Thesis for the degree of Master of Science

Superconformal Theories in Three Dimensions

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Abstract

This thesis consists of six chapters, more than half of which are introductory texts. The main content of the master thesis project is presented in **section** 3.3, **chapter** 5 and **chapter** 6.

The problem investiged in this thesis is related to three-dimensional superconformal gauge field theories. After introducing the superconformal Lie algebras, we turn to D=3 topological supergravity with Chern-Simons terms and then develop an extended pure supergravity which possesses many interesting properties. After that, three-dimensional superconformal theories of the kind originally suggested by J.Schwarz are discussed. The explicit expressions of BLG/ABJM actions are given, as well as the novel three-algebras.

Finally, we couple $\mathcal{N} = 6$ topological supergravity to the ABJM matter action, following standard techniques. Even though we haven't yet finished the verification of SUSY invariance, some arguments are given to explain why this action should be the complete Lagrangian.

The thesis also contains some discussions on Chern-Simons terms and the U(1) gauge field.

Key words:
Superconformal symmetry, supergravities in 3 dimensions, string theory, M-theory, Branes, Chern-Simons terms.

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Contents

Chapter 1

Introduction

1.1 History

The main goal of fundamental physics is to combine the four interactions and all fundamental particles into one theory. After the development of quantum field theory, the Standard Model (SM) was formulated in 1970s and proven to be quite successful. So far, the Standard Model, which describes the electroweak and strong interactions, has been consistent with all experimental results to an astonishing accuracy except for the massive neutrinos. The great achievement of the Standard Model made most physicists believe that the so-called 'final theory' should be described by point-particle quantum field theories. However, for a reasonable fundamental theory of particle physics, there are still some problems with the Standard Model, such as many undetermined parameters and fine-tuning mechanisms. More importantly, it doesn't include gravity.

In 1970s, supersymmetry (SUSY) was discovered and immediately used to address the hierarchy problem as well as some other problems in the Standard Model. Since supersymmetry imposes stringent restrictions on field theories by reducing some arbitrariness in its construction, it is generally expected that nature should respect such symmetries. However, they have to be broken at low energies, according to the fact that no superpartners of these ordinary particles have been observed yet. In 1976, some physicists realized that one could construct so-called 'supergravity' theories to incorporate the principles of general relativity by using local supersymmetry. Then there appeared lots of papers exploring various possible forms of extended supergravities in 1980s, in order to embed gravity into quantum field theory. Among them, supergravity with conformal supersymmetry was of special interest.

However, many shortcomings were soon discovered. One of the crucial problems is that almost all such models were found to be non-renormalizable at the quantum level, which means that a supergravity theory is unlikely to be a fundamental theory. Furthermore, most of them are not phenomenologically acceptable. Take the N=8 maximal supergravity in four-dimensional Minkowski space for example. There are two major problems within it: one is the UV-divergences beyond the 3-loop order[1]; the other is that

[1] In 2007, some theorists found a three-loop cancelation of N=8 supergravity in a novel way, suggesting

it's phenomenologically unacceptable, e.g. chiral fermions are not allowed in it.

Meanwhile, string theory developed rapidly, since it necessarily includes the graviton as a massless spin-2 particle. It first appeared as an attempt to describe the strong interaction, and then was realized to be a possible candidate of quantum gravity. After two "string theory revolutions", the latest version of string theory we call M-theory, after E. Witten proving that the five 10-dimensional superstring theories are actually different special cases of this underlying M-theory. At the same time, he argued that the low-energy limit of M-theory should be described by eleven-dimensional supergravity. It contains M2-branes and M5-branes, but the relationship between them and the exact roles they each play are poorly understood.

In the very end of the last century, the AdS/CFT correspondence was discovered by J.Maldacena, which made people begin to think about the whole theory in a completely new way. The duality of the AdS/CFT correspondence is not only useful in string theory, but can also be applied widely in condense matter physics and strongly coupled QCD. And according to the AdS/CFT correspondence, the solution of M-theory with the most symmetric choice in four dimensions, $AdS_4 \otimes S^7$, should be dual to a three-dimensional gauge theory with the superconformal symmetry $OSp(8|4)$.

In 2007, two research groups, Bagger/Lambert and Gustavsson, independently obtained the classical theory with exactly the symmetry required, making it possible to study M2-branes for the first time. Then another group constructed three-dimensional conformal theories with only six supersymmetries and SU(4) R-symmetry describing stacks of M2-branes. Interestingly, both of them contain a Chern-Simons term. Now these results are known as the BLG and ABJM actions, respectively. Most work presented in this thesis is to study them to explore their possible relevance in M-theory.

1.2 Outline

This thesis, which consists of 6 chapters, has the main purpose to develop a extended pure supergravity theory, and couple it to the superconformal matter theories mentioned above.

Chapter 2 contains an introduction to both bosonic symmetries of spacetime and their supersymmetric extensions, as well as the associated infinitesimal transformations. We also give the definitions of Poincaré and conformal algebras, as well as super-Poincaré and superconformal algebras.

Chapter 3 contains a brief account of the relevant supergravity theories. Beginning with the supersymmetic extension of topologically massive gravity, the three-dimensional superconformal gravities with only Chern-Simons-like terms are discussed for $\mathcal{N} = 1$, $\mathcal{N} = 6$, and $\mathcal{N} = 8$.

Chapter 4 begins with the recent development of three-dimensional superconformal M2 theories. We review the motivations of exploring superconformal Chern-Simons theories, and how they led to the BLG/ABJM actions. Some studies of three-algebras then

that N=8 supergravity may be a perturbatively finite theory of quantum gravity.

follow.

Chapter 5 and **Chapter 6** mainly describe the thesis work done in collaboration with my supervisor Prof. Bengt E.W. Nilsson. We couple the $\mathcal{N} = 6$ topologically pure supergravity to the ABJM matter theory following standard techniques. It may be interesting to study such a gauged three-dimensional theory as the geometric description of M2-branes.

The **Appendix** gives some of the details of the verification of supersymmetry of both the pure conformal supergravity and the gauged ABJM theories.

Chapter 2

Spacetime symmetries

As is well known, a symmetry of a physical system is a mathematical, or physical structure which leaves the system invariant under certain changes. This is usually expressed in terms of an action that is invariant under some transformations. There are two kinds of such symmetries: one is continuous (such as rotation and translations), which can be expressed as Lie groups[1]; the other is discrete (such as parity and time reversal), which can be expressed as finite groups. In quantum field theories, continuous symmetries are further classified as the spacetime symmetries (i.e. continuous spacetime symmetries) and internal symmetries, which are not affecting space and time, e.g. the color symmetry in QCD.

The purpose of this chapter is to briefly introduce spacetime symmetries in four-dimensional quantum field theories. Many of this presentation is following ref.[1], which works in six dimensions. Some other useful review articles on superalgebra in supersymmetry (SUSY) are [2],[3] and [4].

2.1 Bosonic spacetime symmetries

In 1967, Coleman and Mandula [5] investigated how large the spacetime symmetries can be in a relativistic quantum field theory. After restricting themselves to a finite number of different particles in a multiplet and to four dimensions, they stated that the largest Lie algebra of the S-matrix must be a direct product of the Poincaré algebra with a compact internal symmetry algebra (i.e. commuting with the Poincaré algebra). Only if there are no massive particles, the conformal algebra is allowed. The reason is that for a conformally invariant quantum field theory the mass spectra would become continuous according to the scale invariance if there are any massive particles.

Because the Coleman-Mandula theorem came before the appearance of supersymmetry, it only treated bosonic symmetries. Even so, the theorem is very important in particle physics. Symmetries and algebras involved in the theorem will be shown below. To simplify the issue, we only work in four dimensions in this chapter, where coordinates are denoted as x^μ with $\mu = 0, 1, 2, 3$. Of course x^0 is the time coordinate and x^i $(i = 1, 2, 3)$

[1]Especially when looking at only bosonic symmetries.

are spatial coordinates. Also the Einstein summation notation are always used unless otherwise stated.

2.1.1 Poincaré symmetries

As we know, Einstein's special relativity combines space and time into a single concept, namely *space-time*. One of the principles in special relativity is that the speed of light has the same value for all observers in all inertial frames, which implies the proper time interval must be invariant under certain coordinate transformations.

Now, we explore the symmetry which keep the infinitesimal proper time interval

$$d\tau^2 = -dx^\mu dx_\mu = -\eta_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

invariant under the global transformation

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{2.2}$$

$\eta_{\mu\nu}$ is the metric, which is used to lower or raise space-time indices. In the simplest case, i.e. special relativity, the metric is set to be the Minkowski metric, being diagonal with elements (-1,1,1,1) in Cartesian coordinates. The transformation (2.2) is called Lorentz transformation, and the invariance of (2.1) then gives the relationship:

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}. \tag{2.3}$$

The set of all Lorentz transformations (2.2) is called the Poincaré group. Its subgroup with $a^\mu = 0$ is called the Lorentz group (sometimes called the homogeneous Lorentz group), which is isomorphic to the special orthogonal group SO(1,3). By imposing the additional constraints

$$\Lambda^\rho{}_\mu \geq 0; \quad \mathrm{Det}\Lambda = 1 \tag{2.4}$$

on the Lorentz group, one obtains a subgroup, namely the proper Lorentz group. Mostly, we deal with the proper Lorentz group, and unless otherwise stated, all Lorentz transformations are assumed to satisfy the constraint (2.4).

It is easy to find that the Poincaré group is the semidirect product of the Abelian space-time translational group and the Lorentz group. It is obvious that only the latter is non-Abelian. The corresponding Lie algebra generators are then denoted by P_μ and $M_{\mu\nu}$, respectively. In a coordinate representation, they are expressed as

$$P_\mu = \partial_\mu, \tag{2.5}$$

$$M_{\mu\nu} = x_{[\mu}\partial_{\nu]}. \tag{2.6}$$

Hence the infinitesimal coordinate transformations are generated explicitly by the following commutators

$$\delta x^\mu = [a^\nu P_\nu, x^\mu] = a^\nu \tag{2.7}$$

for the translations P_μ; and

$$\delta x^\mu = -[\omega^{\nu\rho} M_{\nu\rho}, x^\mu] = \omega^{\mu\nu} x_\nu \tag{2.8}$$

for the Lorentz rotations $M_{\mu\nu}$. Again, we note that the infinitesimal coordinate transformations of the Poincaré group leave the proper time interval in Minkowski space invariant. Mathematically, one could say that the geometry of Minkowski space is defined by the Poincaré group.

Following the representation above, we introduce the Poincaré algebra, i.e. the Lie algebra of the Poincaré group:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu[\rho}M_{\sigma]\nu} - \eta_{\nu[\rho}M_{\sigma]\mu}, \\
[P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu}P_{\rho]}, \\
[P_\mu, P_\nu] &= 0.
\end{aligned} \tag{2.9}$$

Actually, the last commutator can be modified by an independent term:

$$[P_\mu, P_\nu] = \frac{1}{2R^2}M_{\mu\nu}, \tag{2.10}$$

giving rise to the Anti-de Sitter algebra. When $R \to \infty$, we recover the Poincaré algebra. Furthermore, if P_μ's are considered as 'rotations' by defining $M_{4\mu} = -M_{\mu 4} = RP_\mu$, we will have generators:

$$M_{\hat{\mu}\hat{\nu}} = -M_{\hat{\nu}\hat{\mu}}, \text{with } \hat{\mu} = 0, 1, 2, 3, 4, \tag{2.11}$$

all of which satisfy

$$[M_{\hat{\mu}\hat{\nu}}, M_{\hat{\rho}\hat{\sigma}}] = \eta_{\hat{\mu}[\hat{\rho}}M_{\hat{\sigma}]\hat{\nu}} - \eta_{\hat{\nu}[\hat{\rho}}M_{\hat{\sigma}]\hat{\mu}}, \tag{2.12}$$

with the metric $diag(-,+,+,+,-)$. One then finds that the commutation relations guve exactly the definition of a special orthogonal algebra, $SO(2,3)$.

2.1.2 Conformal symmetry

Now, we consider the conformal symmetry, which is conserved in many physical systems, such as electrodynamics involving only massless particles. Such a symmetry can be derived from relaxing the constraint on the infinitesimal proper time interval slightly leading to a larger set of symmetry transformations.

By definition the conformal symmetry means that $d\tau^2$ (2.1) is invariant up to a scalar factor under the general coordinate transformations (i.e. diffeomorphisms):

$$d\tau^2 \to \Omega(x)d\tau^2,\ \Omega(x) = e^{\omega(x)}. \tag{2.13}$$

The group formed by transformations which respect this property is called the conformal group, where the word *conformal* means it preserves angles.

Starting from the infinitesimal transformation $x^\mu \to x^\mu + \xi^\mu(x)$, we can easily find that the conformal symmetry is determined by the solution to the 'conformal Killing equation':

$$\partial_{(\mu}\xi_{\nu)} - \frac{1}{d}\eta_{\mu\nu}\partial\cdot\xi = 0. \tag{2.14}$$

In dimensions d>2, the conformal algebra is finite-dimensional, while the case of two dimensions is much more complex. As we work only in 4 dimensions in this chapter, our concern is the general solution for higher dimensions:

$$\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + (c^\mu x^2 - 2c\cdot x x^\mu), \tag{2.15}$$

where a^μ is the parameter for the transformation P_μ, while $\omega^\mu{}_\nu$ is related to the Lorentz transformation $M_{\mu\nu}$. Turning to the new ingredients, λ is associated to dilatation D, and c^μ is the parameter of the 'special conformal transformation' K_μ. The full set of conformal transformations is hence expressed as:

$$\delta_C = a^\mu P_\mu + \omega^{\mu\nu} M_{\mu\nu} - \lambda D + c^\mu K^\mu, \tag{2.16}$$

and it follows that the transformation of the infinitesimal proper time interval becomes

$$d\tau^2 \to [1 + 2(\lambda - 2c \cdot x)]d\tau^2, \tag{2.17}$$

which means that $\Omega(x) = 1 + 2(\lambda - 2c \cdot x)$ according to (2.13).

Similarly to the Poincaré symmetry, the dilatation D and the special conformal transformation K_μ can be expressed in a coordinate representation:

$$\begin{aligned} D &= -x \cdot \partial \\ K_\mu &= x \cdot x \partial_\mu - 2x_\mu x \cdot \partial. \end{aligned} \tag{2.18}$$

Using these transformations, one easily derives the commutation relations of the conformal algebra in accordance with the differential expressions mentioned above:

$$\begin{aligned} [P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} P_{\rho]} & [P_\mu, D] &= -P_\mu \\ [K_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} K_{\rho]} & [K_\mu, D] &= K_\mu \\ [P_\mu, K_\nu] &= 4M_{\mu\nu} + 2\eta_{\mu\nu} D & [M_{\mu\nu}, D] &= 0 \\ [P_\mu, P_\nu] &= 0 & [K_\mu, K_\nu] &= 0 \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -\eta_{\rho[\mu} M_{\nu]\sigma} + \eta_{\sigma[\mu} M_{\nu]\rho}, \end{aligned} \tag{2.19}$$

which is actually the SO(2,d) algebra with dimension d=4. In fact, one can define:

$$\mathbf{M}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} M^{\mu\nu} & \frac{1}{4}(P^\mu - K^\mu) & \frac{1}{4}(P^\mu + K^\mu) \\ -\frac{1}{4}(P^\mu - K^\mu) & 0 & -\frac{1}{2}D \\ -\frac{1}{4}(P^\mu + K^\mu) & \frac{1}{2}D & 0 \end{pmatrix}$$

with the metric $\eta^{\mu\nu} = diag(-, +, +, ..., +, -)$. Comparing with the Anti-de Sitter algebra in d+1 dimensions, one finds the duality

$$\text{Conf}_d = \text{AdS}_{d+1}, \tag{2.20}$$

which is an essential ingredient in the AdS/CFT correspondence.

2.2 Supersymmetry

We have discussed all bosonic symmetries which are allowed in quantum field theories according to the Coleman-Mandula theorem. However, the fact that the Coleman-Mandula theorem only involves Lie groups of symmetries implies that fermionic symmetries were not considered and could be added to extend the set of possible 'algebras'. This was later done by Haag, Łopuszańsky and Sohnius, who discussed also supersymmetry, which takes fermions to bosons and bosons to fermions. The idea of supersymmetry was invented by P.Ramond [6] and by two Russian physicists Y.Golfand and E.Likhtman in 1971 [7], but didn't become well known until J.Wess and B.Zumino [8] extended it to four dimensions. For a panoramic overview of supersymmtry, see textbooks [9] and [10].

Now let's turn to fermionic symmetries. To describe fermionic symmetries, Grassmann algebras (i.e. odd Grassmann algebras), such as anticommuting numbers or operators, have to be introduced into the theory.

Define θ^a, a=1,2,3...N, as a set of generators for an algebra satisfying the anticommutative relations:

$$\{\theta^a, \theta^b\} = 0, \quad \text{for all a,b.} \tag{2.21}$$

This algebra is called the (odd) Grassmann algebra and will be denoted as Q (or K in the superconformal case). According to the definition, $(\theta^a)^2$=0 for any a. One can now introduce transformations, as well as associated (anti-)commutators.

Considering

$$\delta(\epsilon) = \epsilon^A Q_A \tag{2.22}$$

for anticommuting parameters ϵ^A and anticommuting operators Q_A, the commutator relation is

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \epsilon_2^B \epsilon_1^A (Q_A Q_B + Q_B Q_A) = \epsilon_2^B \epsilon_1^A \{Q_A, Q_B\}. \tag{2.23}$$

Here one can introduce a general notation to express both commutators and anticommutators:

$$[A, B\} = A \cdot B - (-1)^{AB} B \cdot A. \tag{2.24}$$

In this expression, the [.,.} bracket denotes an anticommutator if both A and B are fermionic, otherwise it means a commutator. In other words, one may take A(or B)=0 in the exponent if the corresponding operator is bosonic, and A(or B)=1 if it is fermionic. Besides, the algebras should also satisfy the super-Jacobi identity, which is a generalization of the bosonic Jacobi identity for Lie algebras:

$$(-1)^{AC}[[A, B\}, C\} + (-1)^{AB}[[B, C\}, A\} + (-1)^{BC}[[C, A\}, B\} = 0, \tag{2.25}$$

or in another easier form:

$$[[A, B\}, C\} - [A, [B, C\}\} + (-1)^{AB}[B, [A, C\}\} = 0. \tag{2.26}$$

The super-Jacobi identity actually leads to a lot of extensions of algebras. One of the most important applications is the result found by Haag, Łopuszańsky and Sohnius.

As a generalization of the Coleman-Mandula result, Haag, Łopuszańsky and Sohnius considered the largest possible symmetries of relativistic field theories again now allowing also for fermionic symmtries. Using both commuting and anticommuting symmetry generators, they obtained the super-Poincaré algebra, and for massless fields only, the superconformal algebra. Another important result is that the super-Poincaré algebra may contain 'central charges', which by definition commute with all algebra generators[2].

The topic of this section is to extend the Coleman-Mandula theorem to construct a complete set of 'superalgebras' including supersymmetry, which are often referred to as the 'Haag-Łopuszańsky-Sohnius theorem'.

[2]However, we won't say much about central charge in this thesis.

2.2.1 Poincaré supersymmetry

By analyzing further the constraints that follow from the super-Jocobi identity, one can now characterize the supersymmetric extension of the bosonic algebras. Firstly, we discuss the super-Poincaré algebra (or Poincaré superalgebra), as well as the Poincaré supersymmetry. The most crucial ingredient of such an algebra is that the odd generators square to the spacetime translations. Such an interplay of fermionic and bosonic algebras acts against the previous Coleman-Mandula theorem, which assumes 'internal symmetries' can't interact with spacetime symmetry algebras.

Introducing the supersymmetric generators $Q^{\alpha i}$, $\bar{Q}^i_{\dot{\alpha}}$ into 4-dimensional consistent quantum field theory, one could have

$$\begin{aligned}[M_{\mu\nu}, Q_{\alpha i}] &= -\frac{1}{4}(\gamma_{\mu\nu})_\alpha^\beta Q_{\beta i}, \\ [M_{\mu\nu}, \bar{Q}^i_{\dot{\alpha}}] &= \frac{1}{4}\bar{Q}^i_{\dot{\beta}}(\gamma_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}},\end{aligned} \tag{2.27}$$

which tell us that the supersymmetries are spinors under the Lorentz group. It follows that the supersymmetry generators carry a representation of the bosonic symmetry algebra. The generators Q's and $\bar{Q}$'s, the supercharges, are odd Grassmann operators.

Here we use μ, ν, ρ... to denote spacetime indices, α, β... to denote left-handed Weyl spinor indices and $\dot{\alpha}$, $\dot{\beta}$... to denote right-handed Weyl spinor indices. Besides, i, j... are indices of the internal symmetry (R-symmetry) which corresponds to some bosonic scalar generators outside of the Poincaré algebra rotating the different supersymmetry generators into each other. Considering that the whole discussion here is in four dimensions, the gamma matrices are denoted as γ_μ.

Since the Hermitian conjugate of a supersymmetry generator is also a similar generator, we can choose the basis to satisfy the reality condition:

$$\bar{Q}^i_{\dot{\alpha}} = (Q_{\alpha i})^+. \tag{2.28}$$

Considering the fact that the Lie superalgebra[3] must have the Poincaré algebra as its subalgebra, the discussions on their representations in [2] gives that the anticommutator $\{Q, \bar{Q}\}$ must be proportional to the energy-momentum operators. That is,

$$\{Q_{\alpha i}, \bar{Q}^j_{\dot{\beta}}\} = 2\delta_i^j(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu, \tag{2.29}$$

where the positive factor of the R.H.S has been absorbed by rescaling Q's while $\sigma^\mu = \gamma^\mu \mathcal{C}^{-1}$ for the four-dimensional case, where $\mathcal{C}$ is the charge conjugation matrix. Then the supersymmetry generators must commute with momenta P's, i.e.

$$[Q, P] = 0, \tag{2.30}$$

to satisfy the super-Jacobi identity.

[3]Here it means a $\mathbb{Z}_2$-graded algebra allowing anti-commutating relations. For mathematical details on Lie superalgebra, see [11].

The Q's may also carry some representation of the compact internal symmetry, so by introducing another kind of bosonic generators, namely the R-symmetry generators, we have:

$$\begin{aligned} [Q_{\alpha i}, T_A] &= (U_A)^j{}_i Q_{\alpha j}, \\ [\bar{Q}^i{}_{\dot{\beta}}, T_A] &= -\bar{Q}^j_{\dot{\beta}} (U_A)^i{}_j. \end{aligned} \tag{2.31}$$

It is easy to check that the super-Jacobi identity of $[TQQ]$ implies:

$$(U)^i{}_j = -(U)_j{}^i = -((U)^j{}_i)^*, \tag{2.32}$$

which means that the largest possible internal symmetry group is U(N), for $i = 1, 2, ...N$.

The last algebraic relation we shall consider is the commutator of two supersymmetry generators of the same chirality. Considering the $[PQQ]$ super-Jacobi identity and the fact that we don't want any new symmetry with Lorentz indices, the most general form of the commutator of two Q's is described by

$$\{Q_{\alpha i}, Q_{\beta j}\} = (\omega^M_{\alpha\beta})_{ij} Z_M = \sum_M (Z^M_{\alpha\beta})_{ij}, \tag{2.33}$$

where Z_M denotes different central charges[4], $\omega_{\alpha\beta}$'s are independent antisymmetric matrices. The corresponding commutator of two $\bar{Q}$'s is its Hermitian conjugation. Generally, p-form central charges $(Z^M_{\alpha\beta})_{ij}$ in the superalgebra correspond to p-dimensional extended objects.

Now we can summarize the commutation relationships of the super-Poincaré algebra in four-dimensions[5], omitting central charges:

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= -\eta_{\mu[\rho} M_{\sigma]\nu} + \eta_{\nu[\rho} M_{\sigma]\mu}; && \\ [P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} P_{\rho]}, & [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, Q_{\alpha i}] &= -\frac{1}{4}(\gamma_{\mu\nu})^\beta_\alpha Q_{\beta i}, & [M_{\mu\nu}, \bar{Q}^i_{\dot{\alpha}}] &= \frac{1}{4}\bar{Q}^i_{\dot{\beta}} (\gamma_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}, \\ \{Q_{\alpha i}, \bar{Q}^j_{\dot{\beta}}\} &= 2\delta^j_i (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, & [Q, P] &= 0, \\ [Q_{\alpha i}, T_A] &= (U_A)^j{}_i Q_{\alpha j}, & [\bar{Q}^i{}_{\dot{\beta}}, T_A] &= -\bar{Q}^j_{\dot{\beta}} (T_A)^i{}_j, \\ [T_A, T_B] &= i f^C_{AB} \cdot T_C & [T_A, M_{\mu\nu}] &= 0. \end{aligned} \tag{2.34}$$

If there is only one supercharge Q_α, i.e. N=1, the theory has the simplest supersymmetry, where the R-symmetry is U(1), denoted by R with commutation relations:

$$[Q, R] = Q; \quad [\bar{Q}, R] = -\bar{Q}. \tag{2.35}$$

The fact that under parity transformation $Q \to \bar{Q}$ and $\bar{Q} \to Q$, R $\to$ -R, implies that the U(1) symmetry is chiral. When $N > 1$, there are additional supersymmetries.

[4]For **higher dimensions**, they are not 'true' central charges. They may not commutate with Lorentz rotations or R-symmetry transformations, since they may carry Lorentz indices and R-symmetry indices [4].

[5]Note that T_A algebras relations don't appear for the $\mathcal{N} = \infty$ super-Poincaré case.

2.2.2 Conformal supersymmetry

As is already mentioned, if all particles are massless, one can add conformal symmetry to the bosonic part of the Lie superalgebra, which together generates the superconformal algebra. To obtain the complete superconformal algebra, we should take dilatation D and the special conformal transformation K_μ into consideration, and define some new generators following the requirements of the super-Jacobi identity.

