and therefore we have to consider Eisenstein series in these representations as well. We begin with the vector representation though. The metric on the internal torus and the background field, $B_{\mu\nu}$, take values in the symmetric space

$$[SO(d,\mathbb{R}) \times SO(d,\mathbb{R})]\backslash SO(d,d,\mathbb{R})/SO(d,d,\mathbb{Z}), \tag{C.10}$$

and in a suitable gauge we can pick the veilbein e such that the mass matrix for the BPS states becomes

$$M(\mathbb{V}) = e^t e = \begin{pmatrix} g^{-1} & g^{-1}B \\ -Bg^{-1} & g - Bg^{-1}B \end{pmatrix}. \tag{C.11}$$

The mass squared reads

$$M^2(\mathbb{V}) = m \cdot e^t e \cdot m = \tilde{m}_i g^{ij} \tilde{m}_j + m^i g_{ij} m^j, \tag{C.12}$$

[2] This is a slight simplification, other symmetries come into play and there is an explanation in terms of the Iwasawa decomposition of the group G but the explanation of this falls outside the scope of this thesis.

where m_i and m^i are Kaluza-Klein momenta and winding modes respectively, and

$$\tilde{m}_i = m_i + B_{ij} m^j. \tag{C.13}$$

The BPS condition in this case reads

$$m_i m^i = 0. \tag{C.14}$$

The spinor case becomes more complex, the mass matrix is given by the spinor or conjugate spinor representation of the group element e. The spinor representation (denoted $\mathbb{S}$) describes charges corresponding to wrapping numbers along odd cycles of T^d and the conjugate spinor representation ($\mathbb{C}$) to even cycles respectively. We give their mass squared and the BPS conditions here (up to a power l_P^{d-8} in the T-symmetric Planck length) only for completeness

$$\begin{aligned}
M^2(\mathbb{S}) &= \frac{1}{V_d}\left[(\tilde{m}^i)^2 + \frac{1}{3!}(\tilde{m}^{ijk})^2 + \frac{1}{5!}(\tilde{m}^{ijklm})^2 + \ldots\right], && \text{(C.15)}\\
\tilde{m}^i &= m^i + \frac{1}{2} m^{jki} B_{jk} + \frac{1}{8} m^{jklmi} B_{jk} B_{lm} + \ldots, && \text{(C.16)}\\
\tilde{m}^{ijk} &= m^{ijk} + \frac{1}{2} m^{lmijk} B_{lm} + \ldots && \text{(C.17)}\\
\tilde{m}^{ijklm} &= m^{ijklm} + \ldots, && \text{(C.18)}
\end{aligned}$$

satisfying the BPS conditions

$$\begin{aligned}
& m^{[i} m^{jkl]} = 0, && \text{(C.19)}\\
& m^{i[jk} m^{lmn]} + m^{i[jklm} m^{n]} = 0, && \text{(C.20)}\\
& m^{ij[k} m^{lmnpq} = 0. && \text{(C.21)}
\end{aligned}$$

And for the conjugate spinor representation

$$\begin{aligned}
M^2(\mathbb{C}) &= \frac{1}{V_d}\left[(\tilde{m})^2 + \frac{1}{2!}(\tilde{m}^{ij})^2 + \frac{1}{4!}(\tilde{m}^{ijkl})^2 + \ldots\right], && \text{(C.22)}\\
\tilde{m} &= m + \frac{1}{2} m^{ij} B_{ij} + \frac{1}{8} m^{ijkl} B_{ij} B_{kl} + \ldots, && \text{(C.23)}\\
\tilde{m}^{ij} &= m^{ij} + \frac{1}{2} m^{klij} B_{kl} + \ldots, && \text{(C.24)}\\
\tilde{m}^{ijkl} &= m^{ijkl} + \ldots, && \text{(C.25)}
\end{aligned}$$

with the BPS conditions

$$\begin{aligned}
& m^{[ij} m^{kl]} + m m^{ijkl} = 0, && \text{(C.26)}\\
& m^{i[j} m^{klmn]} + m m^{ijklmn} = 0, && \text{(C.27)}\\
& m^{ij} m^{klmnpq} + m^{ij[kl} m^{mnpq]} = 0. && \text{(C.28)}
\end{aligned}$$

The Eisenstein series can now be written

$$\mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathcal{R};s} = \hat{\sum_{m}} \delta(m \wedge m) \left[M^2(\mathcal{R})\right]^{-s}, \tag{C.29}$$

with $M^2(\mathcal{R})$ being (C.12), (C.16) and (C.23) for the $\mathbb{V}$, $\mathbb{S}$ and $\mathbb{C}$ representations respectively and $\delta(m \wedge m)$ summarizing the BPS constraints in each case. This series satisfies the eigenvalue equation

$$\Delta_{SO(d,d)}\ \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathcal{R};s} = \Delta(\mathcal{R},s)\ \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathcal{R};s}, \tag{C.30}$$

where

$$\Delta_{SO(d,d)} = \frac{1}{4} g_{ik} g_{jl} \left[\frac{\partial}{\partial g_{ij}} \frac{\partial}{\partial g_{kl}} + \frac{\partial}{\partial B_{ij}} \frac{\partial}{\partial B_{kl}}\right] + \frac{1}{2} g_{ij} \frac{\partial}{\partial g_{ij}}, \tag{C.31}$$

and the eigenvalues are given by

$$\Delta(\mathbb{V},s) = s(s-d+1) \quad , \quad \Delta(\mathbb{S},s) = \Delta(\mathbb{C},s) = \frac{sd(s-d+1)}{4}. \tag{C.32}$$