Recall the two kinds of new algebras we have obtained so far: one is fermionic, Q; the other is bosonic, T. Firstly, one can see that all new bosonic generators should commute with the original conformal group, in particular:

$$[K_\mu, T_A] = 0, \tag{2.36}$$
$$[D, T_A] = 0. \tag{2.37}$$

The commutator of K_μ and the supercharge Q implies a new fermionic generator S following the definition:

$$[K_\mu, Q_{\alpha i}] \equiv (\gamma_\mu)_i^j S_{\alpha j}, \tag{2.38}$$

which is usually denoted as 'the special supersymmetry'. From the $[PDQ]$ super-Jacobi identity, one finds that $[Q, D]$ commutes with P_μ, so this commutator must be a linear combination of several Q's. Further discussions show that

$$[D, Q_{\alpha i}] = -\frac{1}{2} Q_{\alpha j}, \tag{2.39}$$
$$[D, \bar{Q}^i_{\dot\alpha}] = \frac{1}{2} \bar{Q}^i_{\dot\alpha}. \tag{2.40}$$

Now turn to the other kind of fermionic generators S's. Similarly to Q, we have

$$[D, S_{\alpha i}] = \frac{1}{2} S_{\alpha j}, \tag{2.41}$$
$$[D, \bar{S}^i_{\dot\alpha}] = -\frac{1}{2} \bar{S}^i_{\dot\alpha}. \tag{2.42}$$

Substituting K_μ and S for P_μ and Q in (2.34), respectively, one obtains similar commutators:

$$[K_\mu, S] = 0 \tag{2.43}$$
$$[P_\mu, S_{\alpha i}] \propto -(\gamma_\mu)_i^j S_{\alpha j}, \tag{2.44}$$
$$[T_A, S_{\alpha i}] \propto -(U_A)^j{}_i S_{\alpha j}, \tag{2.45}$$

and an anti-commutator:

$$\{S_{\alpha i}, \bar{S}^j_{\dot\beta}\} = -2\delta_i^j \cdot (\sigma^\mu)_{\alpha\dot\beta} K_\mu. \tag{2.46}$$

Finally one can calculate the anticommutator of the two kinds of fermionic generators Q and S, which actually generates the bosonic algebra T (R-symmetry) again,

$$\{Q, S\} \to T + \ D + \ M. \tag{2.47}$$

Generally speaking, such a anticommutator could be completed by adding the other bosonic operators to the right-hand side of the expression as follow:

$$\{Q, S\} \;\Rightarrow\; D + \; M + \; T. \tag{2.48}$$

If we consider such superalgebras in general [12], we find the following supermatrix:

$$\begin{pmatrix} SO(d,2) & Q+S \\ Q-S & T \end{pmatrix}.$$

The maximal dimension where the result is a superconformal algebra is six. The superconformal algebra is widely used in many areas of M-theory, especially in AdS/CFT correspondence.

Chapter 3

D=3 topological supergravity theories

Supergravities (SUGRA) are gauge field theories with local supersymmetries. That is, they are invariant under local supersymmetry transformations. As is well known, supergravities not only include, but also extend general relativity, which makes it possible to merge gravity and particle physics. They are usually formulated in the vielbein formulation (e_μ^α), and contain the gravitations and the gauge fields of supersymmetry (ψ_μ), which are spinors. Many supergravity theories, especially the ones in higher dimensions, also contain another kind of gauge fields $A_{\mu_1\mu_2\ldots\mu_p}$, which are totally antisymmetric in the world indices.

In the end of the 1970s, supergravity was considered as an effective unification of gravity with all other fundamental interactions. However, ordinary extended supergravities were soon found to have a number of defects from both a theoretical and a phenomenological point of view, which ended the SUGRA era when all attempts to fix these problem failed[1]. For reviews of the supergravity history, look at [13] [14].

Recently the interest in supergravities has been increased again due to their relevance to string dualities [15] [16]. Since massless sectors of superstring theories can be described by supergravities, one may obtain some information about string theories indirectly by studying supergravities.

From the previous chapter, we know that supersymmetry can unify spacetime with internal symmetries by adding new fermionic symmetries. According to the Haag-Łopuszański-Sohnius theorem, two kinds of $\mathbb{Z}_2$-graded algebras, i.e. super-Poincaré algebra and superconformal algebra, are of special interest. It follows that they lead to two corresponding classes of supergravities, Poincaré supergravity and conformal supergravity, respectively. In this chapter, we mainly focus on the latter in preparation for the following text.

The conformal supergravity is the extension of the ordinary supergravity to include the Weyl transformation rule:

$$g^{\mu\nu} \to e^{-\lambda(x)} g^{\mu\nu}, \tag{3.1}$$

which makes the whole theory conformally invariant. Equivalently, a conformal supergravity theory can also be considered as the supersymmetrized version of the gravity[2]

[1] Although there have been some articles which indicate that the four-dimensional $\mathcal{N} = 8$ maximal supergravity may be perturbatively finite, it suffers other problems.

[2] By conformal gravity, I mean general gravity theories whose actions are invariant under conformal

with Weyl transformations. It implies that a conformal supergravity can be constructed in this way, as we will do below.

Strictly speaking, we are only interested in situations in three dimensions. In contrast to higher-dimensional supergravities, the vector fields in this case generally appear by introducing Chern-Simons (CS) terms rather than Yang-Mills terms. Due to the novel properties of CS terms, the number of vector fields introduced via CS terms is not determined initially, as well as the dimension of the gauge group. It turns out that much richer phenomena will be generated, which we don't find in higher dimensional cases.

3.1 Minimal topological supergravity

A Chern-Simons term is metric-independent (kind of topological), and has wide applications in many areas of physics. The situation is more interesting in three dimensions, because there could be some observable consequences in condensed matter physics, as well as in higher-temperature limits of our four-dimensional world.

The first time the Chern-Simons term was related to supergravity might be in 1978, when the eleven-dimensional supergravity was constructed by E.Cremmer, B.Julia and J.Scherk [17]. A Chern-Simons term arose there because of the requirement of preserving the local supersymmetry, then such a term appears frequently in lower dimensional supergravity models and has been studied in many articles so far.

In this section, we concentrate our attentions on the simplest three-dimensional supergravities, which include Chern-Simons terms. Such a theory was first found as a supersymmetric extension of a gravitational action in three dimensions. We would like to begin with the topological graviton action in three dimensions [18], then explore for its supersymmetric extension [19].

In 1982, S.Deser etc.[18] constructed a interesting gravitational action by adding a Chern-Simons term to the usual Einstein term:

$$I_G = -\int d^3xeR + \int d^3x\epsilon^{\mu\nu\rho}Tr_\alpha(\omega_\mu\partial_\nu\omega_\rho + \frac{2}{3}\omega_\mu\omega_\nu\omega_\rho), \tag{3.2}$$

where ω is the spin connection, and the trace is over Lorentz indices α in SO(1,2). The antisymmetric tensor density $\epsilon^{\mu\nu\rho}$ with the world line indices $\mu\nu\rho$ is defined as follow:

$$\begin{cases} \epsilon^{\mu\nu\rho} & = 1 \ \text{ if } (\mu\nu\rho) = (012), \\ \epsilon^{\mu\nu\rho} & = e^\mu_\alpha e^\nu_\beta e^\rho_\gamma \epsilon^{\alpha\beta\gamma}, \\ \epsilon^{\mu\nu\rho} & = -e^2\epsilon_{\mu\nu\rho}, \\ e & = \text{det}e^\alpha_\mu = \sqrt{-\text{det}g_{\mu\nu}} = \sqrt{-g}. \end{cases}$$

The first term of the action is the trivial Einstein term I_E, and the second one is the gravitational Chern-Simons term I_{CS}. The whole action actually gives a theory of topologically massive gravity in three dimensions. Though there exist terms of third-derivative order, the action is causal and ghost-free.

transformations. A physical gravity theory should generate general relativity or an alternative to general gravity.

Now turn to the corresponding fermionic action. Recalling that the superpartner of the graviton (χ_α), namely the gravitino, has spin-3/2, it follows that the fermionic action should consist of the trivial Rarita-Schwinger term $I_{3/2}$, and a topological term I_{TF}, analogous to the gravitational action above. This was done in [19].

To construct the local supersymmetry, Rarita-Schwinger terms have to be added to the dreibein-compatible ω in order to obtain its supercovariant version. It's done by defining a supersymmetric version of ω, which is called $\tilde{\omega}$:

$$\tilde{\omega}_{\mu\alpha\beta} = \omega_{\mu\alpha\beta} + K_{\mu\alpha\beta}, \tag{3.3}$$

where

$$\omega_{\mu\alpha\beta} = \frac{1}{2}(\Omega_{\mu\alpha\beta} - \Omega_{\alpha\beta\mu} + \Omega_{\beta\mu\alpha}), \tag{3.4}$$

with

$$\Omega_{\mu\nu\alpha} = \partial_\mu e_\nu{}^\alpha - \partial_\nu e_\mu^\alpha, \tag{3.5}$$

and

$$K_{\mu\alpha\beta} = -\frac{i}{2}(\chi_\mu\gamma_\beta\chi_\alpha - \chi_\mu\gamma_\alpha\chi_\beta - \chi_\alpha\gamma_\mu\chi_\beta). \tag{3.6}$$

One can check that this combination of spin connection and contorsion is supercovariant.

Then covariant derivative of spinors is given by:

$$\tilde{D}_\mu = \partial_\mu + \frac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\sigma^{\alpha\beta}, \tag{3.7}$$

of the Lorentz SO(1,2) gauge group. Following the definition of the covariant derivative, we introduce a useful notation:

$$f^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{D}_\nu\chi_\rho. \tag{3.8}$$

By using only the two gauge fields e_μ^α [spin-2],χ_μ [spin-3/2], S.Deser and J.H.Kay [19] constructed the supersymmetric version of the gravity action mentioned above. However, we are only interested in the gravitational Chern-Simons terms and its corresponding term for the gravitino, so we abandon the Einstein and the Rarita-Schwinger terms, and write the pure supergravity action consisting of only Chern-Simons terms:

$$\begin{aligned} L =& I_{CS} + I_{TF} \\ =& \int d^3x\epsilon^{\mu\nu\rho}Tr_\alpha(\tilde{\omega}_\mu\partial_\nu\tilde{\omega}_\rho + \frac{2}{3}\tilde{\omega}_\mu\tilde{\omega}_\nu\tilde{\omega}_\rho) - 2e^{-1}i\int d^3x\bar{f}^\mu\gamma_\nu\gamma_\mu f^\nu, \end{aligned} \tag{3.9}$$

which was checked to be invariant under supersymmetry transformation rules:

$$\delta e_\mu{}^\alpha = i\bar{\epsilon}\gamma^\alpha\chi_\mu, \quad \delta\chi_\mu = \tilde{D}_\mu\epsilon. \tag{3.10}$$

The second transformation rule comes from the fact that under local gauge transformations, connections always transform into the covariant derivatives of the gauge parameters.

This action consists of two Chern-Simons-like terms, so in some sense it's topological. Actually, it was soon proven to coincide with three-dimensional conformal supergravity by P.van Nieuwenhuizen [20].

In 1988, J. Horne and E.Witten [21][22] indicated that the conformally invariant gravity in three dimensions was equivalent to the gauge theory of the conformal group in three dimensions with a pure Chern-Simons action. It shows that the conformal gravity is exactly solvable in three dimensions. Soon the conclusion was generalized to extended supergravity. In a following article [23] written by U.Lindström and M.Roček, superconformal gravities with Chern-Simons terms were shown to be finite and solvable for arbitrary number of supersymmetries, which implies that $\mathcal{N}$=8 and $\mathcal{N}$=6 pure supergravities involving Chern-Simons terms can be constructed by adding an new gauge field for the R-symmetry group. This statement is explicitly confirmed in the following sections.

3.2 d=3 $\mathcal{N}$=8 topological supergravity

Such a purely topological $\mathcal{N}$=8 supergravity was constructed in a recent paper [24], which is briefly reviewed in this section. The notation here is adopted to the one used above.

To obtain the extended supergravity, gauge field B_μ of spin-1 with indices $i, j...(i, j = 1, 2, ..., 8)$ needs to be added for the gaugings of the R-symmetry. To sum up, now we have three local gauge fields of 'spin' 2, 3/2 and 1, i.e. ${e_\mu}^\alpha[0]$(the general metric), $\chi_\mu[-1/2]$(gravitino), $B^{ij}_\mu[-1]$(gauge field).

The covariant derivative acting on spinors for the $\mathcal{N}$=8 case is described as follow:

$$\tilde{D}_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\epsilon + \frac{1}{4}B_{\mu ij}\Gamma^{ij}\epsilon. \tag{3.11}$$

That is, both the Lorentz $SO(1,2)$ group and the R-symmetry $SO(8)$ groups are gauged. Following the standard notations defined in the last section, the topological supergravity Lagrangian can be constructed from a set of Chern-Simons terms:

$$\begin{aligned} L &= \frac{1}{2}\epsilon^{\mu\nu\rho}Tr_\alpha(\tilde{\omega}_\mu\partial_\nu\tilde{\omega}_\rho + \frac{2}{3}\tilde{\omega}_\mu\tilde{\omega}_\nu\tilde{\omega}_\rho) - \epsilon^{\mu\nu\rho}Tr_i(B_\mu\partial_\nu B_\rho + \frac{2}{3}B_\mu B_\nu B_\rho) \\ & -ie^{-1}\epsilon^{\alpha\mu\nu}\epsilon^{\beta\rho\sigma}(\tilde{D}_\mu\bar{\chi}_\nu\gamma_\beta\gamma_\alpha\tilde{D}_\rho\chi_\sigma), \end{aligned} \tag{3.12}$$

Recalling the SUSY transformation rules

$$\delta {e_\mu}^\alpha = i\bar{\epsilon}\gamma^\alpha\chi_\mu, \;\; \delta\chi_\mu = \tilde{D}_\mu\epsilon, \tag{3.13}$$

the above Lagrangian can be shown to keep the $\mathcal{N} = 8$ supersymmetry if we choose the variation of $B_{\mu ij}$ to be

$$\delta B^{ij}_\mu = -\frac{i}{2}\bar{\epsilon}\Gamma^{ij}\gamma_\nu\gamma_\mu f^\nu, \tag{3.14}$$

which follows from the cancelation of field strength terms. For more details of the verification of the invariance, see [24].

There are also locally scale invariance under:

$$\begin{aligned} \delta_\Delta {e_\mu}^\alpha &= -\phi(x){e_\mu}^\alpha, \\ \delta_\Delta\chi_\mu &= -\tfrac{1}{2}\phi(x)\chi_\mu, \\ \delta_\Delta B^{ij}_\mu &= 0, \end{aligned} \tag{3.15}$$

and $\mathcal{N} = 8$ superconformal symmetry under:

$$\begin{array}{rcl} \delta_S e_\mu{}^\alpha & = & 0, \\ \delta_S \chi_\mu & = & \gamma_\mu \eta, \\ \delta_S B_\mu^{ij} & = & \frac{i}{2}\bar{\eta}\Gamma^{ij}\chi_\mu, \end{array} \tag{3.16}$$

where ϕ and η are the corresponding infinitesimal parameters. Verifying the invariance under these transformation rules, including the SUSY transformations, requires Fierz identities[3].

3.3 d=3 $\mathcal{N}$=6 topological supergravity

Here we come to one of the main aims of this thesis project: to constrcut the $\mathcal{N}$=6 purely topological supergravity. Similarly to the last section, it is obtained by writing an on-shell Lagrangian containing only the same three types of Chern-Simons terms, one for each gauge symmetry.

In accordance with the Bagger-Lambert version [25] of the ABJM matter action [26], the supersymmetry parameter of such an $\mathcal{N}$=6 supergravity is written as ϵ_{AB} with two antisymmetric SU(4) indices in the fundamental representation. Thus they are in the **6** of SU(4) with a vanishing U(1) charge, satisfying the self-duality condition:

$$\epsilon^{AB} = \frac{1}{2}\varepsilon^{ABCD}\epsilon_{AB}, \tag{3.17}$$

in which ϵ^{AB} is the complex conjugate of ϵ_{AB} by definition.

Similarly to the $\mathcal{N} = 8$ case, here we construct the topological supergravity Lagrangian having six supersymmetries:

$$\begin{array}{rcl} \mathcal{L} & = & \frac{1}{2}\epsilon^{\mu\nu\rho}Tr_\alpha(\tilde{\omega}_\mu\partial_\nu\tilde{\omega}_\rho + \frac{2}{3}\tilde{\omega}_\mu\tilde{\omega}_\nu\tilde{\omega}_\rho) - 2\epsilon^{\mu\nu\rho}Tr_A(B_\mu\partial_\nu B_\rho + \frac{2}{3}B_\mu B_\nu B_\rho) \\ & & -ie^{-1}\epsilon^{\alpha\mu\nu}\epsilon^{\beta\rho\sigma}(\tilde{D}_\mu\bar{\chi}_\nu^{AB}\gamma_\beta\gamma_\alpha\tilde{D}_\rho\chi_{\sigma AB}), \end{array} \tag{3.18}$$

where the last term can be written as[4]:

$$-4i(e_\mu{}^\alpha e_\nu{}^\beta e^{-1})\bar{f}^{\mu AB}\gamma_\beta\gamma_\alpha f^\nu_{AB}, \tag{3.19}$$

and the covariant derivative acting on the 2-component spinors, for example ϵ_{AB}, is expressed as:

$$\tilde{D}_\mu\epsilon_{AB} = \partial_\mu\epsilon_{AB} + \tfrac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\epsilon_{AB} - B_\mu{}^C{}_A\epsilon_{CB} - B_\mu{}^C{}_B\epsilon_{AC}, \tag{3.20}$$

in which there is also a supercovariant combination of spin connection and the contorsion:

$$\tilde{\omega}_{\mu\alpha\beta} = \omega_{\mu\alpha\beta} + K_{\mu\alpha\beta}. \tag{3.21}$$

Note that

$$K_{\mu\alpha\beta} = -\frac{i}{2}(\chi_{\mu AB}\gamma_\beta\chi_\alpha^{AB} - \chi_{\mu AB}\gamma_\alpha\chi_\beta^{AB} - \chi_{\alpha AB}\gamma_\mu\chi_\beta^{AB}). \tag{3.22}$$

[3]See **Appendix A.**

[4]The Rarita-Schwinger field strength f is defined to be the same as that of the $\mathcal{N} = 1$ case.

The fact that the action only contains Chern-Simons-like terms, and hence has no propagating degrees of freedom, makes it appropriate for the application in **Chapter** 5. Besides, the infinitesimal supersymmetry transformations that leave the action invariant[5] are given by:

$$\begin{aligned} \delta e_\mu{}^\alpha &= i\bar{\epsilon}^{AB}\gamma^\alpha\chi_{\mu AB}, \\ \delta\chi_{\mu AB} &= \tilde{D}_\mu\epsilon_{AB}, \\ \delta B_\mu{}^A{}_B &= \tfrac{i}{e}(\bar{f}^{AC}_\sigma\gamma_\mu\gamma^\sigma\epsilon_{BC} - \bar{f}^\sigma_{BC}\gamma_\mu\gamma_\sigma\epsilon^{AC}). \end{aligned} \tag{3.23}$$

The transformation of the gauge field B_μ can also be written as

$$\delta B_\mu{}^A{}_B = \tfrac{2i}{e}(\bar{f}^{AC}_\sigma\gamma_\mu\gamma^\sigma\epsilon_{BC} - \tfrac{1}{4}\delta^A_B\bar{f}^{CD}_\nu\gamma_\mu\gamma^\nu\epsilon^{CD}), \tag{3.24}$$

since it is defined to be traceless.

Similarly to the SO(8) case, the theory considered here also has both local scale invariance under

$$\begin{aligned} \delta_\Delta e_\mu{}^\alpha &= -\phi(x)e_\mu{}^\alpha, \\ \delta_\Delta\chi^{AB}_\mu &= -\tfrac{1}{2}\phi(x)\chi^{AB}_\mu, \\ \delta_\Delta B^{ij}_\mu &= 0, \end{aligned} \tag{3.25}$$

and $\mathcal{N} = 6$ superconformal invariance under

$$\begin{aligned} \delta_S e_\mu{}^\alpha &= 0, \\ \delta_S\chi^{AB}_\mu &= \gamma_\mu\eta^{AB}, \\ \delta_S B_\mu{}^A{}_B &= -i(\bar{\eta}^{AB}\chi_{\mu BC} - \bar{\chi}^{AB}_\mu\eta_{BC}). \end{aligned} \tag{3.26}$$

The verification of the superconformal invariance is quite similar to the SO(8) case, while that of the local scale invariance is exactly the same.

[5] The verification of the SUSY invariance is given in **Appendix B.**

Chapter 4

D=3 superconformal matter theories

Recently, superconformal matter theories have become a hot topic due to the AdS/CFT correspondence, which was originally conjectured by Maldacena[27]. Considering the most symmetrical choice of M theory with compactifications involving AdS_4, i.e. the $AdS_4 \times S^7$ solution, it is dual to the three-dimensional gauge field theory with the superconformal symmetry $OSp(8|4)$. If one could find an explicit Lagrangian description of conformally invariant gauge theory with the symmetry $OSp(8|4)$, it will become possible to study the details of interacting M-branes directly.

Since M-theory is the strong coupling limit of Type IIA string theory, it follows that M2-branes could be considered as the strong coupling limit of D2-branes. Considering that the low-energy effective world volume theory of N D2-branes of Type IIA is a maximally supersymmetric Yang-Mills theory in three dimensions, the corresponding world volume theory of N M2-branes should be the strong coupling limit of a maximally supersymmetric Yang-Mills theory in three dimensions. Unfortunately, such a super-Yang-Mills theory seems to have only SO(7) R-symmetry due to rotations of seven transverse coordinates. And it's not conformal because of the dimensionful gauge coupling.

However, in the strong coupling limit, which implies the gauge coupling becomes infinite, corresponding to the increasing coupling constant there arises an extra 8th transverse dimensions, which generates the enhanced SO(8) together with the original seven transverse coordinates. Then appears the conformally invariant theory, which describes the interaction among multiple M2-branes in eleven dimensions.

To sum up, the proposed three-dimensional CFT with $OSp(8|4)$ superconformal symmetry should have eight transverse scalars and eight (two-component) Majorana spinors and preserve the equality of bosonic and fermionic physical degrees of freedom. The construction of a Lagrangian for such a theory is the main content of this chapter.

4.1 Superconformal Chern-Simons theories

To construct a Lagrangian satisfying the requirements above, the key is to find a proper way to introduce the gauge fields to the free theory with global U(N) symmetry. In 2004, J.Schwarz [28] suggested that the kinetic term of gauge field s should be taken to be Chern-Simons type, instead of the F^2 type, to make sure no new propagating degrees of freedom are added. Moreover, since the Chern-Simons term is of dimension three, the coefficient of this term is dimensionless, in accordance with the requirement that the classical theory should be scale invariant.

According to Schwarz's article [28], the pure Chern-Simons action is proportional to

$$\mathcal{L} = tr\left[\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho)\right], \tag{4.1}$$

which has no propagating degrees of freedom on-shell, and hence can have arbitrary number of supersymmetries by assuming that A_μ is simply invariant under supersymmetry transformations.

By adding an auxiliary fermionic field χ to the action above, J.Schwarz constructed three-dimensional gauge theories with both $\mathcal{N}$ supersymmetries and classical scale invariance for $\mathcal{N}$=1, 2. The gauge multiplet he obtained could also be coupled to a 'matter' supermultiplet to construct the corresponding gauged matter Lagrangian. Both of them are based on 'ordinary' gauge theories, where these gauge symmetries are related to Lie algebras.

Under certain assumptions, he also found that there was no Chern-Simons theories with the desired $\mathcal{N} = 8$ supersymmetries. To construct the three-dimensional super-CFT with $OSp(8|4)$ symmetry, one has to change one or some of his assumptions, which was done by Bagger/Lambert and by Gustavsson.

4.2 The BLG action

In 2007, J.Bagger and N.Lambert [29][30] found that one can extend the concept of Lie algebra to provide the Chern-Simons guage theories with the desired $OSp(8|4)$ symmetry. Such an extension of Lie algebra is now called three-algebra[1], which can be described in a vector space with a basis T_a, a=1,2,..,N and a trilinear totally antisymmetric product:

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d, \tag{4.2}$$

where obviously $f^{abc}{}_d$=$f^{[abc]}{}_d$. Furthermore, one could define a symmetric metric to raise or lower indices a, b:

$$h^{ab} = Tr(T^a, T^b). \tag{4.3}$$

Instead of the Jacobi identity of Lie algebras, there is the fundamental identity for the so-called three-algebra:

$$\begin{aligned}[T^a, T^b, [T^c, T^d, T^e]] &= [[T^a, T^b, T^c], T^d, T^e] + [T^c, [T^a, T^b, T^d], T^e] \\ &+ [T^c, T^d, [T^a, T^b, T^e]].\end{aligned} \tag{4.4}$$

Also, another constraint for the 'structure constant' has been found to be necessary:

$$\mathrm{Tr(T^a, [T^b, T^c, T^d]) = -Tr([T^a, T^b, T^c], T^d)}. \tag{4.5}$$

[1]In [31] an equivalent algebraic structure describing multiple M2-branes was suggested by A.Gustavsson at the same time.

All of them together build up the definition of the *original* three-algebra, and actually imply that the structure constant f^{abcd} is totally antisymmetric. Besides, the fundamental identity is described by:

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d, \tag{4.6}$$

in a basis form.

4.2.1 The $\mathcal{N}=8$ explicit Lagrangian

Now, let's turn to the Lagrangian. First of all, the Lagrangian must contain two propagating fields: complex scalars X^i_a and spinors Ψ_a. To construct the symmetry required, one also needs to introduce a gauge field $A_{\mu ab}$ to define the physical 'auxiliary' field:

$$\widetilde{A}_\mu{}^a{}_b = A_{\mu cd} f^{cda}{}_b, \tag{4.7}$$

which is related to the structure constants. Such a field defines the covariant derivative as:

$$D_\mu X^i_a = \partial_\mu X^i_a - \widetilde{A}_\mu{}^b{}_a X^i_b. \tag{4.8}$$

Then the classical Lagrangian, namely the BLG action, takes the form:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(D_\mu X^{ai})(D^\mu X^i_a) + \frac{i}{2}\bar{\Psi}^a \Gamma^\mu D_\mu \Psi_a + \frac{i}{4}\bar{\Psi}_b \Gamma_{ij} X^i_c X^j_d \Psi_a f^{abcd} \\ &\quad -V + \frac{1}{2}\varepsilon^{\mu\nu\lambda}(f^{abcd} A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef}),\end{aligned} \tag{4.9}$$

where $a, b, ..$ are indices of the structure constants, which are connected to the 'auxiliary' gauge field, while the $i, j, ..$ are SO(8) R-symmetry indices, corresponding to the 8 transverse directions. Indices $\mu, \nu, ..$ describe the (2+1) M2-brane world volume, which is flat here. Besides, the 6-scalar potential V has a symmetric form:

$$\begin{aligned}V &= \frac{1}{12} f^{abcd} f^{efg}{}_d X^i_a X^j_b X^k_c X^i_e X^j_f X^k_g \\ &= \frac{1}{2\cdot 3!}\mathrm{Tr}([X^i, X^j, X^k], [X^i, X^j, X^k]).\end{aligned} \tag{4.10}$$

This Lagrangian was shown in [30] to be invariant under gauge tranformation and supersymmetry transformation as follow:

$$\begin{aligned}\delta X^i_a &= i\bar{\epsilon}\Gamma^i \Psi_a \\ \delta \Psi_a &= D_\mu X^i_a \Gamma^\mu \Gamma^i \epsilon - \frac{1}{6} X^i_b X^j_c X^k_d f^{bcd}{}_a \Gamma^{ijk}\epsilon \\ \delta \tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu \Gamma_i X^i_c \Psi_d f^{cdb}{}_a,\end{aligned} \tag{4.11}$$

the algebras of which were also proven to close on shell.

In the BLG action, the spinor ϵ has the opposite chirality from Ψ, i.e.

$$\begin{aligned}\Gamma_{012}\epsilon &= -\,\Gamma^{012}\epsilon = \epsilon \\ \Gamma_{012}\Psi &= -\,\Psi.\end{aligned} \tag{4.12}$$

4.2.2 Further discussions

The Lagrangian given in the last subsection is maximally supersymmetric and classically conformally invariant. Though the Chern-Simons term doesn't conserve the parity by itself, the parity invariance is clarified by defining the parity transformation for those fields in a novel way [32]. So the Lagrangian obtained here is consistent with all the symmetries of M2-branes required in [28]. Furthermore, there are no free parameters apart from an overall constant that is quantized [33], which implies there are no continuous parameters in the theory. All of these advantages make it a proper candidate of a interacting theory of multiple M2-branes.

However, many studies on the BLG action then indicated that it's too difficult to find non-trivial solutions to the fundamental identity. Actually it turns out that there is only one realization of the original three-algebra, namely the $SO(4)$ gauge group, which describes stacks of two M2-branes. By relaxing the assumption of total antisymmetry of the structure constants (such as [34]), the problem can be partly solved. Then many articles came out to construct different 'non-totally-antisymmetric' structure constants to obtain field equations which is the IR limit of those of D2-branes, instead of working out the explicit classical action, which seems to be impossible sometimes. However there are still some theoretical faults in the $\mathcal{N} = 8$ supersymmetric theory.

Fortunately, there is another option, namely to look for theories with a reduced number of supersymmetries. Works have been done to explore the possibility, for example, similar Chern-Simons Lagrangians with $\mathcal{N} = 4$ supersymmtry have been suggested by Witten and others. Among them, the most interesting one with a higher supersymmetry is the so-called ABJM action, which will be discussed in the next section.

4.3 The ABJM action

In June of 2008, Aharony, Bergman, Jafferis and Maldacena (ABJM) [26] constructed a three-dimensional Chern-Simons theory with gauge groups $U(N) \times U(N)$, and proved that this theory has explicitly $\mathcal{N} = 6$ supersymmtry. Following their work, Bagger and Lambert (B-L) [25] rewrote the classical action in the 3-algebra form by relaxing constraints on the original structure constants. They also proved that the BLG action could be a special case of the ABJM action when the supersymmetry is enhanced to $\mathcal{N} = 8$ for levels k=1,2. Later, the authors of [35] redefined the 'generalized' three-algebra, and connected the 'B-L' version of ABJM action to generalized Jordan triple systems. For convenience, we use the notations of the ABJM action in [35] during the whole thesis.

4.3.1 The $\mathcal{N} = 6$ explicit Lagrangian

In the paper [35], the 'generalized' structure constants are defined as $f^{ab}{}_{cd}$ with two upper indices and two lower indices. Furthermore, they are both antisymmetric respectively, i.e.

$$f^{ab}{}_{cd} = f^{[ab]}{}_{cd} = f^{ab}{}_{[cd]} \, . \tag{4.13}$$

while a different fundamental identity is satisfied

$$f^{a[b}{}_{dc} f^{e]d}{}_{gh} = f^{be}{}_{d[g} f^{ad}{}_{h]c}. \tag{4.14}$$

Under complex conjugation there is

$$(f^{ab}{}_{cd})^* = f^{cd}{}_{ab} \equiv f_{ab}{}^{cd}. \tag{4.15}$$

Of course, we have to redefine the physical 'auxiliary' gauge fields, which now are connected to the gauge fields $A_\mu{}^a{}_b$ by

$$\tilde{A}_\mu{}^a{}_b = f^{ac}{}_{bd} A_\mu{}^d{}_c\,. \tag{4.16}$$

Having finished the redefinition of the 'generalized' three-algebra, let's turn to the classical description of the action. In the new form of the ABJM action, the complex scalars and fermions are defined to have the index structure as Z^A_a and Ψ_{Aa}, and their corresponding complex conjugates as $\bar{Z}^a_A$ and Ψ^{Aa}.

Recall the indices AB, which are two antisymmetric SU(4) indices in the fundamental representation. Besides, the real vector representation of $SU(4) = \mathrm{SO}(6)$ when the self-duality condition:

$$\epsilon^{AB} = \frac{1}{2}\varepsilon^{ABCD}\epsilon_{AB} \tag{4.17}$$

is satisfied. Here ϵ^{AB} is defined as the complex conjugate of ϵ_{AB}.

Now, we can rewrite the 'B-L' version of ABJM action as follows:

$$\begin{aligned}\mathcal{L} \;=\; & -(D_\mu Z^A{}_a)(D^\mu \bar{Z}_A{}^a) - i\bar{\Psi}^{Aa}\gamma^\mu D_\mu \Psi_{Aa} \\ & -if^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Aa}Z^B{}_b\bar{Z}_B{}^c + 2if^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Ba}Z^B{}_b\bar{Z}_A{}^c \\ & -\tfrac{i}{2}\epsilon_{ABCD}f^{ab}{}_{cd}\bar{\Psi}^{Ac}\Psi^{Bd}Z^C{}_aZ^D{}_b - \tfrac{i}{2}\epsilon^{ABCD}f^{cd}{}_{ab}\bar{\Psi}_{Ac}\Psi_{Bd}\bar{Z}_C{}^a\bar{Z}_D{}^b \\ & -V + \tfrac{1}{2}\epsilon^{\mu\nu\lambda}(f^{ab}{}_{cd}A_\mu{}^d{}_b\partial_\nu A_\lambda{}^c{}_a + \tfrac{2}{3}f^{bd}{}_{gc}f^{gf}{}_{ae}A_\mu{}^a{}_bA_\nu{}^c{}_dA_\lambda{}^e{}_f)\,,\end{aligned} \tag{4.18}$$

where the covariant derivative is

$$D_\mu Z^A_a = \partial_\mu Z^A_a - \widetilde{A}_\mu{}^b{}_a Z^A_b\,, \tag{4.19}$$

and the potential takes the form

$$V = \tfrac{2}{3}\Upsilon^{CD}{}_{Bd}\bar{\Upsilon}_{CD}{}^{Bd}\,, \tag{4.20}$$

$$\Upsilon^{CD}{}_{Bd} = f^{ab}{}_{cd}Z^C{}_aZ^D{}_b\bar{Z}_B{}^c + f^{ab}{}_{cd}\delta^{[C}{}_BZ^{D]}{}_aZ^E{}_b\bar{Z}_E{}^c\,. \tag{4.21}$$

Finally, there are supersymmetry transformations with respect to the supersymmetry parameters of the $\mathcal{N} = 6$ action, self-dual spinors ϵ^{AB}:

$$\begin{aligned}\delta Z^A{}_a =& i\bar{\epsilon}^{AB}\Psi_{Ba}\,, \\ \delta\Psi_{Bd} =& \gamma^\mu D_\mu Z^A{}_d\epsilon_{AB} + f^{ab}{}_{cd}Z^C{}_aZ^D{}_b\bar{Z}_B{}^c\epsilon_{CD} \\ & - f^{ab}{}_{cd}Z^A{}_aZ^C{}_b\bar{Z}_C{}^c\epsilon_{AB}\,, \\ \delta A_\mu{}^a{}_b =& - i\bar{\epsilon}_{AB}\gamma_\mu\Psi^{Aa}Z^B{}_b + i\bar{\epsilon}^{AB}\gamma_\mu\Psi_{Ab}\bar{Z}_B{}^a\,.\end{aligned} \tag{4.22}$$

The Lagrangian has six supersymmetries under the SUSY transformations above.

4.3.2 Further discussions about the gauge group

Since the constraint on three-algebra is relaxed, the ABJM action allows for a much larger set of models with arbitrary number of M2-branes. It opens the possibility for many variations of $\mathcal{N} = 6$ superconformal theories with $OSp(6|4)$ superconformal symmetry, which can be classified by the associated gauge groups. For example, the BLG action as a special case of the ABJM action, corresponds to the one with gauge group SU(2)× SU(2)[2].

Recall in the 'B-L' version we have used, the gauge group is defined by the three-algebra. The construction of structure constants satisfying the fundamental identity seems to become a crucial concept of the related M2-branes dynamics.

According to the analysis in [31][25], there is a general form of the solution to the fundamental identity, which can be expressed as:

$$f^{ab}{}_{cd} = \sum_{AB} \Omega_{AB}(t^A)^a{}_c(t^B)^b{}_d \tag{4.23}$$

in which $\Omega_{AB} = \Omega_{BA}$, and the T^A is a matrix representation of some Lie algebra. If we substitute the expression into the fundamental identity, further constraints on Ω_{AB} can be found.

In a recent paper [37], the 'generalized' three-algebras was proven to be in one-to-one correspondence to a certain set of Lie superalgebras via Jordan triple system, which may lead to some deeper understanding of the classification of those three-algebras of ABJM type.

[2]For more about the U(N)×U(N) gauge group, especially how the supersymmetry reduces from $\mathcal{N} = 8$ to $\mathcal{N} = 6$, refer to [36].

Chapter 5

The Gauged ABJM theory

Now, both the topologically pure supergravities and the BLG/ABJM superconformal matter theories have been introduced. They can be combined together to give the so-called gauged BLG/ABJM theories. The reason why such a combination is possible is that the topologically pure supergravity doesn't have no propagating degrees of freedom, so the equality of bosonic and fermionic degrees of freedom still holds at least on shell when it couples to the BLG/ABJM matter theories.

While replacing the (2+1)-dimensional Minkowski metric with the general metric $g_{\mu\nu}$, the gauged action we obtain is supposed to be a 'geometric' description of the corresponding world volume theory. A similar geometric description of the world sheet theory, namely the Polyakov string, was studied in [38][39]. According to the importance of Polyakov string to the quantum string, the gauged BLG/ABJM actions may be of some interest for M2-branes.

In this chapter we construct the gauged ABJM theory [40] (i.e. couple the $\mathcal{N}$=6 pure supergravity to the ABJM matter action). Besides, the gauged BLG action [24] has been partly obtained before, but due to some unresolved issues at the one derivative level in $\delta\mathcal{L}$, the entire Lagrangian has not yet been possible to derive.

5.1 The gauged ABJM action

On one hand, the action of $\mathcal{N}=6$ pure topological supergravity L^{conf}_{sugra} has been obtained in **section** 3.3; and on the other hand, the ABJM action L^{cov}_{ABJM} was introduced in **section** 4.3. We are now ready to combine them into the gauged theory. In this section, we give the whole Lagrangian explicitly, as well as its SUSY transformation rules. The process of constructing the action will be discussed in the following sections.

As usual, the covariant derivative is defined before the explicit action is given. As a gauged action, there are three kinds of gauge fields, $\tilde{\omega}_{\mu\alpha\beta}$ for the Lorentz SO(1,2) group, $B^A_{\mu B}$ for the R-symmetry group, $\tilde{A}^a_{\mu b}$ for the gauge group. Besides, it will be shown later that an additional U(1) group, A_μ, is required to conserve SUSY invariance of the gauged action. To sum up, the covariant derivative is defined as follows:

$$\tilde{D}_\mu\psi^{Aa} = \partial_\mu\psi^{Aa} + \tfrac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\psi^{Aa} + B^A_{\mu B}\psi^{Ba} + \tilde{A}^a_{\mu b}\psi^{Ab} + qA_\mu\psi^{Aa}, \tag{5.1}$$

in which $q^2 = \frac{1}{16}$, The value of charge q is derived from the normalization of the Chern-Simons term of the U(1) group.

Now, we can write down the whole Lagrangian of three-dimensional gauged ABJM type (with $A = \pm\sqrt{2}$, which is explained in the next section):

$$\begin{aligned}
L \;=\; & L^{conf}_{sugra} + L^{cov}_{ABJM} && (5.2)\\
& +\frac{1}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho && (5.3)\\
& +iAe\bar{\chi}^{BA}_\mu\gamma^\nu\gamma^\mu\Psi_{Aa}(\tilde{D}_\nu\bar{Z}^a_B - \frac{i}{2}A\bar{\chi}_{\nu BC}\Psi^{Ca}) + c.c. && (5.4)\\
& +i\epsilon^{\mu\nu\rho}(\bar{\chi}^{AC}_\mu\chi_{\nu BC})Z^B_a\tilde{D}_\rho\bar{Z}^a_A + c.c. && (5.5)\\
& -iA(\bar{f}^{\mu AB}\gamma_\mu\Psi_{Aa}\bar{Z}^a_B + \bar{f}^\mu_{AB}\gamma_\mu\Psi^{Aa}Z^B_a) && (5.6)\\
& -\frac{e}{8}\tilde{R}Z^2 && (5.7)\\
& +\frac{i}{2}Z^2\bar{f}^\mu_{AB}\chi^{AB}_\mu && (5.8)\\
& +2ieAf^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\Psi^{d[B})Z^{D]}_a Z^A_b\bar{Z}^c_D + c.c. && (5.9)\\
& -i\epsilon^{\mu\nu\rho}(\bar{\chi}_{\mu AB}\gamma_\nu\chi^{CD}_\rho)(Z^A_aZ^B_b\bar{Z}^c_C\bar{Z}^d_D)f^{ab}{}_{cd}\\
& +\frac{i}{4}\epsilon^{\mu\nu\rho}(\bar{\chi}_{\mu AB}\gamma_\nu\chi^{AB}_\rho)(Z^C_aZ^D_b\bar{Z}^c_C\bar{Z}^d_D)f^{ab}{}_{cd} && (5.10)\\
& -\frac{i}{16}e\epsilon^{ABCD}(\bar{\Psi}_{Aa}\Psi_{Bb})\bar{Z}^a_C\bar{Z}^b_D + c.c.\\
& +\frac{i}{16}e(\bar{\Psi}_{Db}\Psi^{Db})Z^2 - \frac{i}{4}e(\bar{\Psi}_{Db}\Psi^{Bb})\bar{Z}^a_BZ^D_a\\
& +\frac{i}{8}e(\bar{\Psi}_{Db}\Psi^{Da})\bar{Z}^b_BZ^B_a + \frac{3i}{8}e(\bar{\Psi}_{Db}\Psi^{Ba})\bar{Z}^b_BZ^D_a && (5.11)\\
& -\frac{i}{16}eA(\bar{\chi}_{\mu AB}\gamma^\mu\Psi^{Bb})Z^2Z^A_b - \frac{i}{4}eA(\bar{\chi}_{\mu AB}\gamma^\mu\Psi^{Db})Z^A_aZ^B_b\bar{Z}^a_D + c.c && (5.12)\\
& -\frac{i}{4}\epsilon^{\mu\nu\rho}(\bar{\chi}_{\nu AB}\gamma_\rho\chi^{CD}_\mu)Z^A_aZ^B_b\bar{Z}^a_C\bar{Z}^b_D + \frac{i}{64}\epsilon^{\mu\nu\rho}(\bar{\chi}_{\nu AB}\gamma_\rho\chi^{AB}_\mu)Z^4 && (5.13)\\
& +\frac{1}{8}ef^{ab}{}_{cd}Z^2Z^C_aZ^D_b\bar{Z}^c_C\bar{Z}^d_D + \frac{1}{2}ef^{ab}{}_{cd}Z^B_aZ^C_bZ^D_e\bar{Z}^e_B\bar{Z}^c_C\bar{Z}^d_D && (5.14)\\
& +\frac{5}{12\cdot 64}eZ^6 - \frac{1}{32}eZ^2Z^A_bZ^C_a\bar{Z}^b_C\bar{Z}^a_A + \frac{1}{48}eZ^A_aZ^B_bZ^C_d\bar{Z}^b_A\bar{Z}^d_B\bar{Z}^a_C, && (5.15)
\end{aligned}$$

where *c.c.* stands for complex conjugation of the term that is on its left side on the same line. To keep the Lagrangian to be self-conjugate, the Dirac term of the ABJM matter action has to be written in a 'self-conjugate' way:

$$-\frac{1}{2}(ie\bar{\Psi}^{Aa}\gamma^\mu\tilde{D}_\mu\Psi_{Aa} + ie\bar{\Psi}_{Aa}\gamma^\mu\tilde{D}_\mu\Psi^{Aa}), \tag{5.16}$$

which is actually so because the covariant derivative of e^μ_α doesn't vanish automatically.

To make all supersymmetry variation terms of such a complicated Lagrangian vanish, the initial supersymmetry transformations rules have to be modified too. The final transformation rules turn out to be:

$$\delta e_\mu{}^\alpha = i\bar{\epsilon}_{gAB}\gamma^\alpha\chi_\mu^{AB}, \tag{5.17}$$
$$\delta\chi_\mu^{AB} = \tilde{D}_\mu\epsilon_g^{AB}, \tag{5.18}$$

$$\begin{aligned}
\delta B_\mu{}^A{}_B &= \frac{i}{e}(\bar{f}^{\nu AC}\gamma_\mu\gamma_\nu\epsilon_{gBC} - \bar{f}^\nu_{BC}\gamma_\mu\gamma_\nu\epsilon_g^{AC})\\
&+ \tfrac{i}{4}(\bar{\epsilon}_{BD}\gamma_\mu\Psi^{a(D}Z_a^{A)} - \bar{\epsilon}^{AD}\gamma_\mu\Psi_{a(D}\bar{Z}^a_{B)})\\
&- \tfrac{i}{2}(\bar{\epsilon}_g^{AC}\chi_{\mu DC}Z_a^D\bar{Z}^a_B - \bar{\epsilon}_{gBC}\chi_\mu^{DC}Z_a^A\bar{Z}^a_D)\\
&+ \tfrac{i}{8}\delta^A_B(\bar{\epsilon}_g^{EC}\chi_{\mu DC} - \bar{\epsilon}_{gDC}\chi_\mu^{EC})Z_a^D\bar{Z}^a_E\\
&+ \tfrac{i}{8}(\bar{\epsilon}_g^{AD}\chi_{\mu BD} - \bar{\epsilon}_{gBD}\chi_\mu^{AD})Z^2, \qquad (5.19)\\
\delta Z_a^A &= i\bar{\epsilon}^{AB}\Psi_{Ba}, \qquad (5.20)\\
\delta\Psi_{Bd} &= \gamma^\mu\epsilon_{AB}(\tilde{D}_\mu Z_d^A - iA\bar{\chi}_\mu^{AD}\Psi_{Dd})\\
&+ f^{ab}{}_{cd}Z^C{}_aZ_b^D\bar{Z}^c_B\epsilon_{CD} - f^{ab}{}_{cd}Z_a^AZ_b^C\bar{Z}^c_C\epsilon_{AB}\\
&+ \frac{1}{4}Z_c^CZ_d^D\bar{Z}^c_B\epsilon_{CD} + \frac{1}{16}Z^2Z_d^A\epsilon_{AB}, \qquad (5.21)\\
\delta\tilde{A}_\mu{}^c{}_d &= -i(\bar{\epsilon}_{AB}\gamma_\mu\Psi^{Aa}Z_b^B - \bar{\epsilon}^{AB}\gamma_\mu\Psi_{Ab}\bar{Z}^a_B)f^{bc}{}_{ad}\\
&- 2i(\bar{\epsilon}_g^{AD}\chi_{\mu BD} - \bar{\epsilon}_{gBD}\chi_\mu^{AD})Z_b^B\bar{Z}^a_Af^{bc}{}_{ad}, \qquad (5.22)\\
\delta A_\mu &= -iq(\bar{\epsilon}_{AB}\gamma_\mu\Psi^{Aa}Z_a^B - \bar{\epsilon}^{AB}\gamma_\mu\Psi_{Aa}\bar{Z}^a_B)\\
&- 2iq(\bar{\epsilon}_g^{AD}\chi_{\mu BD} - \bar{\epsilon}_{gBD}\chi_\mu^{AD})Z_a^B\bar{Z}^a_A. \qquad (5.23)
\end{aligned}$$

where $\epsilon_m^{AB} = A\epsilon_g^{AB} = \epsilon^{AB}$, and $A = \pm\sqrt{2}$. The verification of the SUSY invariance is mostly provided as an appendix[1], while some standard steps of the coupling are demonstrated in the next section.

5.2 Construction of the gauged ABJM action

Following [24], in this section we demonstrate how the gauged ABJM action can be derived from the two known actions $L^{conf.}_{sugra}$ and $L^{cov.}_{ABJM}$.

We begin with an assumption that the complete gauged action has this form:

$$L = L^{conf.}_{sugra} + L^{cov.}_{ABJM} + L^{cov}_{supercurrent}, \tag{5.24}$$

where the notation $L^{cov}_{supercurrent}$ means all new terms added in order to conserve the SUSY invariance. Meanwhile, the transformation rules have to be modified.

As a demonstration of this procedure, we mostly consider how new terms are added to preserve the invariance of the gauged action at the level of the second order in covariant derivatives. As the whole Lagrangian was obtained following the same method, we'd like not to repeat how the procedure works for the $(D_\mu)^1$ and $(D_\mu)^0$ levels due to their intricacies. This part of the calculation is presented in **Appendix C**.

[1]Note the discussion concerning multiple-fermion variation terms with derivatives has not been completely finished yet, though some arguments are given [40].

5.2.1 Coupling at order $(\tilde{D}_\mu)^2$

Let's begin with the localized ABJM action (of tensor density one):

$$\begin{aligned}\mathcal{L} \;=\; & -e(\tilde{D}_\mu Z^A_a)(\tilde{D}^\mu \bar{Z}^a_A) - ie\bar{\Psi}^{Aa}\gamma^\mu \tilde{D}_\mu \Psi_{Aa} \\ & -ief^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Aa}Z^B_b\bar{Z}^c_B + 2ief^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Ba}Z^B_b\bar{Z}^c_A \\ & -\tfrac{i}{2}e\epsilon_{ABCD}f^{ab}{}_{cd}\bar{\Psi}^{Ac}\Psi^{Bd}Z^C_a Z^D_b - \tfrac{i}{2}e\epsilon^{ABCD}f^{cd}{}_{ab}\bar{\Psi}_{Ac}\Psi_{Bd}\bar{Z}^a_C\bar{Z}^b_D \\ & -eV + \tfrac{1}{2}\epsilon^{\mu\nu\lambda}(f^{ab}{}_{cd}A_\mu{}^d{}_b\partial_\nu A_\lambda{}^c{}_a + \tfrac{2}{3}f^{bd}{}_{gc}f^{gf}{}_{ae}A_\mu{}^a{}_b A_\nu{}^c{}_d A_\lambda{}^e{}_f)\,, \end{aligned} \tag{5.25}$$

which generates new variation terms according to the orginal variations of both vielbeins and gauge field $B_\mu{}^A{}_B$. Note for these fields of the matter sector, the supersymmetry parameter is denoted by ϵ instead of ϵ_m, while for fields of the pure supergravity, the parameter is ϵ_g.

If one only considers terms of the second order in covariant derivatives, a limited number of terms are contributing and we have:

$$\begin{aligned}\delta L_{ABJM}\,|_{D^2} = & -\delta(eg^{\mu\nu})(\tilde{D}_\mu Z^A_a)(\tilde{D}_\nu \bar{Z}^a_A) \\ & - e\tilde{D}_\mu Z^A_a \tilde{D}^\mu(\delta \bar{Z}^a_A) + c.c. \\ & - eg^{\mu\nu}D_\mu Z^A_a(-\delta B_\nu{}^C{}_A\,|_{f\epsilon}\, \bar{Z}^a_C) + c.c. \\ & - \tfrac{i}{2}e(\delta\bar{\Psi}^{Aa}\,|_{DZ}\,\gamma^\mu\tilde{D}_\mu\Psi_{Aa} + \bar{\Psi}^{Aa}\gamma^\mu\tilde{D}_\mu\delta\Psi_{Aa}\,|_{DZ}) + c.c.. \end{aligned} \tag{5.26}$$

The second and fourth lines of the R.H.S. of eq.(5.26) actually give:

$$\begin{aligned} & \tfrac{i}{2}e(\bar{\Psi}^{Ba}\gamma^{\mu\nu}\epsilon_{AB})\tilde{F}_{\mu\nu}{}^b{}_a Z^A_b - \tfrac{i}{2}e(\bar{\Psi}^{Ba}\gamma^{\mu\nu}\epsilon_{AB})G_{\mu\nu}{}^A{}_C Z^C_a \\ & -\tfrac{i}{2}e(\bar{\Psi}^{Aa}\gamma^\mu\gamma^\nu\tilde{D}_\mu\epsilon_{BA})\tilde{D}_\nu Z^B_a + c.c. \end{aligned} \tag{5.27}$$

in which $F_{\mu\nu}{}^b{}_a$ and $G_{\mu\nu}{}^A{}_C$ are the field strengths of $\tilde{A}_\mu{}^a{}_b$ and $B_\mu{}^A{}_B$. The first term has been canceled already in the ungauged ABJM case while the vanishing of the second term requires an additional term added to $\delta B_\mu{}^A{}_B$ in order to derive an opposite term from its Chern-Simons term. Remind the fact $B_\mu{}^A{}_B$'s reverse has to be its conjugate, the new additional term can be expressed as:

$$\delta B_\mu{}^A{}_C\,|_{new-1} = -\tfrac{i}{4}[(\bar{\Psi}^{Ba}\gamma_\mu\epsilon_{CB})Z^A_a - (\bar{\Psi}_{Ba}\gamma_\mu\epsilon^{AB})\bar{Z}^a_C]. \tag{5.28}$$

Now turn to the first line of the R.H.S. of the eq.(5.26). According to the variation of e^μ_α, we have:

$$-\delta(eg^{\mu\nu})\tilde{D}_\mu Z^A_a\tilde{D}_\mu\bar{Z}^a_A = -ie(2\bar{\chi}^{\{\mu}\gamma^{\nu\}}\epsilon_g - g^{\mu\nu}\bar{\chi}\cdot\gamma\epsilon_g)\tilde{D}_\mu Z^A_a\tilde{D}_\mu\bar{Z}^a_A. \tag{5.29}$$

To cancel this variation term, the first new term, namely a supercurrent term, has to be introduced:

$$Aie\bar{\chi}^{BA}_\mu\gamma^\nu\gamma^\mu\Psi_{Aa}(\tilde{D}_\nu\bar{Z}^a_B - \tfrac{i}{2}\hat{A}\bar{\chi}_{\nu BC}\Psi^{Ca}) + c.c., \tag{5.30}$$

in which $\delta\chi$ leads to variation terms which cancel with the last line of (5.27), and the $\delta\Psi|_{DZ}$ gives

$$\begin{aligned} & Ai\epsilon^{\mu\nu\rho}(\bar{\chi}_{\mu AB}\epsilon^{BD})\tilde{D}_\nu Z^A_a\tilde{D}_\rho\bar{Z}^a_D + c.c. \\ & +\tfrac{i}{2}Ae(2\bar{\chi}^{\{\mu}\gamma^{\nu\}}\epsilon - g^{\mu\nu}\bar{\chi}\cdot\gamma\epsilon)\tilde{D}_\mu Z^A_a\tilde{D}_\mu\bar{Z}^a_A, \end{aligned} \tag{5.31}$$

the second line of which should cancel with term(5.29). Both two cancelations here require the relation:

$$A\epsilon^{AB} = 2\epsilon_g^{AB}, \text{ and } \epsilon^{AB} = A\epsilon_g^{AB}, \tag{5.32}$$

which connect the two supersymmetry parameters and imply $A = \pm\sqrt{2}$. Such a relation has been shown to be extremely important during the whole construction.