A perfect example of physic where these series come into play is the calculation of 1/2-BPS one-loop amplitudes in T^d-compactified string theory, a case which we have encountered in chapter 4. These amplitudes can often be written

$$I_d = 2\pi \int_{\mathcal{F}} \frac{d^2\tau}{t_2^2} Z_{d,d}(g;B;\tau), \tag{C.33}$$

where

$$Z_{d,d}(g;B;\tau) = V_d \sum_{m^i,n^i} \exp\left[-\frac{\pi}{\tau_2}(m^i + \tau n^i)(g_{ij} + B_{ij})(m^j + \bar{\tau} n^j)\right], \tag{C.34}$$

is the partition function of the compactification lattice. It can be shown that for $d = 1, 2$ this amplitude can be calculated and rewritten to yield

$$I_d = 2\ \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbb{S};1} + 2\ \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbb{C};1}. \tag{C.35}$$

Equivalently in the $d \geq 3$ case we get

$$I_d = 2\frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-2}}\ \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbb{V};\frac{d}{2}-1} = 2\ \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbb{S};1} = 2\ \mathcal{E}^{SO(d,d,\mathbb{Z})}_{\mathbb{C};1}. \tag{C.36}$$

For some dimensionalities these statements are proved and for others they remain (very well founded) conjectures.

The discussion of Eisenstein series is continued in section 4.2 due to the close relation between U-duality, exceptional Eisenstein series and the rest of the material presented in that chapter.

C.2 Additions concerning $\mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s}(\tau)$

We shall expand upon the Eisenstein series $\mathcal{E}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s}(\tau)$ a bit here. We begin by proving the relation (4.88). It is easily realized that $\mu_s(N)$ is multiplicative, i.e. that $\mu_s(MN) = \mu_s(M)\mu_s(N)$ (just calculate for some MN and s to see this), this along with the fact that $N \in \mathbb{Z}$ (meaning that N is a product of prime numbers) means that we only have to prove (4.88) for $N = p^k$, a power of some prime p. We get

$$\mu_s(p^k) = \sum_{n|p^k} n^{-2s+1} = \sum_{n=0}^{k} (p^n)^{-2s+1} = \sum_{n=0}^{k} p^{nt}, \tag{C.37}$$

where we have set $t = -2s+1$. Turning our attention to the left-hand side of (4.88) we see that the only contributing terms are those where $N \int \mathbb{Q}$ is a p-adic integer for all p, the set of N's satisfying this it $\mathbb{Z}$, we can also skip the infinite product over p since the only contribution comes from the prime p whose power we have fixed N to. Thus we can write

$$\sum_{N\in\mathbb{Z}} \sum_{n=0}^{k} p^{nt} = \sum_{N\in\mathbb{Z}} \frac{1 - p^t \left|p^k\right|_p^{-t}}{1 - p^{-t}}, \tag{C.38}$$

implicating

$$\sum_{n=0}^{k} p^{nt} = \frac{1 - p^{t(1+k)}}{1 - p^{-t}}, \tag{C.39}$$

calculating the geometric series on the left-hand side we see that this is indeed so.

Next we turn to a representation in which to construct this Eisenstein series which is better suited for those less physically inclined or who simply have not come in touch with Eisenstein series as physical amplitudes before.

By taking the intuitive (yet reducible) representation (4.77) and constraining it to homogeneous even functions

$$\phi(x, y) = \lambda^{2s}\phi(\lambda x, \lambda y), \tag{C.40}$$

and setting $y = 1$ we obtain an irreducible representation

$$\rho \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \phi(x) \to \phi(x + t), \tag{C.41}$$

$$\rho \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} : \phi(x) \to e^{-2st}\phi(e^{-2t}x), \tag{C.42}$$

$$\rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \phi(x) \to x^{-2s}\phi(-1/x), \tag{C.43}$$

in which the spherical vector becomes

$$f_s(x) = (x^2+1)^{-s}. \tag{C.44}$$

The action of the representation element corresponding to the group element g_τ on the spherical vector is

$$\begin{aligned}(x^2+1)^{-s} \to \tau_2^{-s}((x\tau_2^{-1})^2+1)^{-s} \to \\ \to \tau_2^{-s}((x+\tau_1)^2\tau_2^{-2}+1)^{-s} = \left[\frac{\tau_2}{(x+\tau_1)^2+\tau_2^2}\right]^{-s},\end{aligned} \tag{C.45}$$