Now, the remaining variation terms are:

$$\begin{aligned}
&-eg^{\mu\nu} D_\mu Z_a^A(-\delta B_\nu{}^C{}_A \mid_{f\epsilon} \bar{Z}_C^a) + c.c. \\
&+Ai\epsilon^{\mu\nu\rho}(\bar{\chi}_{\mu AB}\epsilon^{BD})\tilde{D}_\nu Z_a^A \tilde{D}_\rho \bar{Z}_D^a + c.c.,
\end{aligned} \tag{5.33}$$

which require adding another new term

$$iA'\epsilon^{\mu\nu\rho}\bar{\chi}_\mu^{AC}\chi_{\nu BC} Z_A^a \tilde{D}_\rho \bar{Z}_a^B + c.c.. \tag{5.34}$$

It is easily seen that its $(D_\mu)^2$ variation terms can only be derived from varying the two χ's in it. By using integration by parts, the result turns out to be

$$\begin{aligned}
&-2A'i[(\bar{\epsilon}_{gAB} f^{\rho AC}) - (\bar{\epsilon}_g^{AC} f_{AB}^\rho)]\bar{Z}_C^a \tilde{D}_\rho Z_a^B + c.c \\
&-A'i\epsilon^{\mu\nu\rho}[(\bar{\epsilon}_{gAB}\chi_\nu^{AC}) - (\bar{\epsilon}_g^{AC}\chi_{\nu AB})]\tilde{D}_\rho Z_a^B \tilde{D}_\mu \bar{Z}_C^a + c.c. \\
&+\tfrac{i}{2}A'[(\bar{\epsilon}_{gAB}\chi_\nu^{AC}) - (\bar{\epsilon}_g^{AC}\chi_{\nu AB})]\bar{Z}_C^a \tilde{F}_{\mu\rho}{}^d{}_a Z_d^B + c.c. \\
&-\tfrac{i}{2}A'[(\bar{\epsilon}_{gAB}\chi_\nu^{AC}) - (\bar{\epsilon}_g^{AC}\chi_{\nu AB})]\bar{Z}_C^a G_{\mu\rho}{}^B{}_D Z_a^D + c.c.,
\end{aligned} \tag{5.35}$$

in which the second half implies new terms of both $\delta A_\mu{}^a{}_b$ and $\delta B_\mu{}^A{}_B$, similarly to the eq.(5.28). That is,

$$\begin{aligned}
\delta A_\mu{}^b{}_a \mid_{new-1} &= 2iA'[(\bar{\chi}_{\mu CD}\epsilon_g^{DB}) - (\bar{\chi}_\mu^{BD}\epsilon_{gDC})Z_a^C]\bar{Z}_B^b, \\
\delta B_\mu{}^A{}_C \mid_{new-2} &= \tfrac{i}{2}A'[(\bar{\chi}_{\mu BD}\epsilon_g^{DA})Z_a^B \bar{Z}_C^a - (\bar{\chi}_\mu^{BD}\epsilon_{gDC})Z_a^A \bar{Z}_B^a],
\end{aligned} \tag{5.36}$$

which affect the calculation of $(D_\mu)^2$ variation terms because of their corresponding Chern-Simons terms[2].

Using the relation $A\epsilon^{AB} = 2\epsilon_g^{AB}$, one could find if $A' = 1$, the second line of term(5.35) cancels the second line of the remaining terms mentioned above. More interestingly, according to the fact

$$\delta B_\mu{}^A{}_B|_{orginal} = i(\bar{f}^{\nu AC}\gamma_\mu\gamma_\nu\epsilon_{gBC} - \bar{f}_{BC}^\nu\gamma_\mu\gamma_\nu\epsilon_g^{AC}), \tag{5.37}$$

their first lines also add and hence give a simpler expression:

$$\begin{aligned}
&2i(\bar{f}^{\nu CB}\gamma_\nu\gamma^\mu\epsilon_{gBA})\tilde{D}_\mu Z_a^A \bar{Z}_C^a + c.c. \\
&+\tfrac{i}{2}(\bar{f}^{\nu AB}\gamma_\nu\gamma^\mu\epsilon_{gAB})\tilde{D}_\mu Z_a^C \bar{Z}_C^a + c.c.,
\end{aligned} \tag{5.38}$$

which are all variation terms left of second order in covariant derivatives up to now.

The next term added to the Lagrangian is assumed to be

$$L_{A''} = iA''\bar{f}^{AB}\cdot\gamma\Psi_{Aa}\bar{Z}_B^a + c.c., \tag{5.39}$$

[2] Here when refering to '$(D_\mu)^2$ terms' we include also field strength terms.

in which the variation of Ψ generates exactly the same variation terms to the first part of term(5.38) but with opposite sign when we set $A'' \equiv -A$. Meanwhile, by varying χ in its field strength f^{AB}, new $G_{\mu\nu}{}^{B}{}_{A}$ terms arise due to the appearance of $[\tilde{D}_\mu, \tilde{D}_\nu]$, which means there exist new variation terms in $\delta B_\mu{}^A{}_C$:

$$\delta B_\mu{}^A{}_C \mid_{new-3} = -\tfrac{i}{4}A''[(\bar{\Psi}^{[Aa}\gamma_\mu\epsilon_{gBC})Z^{B]}_a - (\bar{\Psi}_{[Ca}\gamma_\mu\epsilon_g^{BA})\bar{Z}^a_{B]}], \tag{5.40}$$

Furthermore, $[\tilde{D}_\mu, \tilde{D}_\nu]$ acting on spinors also generates $\tilde{R}_{\mu\nu\alpha\beta}$ terms, which is

$$-\tfrac{A''}{4}\tilde{R}^{**\mu,\gamma}(\bar{\Psi}^{Aa}\gamma_\mu\gamma_\gamma\epsilon_{gAB})Z^B_a + c.c., \tag{5.41}$$

in which $R^{**\mu,\gamma}$ is the double dual of curvature tensor. Remind the triple dual can be shown as

$$\tilde{R}^{***}_\mu = i\chi_{\nu AB}\gamma_\mu f^{\nu AB}, \tag{5.42}$$

the term (5.41) can be divided into two parts:

$$\begin{aligned}&-\tfrac{A''}{4}\tilde{R}^{**}(\bar{\Psi}^{Aa}\epsilon_{gAB})Z^B_a + c.c.\\ &+\tfrac{iA''}{2}\chi_{\mu AB}\gamma_\nu f^{\mu AB}(\bar{\Psi}^{Ca}\gamma_\nu\epsilon_{gCD})Z^D_a + c.c.,\end{aligned} \tag{5.43}$$

in which the first part is canceled later in this section while the other part remains until we calculate the $(D_\mu)^1$ variation terms.

Required by the local scale invariance of the scale fields Z, there has to be a gravity term in the action:

$$L_{RZ^2} = -\tfrac{e}{8}\tilde{R}Z^2. \tag{5.44}$$

All variation terms generated by such a gravity term have been worked out to be:

$$\begin{aligned}\delta\tilde{L}_{RZ^2} \;=\;& \tfrac{i}{2e}\epsilon^{\mu\nu\rho}[K_{\sigma\mu}{}^\sigma\bar{\epsilon}_g^{AB}\gamma_\nu f_{\rho AB} + K_{\mu\nu}{}^\sigma(\bar{\epsilon}_g^{AB}\gamma_\sigma f_{\rho AB} - \tfrac{1}{2}g_{\sigma\rho}\bar{\epsilon}_g^{AB}\gamma\cdot f_{AB})]Z^2\\ &+\tfrac{i}{4e}Z^2\tilde{R}^{**}_{\mu,\nu}\bar{\epsilon}_g^{AB}\gamma^\nu\chi^\mu_{AB}\\ &-\tfrac{i}{4e}\tilde{R}^{**}(\bar{\epsilon}_m^{AB}\Psi_{Aa}\bar{Z}^a_B + \bar{\epsilon}_{mAB}\Psi^{Aa}Z^B_a)\\ &+\tfrac{i}{2e}\epsilon^{\mu\nu\rho}\tilde{D}_\mu Z^2\bar{\epsilon}_g^{AB}\gamma_\nu f_{\rho AB},\end{aligned} \tag{5.45}$$

in which the third line cancels with the first line of term(5.43) according to $A''\epsilon_g = -\epsilon$. Because the first line are terms linear to the covariant derivative, we don't take them into our consideration in this section.

Directly, we add the last term when concerning only the $(D_\mu)^2$ cancelation:

$$L_{Z^2 f\chi} = iA'''Z^2\bar{f}^\mu_{AB}\chi^{AB}_\mu. \tag{5.46}$$

Varying the first χ, i.e. the χ in f, it gives:

$$-\tfrac{i}{4}A'''Z^2(\tilde{R}^{**\mu,\nu}\bar{\epsilon}_{gAB}\gamma_\nu\chi^{AB}_\mu - 2\epsilon^{\mu\nu\rho}G_{\mu\nu}{}^A{}_C\bar{\epsilon}_g^{CB}\chi_{\rho AB}), \tag{5.47}$$

and if varying the second χ, we obtain:

$$iA'''Z^2\bar{f}^{\mu AB}\tilde{D}_\mu\epsilon_{gAB}. \tag{5.48}$$

Both variation terms involving R^{**} cancel now, hence what we have to fight are the second line of term (5.38), the first and fourth lines of term (5.45), and term (5.47) and (5.48). They are gathered to give:

$$\begin{aligned}&\tfrac{i}{2}(\bar{f}^{\mu AB}\epsilon_{gAB})\tilde{D}_\mu Z^2\\&+iA'''Z^2\bar{f}^{\mu AB}\tilde{D}_\mu\epsilon_{gAB}\\&-\tfrac{i}{4}(A'''-1)Z^2\tilde{R}^{**\mu,\nu}(\bar{\epsilon}_{gAB}\gamma_\nu\chi_\mu^{AB})\\&+\tfrac{i}{2}A'''Z^2\epsilon^{\mu\nu\rho}G_{\mu\nu}{}^A{}_C\bar{\epsilon}_g^{CB}\chi_{\rho AB}.\end{aligned}\tag{5.49}$$

Using the integration by parts and defining $A'''=\frac{1}{2}$, we can easily find the sum of the first three lines is equal to the last line, that is,

$$+\tfrac{i}{2}Z^2\epsilon^{\mu\nu\rho}G_{\mu\nu}{}^A{}_C\bar{\epsilon}_g^{CB}\chi_{\rho AB}.\tag{5.50}$$

Such a term always means new variation terms in $\delta B_\mu{}^A{}_B$:

$$\delta B_\mu{}^A{}_C\mid_{new-4}=\tfrac{i}{8}[(\bar{\chi}_\mu^{AB}\epsilon_{gBC})-(\bar{\chi}_{\mu BC}\epsilon_g^{AB})]Z^2.\tag{5.51}$$

Now, all $(\tilde{D}_\mu)^2$ variation terms vanish, as well as field strength terms. To sum up what we have obtained so far, there are the modified Lagrangian:

$$\begin{aligned}L\mid_{\tilde{D}^3,\tilde{D}^2} \;=\;& L_{sugra}^{conf}+L_{BLG}^{cov}+iA(ee_\alpha{}^\mu e_\beta{}^\nu)\bar{\chi}_\mu^{AB}\gamma^\beta\gamma^\alpha\Psi_{Aa}(\tilde{D}_\nu\bar{Z}_B^a-\tfrac{i}{2}\hat{A}\bar{\chi}_{\nu BC}\Psi^{Ca})+c.c.\\&+i\epsilon^{\mu\nu\rho}\bar{\chi}_\mu^{AC}\chi_{\nu BC}Z_A^a\tilde{D}_\rho\bar{Z}_a^B+c.c.\\&-iA(\bar{f}^{\mu AB}\gamma_\mu\Psi_{Aa}\bar{Z}_B^a+c.c.\\&-\tfrac{e}{8}\tilde{R}Z^2+\tfrac{i}{2}Z^2\bar{f}_{AB}^\mu\chi_\mu^{AB}\end{aligned}\tag{5.52}$$

and modified transformation rules:

$$\begin{aligned}\delta e_\mu{}^\alpha &= i\bar{\epsilon}_{gAB}\gamma^\alpha\chi_\mu^{AB},\\ \delta\chi_\mu^{AB} &= i\tilde{D}_\mu\epsilon_g^{AB},\\ \delta B_\mu{}^A{}_B &= i(\bar{f}^{\nu AC}\gamma_\mu\gamma_\nu\epsilon_{gBC}-\bar{f}_{BC}^\nu\gamma_\mu\gamma_\nu\epsilon_g^{AC})\\&\quad+\tfrac{i}{4}(\bar{\epsilon}_{mBD}\gamma_\mu\Psi^{a(D}Z_a^{A)}-\bar{\epsilon}_m^{AD}\gamma_\mu\Psi_{a(D}\bar{Z}_{B)}^a)\\&\quad-\tfrac{i}{4}(\bar{\epsilon}_g^{AC}\chi_{\mu DC}-\bar{\epsilon}_{gDC}\chi_\mu^{AC})Z_a^D\bar{Z}_B^a-\tfrac{i}{4}(\bar{\epsilon}_g^{DC}\chi_{\mu BC}-\bar{\epsilon}_{gBC}\chi_\mu^{DC})Z_a^A\bar{Z}_D^a-trace\\&\quad+\tfrac{i}{8}(\bar{\epsilon}_g^{AD}\chi_{\mu BD}-\bar{\epsilon}_{gBD}\chi_\mu^{AD})Z^2,\\ \delta Z_a^A &= i\bar{\epsilon}_m^{AB}\Psi_{Ba},\\ \delta\Psi_{Bd} &= \gamma^\mu\epsilon_{mAB}(\tilde{D}_\mu Z^A{}_d-i\hat{A}\bar{\chi}_\mu^{AD}\Psi_{Dd})\\&\quad+f^{ab}{}_{cd}Z^C{}_aZ^D{}_b\bar{Z}_B{}^c\epsilon_{mCD}-f^{ab}{}_{cd}Z^A{}_aZ^C{}_b\bar{Z}_C{}^c\epsilon_{mAB},\\ \delta A_\mu{}^a{}_b &= -i(\bar{\epsilon}_{mAB}\gamma_\mu\Psi^{Aa}Z_b^B-\bar{\epsilon}_m^{AB}\gamma_\mu\Psi_{Ab}\bar{Z}_B^a)\\&\quad-2i(\bar{\epsilon}_g^{AD}\chi_{\mu BD}-\bar{\epsilon}_{gBD}\chi_\mu^{AD})Z_b^B\bar{Z}_A^a.\end{aligned}\tag{5.53}$$

The coefficient A has been worked out to be $\pm\sqrt{2}$ according to the combination of relations $A\epsilon_g=\epsilon$ and $2\epsilon_g=A\epsilon$. Another coefficient $\hat{A}$ is $\pm\sqrt{2}$, determined by the requirement that $\delta\Psi$ should be supercovariant. We also add an additional *trace* term to $\delta B_\mu{}^A{}_B$, because it is believed to be traceless[3].

[3]This conclusion was obtained in the section about $\mathcal{N}=6$ pure supergravity.

5.2.2 Coupling at order $(\tilde{D}_\mu)^1$ and $(\tilde{D}_\mu)^0$

In this subsection, we continue constructing the supersymmetric Lagrangian by considering the cancelation of all variation terms which are linear to, or independent of, the covariant derivative. Following standard techniques used above, one finds that new terms of the form:

$$(\bar{\chi}_\mu \gamma^\mu \Psi) Z^3,\ (\bar{\chi}_\mu \gamma^{\rho\mu} \chi_\rho) Z^4,\ \text{and } Z^6 \tag{5.54}$$

need to be added to the coupled Lagrangian. To conserve the SUSY invariance, we also need modify the variations of those fields. In particular, we need to add

$$\delta \Psi_{Bd}\mid_{new} = \tfrac{1}{4} Z^C_c Z^D_d \bar{Z}^c_B \epsilon_{CD} + \tfrac{1}{16} Z^2 Z^A_d \epsilon_{AB}, \tag{5.55}$$

which then implies that the complete Lagrangian should contain $(\bar{\Psi}\Psi)Z^2$ without structure constants, in accordance with the existence of Z^6 term which is also independent of structure constants.

Finally, the whole Lagrangian could be obtained. The verification of its SUSY invariance is discussed in the **Appendix**.C.

5.3 The additional U(1) gauge field

In this section, we would like to comment on the the abelian gauge field that is written out explicitly in both the covariant derivative (5.1) and the Chern-Simons term (5.3) of the Lagrangian.

If the ABJM matter was coupled to the pure supergravity in the simplest way, the covariant derivative acting on scalar fields would be defined as:

$$\tilde{D}_\mu \bar{Z}^a_A = \partial_\mu \bar{Z}^a_A - B^B_{\mu A} \bar{Z}^a_B + \tilde{A}^a_{\mu b} \bar{Z}^b_A. \tag{5.56}$$

However, it doesn't preserve the local supersymmetry of the gauged action, e.g. there are terms like

$$\begin{aligned} &\tfrac{i}{8} e \bar{Z}^a_A \tilde{D}_\mu Z^A_a [(\bar{\epsilon}_{gDC} \chi^{\nu BD}) - (\bar{\epsilon}^{BD}_g \chi^\nu_{DC})] \bar{Z}^b_B Z^C_b + c.c. \\ &\tfrac{i}{16} e \bar{Z}^a_A \tilde{D}_\mu Z^A_a [(\bar{\epsilon}_{BD} \gamma^\mu \Psi^{Db}) \bar{Z}^B_b - (\bar{\epsilon}^{BD} \gamma^\mu \Psi_{Db})] + c.c. \end{aligned} \tag{5.57}$$

that remain when calculating the variation terms linear in the covariant derivative. To keep the supersymmetry, introducing an additional U(1) gauge group is necessary. Then the covariant derivative becomes

$$\tilde{D}_\mu \bar{Z}^a_A = \partial_\mu \bar{Z}^a_A - B^B_{\mu A} \bar{Z}^a_B + \tilde{A}^a_{\mu b} \bar{Z}^b_A + q \tilde{A}_\mu \bar{Z}^a_A, \tag{5.58}$$

and the term (5.57) vanishes as a consequence of the variation of the new U(1) field:

$$\begin{aligned} \delta A_\mu = & -iq(\bar{\epsilon}_{AB} \gamma_\mu \Psi^{Aa} Z^B_a - \bar{\epsilon}^{AB} \gamma_\mu \Psi_{Aa} \bar{Z}^a_B) \\ & - 2iq(\bar{\epsilon}^{AD}_g \chi_{\mu BD} - \bar{\epsilon}_{gBD} \chi^{AD}_\mu) Z^B_a \bar{Z}^a_A, \end{aligned} \tag{5.59}$$

where $q^2 = \frac{1}{16}$.

Furthermore, by adding to the Lagrangian a normalized Chern-Simons term of the new gauge field A_μ:

$$\frac{1}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \tag{5.60}$$

all relevant variation terms of the second order in covariant derivatives vanish in the same way as for $\tilde{A}_\mu{}^a{}_b$.

When the covariant derivative acts on spinors

$$\tilde{D}_\mu\psi^{Aa} = \partial_\mu\psi^{Aa} + \tfrac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\psi^{Aa} + B^A_{\mu B}\psi^{Ba} + \tilde{A}^a_{\mu b}\psi^{Ab} + qA_\mu\psi^{Aa}, \tag{5.61}$$

the U(1) field is also supported by the fact that variation terms in the form of $(\epsilon\Psi)(\Psi\Psi)Z$ without structure constant in $\delta\mathcal{L}$ vanish if $q^2 = \frac{1}{16}$.

Interestingly, the additional gauge field rises the question whether or not the U(1) gauge group is related to the structure constant $f^{ab}{}_{cd}$ since one may notice that the variation of the new gauge field is similar to that of $\tilde{A}_\mu{}^a{}_b$. Although $f^{ab}{}_{cd} = \delta^{ab}_{cd}$ is one solution of the fundamental identity, it is not clear if the U(1) gauge field can be absorbed into the structure constant.

Chapter 6

Outlook

Here we have reviewed both conformal supergravities and superconformal matter theories have been reviewed. Based on them we constructed the matter $\mathcal{N}=6$ three-dimensional superconformal action coupled to the Chern-Simons supergravity with the same supersymmetries, namely the gauged ABJM action. Recalling that the ABJM theory is believed to describe the interaction of multiple M2-branes at the infrared fixpoint, such a coupling may be of some interest.

As mentioned before, the $\mathcal{N}=6$ Chern-Simons supergravity is topological, and hence reduces to a superconformal theory when coupled to conformal matter. More interestingly, all gauge fields of the gauged ABJM theory are introduced by adding Chern-Simons terms. Considering that a Chern-Simons term doesn't introduce any new propagating degrees of freedom, what we have done here may hint at further interesting questions of the effect of Chern-Simons-like terms in gaugings of three-dimensional superconformal theories, for example, the possibility of multiple level number k.

At last, we'd like to point out that the possibility of decoupling the gauged action to obtain an generalization of the original ABJM matter action, with three gauge fields $\tilde{A}_\mu{}^a{}_b$, A_μ and $B^A_{\mu B}$ remaining. Actually, if we set $e\equiv 1$, $\tilde{\omega}\equiv 0$ and $\chi_{AB}\equiv 0$, and adopt the Minkowski metric in the gauged ABJM action, the action would again concern only the matter section, and the only non-vanishing term of δL under the modified SUSY transformations comes from the second term of $\delta B^A_{\mu B}$, i.e.

$$\frac{i}{4}(\bar{\epsilon}_{BD}\gamma_\mu\Psi^{a(D}Z^{A)}_a-\bar{\epsilon}^{AD}\gamma_\mu\Psi_{a(D}\bar{Z}^a_{B)}). \tag{6.1}$$

Though there may be little chance to work it out directly, it implies the possibility of introducing a Chern-Simons term with the R-symmetry indices into the ordinary superconformal Chern-Simons matter theories.

Appendix **A**

Fierz identities

In this appendix, we only discuss the Fierz identity in SO(1,10) case, i.e. Clifford algebra for the BLG action. Such a identity is very useful in the verification of the invariant of either the $\mathcal{N}=8$ pure supergravity Lagrangian or the BLG matter Lagrangian. The situation in any other dimensions is quite similar to the case discussed here. For a general discussion in arbitrary dimensions, refer to [41],[42] etc.

A.1 The general form of the Fierz identity

Considering the fact that all spinorial quantities are those of the eleven-dimensional Clifford algebra, one produces the $\Gamma^m (m = 0, 1, 2, ..., 10.)$ matrices as

$$\begin{cases} \Gamma^\mu = \gamma^\mu \otimes \Gamma^9, & \mu = 0, 1, 2; \\ \Gamma^I = I_{2\times 2} \otimes \Gamma^i, & I = i = 3, 4, \ldots, 10. \end{cases}$$

which, apparently, satisfy the condition $\Gamma^\mu \Gamma^I = -\Gamma^I \Gamma^\mu$.

Defining

$$\begin{cases} C_{ab} = C = \gamma^0 \otimes \Gamma^9, \\ C^{ab} = C^{-1} = -C \\ \Gamma^m = (\Gamma^m)^c{}_d\,, \end{cases}$$

in which C is considered as a metric to raise or lower indices. Then according to the definition, we can easily find a basis

$$\begin{cases} I, \Gamma^I, \Gamma^{IJ}, \Gamma^{IJK}, ..., \Gamma^{I_1 I_2 \ldots I_8} \\ C\Gamma^\mu, C\Gamma^\mu\Gamma^I, C\Gamma^\mu\Gamma^{IJ}, C\Gamma^\mu\Gamma^{IJK}, ..., C\Gamma^\mu\Gamma^{I_1 I_2 \ldots I_8}, \end{cases}$$

which can be used to expand any 32×32 matrix in series, such as the direct product of two 32-dimensional spinors ϵ, η.

However, if ϵ, η are Weyl spinors, i.e. $\Gamma^9 \epsilon = \epsilon$, $\Gamma^9 \eta = \eta$, the situation would be simplified. Recall the relationship

$$\Gamma_{I_1 \ldots I_p} = \frac{1}{(8-p)!} \Gamma_{I_1 \ldots I_p I_{p+1} \ldots I_8} \Gamma^{I_{p+1} \ldots I_8} \Gamma^9 \qquad \text{(A.1)}$$

for the SO(1,10) Clifford algebra, it follows that all terms left from the basis we mentioned above that contribute to the direct product of two weyl spinors $\epsilon_a \otimes \eta_b$ have an even number of indices, and $p \leq 4$, that is,

$$\begin{aligned}\epsilon_a \otimes \eta_b =& Z_1 \bar{\epsilon}\eta C_{ab} + Z_2 \bar{\epsilon}\Gamma_\mu \eta (\Gamma^\mu C)_{ab} \\ &+ Z_3 \bar{\epsilon}\Gamma^{AB}\eta(\Gamma^{AB}C)_{ab} + Z_4 \bar{\epsilon}\Gamma_\mu \Gamma_{AB}\eta(\Gamma^\mu \Gamma^{AB} C)_{ab} \\ &+ Z_5 \bar{\epsilon}\Gamma^{ABCD}\eta(\Gamma^{ABCD}C)_{ab} + Z_6 \bar{\epsilon}\Gamma_\mu \Gamma_{ABCD}\eta(\Gamma^\mu \Gamma^{ABCD} C)_{ab}\end{aligned} \tag{A.2}$$

A.2 How to determine the coefficients

By multiplying both sides of eq.(A.2) by same factors and then using the contraction of indices, those coefficients can be easily determined.

1. For the first coefficient Z_1,

$$\begin{aligned}C^{ab} * L.H.S. =& \bar{\epsilon}\eta \\ C^{ab} * R.H.S. =& Z_1 \bar{\epsilon}\eta C_{ab} C^{ab} + Z_4 \bar{\epsilon}\Gamma_\mu \Gamma_{AB}\eta(\Gamma^\mu \Gamma^{AB} C)_{ab} C^{ab} + Z_5 \bar{\epsilon}\Gamma^{ABCD}\eta(\Gamma^{ABCD}C)_{ab}C^{ab} \\ =& - Z_1 \bar{\epsilon}\eta \cdot Tr1.\end{aligned}$$

For weyl spinor, there is Tr1=16, then

$$Z_1 = -\frac{1}{16}. \tag{A.3}$$

2. For the second coefficient Z_2,

$$\begin{aligned}(C\Gamma^\nu)^{ab} * L.H.S. =& \bar{\epsilon}\Gamma^\nu \eta \\ (C\Gamma^\nu)^{ab} * R.H.S. =& Z_2 \bar{\epsilon}\Gamma_\mu \eta (\Gamma^\mu C)_{ab} (C\Gamma^\nu)^{ab} + Z_3 \bar{\epsilon}\Gamma_{AB}\eta(\Gamma^{AB}C)_{ab} (C\Gamma^\nu)^{ab} \\ &+ Z_6 \bar{\epsilon}\Gamma_\mu \Gamma_{AB}\eta(\Gamma^\mu \Gamma^{ABCD} C)_{ab} (C\Gamma^\nu)^{ab} \\ =& 16 Z_2 \bar{\epsilon}\Gamma^\nu \eta.\end{aligned}$$

Then

$$Z_2 = \frac{1}{16}. \tag{A.4}$$

3. For the third coefficient Z_3,

$$\begin{aligned}\left(C\Gamma^{EF}\right)^{ab} * L.H.S. =& \bar{\epsilon}\Gamma^{EF}\eta \\ \left(C\Gamma^{EF}\right)^{ab} * R.H.S. =& Z_2 \bar{\epsilon}\Gamma_\mu \eta (\Gamma^\mu C)_{ab} \left(C\Gamma^{EF}\right)^{ab} + Z_3 \bar{\epsilon}\Gamma_{AB}\eta(\Gamma^{AB}C)_{ab} \left(C\Gamma^{EF}\right)^{ab} \\ &+ Z_6 \bar{\epsilon}\Gamma_\mu \Gamma_{AB}\eta(\Gamma^\mu \Gamma^{ABCD} C)_{ab} \left(C\Gamma^{EF}\right)^{ab} \\ =& - 32 Z_3 \bar{\epsilon}\Gamma_{EF}\eta.\end{aligned}$$

Then

$$Z_3 = -\frac{1}{32}. \tag{A.5}$$

4. For the fourth coefficient Z_4,

$$
\begin{aligned}
\left(C\Gamma^{\nu}\Gamma^{CD}\right)^{ab} * L.H.S. =&\bar{\epsilon}\Gamma^{\nu}\Gamma^{CD}\eta,\\
C^{ab} * R.H.S. =&Z_1\bar{\epsilon}\eta C_{ab}\left(C\Gamma^{\nu}\Gamma^{CD}\right)^{ab} + Z_4\bar{\epsilon}\Gamma_{\mu}\Gamma_{AB}\eta(\Gamma^{\mu}\Gamma^{AB}C)_{ab}\left(C\Gamma^{\nu}\Gamma^{CD}\right)^{ab}\\
&+ Z_5\bar{\epsilon}\Gamma^{ABCD}\eta(\Gamma^{ABCD}C)_{ab}\left(C\Gamma^{\nu}\Gamma^{CD}\right)^{ab}\\
=&32Z_4\bar{\epsilon}\Gamma^{\nu}\Gamma^{CD}\eta.
\end{aligned}
$$

Then

$$
Z_4 = \frac{1}{32}. \tag{A.6}
$$

5. For the fifth coefficient Z_5,

$$
\begin{aligned}
\left(C\Gamma^{EFGH}\right)^{ab} * L.H.S. =&\bar{\epsilon}\Gamma^{EFGH}\eta = \bar{\epsilon}\Gamma^{EFGH^+}\eta,\\
\left(C\Gamma^{EFGH}\right)^{ab} * R.H.S. =&Z_1\bar{\epsilon}\eta C_{ab}\left(C\Gamma^{EFGH}\right)^{ab} + Z_4\bar{\epsilon}\Gamma_{\mu}\Gamma_{AB}\eta(\Gamma^{\mu}\Gamma^{AB}C)_{ab}\left(C\Gamma^{EFGH}\right)^{ab}\\
&+ Z_5\bar{\epsilon}\Gamma^{ABCD}\eta(\Gamma^{ABCD}C)_{ab}\left(C\Gamma^{EFGH}\right)^{ab}\\
=&- Z_5\bar{\epsilon}\frac{1}{2}\left(\delta^{EFGH}_{ABCD} + \frac{1}{4!}\varepsilon_{ABCD}{}^{EFGH}\right)\Gamma^{ABCD}\eta\cdot 2\cdot 4! Tr1\\
=&- 32\cdot 4!\cdot Z_5\bar{\epsilon}\Gamma^{EFGH^+}\eta \;\; \text{for Weyl spinor.}
\end{aligned}
$$

Then

$$
Z_5 = -\frac{1}{32\cdot 4!}. \tag{A.7}
$$

$$
\left(C\Gamma^{\nu}\Gamma^{EFGH}\right)^{ab} * L.H.S. =\bar{\epsilon}\Gamma^{\nu}\Gamma^{EFGH}\eta = \bar{\epsilon}\Gamma^{\nu}\Gamma^{EFGH^+}\eta.
$$

Similarly,

$$
\left(C\Gamma^{\nu}\Gamma^{EFGH}\right)^{ab} * R.H.S. =32\cdot 4!\cdot Z_5\bar{\epsilon}\Gamma^{\nu}\Gamma^{EFGH^+}\eta,
$$

for weyl spinor. Then

$$
Z_6 = \frac{1}{32\cdot 4!}. \tag{A.8}
$$

A.3 The Fierz identity

According to these coefficients above, the Fierz identity of Weyl spinors can be written as:

$$
\begin{aligned}
\epsilon_a\otimes\eta_b =& -\frac{1}{16}\bar{\epsilon}\eta C_{ab} + \frac{1}{16}\bar{\epsilon}\Gamma_{\mu}\eta(\Gamma^{\mu}C)_{ab}\\
&-\frac{1}{32}\bar{\epsilon}\Gamma^{AB}\eta(\Gamma^{AB}C)_{ab} + \frac{1}{32}\bar{\epsilon}\Gamma_{\mu}\Gamma_{AB}\eta(\Gamma^{\mu}\Gamma^{AB}C)_{ab}\\
&-\frac{1}{32\cdot 4!}\bar{\epsilon}\Gamma^{ABCD}\eta(\Gamma^{ABCD}C)_{ab} + \frac{1}{32\cdot 4!}\bar{\epsilon}\Gamma_{\mu}\Gamma_{ABCD}\eta(\Gamma^{\mu}\Gamma^{ABCD}C)_{ab}
\end{aligned} \tag{A.9}
$$

If both sides of the equ.A.9 are multiplied by C^{ab} on the right, one would obtain

$$\begin{aligned}\epsilon\otimes\bar{\eta}=&-\frac{1}{16}\bar{\epsilon}\eta\mathbb{I}+\frac{1}{16}\bar{\epsilon}\Gamma_\mu\eta\Gamma^\mu\\&-\frac{1}{32}\bar{\epsilon}\Gamma^{AB}\eta\Gamma^{AB}+\frac{1}{32}\bar{\epsilon}\Gamma_\mu\Gamma_{AB}\eta\Gamma^\mu\Gamma^{AB}\\&-\frac{1}{32\cdot 4!}\bar{\epsilon}\Gamma^{ABCD}\eta\Gamma^{ABCD}+\frac{1}{32\cdot 4!}\bar{\epsilon}\Gamma_\mu\Gamma_{ABCD}\eta\Gamma^\mu\Gamma^{ABCD}\\=&+\frac{1}{16}\bar{\epsilon}\Gamma_\mu\eta\Gamma^\mu-\frac{1}{32}\bar{\epsilon}\Gamma^{AB}\eta\Gamma^{AB}+\frac{1}{32\cdot 4!}\bar{\epsilon}\Gamma_\mu\Gamma_{ABCD}\eta\Gamma^\mu\Gamma^{ABCD}\ (a)\\&-\frac{1}{16}\bar{\epsilon}\eta\mathbb{I}+\frac{1}{32}\bar{\epsilon}\Gamma_\mu\Gamma_{AB}\eta\Gamma^\mu\Gamma^{AB}-\frac{1}{32\cdot 4!}\bar{\epsilon}\Gamma^{ABCD}\eta\Gamma^{ABCD}\qquad (s)\end{aligned}\tag{A.10}$$

It is easy to prove that the expansion is also right for two anti-weyl spinors, such as, Ψ's in the BLG action.

A.4 Gamma matrices

Here is some identities used above, most of which have been proven in [42]:

$$\begin{aligned}Tr(\Gamma_\mu)=&0,\\Tr(\Gamma_I)=&0,\\Tr(\Gamma_{IJ})=&0,\\Tr(\Gamma_\mu\Gamma^\nu)=&g^{\mu\nu}Tr1,\\Tr(\Gamma^{ABCD})=&Tr(\Gamma^{[ABC}\Gamma^{D]}),\\=&0\\Tr(\Gamma^\mu\Gamma^{AB..})=&Tr\left((\gamma^\mu\otimes\Gamma^9)(\mathbb{I}\otimes\Gamma^{ab..})\right),\\=&Tr\left(\gamma^\mu\otimes(\Gamma^9\cdot\Gamma^{ab..})\right),\\=&0\\Tr(\Gamma^{ABCD}\Gamma^{EFGH})=&Tr(\Gamma_{ABCD}{}^{EFGH}+4!(-1)^6\cdot Tr1\cdot\delta^{EFGH}_{ABCD},\\&+M_1\cdot\Gamma_{ABC}{}^{EFG}+M_2\cdot\Gamma_{ABC}{}^{EF}+M_3\cdot\Gamma_A{}^E),\\=&Tr(\Gamma_{ABCD}{}^{EFGH}+4!(-1)^6\cdot Tr1\cdot\delta^{EFGH}_{ABCD}).\end{aligned}$$

Appendix B

SUSY invariance of $\mathcal{N} = 6$ pure supergravity

B.1 The variation of the Lagrangian

The lagrangian of the pure topological $\mathcal{N}$=6 conformal supergravity is:

$$\begin{aligned}\mathcal{L}^{conf}_{sugra} = &\frac{1}{2}\varepsilon^{\mu\nu\rho}Tr_\alpha(\widetilde{\omega}_\mu\partial_\nu\widetilde{\omega}_\rho + \frac{2}{3}\widetilde{\omega}_\mu\widetilde{\omega}_\nu\widetilde{\omega}_\rho)\\ &- 2\varepsilon^{\mu\nu\rho}Tr_A(B_\mu\partial_\nu B_\rho + \frac{2}{3}B_\mu B_\nu B_\rho)\\ &- 4i(e^{-1}e^\alpha{}_\mu e^\beta{}_\nu)(\overline{f}^{\mu AB}\gamma_\beta\gamma_\alpha f^\mu{}_{AB}).\end{aligned}$$

and the SUSY transformations then take the form:

$$\begin{aligned}\delta e^\alpha{}_\mu &= i\bar{\epsilon}^{AB}\gamma^\alpha\chi_{\mu AB}\\ \delta\chi_{\mu\ AB} &= \widetilde{D}_\mu\epsilon_{AB}\\ \delta B_\mu{}^A{}_C &= \frac{1}{e}[\bar{\epsilon}^{AB}\gamma^\alpha\gamma^\mu f^\alpha_{BC} - \overline{f}^{\alpha\ AB}\gamma^\mu\gamma^\alpha\epsilon_{BC}]\end{aligned} \tag{B.1}$$

Substituting these variations into $\delta\mathcal{L}$, it gives:

$$\delta\mathcal{L}_1 = 4(\bar{\epsilon}^{AB}\gamma_\alpha\gamma_\beta f^\alpha{}_{AB})(\overline{f}^\mu{}_{CD}\gamma^\beta\chi_\mu{}^{CD}) \tag{B.2}$$

$$\delta\mathcal{L}_2 = 8(\overline{f}^\mu{}_{AB}\gamma_\nu\gamma_\tau f^{\nu AB})(\bar{\epsilon}_{CD}\gamma^\tau\chi_\mu^{CD}) \tag{B.3}$$

$$- 4(\overline{f}^\mu{}_{AB}\gamma_\nu\gamma_\mu f^{\nu\ AB})(\bar{\epsilon}_{CD}\gamma^\sigma\chi_\sigma{}^{CD}) \tag{B.4}$$

$$\delta\mathcal{L}_3 = 4(\overline{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\gamma\chi_\lambda{}^{AB})(\bar{\epsilon}^{CD}\gamma_\rho f^\gamma{}_{CD})\epsilon^{\beta\lambda\rho} \tag{B.5}$$

$$- 2(\overline{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\rho\chi_\lambda{}^{AB})(\bar{\epsilon}^{CD}\gamma_\nu f^\nu{}_{CD})\epsilon^{\beta\lambda\rho} \tag{B.6}$$

$$\delta\mathcal{L}_4 = 8(\overline{f}^\alpha{}_{AB}\gamma_\nu\gamma_\alpha\chi_\lambda{}^{CB})(\bar{\epsilon}^{AD}\gamma_\mu\gamma_\rho f^\mu{}_{DC})\epsilon^{\nu\rho\lambda} \tag{B.7}$$

$$- 8(\overline{f}^\alpha{}_{AB}\gamma_\nu\gamma_\alpha\chi_\lambda{}^{CB})(\bar{\epsilon}_{DC}\gamma_\mu\gamma_\rho f^{\mu\ AD})\epsilon^{\nu\rho\lambda} \tag{B.8}$$

B.2 Fierzing in the SO(2, 1) case

Then each term can be Fierzed into two kinds of terms:

$$\begin{aligned}
\delta\mathcal{L}_1 &= 4(\bar{\epsilon}^{AB}\gamma_\alpha\gamma_\beta f^\alpha{}_{AB})(\bar{f}^\mu{}_{CD}\gamma^\beta\chi_\mu{}^{CD}) \\
&= -2[(\bar{f}^\mu{}_{CD}\gamma^\beta\gamma_\alpha\gamma_\beta f^\alpha{}_{AB})(\bar{\epsilon}^{AB}\chi_\mu{}^{CD}) && \text{(B.9)}\\
&\quad + (\bar{f}^\mu{}_{CD}\gamma^\beta\gamma^\delta\gamma_\alpha\gamma_\beta f^\alpha{}_{AB})(\bar{\epsilon}^{AB}\gamma_\delta\chi_\mu{}^{CD})] && \text{(B.10)}\\
\delta\mathcal{L}_2 &= 8(\bar{f}^\mu{}_{AB}\gamma_\nu\gamma_\tau f^{\nu AB})(\bar{\epsilon}_{CD}\gamma^\tau\chi_\mu^{CD}) && \text{(B.11)}\\
&\quad - 4(\bar{f}^\mu{}_{AB}\gamma_\nu\gamma_\mu f^{\nu\ AB})(\bar{\epsilon}_{CD}\gamma^\sigma\chi_\sigma{}^{CD}) && \text{(B.12)}\\
\text{the first half of } \delta\mathcal{L}_3 &= 4(\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\gamma\chi_\lambda{}^{AB})(\bar{\epsilon}^{CD}\gamma_\rho f^\gamma{}_{CD})\epsilon^{\beta\lambda\rho} \\
&= -2\epsilon^{\beta\lambda\rho}[(\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\gamma\gamma_\rho f^\gamma{}_{CD})(\bar{\epsilon}^{CD}\chi_\lambda{}^{AB}) && \text{(B.13)}\\
&\quad + (\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\gamma\gamma^\delta\gamma_\rho f^\gamma{}_{CD})(\bar{\epsilon}^{CD}\gamma_\delta\chi_\lambda{}^{AB})] && \text{(B.14)}\\
\text{the second half of } \delta\mathcal{L}_3 &= -2(\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\rho\chi_\lambda{}^{AB})(\bar{\epsilon}^{CD}\gamma_\nu f^\nu{}_{CD})\epsilon^{\beta\lambda\rho} \\
&= \epsilon^{\beta\lambda\rho}[(\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\rho\gamma_\nu f^\nu{}_{CD})(\bar{\epsilon}^{CD}\chi_\lambda{}^{AB}) && \text{(B.15)}\\
&\quad + (\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\rho\gamma^\delta\gamma_\nu f^\nu{}_{CD})(\bar{\epsilon}^{CD}\gamma_\delta\chi_\lambda{}^{AB})] && \text{(B.16)}\\
\text{the first half of } \delta\mathcal{L}_4 &= 8(\bar{f}^\alpha{}_{AB}\gamma_\nu\gamma_\alpha\chi_\lambda{}^{CB})(\bar{\epsilon}^{AD}\gamma_\mu\gamma_\rho f^\mu{}_{DC})\epsilon^{\nu\rho\lambda} \\
&= -4\epsilon^{\beta\lambda\rho}(\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma^\delta\gamma_\mu\gamma_\rho f^\mu{}_{CD})(\bar{\epsilon}^{AD}\gamma_\delta\chi_\lambda{}^{CB}) && \text{(B.17)}\\
\text{the second half of } \delta\mathcal{L}_4 &= -8(\bar{f}^\alpha{}_{AB}\gamma_\nu\gamma_\alpha\chi_\lambda{}^{CB})(\bar{\epsilon}_{DC}\gamma_\mu\gamma_\rho f^{\mu\ AD})\epsilon^{\nu\rho\lambda} \\
&= 4\epsilon^{\beta\lambda\rho}[(\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\mu\gamma_\rho f^{\mu\ AD})(\bar{\epsilon}\ {}_{CD}\chi_\lambda{}^{CB}) && \text{(B.18)}\\
&\quad + (\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma^\delta\gamma_\mu\gamma_\rho f^{\mu\ AD})(\bar{\epsilon}\ {}_{CD}\gamma_\delta\chi_\lambda{}^{CB})] && \text{(B.19)}
\end{aligned}$$

B.3 The first half of the Fierzing results

Considering the R.H.S. of the Fierz identity for the 2-component spinors, its first half contributes $(\bar{\epsilon}\cdot\chi)$-terms to the total variation:

such terms from $\delta\mathcal{L}_1$

$$= -2(\bar{f}^\alpha{}_{CD}\gamma_\alpha f^\lambda{}_{AB})(\bar{\epsilon}^{CD}\chi_\lambda{}^{AB}) \tag{B.20}$$

such terms from the first half of $\delta\mathcal{L}_3$

$$= -4\epsilon^{\lambda\gamma\alpha}(\bar{f}_{\alpha\ AB}f_{\gamma\ CD})(\bar{\epsilon}^{CD}\chi_\lambda{}^{AB}) \tag{B.21}$$

$$-4(\bar{f}^\gamma{}_{AB}\gamma^\lambda f_{\gamma\ CD})(\bar{\epsilon}^{CD}\chi_\lambda{}^{AB}) \tag{B.22}$$

such terms from the second half of $\delta\mathcal{L}_3$

$$= -2(\bar{f}^\lambda{}_{AB}\gamma^\nu f_{\nu\ CD})(\bar{\epsilon}^{CD}\chi_\lambda{}^{AB}) \tag{B.23}$$

such terms from $\delta\mathcal{L}_4$ (only the second half contributes)

$$=4\epsilon^{\beta\rho\lambda}(\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\mu\gamma_\rho f^{\mu\ DA})(\bar{\epsilon}\ {}_{CD}\chi_\lambda{}^{CB}) \tag{B.24}$$

$$[=4\cdot\frac{1}{2}\epsilon^{DAEF}\epsilon^{\beta\rho\lambda}(\bar{f}^\alpha{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\mu\gamma_\rho f^\mu{}_{EF})(\bar{\epsilon}\ {}_{CD}\chi_\lambda{}^{CB})]$$

By expanding the L.H.S of the equation below,

$$10 \cdot \frac{1}{2}\epsilon^{DAEF}\epsilon^{\beta\rho\lambda}(\overline{f}^{\alpha}{}_{A[B}\gamma_\beta\gamma_\alpha\gamma_\mu\gamma_\rho f^{\mu}{}_{EF})(\overline{\epsilon}{}_{\ CD]}\chi_\lambda{}^{CB}) = 0$$

we could easily obtain such a relationship:

$$\epsilon^{\beta\rho\lambda}(\overline{f}^{\alpha}{}_{AB}\gamma_\beta\gamma_\alpha\gamma_\mu\gamma_\rho f^{\mu\ DA})(\overline{\epsilon}{}_{\ CD}\chi_\lambda{}^{CB}) = \frac{1}{2}\cdot\epsilon^{\beta\rho\lambda}(\overline{f}^{\alpha}{}_{CD}\gamma_\beta\gamma_\alpha\gamma_\mu\gamma_\rho f^{\mu}{}_{AB})(\overline{\epsilon}{}^{\ CD}\chi_\lambda{}^{AB}),$$

then, using this relationship, those terms from $\delta\mathcal{L}_4$

$$=4\cdot\epsilon^{\lambda\mu\alpha}(\overline{f}_{\alpha\ AB} f_{\mu\ CD})(\overline{\epsilon}^{CD}\chi_\lambda{}^{AB}) \tag{B.25}$$

$$+\,4(\overline{f}^{\mu}{}_{AB}\gamma^\lambda f_{\mu\ CD})(\overline{\epsilon}^{CD}\chi_\lambda{}^{AB}) \tag{B.26}$$

To sum up, these terms from $\delta\mathcal{L}_1$ cancel similar terms from the second half of $\delta\mathcal{L}_3$, and these terms from $\delta\mathcal{L}_4$ cancel similar ones from the first half of $\delta\mathcal{L}_4$. That is, all $(\overline{\epsilon}\cdot\chi)$-terms in the variation cancel.

B.4 The second half of the Fierzing results

Considering the R.H.S. of the Fierz identity for the 2-component spinors, its second half contributes $(\overline{\epsilon}\cdot\gamma\cdot\chi)$-terms. Let's expand these terms, which actually are the rest of the variation.

1.All such terms

such terms from $\delta\mathcal{L}_1$

$$=-\,6(\overline{f}^{\mu}{}_{AB} f_{\alpha\ CD})(\overline{\epsilon}^{CD}\gamma^\alpha\chi_\mu{}^{AB}) \tag{B.27}$$

$$-\,2(\overline{f}^{\mu}{}_{AB}\gamma^{\alpha\delta} f_{\alpha\ CD})(\overline{\epsilon}^{CD}\gamma_\delta\chi_\mu{}^{AB}) \tag{B.28}$$

such terms from the first half of $\delta\mathcal{L}_2$

$$=8(\overline{f}^{\mu}{}_{AB} f_\nu{}^{AB})(\overline{\epsilon}^{CD}\gamma^\nu\chi_{\mu\ CD}) \tag{B.29}$$

$$+\,8(\overline{f}^{\mu}{}_{AB}\gamma^{\nu\tau} f_\nu{}^{AB})(\overline{\epsilon}^{CD}\gamma_\tau\chi_{\mu\ CD}) \tag{B.30}$$

such terms from the second half of $\delta\mathcal{L}_2$

$$=-\,4(\overline{f}^{\mu}{}_{AB} f_\mu{}^{AB})(\overline{\epsilon}_{CD}\gamma^\sigma\chi_\sigma{}^{CD}) \tag{B.31}$$

$$-\,4(\overline{f}^{\mu}{}_{AB}\gamma_{\nu\mu} f^{\nu\ AB})(\overline{\epsilon}_{CD}\gamma^\sigma\chi_\sigma{}^{CD}) \tag{B.32}$$

such terms from the first half of $\delta\mathcal{L}_3$

$$=-4(\overline{f}^{\delta}{}_{AB}f^{\lambda}{}_{CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB}) \tag{B.33}$$
$$+4(\overline{f}^{\lambda}{}_{AB}f_{\lambda\ CD})(\overline{\epsilon}^{CD}\gamma^{\delta}\chi_{\delta}{}^{AB}) \tag{B.34}$$
$$+4(\overline{f}^{\lambda}{}_{AB}f^{\delta}{}_{CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB}) \tag{B.35}$$
$$-4(\overline{f}^{\lambda}{}_{AB}\gamma^{\gamma\delta}f_{\gamma\ CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB}) \tag{B.36}$$
$$-4(\overline{f}_{\alpha\ AB}\gamma^{\delta\alpha}f^{\lambda}{}_{CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB}) \tag{B.37}$$
$$+4(\overline{f}_{\alpha\ AB}\gamma^{\gamma\alpha}f_{\gamma\ CD})(\overline{\epsilon}^{CD}\gamma^{\lambda}\chi_{\lambda}{}^{AB}) \tag{B.38}$$

such terms from the second half of $\delta\mathcal{L}_3$

$$=-2(\overline{f}^{\lambda}{}_{AB}f^{\delta}{}_{CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB}) \tag{B.39}$$
$$+2(\overline{f}^{\lambda}{}_{AB}\gamma^{\gamma\delta}f_{\gamma\ CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB}) \tag{B.40}$$

such terms from the first half of $\delta\mathcal{L}_4$

$$=-8(\overline{f}^{\gamma}{}_{AB}f_{\gamma\ CD})(\overline{\epsilon}^{AD}\gamma^{\lambda}\chi_{\lambda}{}^{CB}) \tag{B.41}$$
$$+8(\overline{f}^{\lambda}{}_{AB}f^{\delta}{}_{CD})(\overline{\epsilon}^{AD}\gamma_{\delta}\chi_{\lambda}{}^{CB}) \tag{B.42}$$
$$+8(\overline{f}^{\delta}{}_{AB}f^{\lambda}{}_{CD})(\overline{\epsilon}^{AD}\gamma_{\delta}\chi_{\lambda}{}^{CB}) \tag{B.43}$$
$$-8(\overline{f}_{\alpha\ AB}\gamma^{\gamma\alpha}f_{\gamma\ CD})(\overline{\epsilon}^{AD}\gamma^{\lambda}\chi_{\lambda}{}^{CB}) \tag{B.44}$$
$$+8(\overline{f}_{\gamma\ AB}\gamma^{\delta\gamma}f^{\lambda}{}_{CD})(\overline{\epsilon}^{AD}\gamma_{\delta}\chi_{\lambda}{}^{CB}) \tag{B.45}$$
$$+8(\overline{f}^{\lambda}{}_{AB}\gamma^{\gamma\delta}f_{\gamma\ CD})(\overline{\epsilon}^{AD}\gamma_{\delta}\chi_{\lambda}{}^{CB}) \tag{B.46}$$

such terms from the second half of $\delta\mathcal{L}_4$

$$=8(\overline{f}^{\gamma}{}_{AB}f_{\gamma}{}^{AD})(\overline{\epsilon}_{CD}\gamma^{\lambda}\chi_{\lambda}{}^{CB}) \tag{B.47}$$
$$-8(\overline{f}^{\lambda}{}_{AB}f^{\delta\ AD})(\overline{\epsilon}_{CD}\gamma_{\delta}\chi_{\lambda}{}^{CB}) \tag{B.48}$$
$$-8(\overline{f}^{\delta}{}_{AB}f^{\lambda\ AD})(\overline{\epsilon}_{CD}\gamma_{\delta}\chi_{\lambda}{}^{CB}) \tag{B.49}$$
$$+8(\overline{f}_{\alpha\ AB}\gamma^{\gamma\alpha}f_{\gamma}{}^{AD})(\overline{\epsilon}_{CD}\gamma^{\lambda}\chi_{\lambda}{}^{CB}) \tag{B.50}$$
$$-8(\overline{f}_{\gamma\ AB}\gamma^{\delta\gamma}f^{\lambda\ AD})(\overline{\epsilon}_{CD}\gamma_{\delta}\chi_{\lambda}{}^{CB}) \tag{B.51}$$
$$-8(\overline{f}^{\lambda}{}_{AB}\gamma^{\gamma\delta}f_{\gamma}{}^{AD})(\overline{\epsilon}_{CD}\gamma_{\delta}\chi_{\lambda}{}^{CB}) \tag{B.52}$$

2.$(\overline{f}^{\mu}f_{\mu})(\overline{\epsilon}\gamma^{\lambda}\chi_{\lambda})$-terms

Terms in this form are Term.(B.31), Term.(B.34), Term.(B.41) and Term.(B.47). By expanding the L.H.S of the equation below,

$$5\cdot\frac{1}{2}\epsilon_{EFCD}(\overline{f}^{\gamma}{}_{AB}f_{\gamma}{}^{[EF})(\overline{\epsilon}^{\ AD}\gamma^{\lambda}\chi_{\lambda}{}^{C]B})=0$$

we could easily obtain such a relationship:

$$\begin{aligned}-2(\overline{f}^{\gamma}{}_{AB}f_{\gamma\ CD})(\overline{\epsilon}\ ^{AD}\gamma^{\lambda}\chi_{\lambda}{}^{CB})+(\overline{f}^{\gamma}{}_{AB}f_{\gamma\ CD})(\overline{\epsilon}\ ^{CD}\gamma^{\lambda}\chi_{\lambda}{}^{AB})\\ =2(\overline{f}^{\gamma}{}_{AB}f_{\gamma}{}^{AD})(\overline{\epsilon}\ _{CD}\gamma^{\lambda}\chi_{\lambda}{}^{CB})\end{aligned} \tag{B.53}$$

That is,

$$Term.(B.34)+Term.(B.41)=Term.(B.47), \tag{B.54}$$

then the sum of these terms can be expressed only as [Term.(B.31)+2Term.(B.47)]. Again by expanding the L.H.S of the equation below,

$$5\cdot\frac{1}{2}\epsilon_{CDEF}(\overline{f}^{\mu}{}_{AB}f_{\mu}{}^{A[B})(\overline{\epsilon}\ ^{EF}\gamma^{\sigma}\chi_{\sigma}{}^{CD]})=0$$

we could easily obtain such a relationship:

$$(\overline{f}^{\mu}{}_{AB}f_{\mu}{}^{AB})(\overline{\epsilon}\ _{CD}\gamma^{\sigma}\chi_{\sigma}{}^{CD})=4(\overline{f}^{\mu}{}_{AB}f_{\mu}{}^{AD})(\overline{\epsilon}\ _{CD}\gamma^{\sigma}\chi_{\sigma}{}^{CD}),$$

which means

$$Term.(B.31)+2Term.(B.47)=0. \tag{B.55}$$

So all $(\overline{f}^{\mu}f_{\mu})(\overline{\epsilon}\gamma^{\lambda}\chi_{\lambda})$-terms cancel.