and from the form of the Eisenstein series 4.76 we see that the summation measure should be

$$\mu_s(n) = \sum_{n\neq 0} n^{-2s}. \tag{C.46}$$

This gives us, according to our construction, the series

$$\int dx \sum_{m\in\mathbb{Z}} \mu_s(n)\delta(x-m/n)\left[\frac{\tau_2}{(x+\tau_1)^2+\tau_2^2}\right]^{-s}, \tag{C.47}$$

an expression in which the observant reader notices that we lack some of the terms that are in the original Eisenstein series. This is due to the fact that we loose information when we restrict the functions of the representation to homogeneous functions and set $y=1$. This summation measure can be gotten from the p-adic spherical vector corresponding to (C.44) and turns out to be

$$f_{p,s}(x) = \max(1, |x|_p)^{-s}. \tag{C.48}$$

That this is so can be proved by methods similar to those used in the previous example.

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Notation

The symbols are listed here, first in accordance to the section in which they first appear and then in the order they appear within the section. After each description the equation numbers are listed when a strict definition exists.

2 The Supermembrane I: General Theory and Problems

2.1 The Bosonic Membrane

ξ^i	:	Worldsheet coordinates
X^μ	:	Embedding fields
D	:	Dimension of spacetime
T	:	Tension of the membrane
g_{ij}	:	Induced worldvolume metric
g	:	Determinant of induced metric
ϕ_i	:	Classical constraints
$\{\cdot,\cdot\}$	:	Poisson bracket (2.11)
X^+, X^-	:	Light-cone coordinates (2.19)
$\vec{X}, X^a$	:	Residual coordinates (2.20)
g_{rs}	:	Induced metric w.r.t. $\vec{X}$ (2.22)
u_r	:	First column in full (light-cone) metric g_{ij} (2.22)
g'	:	Determinant of residual metric (2.25)
$\{\cdot,\cdot\}'$	:	Poisson bracket (2.36))
Y_α	:	Basis functions of the bosonic membrane
$f_{\alpha\beta\gamma}$	:	Structure constants of APD algebra (2.44)
$T_a^{(N)}$	:	Basis matrices of $SU(N)$
$f_{abc}^{(N)}$	:	Structure constants of $SU(N)$ (2.45)
$\Delta_{(N)}$	:	Differential operator
$Z_Q(t)$	:	Partition function of quantized Hamiltionian (2.50)
$Z_{cl}(t)$	:	Partition function of classical Hamiltionian (2.50)
Δ_j^λ	:	Cube in $\mathbb{R}^\nu$

2.2 Adding Supersymmetry

ψ^μ : Worldvolume spinor
χ_i : Rarita-Schwinger spinor
D_i : Covariant derivative (2.63)
γ^i : Gamma matrices (2.61)
ϵ^{ijk} : Totally anti-symmetric tensor
e_i^a : Worldwolume dreibeins
e : Determinant of dreibein e_i^a
$\bar{\psi}$: Adjoint spinor
ω_i^a : Connection field (2.64)
Z^M : Superspace coordinates (2.72)
θ^α : Spacetime spinors
Γ^μ : Spacetime Dirac matrices
ϵ : Constant anti-commuting spacetime spinor
Π^A : Flat superspace one-forms (2.74)
e_M^A : Flat superspace veilbein
${}^*\Pi^A$: Pull-back of Π^A (2.75)
Π^μ, Π^α : Bosonic and fermionic part of Π^A resp. (2.76)
Π_i^μ, Π_i^α : Bosonic and fermionic part of pullback (2.77)
$\mathbf{h}$: Flat superspace 4-form (2.79)
${}^*\mathbf{b}$: Worldvolume 3-form (2.80)
b_{ijk} : Worldvolume 3-form coefficients (2.82)
$\kappa^\alpha(\xi)$: Parameter in the local fermionic symmetry
Γ : Matrix, part of projection factor in action (2.87)
E_i^A : SUSY invariant worldvolume field in curved spacetime (2.93)
$\mathbf{H}$: Curved superspace 4-form
T_{AB}^C : Supergravity torsion tensor
$X^\pm$: Supermembrane light-cone coordinates (2.99)
$\Gamma^\pm$: Light-cone Dirac matrices (2.101)
$g_{rs}, \bar{g}_r s$: Supermembrane induced residual metric (2.104)
$\bar{g}$: Determinant of induced residual metric
Δ : Light-cone part of determinant (2.105)
$\{\cdot, \cdot\}$: Poisson bracket (2.108)
$\omega(\xi)$: Metric on the membrane

$Y_A(\xi)$: Basis functions of embedding fields

f_{AB}^{C} : Area-preserving algebra structure constants (2.114)

G_Λ : Finite Lie group

ξ_F : Spinor component of toy-model/membrane wave function

ψ_0 : Ground state toy-model/membrane wave-function

χ : Tuning-function of the toy-model/membrane wave function