3.$(\overline{f}^{\mu}\gamma_{\nu\mu}f^{\nu})(\overline{\epsilon}\gamma^{\lambda}\chi_{\lambda})$-terms

Terms in this form are Term.(B.32), Term.(B.38), Term.(B.44) and Term.(B.50). Similarly to the above case, we also obtain:

$$Term.(B.38)+Term.(B.44)=Term.(B.50), \tag{B.56}$$

and

$$Term.(B.32)+2Term.(B.50)=0, \tag{B.57}$$

(to get the later equation, one would find $\overline{f}^{\mu}{}_{AB}\gamma_{\nu\mu}f^{\nu\ AD}=\overline{f}^{\nu\ AD}\gamma_{\mu\nu}f^{\mu}{}_{AB}$ useful.)

Then, of course, all $(\overline{f}^{\mu}\gamma_{\nu\mu}f^{\nu})(\overline{\epsilon}\gamma^{\lambda}\chi_{\lambda})$-terms cancel.

4.$(\overline{f}f)(\overline{\epsilon}\gamma^{\sigma}\chi_{\lambda})$**-terms**

Terms in this form are Term.(B.27), Term.(B.29), Term.(B.33), Term.(B.35), Term.(B.39), Term.(B.42), Term.(B.43), Term.(B.48) and Term.(B.49).

$$Term.(B.27)+Term.(B.35)+Term.(B.39)=-\,4(\overline{f}^{\lambda}{}_{AB}f^{\delta}{}_{CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB}) \tag{B.58}$$

$$Term.(B.33)=-\,4(\overline{f}^{\delta}{}_{AB}f^{\lambda}{}_{CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB}) \tag{B.59}$$

Similarly to the above case of Equ.(B.53),

$$\begin{aligned}&(\overline{f}^{\lambda}{}_{AB}f^{\delta}{}_{CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB})-2(\overline{f}^{\lambda}{}_{AB}f^{\delta}{}_{CD})(\overline{\epsilon}^{AD}\gamma_{\delta}\chi_{\lambda}{}^{CB})\\&\qquad=2(\overline{f}^{\lambda}{}_{AB}f^{\delta\ AD})(\overline{\epsilon}_{CD}\gamma_{\delta}\chi_{\lambda}{}^{CB}),\end{aligned}\tag{B.60}$$

which means

$$[Term.(B.27)+Term.(B.35)+Term.(B.39)]+Term.(B.42)=Term.(B.48)$$
$$[Term.(B.27)+Term.(B.35)+Term.(B.39)]+Term.(B.42)+Term.(B.48)=2Term.(B.48)$$

$$Term.(B.33)+Term.(B.43)=Term.(B.49)$$
$$Term.(B.33)+Term.(B.43)+Term.(B.49)=2Term.(B.49)$$

And apparently,according to the equation below,

$$5\cdot\frac{1}{2}\epsilon_{CDEF}(\overline{f}^{\mu}{}_{AB}f^{\nu\ A[B})(\overline{\epsilon}^{\ EF}\gamma_{\nu}\chi_{\mu}{}^{CD]})=0$$

we could easily obtain such a relationship:

$$\begin{aligned}&(\overline{f}^{\mu}{}_{AB}f^{\nu\ AB})(\overline{\epsilon}_{\ CD}\gamma_{\nu}\chi_{\mu}{}^{CD})\\&\qquad=2(\overline{f}^{\mu}{}_{AD}f^{\nu\ AB})(\overline{\epsilon}^{\ CD}\gamma_{\nu}\chi_{\mu\ CB})\\&\qquad+2(\overline{f}^{\mu}{}_{AB}f^{\nu\ AD})(\overline{\epsilon}_{\ CD}\gamma_{\nu}\chi_{\mu}{}^{CB}),\end{aligned}$$

Then

$$Term.(B.29)+2Term.(B.48)+2Term.(B.49)=0\tag{B.61}$$

That is, all $(\overline{f}f)(\overline{\epsilon}\gamma^{\sigma}\chi_{\lambda})$-terms cancel.

5.$(\overline{f}\gamma_{\mu\delta}f)(\overline{\epsilon}\gamma^{\delta}\chi_{\lambda})$-terms

For Term.(B.28), Term.(B.30), Term.(B.36), Term.(B.37), Term.(B.40), Term.(B.45), Term.(B.46), Term.(B.51) and Term.(B.52).

$$Term.(B.28)+Term.(B.36)+Term.(B.40)=-4(\overline{f}_{\lambda\ AB}\gamma^{\gamma\delta}f_{\gamma\ CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB})$$
$$Term.(B.37)=-4(\overline{f}_{\gamma\ AB}\gamma^{\delta\gamma}f^{\lambda}_{CD})(\overline{\epsilon}^{CD}\gamma_{\delta}\chi_{\lambda}{}^{AB})$$

Similarly to the above case of Equ.(B.53),

$$[Term.(B.28)+Term.(B.36)+Term.(B.40)]+Term.(B.46)=Term.(B.52)$$
$$[Term.(B.28)+Term.(B.36)+Term.(B.40)]+Term.(B.46)+Term.(B.52)=2Term.(B.52)$$

$$Term.(B.37)+Term.(B.45)=Term.(B.51)$$
$$Term.(B.37)+Term.(B.45)+Term.(B.51)=2Term.(B.51)$$

And according to the equation below,

$$5 \cdot \frac{1}{2}\epsilon_{CDEF}(\overline{f}^{\lambda}{}_{AB}\gamma^{\gamma\delta}f_{\gamma}{}^{A[B})(\overline{\epsilon}^{\ EF}\gamma_{\delta}\chi_{\lambda}{}^{CD]}) = 0$$

we could easily obtain such a relationship:

$$\begin{aligned}
&(\overline{f}^{\lambda}{}_{AB}\gamma^{\gamma\delta}f_{\gamma}{}^{AB})(\overline{\epsilon}_{\ CD}\gamma_{\delta}\chi_{\lambda}{}^{CD}) \\
&\qquad =2(\overline{f}^{\lambda}{}_{AB}\gamma^{\gamma\delta}f_{\gamma}{}^{AD})(\overline{\epsilon}^{\ CD}\gamma_{\delta}\chi_{\lambda\ CB}) + 2(\overline{f}^{\lambda}{}_{AD}\gamma^{\gamma\delta}f_{\gamma}{}^{AB})(\overline{\epsilon}^{\ CD}\gamma_{\delta}\chi_{\lambda\ CB}) \\
&\qquad =2(\overline{f}^{\lambda}{}_{AB}\gamma^{\gamma\delta}f_{\gamma}{}^{AD})(\overline{\epsilon}^{\ CD}\gamma_{\delta}\chi_{\lambda\ CB}) + 2(\overline{f}_{\gamma}{}^{AB}\gamma^{\delta\gamma}f^{\lambda}{}_{AD})(\overline{\epsilon}^{\ CD}\gamma_{\delta}\chi_{\lambda\ CB})
\end{aligned}$$

Then

$$Term.(B.30) + 2Term.(B.52) + 2Term.(B.51) = 0 \tag{B.62}$$

That is, all terms in the form of $(\overline{f}\gamma_{\mu\delta}f)(\overline{\epsilon}\gamma^{\delta}\chi_{\lambda})$ cancel.

To sum up, all terms cancel, which gives a *zero* variation, so the Lagrangian is SUSY invariant.

Appendix C

SUSY invariance of gauged ABJM action

Let's check the supersymmetry[1] of the gauged ABJM action, whose explicit expression has been given in **Chapter** 5.1 with the corresponding supersymmetric transformation rules. It's easier to do it separately, i.e. to compute each kind of variation terms at a time, instead of writing down all variation terms at the beginning.

In this appendix, we analyze most of the variation terms and prove them to be vanish. Then follows a final discussion on terms remaining. Variation terms which have vanished either in the original ABJM action or in the pure supergravity are not discussed in details, though they are mentioned sometimes.

A trick we use here is, we still consider the original form of the Dirac term as the actual one when dealing with its SUSY variation terms including only two fermionic variables. It's because under SUSY transformations the self-conjugate form of the Dirac term

$$-\frac{1}{2}(ie\bar{\Psi}^{Aa}\gamma^{\mu}\tilde{D}_{\mu}\Psi_{Aa} + ie\bar{\Psi}_{Aa}\gamma^{\mu}\tilde{D}_{\mu}\Psi^{Aa}) \tag{C.1}$$

actually gives

$$-\delta_{SUSY}(iee^{\mu}_{\beta}\bar{\Psi}^{Aa}\gamma^{\beta}\tilde{D}_{\mu}\Psi_{Aa}) - \frac{i}{2}\delta_{SUSY}(e\tilde{D}_{\mu}e^{\mu}_{\beta}\bar{\Psi}^{Aa}\gamma^{\beta}\Psi_{Aa}), \tag{C.2}$$

where the second part doesn't contribute $(D_\mu)^2$ terms. The fact can be easily proven by using integration by parts. Of course, when discussing the $D(\bar{\Psi}\Psi)(\epsilon\chi)$ variation terms,

$$\frac{i}{2}\delta_{SUSY}(ee^{\mu}_{\beta})\tilde{D}_{\mu}\bar{\Psi}_{Aa}\gamma^{\beta}\Psi^{Aa} \tag{C.3}$$

has to be taken into consideration.

Note the notation '$\cdot f$' in these formulas below means there exists a factor of structure constant $f^{ab}{}_{cd}$. And when the D_μ appear in headings, it may act on any field in the term, though it's written on the left of a certain field.

At last, keep in mind the relation $\epsilon^{AB}_m = A\epsilon^{AB}_g = \epsilon^{AB}$, where $A = \pm\sqrt{2}$, as we may not mention it all the time when using it.

[1]For a literal description of the verification, refer to [40].

C.1 Variation terms at order $(\tilde{D}_\mu)^2$

Terms in this form have vanished in **Chapter** 5.2 while the gauged ABJM action was constructed at this order. So do the $G_{\mu\nu}$-terms and $F_{\mu\nu}$-terms.

C.2 Variation terms linear in $\tilde{D}_\mu$

C.2.1 Terms with two fermions

C.2.1.1 the $e(\bar{\epsilon}\gamma^\mu\Psi)\tilde{D}_\mu Z^3 \cdot f$-terms

Such variation terms come from the $\delta\Psi$ of the fermionic kinetic term (i.e. the Dirac term) and the χ of the $(\bar{\chi}_\mu\gamma^\mu\Psi)Z^3$ term.

Let's begin with the kinetic term of fermions, i.e.$-ie(\bar{\Psi}^{Aa}\gamma^\mu D_\mu\Psi_{Aa})$. Obviously,

$$\begin{aligned}
-ie\bar{\Psi}^{Aa}&\gamma^\mu D_\mu(\delta\Psi_{Aa}\mid_{Z^3\cdot f})\\
&=-ie\bar{\Psi}^{Aa}\gamma^\mu D_\mu(f^{cd}{}_{ba}\epsilon_{BD}Z^B_cZ^D_d\bar{Z}^b_A - f^{cd}{}_{ba}\epsilon_{BA}Z^B_cZ^D_d\bar{Z}^b_D)\\
&=-ie\bar{\Psi}^{Aa}\gamma^\mu(f^{cd}{}_{ba}D_\mu\epsilon_{BD}Z^B_cZ^D_d\bar{Z}^b_A - f^{cd}{}_{ba}D_\mu\epsilon_{BA}Z^B_cZ^D_d\bar{Z}^b_D)\\
&\quad-\Psi^{Aa}\gamma^\mu f^{cd}{}_{ba}\epsilon_{BD}D_\mu(Z^B_cZ^D_d\bar{Z}^b_A) + f^{cd}{}_{ba}\epsilon_{BA}D_\mu(Z^B_cZ^D_d\bar{Z}^b_D)
\end{aligned}$$

in which the last two terms have been canceled in the ABJM action, and the first term and its complex conjugate cancel with the variation of χ from the $(\chi\Psi)Z^3\cdot f$ term (5.9), just as mentioned.

Actually all other $(\bar{\epsilon}\gamma^\mu\Psi)D_\mu Z^3\cdot f$ terms have been proved to vanish in the original ABJM theory.

C.2.1.2 The $e(\bar{\epsilon}\gamma^\mu\Psi)\tilde{D}_\mu Z^3$-terms

Variation terms considered here are generated by these terms below. Firstly, because we add two new terms Z^3 without structure constants to the variation of Ψ, two corresponding $D_\mu\epsilon$ terms will arise from the kinetic term of the fermion. Similarly to the previous case with structure constant, they should cancel with the $\delta\chi$ of the $(\chi\Psi)Z^3$ term(5.12). What's left now is:

$$\begin{aligned}
&ie\frac{1}{4}(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BA})D_\mu(Z^D_a\bar{Z}^a_B\bar{Z}^b_A)+c.c.\\
&+ie\frac{1}{16}(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BD})D_\mu(Z^2\bar{Z}^b_B)+c.c.
\end{aligned}\tag{C.4}$$

Terms in the same form are also from the $(\Psi\Psi)Z^2$ term(5.11) of the Lagrangian:

$$\begin{aligned}
ie\tfrac{1}{8}\epsilon^{ABCD}&(\bar{\Psi}_{Db}\delta\Psi_{Ca}\mid_{DZ})\bar{Z}^b_A\bar{Z}^a_B+c.c.\\
&=+ie\tfrac{1}{8}(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BA})D_\mu Z^D_a\bar{Z}^b_A\bar{Z}^a_B+c.c.\\
&\quad-ie\tfrac{1}{8}(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BD})D_\mu Z^A_a\bar{Z}^b_A\bar{Z}^a_B+c.c.\\
&\quad+ie\tfrac{1}{8}(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BD})D_\mu Z^A_a\bar{Z}^b_B\bar{Z}^a_A+c.c.
\end{aligned}\tag{C.5}$$

$$
\begin{aligned}
ie\tfrac{1}{16}(\bar{\Psi}_{Db}\delta\Psi^{Db}\mid_{DZ})Z^2 + c.c.& \\
&= ie\tfrac{1}{16}(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BD})D_\mu\bar{Z}^b_B Z^2 + c.c. && \text{(C.6)}\\
ie\tfrac{1}{8}(\bar{\Psi}_{Db}\delta\Psi^{Da}\mid_{DZ})\bar{Z}^b_A Z^A_a + c.c.& \\
&= ie\tfrac{1}{8}(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BD})D_\mu\bar{Z}^a_B\bar{Z}^b_A\bar{Z}^a_A + c.c. && \text{(C.7)}\\
-ie\tfrac{1}{4}(\bar{\Psi}_{Db}\delta\Psi^{Bb}\mid_{DZ})\bar{Z}^a_B Z^D_a + c.c.& \\
&= ie\tfrac{1}{4}(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BA})D_\mu\bar{Z}^b_A\bar{Z}^a_B Z^D_a + c.c. && \text{(C.8)}\\
ie\tfrac{3}{8}(\bar{\Psi}_{Db}\delta\Psi^{Ba}\mid_{DZ})\bar{Z}^b_B Z^D_b + c.c.& \\
&= ie\tfrac{3}{8}(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BA})D_\mu\bar{Z}^a_B\bar{Z}^b_A Z^D_a + c.c. && \text{(C.9)}
\end{aligned}
$$

And $-eg^{\mu\nu}D_\mu\bar{Z}^a_A Z^B_a\delta B_\nu{}^A{}_B + c.c.$ of the kinetic term of the scalar fields contribute:

$$
\begin{aligned}
&\frac{i}{8}e(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BD})D_\mu Z^A_a\bar{Z}^b_A\bar{Z}^a_B + c.c.\\
&-\frac{i}{8}e(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BD})D_\mu\bar{Z}^a_B\bar{Z}^b_A Z^A_a + c.c.\\
&-\frac{i}{8}e(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BA})D_\mu\bar{Z}^a_B\bar{Z}^b_A Z^D_a + c.c.\\
&+\frac{i}{8}e(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BA})D_\mu Z^D_a\bar{Z}^b_A\bar{Z}^a_B + c.c. \qquad \text{(C.10)}
\end{aligned}
$$

The last term in this form comes from $\frac{1}{16}eg^{\mu\nu}D_\mu\bar{Z}^a_A Z^A_a\delta A_\nu + c.c.$, i.e. the additional U(1) gauge field of the Klein-Gordon term:

$$
\begin{aligned}
&+\frac{i}{16}e(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BD})D_\mu\bar{Z}^a_A Z^A_a\bar{Z}^a_B + c.c.\\
&-\frac{i}{16}e(\bar{\Psi}_{Db}\gamma^\mu\epsilon^{BD})D_\mu Z^A_a\bar{Z}^a_A\bar{Z}^b_B + c.c. \qquad \text{(C.11)}
\end{aligned}
$$

We will find the sum of these variation terms above equals to *zero*.

C.2.1.3 The $(\bar{\epsilon}\cdot\chi)DZ^4\cdot f$-terms

The variation $\delta\Psi\mid_{DZ}$ in the previous term $(\chi\Psi)Z^3\cdot f(5.9)$ (namely $L_{\hat{B}}$) gives

$$
\begin{aligned}
&-ieAf^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\gamma^\nu\epsilon^{CD})Z^A_a Z^B_b\bar{Z}^c_D D_\nu\bar{Z}^d_C + c.c.\\
&+ieAf^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\gamma^\nu\epsilon^{CB})Z^A_a Z^D_b\bar{Z}^c_D D_\nu\bar{Z}^d_C + c.c.\\
&\quad= -ieAf^{ab}{}_{cd}(\bar{\chi}^\nu_{AB}\epsilon^{CD})Z^A_a Z^B_b\bar{Z}^c_D D_\nu\bar{Z}^d_C + c.c.\\
&\qquad+ieAf^{ab}{}_{cd}(\bar{\chi}^\nu_{AB}\epsilon^{CB})Z^A_a Z^D_b\bar{Z}^c_D D_\nu\bar{Z}^d_C + c.c.\\
&\qquad+ieAf^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{CD})Z^A_a Z^B_b\bar{Z}^c_D D_\nu\bar{Z}^d_C + c.c.\\
&\qquad-ieAf^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{CB})Z^A_a Z^D_b\bar{Z}^c_D D_\nu\bar{Z}^d_C + c.c.
\end{aligned}
$$

Then considering the variation $\delta\Psi\mid_{Z^3\cdot f}$ in $(\chi\Psi)DZ$ term (i.e. the first half of (5.4)) which gives:

$$
\begin{aligned}
&ieA(\bar{\chi}_{\mu AB}\gamma^\nu\gamma^\mu\epsilon^{CD})f^{ab}{}_{cd}D_\nu Z^A_a Z^B_b \bar{Z}^c_D \bar{Z}^d_C + c.c.\\
&-ieA(\bar{\chi}_{\mu AB}\gamma^\nu\gamma^\mu\epsilon^{CB})f^{ab}{}_{cd}D_\nu Z^A_a Z^D_b \bar{Z}^c_D \bar{Z}^d_C + c.c.\\
&\quad = + ieA(\bar{\chi}^\nu_{AB}\epsilon^{CD})f^{ab}{}_{cd}D_\nu Z^A_a Z^B_b \bar{Z}^c_D \bar{Z}^d_C + c.c.\\
&\quad\quad - ieA(\bar{\chi}^\nu_{AB}\epsilon^{CB})f^{ab}{}_{cd}D_\nu Z^A_a Z^D_b \bar{Z}^c_D \bar{Z}^d_C + c.c.\\
&\quad\quad + ieA(\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{CD})f^{ab}{}_{cd}D_\nu Z^A_a Z^B_b \bar{Z}^c_D \bar{Z}^d_C + c.c.\\
&\quad\quad - ieA(\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{CB})f^{ab}{}_{cd}D_\nu Z^A_a Z^D_b \bar{Z}^c_D \bar{Z}^d_C + c.c.
\end{aligned}
\tag{C.12}
$$

Also, we have some similar terms from $-ieg^{\mu\nu}D_\mu Z^D_a \bar{Z}^c_D \delta\widetilde{A}_\nu{}^a{}_c + c.c.$ in the kinetic term of the scalar fields:

$$
\begin{aligned}
&- ieA(\bar{\chi}^\mu_{AB}\epsilon^{CB})Z^A_a D_\mu Z^D_b \bar{Z}^c_D Z^d_C f^{ab}{}_{cd} + c.c.\\
&+ ieA(\bar{\chi}^{\mu CB}\epsilon_{AB})Z^A_a D_\mu Z^D_b \bar{Z}^c_D Z^d_C f^{ab}{}_{cd} + c.c.
\end{aligned}
\tag{C.13}
$$

in which,$2\epsilon_g = A\epsilon$ has been used.

1) $(\bar{\epsilon}\gamma^{\mu\nu}\chi_\mu)D_\nu Z^4 \cdot f$

To sum up, considering all $(\bar{\chi}_\nu\gamma^{\mu\nu}\epsilon)D_\mu Z^4 \cdot f$-terms first, the sum will be written as

$$
\begin{aligned}
&-\frac{i}{2}Ae(\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{CD})D_\nu(Z^A_a Z^B_b \bar{Z}^c_D \bar{Z}^d_C)f^{ab}{}_{cd} + c.c.\\
&-\frac{i}{2}Ae(\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{CB})D_\nu Z^A_a Z^D_b \bar{Z}^c_D \bar{Z}^d_C f^{ab}{}_{cd} + c.c.\\
&-\frac{i}{2}Ae(\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}\epsilon_{CB})Z^A_a Z^D_b \bar{Z}^c_D D_\nu \bar{Z}^d_C f^{ab}{}_{cd} + c.c.\\
&= + \frac{i}{2}Ae(\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{CD})D_\nu(Z^A_a Z^B_b \bar{Z}^c_C \bar{Z}^d_D)f^{ab}{}_{cd} + c.c.\\
&\quad + \frac{i}{4}Ae(\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{AB})D_\nu(Z^C_a Z^D_b \bar{Z}^c_C \bar{Z}^d_D)f^{ab}{}_{cd}
\end{aligned}
\tag{C.14}
$$

the $\delta\Psi\mid_{Z^3\cdot f}$ of the $(f\Psi)Z$ term (5.6) (i.e. $L_{A''}$) gives:

$$
\begin{aligned}
&-\frac{i}{2}Ae(D_\nu\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{CD})(Z^A_a Z^B_b \bar{Z}^c_C \bar{Z}^d_D)f^{ab}{}_{cd} + c.c.\\
&-\frac{i}{4}Ae(D_\nu\bar{\chi}_{\mu AB}\gamma^{\nu\mu}\epsilon^{AB})(Z^C_a Z^D_b \bar{Z}^c_C \bar{Z}^d_D)f^{ab}{}_{cd}
\end{aligned}
\tag{C.15}
$$

Also, according to the complete differential, the $\delta\chi$ of another term $(\chi\chi)Z^4 \cdot f(5.10)$ is also needed to give the last part of the complete differential in order to make all such terms cancel according to the boundary condition. Note $2\epsilon_g = A\epsilon$ has also been used here.

2) $(\bar{\epsilon}\chi^\mu)D_\mu Z^4 \cdot f$

Then we write all $(\bar{\chi}^\mu\epsilon)D_\mu Z^4 \cdot f$-terms together, and find it can be written as another form:

$$
\begin{aligned}
=&-ieAf^{ab}{}_{cd}(\bar{\chi}^{\nu CD}\epsilon_{AB})D_\nu Z^A_a Z^B_b \bar{Z}^c_D \bar{Z}^d_C + c.c.\\
&+ieAf^{ab}{}_{cd}(\bar{\chi}^{\nu}_{AB}\epsilon^{CD})D_\nu Z^A_a Z^B_b \bar{Z}^c_D \bar{Z}^d_C + c.c.\\
&-2ieAf^{ab}{}_{cd}(\bar{\chi}^{\nu}_{AB}\epsilon^{CB})D_\nu Z^A_a Z^D_b \bar{Z}^c_D \bar{Z}^d_C + c.c.\\
&-2ieAf^{ab}{}_{cd}(\bar{\chi}^{\nu}_{AB}\epsilon^{CB})D_\nu Z^D_b Z^A_a \bar{Z}^c_D \bar{Z}^d_C + c.c.\\
&+ieAf^{ab}{}_{cd}(\bar{\chi}^{\nu}_{AB}\epsilon^{AB})D_\nu Z^C_a Z^D_b \bar{Z}^c_D \bar{Z}^d_C + c.c.,
\end{aligned}
\tag{C.16}
$$

which vanish due to the duality equation

$$
\chi^{\nu CD}\epsilon_{AB} = \chi^{\nu}_{MN}\cdot 6\delta^{[MN}_{AB}\epsilon^{CD]}. \tag{C.17}
$$

C.2.1.4 The $(\bar{\epsilon}\chi)DZ^4$-terms

Firstly, the $\delta\Psi\mid_{DZ}$ of the $(\chi\Psi)Z^3$ term (5.12) gives:

$$
\begin{aligned}
&-ieA\tfrac{1}{4}(\bar{\chi}^{BA}_\mu\gamma^\mu\gamma^\nu\epsilon_{CD})\bar{Z}^a_B\bar{Z}^b_A Z^D_a D_\nu Z^C_b + c.c.\\
&-ieA\tfrac{1}{16}(\bar{\chi}^{BD}_\mu\gamma^\mu\gamma^\nu\epsilon_{CD})Z^2\bar{Z}^b_B D_\nu Z^C_b + c.c.\\
&\qquad=-ieA\tfrac{1}{4}(\bar{\chi}^{\nu BA}\epsilon_{CD})\bar{Z}^a_B\bar{Z}^b_A Z^D_a D_\nu Z^C_b + c.c.\\
&\qquad\quad-ieA\tfrac{1}{16}(\bar{\chi}^{\nu BD}\epsilon_{CD})Z^2\bar{Z}^b_B D_\nu Z^C_b + c.c.\\
&\qquad\quad+ieA\tfrac{1}{4}(\bar{\chi}^{BA}_\mu\gamma^{\nu\mu}\epsilon_{CD})\bar{Z}^a_B\bar{Z}^b_A Z^D_a D_\nu Z^C_b + c.c.\\
&\qquad\quad+ieA\tfrac{1}{16}(\bar{\chi}^{BD}_\mu\gamma^{\nu\mu}\epsilon_{CD})Z^2\bar{Z}^b_B D_\nu Z^C_b + c.c.
\end{aligned}
\tag{C.18}
$$

Secondly, the $\delta\Psi\mid_{Z^3}$ of the first part of the term (5.4), $(\chi\Psi)DZ$ gives:

$$
\begin{aligned}
&+ieA\tfrac{1}{4}(\bar{\chi}^{AB}_\mu\gamma^\nu\gamma^\mu\epsilon_{DC})Z^D_b Z^C_a\bar{Z}^b_B D_\nu\bar{Z}^a_A + c.c.\\
&+ieA\tfrac{1}{16}(\bar{\chi}^{AB}_\mu\gamma^\nu\gamma^\mu\epsilon_{DB})Z^2 Z^D_a D_\nu\bar{Z}^a_A + c.c.\\
&\qquad=ieA\tfrac{1}{4}(\bar{\chi}^{\nu AB}\epsilon_{DC})Z^D_b Z^C_a\bar{Z}^b_B D_\nu\bar{Z}^a_A + c.c.\\
&\qquad\quad+ieA\tfrac{1}{16}(\bar{\chi}^{\nu AB}\epsilon_{DB})Z^2 Z^D_a D_\nu\bar{Z}^a_A + c.c.\\
&\qquad\quad+ieA\tfrac{1}{4}(\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}\epsilon_{DC})Z^D_b Z^C_a\bar{Z}^b_B D_\nu\bar{Z}^a_A + c.c.\\
&\qquad\quad+ieA\tfrac{1}{16}(\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}\epsilon_{DB})Z^2 Z^D_a D_\nu\bar{Z}^a_A + c.c.
\end{aligned}
\tag{C.19}
$$

then they can be divided into two kinds of variation terms, and then calculated separately.

1) $(\bar{\epsilon}\gamma^{\mu\nu}\chi_\mu)D_\nu Z^4$

From above, what we have now are:

$$
\begin{aligned}
&+ieA\tfrac{1}{4}(\bar{\chi}^{BA}_\mu\gamma^{\nu\mu}\epsilon_{CD})\bar{Z}^a_B\bar{Z}^b_A Z^D_a D_\nu Z^C_b + c.c.\\
&+ieA\tfrac{1}{16}(\bar{\chi}^{BD}_\mu\gamma^{\nu\mu}\epsilon_{CD})Z^2\bar{Z}^b_B D_\nu Z^C_b + c.c.\\
&+ieA\tfrac{1}{4}(\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}\epsilon_{DC})Z^D_b Z^C_a\bar{Z}^b_B D_\nu\bar{Z}^a_A + c.c.\\
&+ieA\tfrac{1}{16}(\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}\epsilon_{DB})Z^2 Z^D_a D_\nu\bar{Z}^a_A + c.c.,
\end{aligned}
\tag{C.20}
$$

which finally give:

$$-\tfrac{i}{8}eA(\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}\epsilon_{DC})D_\nu(Z^D_aZ^C_b\bar{Z}^a_B\bar{Z}^b_A)+c.c.$$
$$-\tfrac{i}{64}eA(\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}\epsilon_{AB})D_\nu Z^4 \tag{C.21}$$

Then the $\delta\Psi\mid_{Z^3}$ of the $(f\Psi)Z$ term (5.6) gives

$$-\tfrac{i}{8}eA(D_\nu\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}\epsilon_{DC})(Z^D_aZ^C_b\bar{Z}^a_B\bar{Z}^b_A)+c.c.$$
$$-\tfrac{i}{64}eA(D_\nu\bar{\chi}^{AB}_{\nu\mu}\gamma^{\mu}\epsilon_{AB})Z^4 \tag{C.22}$$

and the $(\chi\chi)Z^4$ term(5.13),which gives

$$-\tfrac{i}{8}eA(\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}D_\nu\epsilon_{DC})(Z^D_aZ^C_b\bar{Z}^a_B\bar{Z}^b_A)+c.c.$$
$$-\tfrac{i}{64}eA(\bar{\chi}^{AB}_\mu\gamma^{\nu\mu}D_\nu\epsilon_{AB})Z^4 \tag{C.23}$$

Recall the complete differential for integration, all such terms here cancel according to the boundary condition. Note $2\epsilon_g = A\epsilon$ has also been used.

2) $(\bar{\epsilon}\chi^\mu)D_\mu Z^4$

From above, what we have now are:

$$-ieA\tfrac{1}{4}(\bar{\chi}^{BA}_\mu\gamma^\mu\gamma^\nu\epsilon_{CD})\bar{Z}^a_B\bar{Z}^b_AZ^D_aD_\nu Z^C_b+c.c.$$
$$-ieA\tfrac{1}{16}(\bar{\chi}^{BD}_\mu\gamma^\mu\gamma^\nu\epsilon_{CD})Z^2\bar{Z}^b_BD_\nu Z^C_b+c.c.$$
$$\begin{aligned}=&+ieA\tfrac{1}{4}(\bar{\chi}^{\nu AB}\epsilon_{DC})Z^D_bZ^C_a\bar{Z}^b_BD_\nu\bar{Z}^a_A+c.c.\\&+ieA\tfrac{1}{16}(\bar{\chi}^{\nu AB}\epsilon_{DB})Z^2Z^D_aD_\nu\bar{Z}^a_A+c.c.\\&+ieA\tfrac{1}{4}(\bar{\chi}^{BA}_\mu\gamma^{\nu\mu}\epsilon_{CD})\bar{Z}^a_B\bar{Z}^b_AZ^D_aD_\nu Z^C_b+c.c.\\&+ieA\tfrac{1}{4}(\bar{\chi}^{BD}_\mu\gamma^{\nu\mu}\epsilon_{CD})Z^2\bar{Z}^b_BD_\nu Z^C_b+c.c.\end{aligned} \tag{C.24}$$

Also we have $-eg^{\mu\nu}D_\mu\bar{Z}^a_AZ^B_a\delta B_\nu{}^A{}_B+c.c.$ from the Klein-Gordon term, which contribute:

$$\begin{aligned}&-\frac{i}{2}e(\bar{\chi}^{\nu DC}\epsilon_{gAD})D_\nu Z^A_a\bar{Z}^a_BZ^B_b\bar{Z}^b_C+c.c.\\&+\frac{i}{2}e(\bar{\chi}^\nu_{DC}\epsilon^{BD}_g)D_\nu Z^A_a\bar{Z}^a_BZ^C_b\bar{Z}^b_A+c.c.\\&+\frac{i}{8}e(\bar{\chi}^{\nu DC}\epsilon_{gBD})D_\nu Z^A_a\bar{Z}^a_AZ^B_b\bar{Z}^b_C+c.c.\\&-\frac{i}{8}e(\bar{\chi}^\nu_{DC}\epsilon^{BD}_g)D_\nu Z^A_a\bar{Z}^a_AZ^C_b\bar{Z}^b_B+c.c.\\&+\frac{i}{8}e(\bar{\chi}^{\nu DB}\epsilon_{gAD})D_\nu Z^A_a\bar{Z}^a_BZ^2+c.c.\\&-\frac{i}{8}e(\bar{\chi}^\nu_{DA}\epsilon^{BD}_g)D_\nu Z^A_a\bar{Z}^a_BZ^2+c.c.\end{aligned} \tag{C.25}$$

The last terms in this form come from $\frac{1}{16}eg^{\mu\nu}D_\mu\bar{Z}^a_AZ^A_a\delta A_\nu{}^b{}_b+c.c.$ of the kinetic term of scalar field:

$$\begin{aligned}&-\frac{i}{8}e(\bar{\chi}^{\nu BD}\epsilon_{gDC})D_\nu Z^A_a\bar{Z}^a_AZ^C_b\bar{Z}^b_B+c.c.\\&+\frac{i}{8}e(\bar{\chi}^\nu_{DC}\epsilon^{BD}_g)D_\nu Z^A_a\bar{Z}^a_AZ^C_b\bar{Z}^b_B+c.c.\end{aligned} \tag{C.26}$$

If we add them together, we would find all such terms vanish.

C.2.2 Terms with more than two fermions

C.2.2.1 The $e(\Psi\Psi)(\epsilon\tilde{D}_\mu\chi)$-terms

the Dirac term is the first source of such variation terms. According to integration by parts, $\delta(ee^\mu_\alpha)$ of the Dirac term gives

$$-i\delta(ee^\mu_\alpha)(\bar{\Psi}^{Aa}\gamma^\alpha\tilde{D}_\mu\Psi_{Aa})+\frac{i}{2}\delta(ee^\mu_\alpha)\tilde{D}_\mu(\bar{\Psi}^{Aa}\gamma^\alpha\Psi_{Aa}). \tag{C.27}$$

Substituting the transformation rule of δe^μ_α into the term above, we obtain

$$\frac{1}{2}Ae(\bar{\epsilon}^{BC}\gamma^{[\nu}\chi_{\nu BC})(\bar{\Psi}^{Aa}\gamma^{\mu]}D_\mu\Psi_{Aa})+c.c., \tag{C.28}$$

in which the relation $2\epsilon^{AB}_g=A\epsilon^{AB}$ has been used.

The $\delta\Psi\mid_{(\chi\Psi)\epsilon}$ of the Dirac term contributes

$$eA(\tilde{D}_\mu\bar{\Psi}^{Aa}\gamma^\mu\gamma^\nu\epsilon_{BA})(\bar{\chi}^{BD}\Psi_{Da})+c.c., \tag{C.29}$$

which is fierzed to be

$$\begin{aligned}&-\tfrac{1}{2}eA(\tilde{D}_\mu\bar{\Psi}^{Aa}\Psi_{Da})(\bar{\chi}^{BD}\gamma^\mu\gamma^\nu\epsilon_{BA})+c.c.\\&-\tfrac{1}{2}eA(\tilde{D}_\mu\bar{\Psi}^{Aa}\gamma_\rho\Psi_{Da})(\bar{\chi}^{BD}\gamma^\rho\gamma^\mu\gamma^\nu\epsilon_{BA})+c.c.\end{aligned} \tag{C.30}$$

Also, the variation of the Dirac term contains $\delta B_\mu{}^B{}_A$ in D_μ, which is found to be

$$-\tfrac{A}{2}(\bar{\Psi}^{Aa}\gamma^\mu\Psi_{Ba})(\bar{\epsilon}^{BD}\gamma_\nu\gamma_\mu f^\nu{}_{DA})+c.c. \tag{C.31}$$

The last variation term of this kind from the Dirac term is not that explicit. It derives from $-\frac{i}{2}(\bar{\Psi}^{Aa}\gamma^\mu\frac{1}{4}\delta\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\Psi_{Aa})+c.c.$. Recall that

$$\delta\tilde{\omega}^*_{\mu,\nu}=-2i(\bar{\epsilon}^{AB}\gamma_\mu f_{\nu AB}-\tfrac{1}{2}g_{\mu\nu}\bar{\epsilon}^{AB}\gamma^\rho f_{\rho AB}), \tag{C.32}$$

the term above can be simplified as

$$\tfrac{A}{4}e(\bar{\Psi}^{Aa}\Psi_{Aa})(\bar{\epsilon}^{BC}\gamma^\nu f_{\nu BC}). \tag{C.33}$$

Then turn to $iAe(\bar{\chi}_{\mu AB}\gamma^\nu\gamma^\mu\Psi^{Ba})D_\nu(Z^A_a-\frac{iA}{2}(\bar{\chi}^{AD}_\nu\Psi_{Da}))+c.c.$, of which the $\delta D_\nu Z^A_a$ and $\delta\chi$ generate terms we look for. All of them have to be fierzed into $(\Psi\Psi)(\chi\epsilon)$-form and the result turns out to be

$$\begin{aligned}&\tfrac{Ae}{2}(\bar{\Psi}^{Ba}\gamma^\mu\Psi_{Da})[(D_\nu\bar{\epsilon}^{AD}\gamma^\nu\chi_{\mu AB})+(D_\mu\bar{\epsilon}_{AB}\gamma^\nu\chi_\nu{}^{AD})]+c.c.\\+&\tfrac{Ae}{2}(\bar{\Psi}^{Ba}D_\nu\Psi_{Da})[(\bar{\epsilon}^{AD}\gamma^\mu\gamma^\nu\chi_{\mu AB})+c.c.\\+&\tfrac{Ae}{2}(\bar{\Psi}^{Ba}\gamma_\rho D_\nu\Psi_{Da})[(\bar{\epsilon}^{AD}\gamma_\rho\gamma^\mu\gamma^\nu\chi_{\mu AB})+c.c.\end{aligned} \tag{C.34}$$

Finally, the δZ^B_a of the term $-iA(\bar{f}^\mu_{AB}\gamma_\mu\Psi^{Aa})\delta Z^B_a+c.c.$ goes as follow:

$$\begin{aligned}&-\tfrac{A}{2}e(\bar{\Psi}^{Aa}\gamma^{\nu\rho}D_\nu\chi_{\rho AB})(\bar{\epsilon}^{BD}\Psi_{Da})+c.c.\\&=\tfrac{A}{4}e(\bar{\Psi}^{Aa}\Psi_{Da})(\bar{\epsilon}^{BD}\gamma^{\nu\rho}D_\nu\chi_{\rho AB})+c.c.\\&-\tfrac{A}{4}e(\bar{\Psi}^{Aa}\gamma_\mu\Psi_{Da})(\bar{\epsilon}^{BD}\gamma_\mu\gamma^{\nu\rho}D_\nu\chi_{\rho AB})+c.c.\end{aligned} \tag{C.35}$$

These are all terms we are interested in at this step. On one hand, terms containing $(\bar{\Psi}\Psi)$ without gamma matrix inside can easily be seen to cancel; On the other hand, the sum of terms with the factor $(\bar{\Psi}\cdot\gamma\cdot\Psi)$ is also zero according to the fact $\int d(..)\equiv 0$ when the integration range is the whole space. That is, all of them vanish.

C.2.2.2 The $e(\chi\chi)(\epsilon\Psi)D_\mu Z$-terms

Let's consider the terms with the derivative acting on the scalar fields first.

The $\delta\Psi\mid_{DZ}$ variation of $\frac{A^2}{2}e(\bar{\chi}_{\mu AB}\gamma^\nu\gamma^\mu\Psi^{Ba})(\bar{\chi}^{AD}_\mu\Psi_{Da})+c.c.$ gives:

$$\begin{aligned}
&-\tfrac{1}{2}e(\bar{\Psi}^{Ba}\gamma^\mu\gamma^\nu\gamma^\rho\epsilon_{CD})(\bar{\chi}^{AD}_\nu\chi_{\mu AB})\tilde{D}_\rho Z^C_a+c.c.\\
&-\tfrac{1}{2}e(\bar{\Psi}^{Ba}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\epsilon_{CD})(\bar{\chi}^{AD}_\nu\gamma^\lambda\chi_{\mu AB})\tilde{D}_\rho Z^C_a+c.c.\\
&-\tfrac{1}{2}e(\bar{\Psi}^{Da}\gamma^\nu\gamma^\mu\gamma^\rho\epsilon_{CB})(\bar{\chi}^{AB}_\mu\chi_{\nu AD})\tilde{D}_\rho Z^C_a+c.c.\\
&-\tfrac{1}{2}e(\bar{\Psi}^{Da}\gamma^\lambda\gamma^\nu\gamma^\mu\gamma^\rho\epsilon_{CB})(\bar{\chi}^{AB}_\mu\gamma^\lambda\chi_{\nu AD})\tilde{D}_\rho Z^C_a+c.c..
\end{aligned}\tag{C.36}$$

The second variation term of this kind comes from the $\delta\Psi\mid_{(\Psi\chi)\epsilon}$ of $+iAe\bar{\chi}^{BA}_\mu\gamma^\nu\gamma^\mu\Psi_{Aa}\tilde{D}_\nu\bar{Z}^a_B+$ $c.c.$, which equals to

$$\begin{aligned}
&-e(\bar{\chi}_{\mu CB}\chi_{\rho AD})(\bar{\Psi}^{Da}\gamma^\nu\gamma^\mu\gamma^\rho\epsilon^{AB})\tilde{D}_\nu Z^C_a+c.c.\\
&-e(\bar{\chi}_{\mu CB}\gamma_\lambda\chi_{\rho AD})(\bar{\Psi}^{Da}\gamma^\lambda\gamma^\nu\gamma^\mu\gamma^\rho\epsilon^{AB})\tilde{D}_\nu Z^C_a+c.c.,
\end{aligned}\tag{C.37}$$

while the third one is derived from the same term by varying the $ee^\mu_\alpha e^\nu_\beta$ and gives

$$\begin{aligned}
&-e\tfrac{1}{2}(\bar{\chi}^{AB}_\rho\chi_{\mu CD})(\bar{\Psi}^{Da}\gamma^\mu\gamma^\nu\gamma^\rho\epsilon_{AB})\tilde{D}_\nu Z^C_a+c.c.\\
&-e\tfrac{1}{2}(\bar{\chi}^{AB}_\rho\gamma_\lambda\chi_{\mu CD})(\bar{\Psi}^{Da}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\epsilon_{AB})\tilde{D}_\nu Z^C_a+c.c.\\
&+e\tfrac{1}{2}(\bar{\chi}^{AB}_\rho\chi_{\mu CD})(\bar{\Psi}^{Da}\gamma^\mu\gamma^\rho\gamma^\nu\epsilon_{AB})\tilde{D}_\nu Z^C_a+c.c.\\
&+e\tfrac{1}{2}(\bar{\chi}^{AB}_\rho\gamma_\lambda\chi_{\mu CD})(\bar{\Psi}^{Da}\gamma^\mu\gamma^\rho\gamma^\lambda\gamma^\nu\epsilon_{AB})\tilde{D}_\nu Z^C_a+c.c.\\
&+e\tfrac{1}{2}(\bar{\chi}^{AB}_\rho\chi_{\mu CD})(\bar{\Psi}^{Da}\gamma^\rho\gamma^\nu\gamma^\mu\epsilon_{AB})\tilde{D}_\nu Z^C_a+c.c.\\
&+e\tfrac{1}{2}(\bar{\chi}^{AB}_\rho\chi_{\mu CD})(\bar{\Psi}^{Da}\gamma^\rho\gamma^\nu\gamma^\lambda\gamma^\mu\epsilon_{AB})\tilde{D}_\nu Z^C_a+c.c..
\end{aligned}\tag{C.38}$$

Then comes the fourth variation term:

$$\begin{aligned}
&i\epsilon^{\mu\nu\rho}(\bar{\chi}^{AC}_\mu\chi_{\nu BC})\delta Z^B_a\tilde{D}_\rho\bar{Z}^a_A+c.c.\\
&=-\epsilon^{\mu\nu\rho}(\bar{\chi}^{AC}_\mu\chi_{\nu BC})(\bar{\Psi}_{Da}\epsilon^{BD})\tilde{D}_\rho\bar{Z}^a_A+c.c.
\end{aligned}\tag{C.39}$$

The last one is related to the $\Psi\mid_{\epsilon DZ}$ in the self-conjugate Dirac term, where the factor $D_\nu e^\mu_\alpha$ obviously leads to some variation terms which have to be take into our consideration[2]. For the first half of the Dirac term,

$$\begin{aligned}
&-\tfrac{i}{2}\bar{\Psi}^{Aa}\gamma^\alpha\tilde{D}_\mu(\delta\Psi_{Aa})e^\mu_\alpha\rightarrow-\tfrac{i}{2}(\bar{\Psi}^{Aa}\gamma^\alpha\gamma^\beta\epsilon_{BA})e^\mu_\alpha\tilde{D}_\mu e^\nu_\beta\tilde{D}_\nu Z^B_a,\\
&-\tfrac{i}{2}(\delta\bar{\Psi}^{Aa})\gamma^\alpha\tilde{D}_\mu\Psi_{Aa}e^\mu_\alpha\rightarrow-\tfrac{i}{2}(\bar{\Psi}_{Aa}\gamma^\alpha\gamma^\beta\epsilon^{BA})\tilde{D}_\mu(e^\mu_\alpha e^\nu_\beta)\tilde{D}_\nu\bar{Z}^a_B,
\end{aligned}\tag{C.40}$$

the second half are just their adjoints. Obviously, both of them only contribute to variation terms with the factor $(\chi\gamma\chi)$.

Now we sum them up, the result is

$$-\epsilon^{\mu\nu\rho}(\bar{\chi}^{AB}_\mu\chi_{\nu AC})(\bar{\Psi}^{Da}\epsilon_{BD})\tilde{D}_\rho Z^C_a+c.c.,\tag{C.41}$$

[2]There is no such contributions from K-G term due to the fact $D_\rho g^{\mu\nu}=0$.

in which the derivative is acting on Z. Meanwhile,

$$i\epsilon^{\mu\nu\rho}(\bar{\chi}^{AB}_\mu \chi_{\nu AC}) Z^C_a \tilde{D}_\rho(\delta \bar{Z}^a_B) + c.c.. \tag{C.42}$$

give the corresponding variation term with the derivative acting on $(\Psi\epsilon)$. According to the complete differential, what we need to cancel all terms above is a similar term with the derivative acting on $(\chi\chi)$, which can be expressed by the field strength $f^{\mu AB}$ as:

$$[-4(\bar{f}^{\mu AB}\chi_{\mu AD})(\bar{\Psi}^{Ca}\epsilon_{BC})Z^D_a - (\bar{f}^{\mu AB}\chi_{\mu AB})(\bar{\Psi}^{Ca}\epsilon_{CD})Z^D_a] + c.c.. \tag{C.43}$$

These terms are obtained by the variations below:

1)the variation $\delta B\mid_{(\Psi\epsilon)Z}$ of the χ's Chern-Simons term, which gives:

$$\begin{aligned}
&-\tfrac{1}{2}(\bar{f}^{\mu AB}\gamma_\nu\gamma_\mu\chi_{\rho BC})\epsilon^{\nu\rho\lambda}(\bar{\epsilon}^{CD}\gamma_\rho\Psi_{Da})\bar{Z}^a_A + c.c.\\
&-\tfrac{1}{2}(\bar{f}^{\mu AB}\gamma_\nu\gamma_\mu\chi_{\rho BC})\epsilon^{\nu\rho\lambda}(\bar{\epsilon}^{CD}\gamma_\rho\Psi_{Aa})\bar{Z}^a_D + c.c.\\
&-\tfrac{1}{2}(\bar{f}^{\mu AB}\gamma_\nu\gamma_\mu\chi_{\rho BC})\epsilon^{\nu\rho\lambda}(\bar{\epsilon}_{DA}\gamma_\rho\Psi^{Da})Z^C_a + c.c.\\
&-\tfrac{1}{2}(\bar{f}^{\mu AB}\gamma_\nu\gamma_\mu\chi_{\rho BC})\epsilon^{\nu\rho\lambda}(\bar{\epsilon}_{DA}\gamma_\rho\Psi^{Ca})Z^D_a + c.c.
\end{aligned} \tag{C.44}$$

2)the variation of $B_\nu{}^A{}_C$ of the $iAe\bar{\chi}^{BA}_\mu\gamma^\nu\gamma^\mu\Psi_{Aa}\tilde{D}_\nu\bar{Z}^a_B + c.c.$, which gives:

$$\begin{aligned}
&\tfrac{1}{2}(\bar{f}^{\rho CD}\gamma_\nu\gamma_\rho\epsilon_{CD})(\bar{\Psi}^{Ba}\gamma^\mu\gamma^\nu\chi_{\mu AB})Z^A_a + c.c.\\
&+(\bar{f}^{\rho DA}\gamma_\nu\gamma_\rho\epsilon_{CD})(\bar{\Psi}^{Ba}\gamma^\mu\gamma^\nu\chi_{\mu AB})Z^C_a + c.c.\\
&-(\bar{f}^{\rho AD}\gamma_\nu\gamma_\rho\epsilon_{CD})(\bar{\Psi}^{Ba}\gamma^\mu\gamma^\nu\chi_{\mu AB})Z^C_a + c.c.
\end{aligned} \tag{C.45}$$

3)the variation of the supersymmetric 'spin connection' in the term $-iA\bar{f}^\mu_{AB}\gamma_\mu\Psi^{Aa}Z^B_a +$ $c.c.$, which gives:

$$\begin{aligned}
&-\tfrac{1}{2}(\bar{f}^{\mu CD}\gamma_\nu\epsilon_{CD})(\bar{\Psi}^{Aa}\gamma^{\nu\rho}\gamma_\mu\chi_{\rho AB})Z^B_a + c.c.\\
&+\tfrac{1}{4}(\bar{f}^{\mu AD}\gamma_\mu\epsilon_{CD})(\bar{\Psi}^{Aa}\gamma^{\nu\rho}\gamma_\nu\chi_{\rho AB})Z^B_a + c.c.
\end{aligned} \tag{C.46}$$

Then the variation of $B_\nu{}^A{}_C$ in the same term gives:

$$\begin{aligned}
&-(\bar{f}^{\mu CD}\gamma_\nu\gamma_\mu\epsilon_{AD})(\bar{\Psi}^{Aa}\gamma^{\nu\rho}\chi_{\rho BC})Z^B_a + c.c.\\
&+\tfrac{1}{4}(\bar{f}^{\mu CD}\gamma_\mu\epsilon_{CD})(\bar{\Psi}^{Aa}\gamma^{\nu\rho}\chi_{\rho BA})Z^B_a + c.c.\\
&+(\bar{f}^{\mu CD}\gamma_\nu\gamma_\mu\epsilon_{AD})(\bar{\Psi}^{Ba}\gamma^{\nu\rho}\chi_{\rho BC})Z^A_a + c.c.\\
&-\tfrac{1}{4}(\bar{f}^{\mu CD}\gamma_\mu\epsilon_{CD})(\bar{\Psi}^{Ba}\gamma^{\nu\rho}\chi_{\rho BA})Z^A_a + c.c.
\end{aligned} \tag{C.47}$$

Also regarding the same term, the $\delta\Psi\mid_{(\chi\Psi)\epsilon}$ gives:

$$-2(\bar{f}^\mu_{AB}\gamma_\mu\gamma^\rho\epsilon^{CA})(\bar{\Psi}^{Da}\chi_{\rho CD})Z^B_a + c.c., \tag{C.48}$$

and the δe^μ_α gives:

$$(\bar{f}^\mu_{AB}\gamma_\mu\Psi^{Aa})Z^B_a(\bar{\epsilon}_{CD}\gamma^\mu\chi^{CD}_\mu) + c.c.. \tag{C.49}$$

4)the variation of the scalar field in $\frac{i}{2}|Z|^2\bar{f}^\mu_{AB}\chi^{AB}_\mu$ gives:

$$-\tfrac{1}{2}(\bar{f}^\mu_{AB}\chi^{AB}_\mu)(\Psi\chi^{Da}\epsilon_{CD})Z^C_a + c.c. \tag{C.50}$$

5)the final one is a little tricky. When calculating the $\delta\chi$ of the term $-iA\bar{f}^\mu_{AB}\gamma_\mu\Psi^{Aa}Z^B_a+$ $c.c.$, we obtain

$$\tfrac{i}{16}\epsilon^{\mu\nu\rho}\tilde{R}_{\nu\rho\alpha\beta}(\bar{\Psi}^{Aa}\gamma_\mu\gamma^{\alpha\beta}\epsilon_{AB})Z^B_a + c.c., \tag{C.51}$$

which in fact can be transformed to be

$$\tfrac{i}{2}\tilde{R}^{**\mu,\gamma}(\bar{\Psi}^{Aa}(g_{\mu\gamma}+\gamma_{\mu\gamma}\epsilon_{AB})Z^B_a + c.c.. \tag{C.52}$$

While the $g^{\mu\gamma}$ part has vanished in the $(D_\mu)^2$ calculation, the $\gamma_{\mu\gamma}$ part generates

$$\tfrac{1}{2}(\bar{f}^{\mu AB}\gamma_\lambda\chi_{\mu AB})(\bar{\Psi}^{Ca}\gamma^\nu\epsilon_{CD})Z^D_a + c.c. \tag{C.53}$$

Gathering all these terms from the five sources above[3], we find exactly the term.(C.43). That is, all terms in the form $e(\chi\chi)(\epsilon\Psi)D_\mu Z$ vanish.

C.2.2.3 The $(f\chi)(\chi\chi)Z$-terms

Note to make the notations simpler, during the discussion on such terms, we use ϵ_g instead of ϵ.

The first one is from the variation of $B_\mu{}^A{}_B\mid_{(\chi\epsilon)Z^2}$ in χ's Chern-Simons term, which contains 6 terms:

$$\begin{aligned}
&2(\bar{f}^{\mu AB}\gamma_\nu\gamma_\mu\chi_{\lambda BC})[(\epsilon^C_g D\chi_{\rho DE})Z^E_a\bar{Z}^a_A - (\epsilon_{gAE}\chi^{ED}_\rho)Z^C_a\bar{Z}^a_D]\epsilon^{\nu\rho\lambda} + c.c.\\
&+\tfrac{1}{2}\bar{f}^{\mu AB}\gamma_\nu\gamma_\mu\chi_{\lambda BC})[(\epsilon^C_g D\chi_{\rho AD})Z^2 - (\epsilon_{gAD}\chi^{CD}_\rho)Z^2]\epsilon^{\nu\rho\lambda} + c.c.\\
&+\tfrac{1}{2}(\bar{f}^{\mu AB}\gamma_\nu\gamma_\mu\chi_{\lambda AB})[(\epsilon^C_g D\chi_{\rho DE})Z^E_a\bar{Z}^a_C - (\epsilon_{gCD}\chi^{DE}_\rho)Z^C_a\bar{Z}^a_E]\epsilon^{\nu\rho\lambda} + c.c.,
\end{aligned} \tag{C.54}$$

all of which have to be fierzed into the $(\chi\chi)$-part and $(\chi\gamma\chi)$-part.

The second is from the variation of $B_\mu{}^A{}_B\mid_{f\cdot\epsilon}$ in $i\epsilon^{\mu\nu\rho}(\bar{\chi}^{AC}_\mu\chi_{\nu BC})Z^B_a\tilde{D}_\rho\bar{Z}^a_A + c.c.$, which is:

$$\begin{aligned}
&-\epsilon^{\mu\nu\rho}(\bar{\chi}_{\mu AB}\chi^{AB}_\nu)(\bar{f}^{\lambda DE}\gamma_\rho\gamma_\lambda\epsilon_{gCE})Z^B_a\bar{Z}^a_D + c.c.\\
&-\epsilon^{\mu\nu\rho}(\bar{\chi}_{\mu AB}\chi^{AC}_\nu)(\bar{f}^{\lambda BE}\gamma_\rho\gamma_\lambda\epsilon_{gDE})Z^D_a\bar{Z}^a_C + c.c..
\end{aligned} \tag{C.55}$$

The third one is from the variation of $B_\mu{}^A{}_B\mid_{f\cdot\epsilon}$ in $\frac{i}{2}|Z|^2\bar{f}^\mu_{AB}\chi^{AB}_\mu$, and can be expressed as:

$$\tfrac{1}{2}\epsilon^{\mu\nu\rho}(\bar{\chi}^{AB}_\mu\chi_{\rho BC})(\bar{f}^{\lambda CD}\gamma_\nu\gamma_\lambda\epsilon_{gDA})Z^2 + c.c. \tag{C.56}$$

The fourth one is from the $\delta\tilde{\omega}_{\mu\alpha\beta}$ of $\frac{i}{2}|Z|^2\bar{f}^\mu_{AB}\chi^{AB}_\mu$:

$$\begin{aligned}
&-\tfrac{1}{4}\epsilon^{\mu\nu\rho}(\bar{\chi}^{AB}_\mu\gamma_\lambda\chi_{\rho AB})(\bar{f}^{\lambda CD}\gamma_\nu\epsilon_{gCD})Z^2 + c.c.\\
&+\tfrac{1}{8}\epsilon^{\mu\nu\rho}(\bar{\chi}^{AB}_\mu\gamma_\nu\chi_{\rho AB})(\bar{f}^{\lambda CD}\gamma_\lambda\epsilon_{gCD})Z^2 + c.c..
\end{aligned} \tag{C.57}$$

The fifth one is the last term of $\delta\tilde{L}_{RZ^2}$:

$$-\tfrac{i}{2e}\epsilon^{\mu\nu\rho}K_{\mu\nu}{}^\sigma[(\bar{f}_{\rho AB}\gamma_\sigma\epsilon^{AB}_g)Z^2 - \tfrac{1}{2}g_{\sigma\rho}(\bar{f}_{\lambda AB}\gamma^\lambda\epsilon^{AB}_g)Z^2] + c.c., \tag{C.58}$$

[3]We have used Fierz identity to transform these terms into the same form $(f\chi)(\Psi\epsilon)$ while using ϵ^{ABCD} to raise or lower some indices.

which actually is:

$$
\begin{aligned}
&-\tfrac{1}{4}\epsilon^{\mu\nu\rho}(\bar{\chi}_\mu^{CD}\gamma^\sigma\chi_{\nu CD})((\bar{f}_{\rho AB}\gamma_\sigma\epsilon_g^{AB})Z^2 + c.c. \\
&+\tfrac{1}{8}\epsilon^{\mu\nu\rho}(\bar{\chi}_\mu^{CD}\gamma_\rho\chi_{\nu CD})(\bar{f}_{\lambda AB}\gamma^\lambda\epsilon_g^{AB})Z^2 + c.c..
\end{aligned}
\tag{C.59}
$$

Besides, there is also the sixth variation term from $\delta\tilde{L}_{RZ^2}$:

$$
-\frac{i}{4}(\bar{\chi}^\mu_{AB}\gamma^\nu\epsilon_g^{AB})R^{**}_{\mu,\nu}Z^2, \tag{C.60}
$$

Similarly to eq.(C.53), $\frac{i}{2}|Z|^2\bar{f}^\mu_{AB}\chi_\mu^{AB}$ also generates variation terms involving the Ricci tensor, which is:

$$
\frac{i}{4}(\bar{\chi}_\mu^{AB}\gamma_\gamma\epsilon_{gAB})R^{**\mu,\gamma}Z^2. \tag{C.61}
$$

We can see it cancels with the sixth one.

By cycling the SU(4) indices [ABCDE], i.e. using

$$
\epsilon^{\mu\nu\rho}(\bar{\chi}_\rho^{AB}\chi_{\lambda[AB})(\bar{f}^{\mu CD}\gamma_\nu\gamma_\mu\epsilon_{gCD})Z_a^E\bar{Z}^a_{E]} + c.c. = 0, \tag{C.62}
$$

it can be easily proven that the $(\chi\chi)$-part of the first one cancels with the second and third ones. Following the same procedures, we find $(\chi\gamma\chi)$-part of the first variation term cancels with itself. The fourth one cancels with the fifth one by cycling the world line indices $[\mu\nu\rho\lambda]$, which actually is

$$
\tfrac{1}{4}\epsilon^{[\mu\nu\rho}(\bar{\chi}_\nu^{AB}\gamma^{\lambda]}\chi_{\rho AB})(\bar{f}_\lambda^{CD}\gamma_\mu\epsilon_{gCD})Z^2 + c.c. = 0. \tag{C.63}
$$

To sum up, all such terms vanishes.

C.3 Variation terms independent of D_μ

In this section, we discuss all variation terms which don't contain the covariant derivative, or ${f_\mu}^{AB}$. We also begin with terms of second order in fermionic variables. For terms including multiple fermions, such as $(\chi\chi)(\epsilon\Psi)Z^2$, the verification is incomplete.

C.3.1 Terms with two fermions

C.3.1.1 The $(\Psi\epsilon)Z^5 \cdot f^2$-terms

Such terms vanish just like what happens in the original ABJM theory, since no new contributions to them are given.

C.3.1.2 The $(\Psi\epsilon)Z^5 \cdot f$-terms

the new defined variation $\Psi\mid_{Z^3}$ of the original interacting terms in ABJM action gives:

$$
\begin{aligned}
&-if^{ab}{}_{cd}(\bar{\Psi}_{Aa}\delta\Psi^{Ad}\mid_{Z^3})Z_b^B\bar{Z}^c_B + c.c. \\
&\quad= -i\tfrac{1}{4}f^{ab}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{BD})Z_e^A Z_b^C\bar{Z}^e_B\bar{Z}^d_D\bar{Z}^c_C + c.c. \\
&\qquad -i\tfrac{1}{16}f^{ab}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{BA})Z^2 Z_b^E\bar{Z}^d_B\bar{Z}^c_E + c.c.
\end{aligned}
\tag{C.64}
$$

$$
\begin{aligned}
&2if^{ab}{}_{cd}(\bar{\Psi}_{Aa}\delta\Psi^{Bd}\mid_{Z^3})Z^A_b\bar{Z}^c_B + c.c.\\
&\quad= - i\tfrac{1}{4}f^{ab}{}_{cd}\epsilon^{ABCD}(\bar{\Psi}_{Aa}\epsilon_{EF})Z^E_e Z^F_b\bar{Z}^e_B\bar{Z}^d_D\bar{Z}^c_C + c.c.\\
&\qquad - i\tfrac{1}{16}f^{ab}{}_{cd}\epsilon^{ABCD}(\bar{\Psi}_{Aa}\epsilon_{EB})Z^2 Z^E_b\bar{Z}^d_D\bar{Z}^c_C + c.c.
\end{aligned}
\tag{C.65}
$$

And similar variation terms also come from the $\delta\Psi\mid_{Z^3\cdot f}$ of the new added terms $(\Psi\Psi)Z^2$(5.11)of the Lagrangian:

$$
\begin{aligned}
&ie\tfrac{1}{8}\epsilon^{ABCD}(\bar{\Psi}_{Aa}\delta\Psi_{Bd}\mid_{Z^3\cdot f})\bar{Z}^d_C\bar{Z}^a_D + c.c.\\
&\quad=ie\tfrac{1}{8}\epsilon^{ABCD}(\bar{\Psi}_{Aa}\epsilon_{EF})Z^E_e Z^F_b\bar{Z}^c_B\bar{Z}^d_C\bar{Z}^a_D + c.c.\\
&\qquad - ie\tfrac{1}{8}\epsilon^{ABCD}(\bar{\Psi}_{Aa}\epsilon_{EB})Z^E_e Z^F_b\bar{Z}^c_F\bar{Z}^d_C\bar{Z}^a_D + c.c.\\
&\quad= - ie\tfrac{1}{8}(\bar{\Psi}_{Aa}\cdot 3\delta^{[A}_E\epsilon^{CD]})Z^E_e Z^F_b\bar{Z}^c_F\bar{Z}^d_C\bar{Z}^a_D + c.c.\\
&\qquad + ie\tfrac{1}{8}(\bar{\Psi}_{Aa}\cdot 12\delta^{[AB}_{EF}\epsilon^{CD]})Z^E_e Z^F_b\bar{Z}^c_B\bar{Z}^d_C\bar{Z}^a_D + c.c.
\end{aligned}
\tag{C.66}
$$

$$
\begin{aligned}
&ie\tfrac{1}{16}(\bar{\Psi}_{Aa}\delta\Psi^{Aa}\mid_{Z^3\cdot f})Z^2 + c.c.\\
&\quad= - ie\tfrac{1}{16}f^{ab}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{CD})Z^2 Z^A_b\bar{Z}^c_C\bar{Z}^d_D + c.c.\\
&\qquad + ie\tfrac{1}{16}f^{ab}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{DA})Z^2 Z^C_b\bar{Z}^c_D\bar{Z}^d_C + c.c.
\end{aligned}
\tag{C.67}
$$

$$
\begin{aligned}
&ie\tfrac{1}{8}(\bar{\Psi}_{Aa}\delta\Psi^{Ab}\mid_{Z^3\cdot f})\bar{Z}^a_B Z^B_b + c.c.\\
&\quad=ie\tfrac{1}{8}f^{eb}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{CD})Z^A_e Z^B_b\bar{Z}^c_C\bar{Z}^a_B\bar{Z}^d_D + c.c.\\
&\qquad - ie\tfrac{1}{8}f^{eb}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{DA})Z^C_e Z^B_a\bar{Z}^c_D\bar{Z}^a_B\bar{Z}^d_C + c.c.
\end{aligned}
\tag{C.68}
$$

$$
\begin{aligned}
&-ie\tfrac{1}{4}(\bar{\Psi}_{Aa}\delta\Psi^{Ba}\mid_{Z^3\cdot f})\bar{Z}^b_B Z^A_b + c.c.\\
&\quad= + ie\tfrac{1}{4}f^{ab}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{CD})Z^A_e Z^B_b\bar{Z}^c_C\bar{Z}^e_B\bar{Z}^d_D + c.c.\\
&\qquad - ie\tfrac{1}{4}f^{eb}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{CB})Z^A_e Z^D_b\bar{Z}^c_D\bar{Z}^e_B\bar{Z}^d_C + c.c.
\end{aligned}
\tag{C.69}
$$

$$
\begin{aligned}
&ie\tfrac{3}{8}(\bar{\Psi}_{Aa}\delta\Psi^{Bb}\mid_{Z^3\cdot f})\bar{Z}^a_B Z^A_b + c.c.\\
&\quad= - ie\tfrac{3}{8}f^{eb}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{CD})Z^A_e Z^B_b\bar{Z}^c_C\bar{Z}^a_B\bar{Z}^d_D + c.c.\\
&\qquad + ie\tfrac{3}{8}f^{eb}{}_{cd}(\bar{\Psi}_{Aa}\epsilon^{CB})Z^A_b Z^E_e\bar{Z}^c_D\bar{Z}^a_B\bar{Z}^d_C + c.c.
\end{aligned}
\tag{C.70}
$$

We find there are five kinds of $(\bar{\Psi}_{Db}\epsilon^{AB})Z^5\cdot f$ terms left, whose coefficients satisfy the relations to cancel themselves.

For variation terms like $(\bar{\Psi}_{Db}\epsilon^{AB})Z^5\cdot f$, which is the rest, will cancel the variation of the scalar fields Z in the potential term with one structure constant(5.14).

C.3.1.3 The $(\Psi\epsilon)Z^5$-terms

The variation of $\Psi\mid_{Z^3}$ of the $(\Psi\Psi)Z^2$ term (5.11) gives:

$$
\begin{aligned}
ie\tfrac{1}{8}\epsilon^{ABCD}(\bar{\Psi}_{Aa}\delta\Psi_{Bd}\mid_{Z^3})\bar{Z}^d_C\bar{Z}^a_D + c.c. =&ie\tfrac{1}{32}(\bar{\Psi}_{Db}\cdot 12\delta^{[AB}_{EF}\epsilon^{CD]})Z^E_d Z^F_a\bar{Z}^d_C\bar{Z}^b_A\bar{Z}^a_B + c.c.\\
&- ie\tfrac{1}{128}(\bar{\Psi}_{Db}\cdot 3\delta^{[A}_E\epsilon^{BD]})Z^2 Z^E_a\bar{Z}^b_A\bar{Z}^a_B + c.c.
\end{aligned}
\tag{C.71}
$$

$$
\begin{aligned}
ie\tfrac{1}{16}(\bar\Psi_{Aa}\delta\Psi^{Aa}\mid_{Z^3})Z^2 + c.c. =& - ie\tfrac{1}{64}(\bar\Psi_{Db}\epsilon^{BA})Z^2 Z^D_a \bar Z^a_A \bar Z^b_B + c.c.\\
&+ ie\tfrac{1}{256}(\bar\Psi_{Db}\epsilon^{AD})Z^2 Z^2 \bar Z^b_A + c.c. \qquad (C.72)\\
ie\tfrac{1}{8}(\bar\Psi_{Aa}\delta\Psi^{Ab}\mid_{Z^3})\bar Z^a_B Z^B_b + c.c. =& ie\tfrac{1}{32}(\bar\Psi_{Db}\epsilon^{BA})Z^D_a Z^C_d \bar Z^a_B \bar Z^b_C \bar Z^d_A + c.c.\\
&+ ie\tfrac{1}{128}(\bar\Psi_{Db}\epsilon^{AD})Z^2 Z^C_a \bar Z^a_A \bar Z^b_C + c.c. \qquad (C.73)\\
-ie\tfrac{1}{4}(\bar\Psi_{Aa}\delta\Psi^{Ba}\mid_{Z^3})\bar Z^b_B Z^A_b + c.c. =& - ie\tfrac{1}{16}(\bar\Psi_{Db}\epsilon^{BA})Z^C_a Z^D_d \bar Z^a_B \bar Z^b_A \bar Z^d_C + c.c.\\
&+ ie\tfrac{1}{64}(\bar\Psi_{Db}\epsilon^{BA})Z^2 Z^D_a \bar Z^b_A \bar Z^a_B + c.c. \qquad (C.74)\\
ie\tfrac{3}{8}(\bar\Psi_{Aa}\delta\Psi^{Bb}\mid_{Z^3})\bar Z^a_B Z^A_b + c.c. =& ie\tfrac{3}{32}(\bar\Psi_{Db}\epsilon^{BA})Z^C_d Z^D_a \bar Z^a_B \bar Z^d_A \bar Z^b_C + c.c.\\
&- ie\tfrac{3}{128}(\bar\Psi_{Db}\epsilon^{BA})Z^2 Z^D_a \bar Z^a_A \bar Z^b_B + c.c. \qquad (C.75)
\end{aligned}
$$

On one hand, all of terms like $(\bar\Psi_{Db}\epsilon^{AB})Z^5$ above vanish; On the other hand, terms like $(\bar\Psi_{Db}\epsilon^{AD})Z^5$, which are the rest, cancel the variation of the scalar fields Z in the potential term without structure constant(5.15).

C.3.1.4 The $(\bar\chi\cdot\epsilon)Z^6\cdot f^2$-terms

The $\delta\Psi\mid_{Z^3 f}$ in the $L_{\hat B}$ term $(\chi\Psi)Z^3\cdot f(5.9)$ generates

$$
\begin{aligned}
&Aie(\bar\chi^{AB}_\mu\gamma^\mu\epsilon_{EF}Z^E_m Z^F_n \bar Z^e_B - \bar\chi^{AB}_\mu\gamma^\mu\epsilon_{EB}Z^E_m Z^F_n \bar Z^e_F)f^{mn}{}_{ed}f^{cd}{}_{ab}Z^D_c \bar Z^a_A \bar Z^b_D + c.c.\\
&-Aie(\bar\chi^{AB}_\mu\gamma^\mu\epsilon_{EF}Z^E_m Z^F_n \bar Z^e_D - \bar\chi^{AB}_\mu\gamma^\mu\epsilon_{ED}Z^E_m Z^F_n \bar Z^e_F)f^{mn}{}_{ed}f^{cd}{}_{ab}Z^D_c \bar Z^a_A \bar Z^b_B + c.c.\\
&= - 3Aie(\bar\chi^{AB}_\mu\gamma^\mu\epsilon_{EF} + \bar\chi_{\mu EF}\gamma^\mu\epsilon^{AB})f^{ab}{}_{cd}f^{ed}{}_{mn}Z^E_a Z^F_b Z^D_e \bar Z^m_A \bar Z^n_B \bar Z^c_D\\
&\quad - \tfrac{i}{2}Ae(\bar\chi^{AB}_\mu\gamma^\mu\epsilon_{AB})f^{ab}{}_{cd}f^{ed}{}_{mn}Z^E_a Z^F_b Z^D_e \bar Z^m_E \bar Z^c_F \bar Z^n_D\\
&= - \tfrac{i}{3}Ae(\bar\chi^{AB}_\mu\gamma^\mu\epsilon_{AB})f^{ab}{}_{cd}f^{ed}{}_{mn}Z^E_a Z^F_b Z^D_e \bar Z^m_E \bar Z^n_F \bar Z^c_D\\
&\quad + \tfrac{i}{6}Ae(\bar\chi^{AB}_\mu\gamma^\mu\epsilon_{AB})f^{ab}{}_{cd}f^{ed}{}_{mn}Z^E_a Z^F_b Z^D_e \bar Z^m_E \bar Z^c_F \bar Z^n_D, \qquad (C.76)
\end{aligned}
$$

which is exactly canceled by the variation of the dreibein in the potential term of the original ABJM action.

Also note, to get the result above we need to prove the equation

$$
\begin{aligned}
&9Aie(\bar\chi^{AB}_\mu\gamma^\mu\epsilon_{EF} + \bar\chi_{\mu EF}\gamma^\mu\epsilon^{AB})f^{ab}{}_{cd}f^{ed}{}_{mn}Z^E_a Z^F_b Z^D_e \bar Z^m_A \bar Z^n_B \bar Z^c_D\\
&= iAe(\bar\chi^{AB}_\mu\gamma^\mu\epsilon_{AB})f^{ab}{}_{cd}f^{ed}{}_{mn}Z^E_a Z^F_b Z^D_e \bar Z^m_E \bar Z^n_F \bar Z^c_D\\
&\quad - 2iAe(\bar\chi^{AB}_\mu\gamma^\mu\epsilon_{AB})f^{ab}{}_{cd}f^{ed}{}_{mn}Z^E_a Z^F_b Z^D_e \bar Z^m_E \bar Z^c_F \bar Z^n_D
\end{aligned}
$$

by circling [ABCDE] and using the fundamental identity of the structure constants.

C.3.1.5 The $(\bar\chi\cdot\epsilon)Z^6\cdot f$-terms

The two new terms $(\chi\Psi)Z^3(5.12)$ contribute to terms with structure constants, due to the $\delta\Psi\mid_{Z^3.f}$:

$$
\begin{aligned}
&+ ieA\tfrac{1}{4}f^{ab}{}_{cd}(\bar\chi_{\mu AB}\gamma^\mu\epsilon^{EF})Z^A_b Z^B_e Z^D_a \bar Z^a_D \bar Z^c_E \bar Z^d_F + c.c.\\
&- ieA\tfrac{1}{4}f^{ab}{}_{cd}(\bar\chi_{\mu AB}\gamma^\mu\epsilon^{EF})Z^A_b Z^B_e Z^D_a \bar Z^d_D \bar Z^c_E \bar Z^e_F + c.c. \qquad (C.77)
\end{aligned}
$$

and

$$\begin{aligned}&+ieA\tfrac{1}{16}f^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EF})Z^2Z^A_aZ^B_b\bar{Z}^c_E\bar{Z}^d_F+c.c.\\&+ieA\tfrac{1}{16}f^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EB})Z^2Z^A_bZ^F_a\bar{Z}^c_E\bar{Z}^d_F+c.c.\end{aligned}\tag{C.78}$$

Also, the $(\chi\Psi)Z^3\cdot f$ term $L_{\hat{B}}$ generates some similar terms, due to the $\delta\Psi\mid_{Z^3}$:

$$\begin{aligned}&-ieA\tfrac{1}{4}f^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EF})Z^A_aZ^B_eZ^D_b\bar{Z}^c_D\bar{Z}^e_E\bar{Z}^d_F+c.c.\\&+ieA\tfrac{1}{16}f^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EB})Z^A_aZ^2Z^D_b\bar{Z}^d_E\bar{Z}^c_D+c.c.\end{aligned}\tag{C.79}$$

and

$$\begin{aligned}&-ieA\tfrac{1}{4}f^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EF})Z^A_aZ^B_bZ^D_e\bar{Z}^c_D\bar{Z}^e_E\bar{Z}^d_F+c.c.\\&+ieA\tfrac{1}{16}f^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EB})Z^2Z^A_aZ^B_b\bar{Z}^d_E\bar{Z}^c_F+c.c.\end{aligned}\tag{C.80}$$

Circling the SU(4) indices [ABDEF], we could find that the R.H.S of the four equations above cancel the variation of the dreibein in the potential term with one structure constant(5.14). Of course, the relations among these coefficients have been used.

C.3.1.6 The $(\bar{\chi}\cdot\epsilon)Z^6$-terms

The $\delta\Psi\mid_{Z^3}$ variation of the $(\chi\Psi)Z^3$ term(5.12) gives:

$$\begin{aligned}&ieA\tfrac{1}{16}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EF})Z^A_aZ^B_bZ^D_d\bar{Z}^a_D\bar{Z}^b_E\bar{Z}^d_F+c.c.\\&-ieA\tfrac{1}{64}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EF})Z^2Z^A_aZ^B_b\bar{Z}^a_E\bar{Z}^b_F+c.c.\end{aligned}\tag{C.81}$$

and

$$\begin{aligned}&ieA\tfrac{1}{64}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EF})Z^2Z^A_aZ^B_b\bar{Z}^a_F\bar{Z}^b_E+c.c.\\&-ieA\tfrac{1}{256}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{AE})Z^4Z^B_b\bar{Z}^b_E+c.c.\\=&ieA\tfrac{1}{64}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{EF})Z^2Z^A_aZ^B_b\bar{Z}^a_F\bar{Z}^b_E+c.c.\\&-ieA\tfrac{1}{512}(\bar{\chi}_{\mu AB}\gamma^\mu\epsilon^{AB})Z^6\end{aligned}\tag{C.82}$$

Circling the SU(4) indices [ABDEF], we will find, the two terms above cancel the variation of the dreibein in the potential term without structure constant(5.15).

C.3.2 Terms with more than two fermions

When considering variation terms independent of D_μ with multiple-fermion, there are

$$\bar{\epsilon}\chi\chi^6,\ \bar{\epsilon}\chi\chi^4Z^2,\ \bar{\epsilon}\Psi\chi^4Z,\ \bar{\epsilon}\chi\chi^2\Psi^2,\ \bar{\epsilon}\chi\chi^2Z^4,\ \bar{\epsilon}\Psi\chi^2Z^3,\ \bar{\epsilon}\chi\Psi^2Z^2,\ \bar{\epsilon}\Psi\Psi^2Z\tag{C.83}$$

(with structure constants or not), which are all kinds of terms left in δL. To finalize the proof of supersymmetry, they have to be proven to vanish.

The first kind, $\bar{\epsilon}\chi\chi^6$, is irrelevant to the matter sector, so it has to disappear even in the pure supergravity case. For the second and third terms $\bar{\epsilon}\chi\chi^4Z^2$, $\bar{\epsilon}\Psi\chi^4Z$, they are absorbed into the contorsion in D_μ, and hence vanish while all variation terms including covariant derivative are canceled.

Then jump to the last one, $\bar{\epsilon}\Psi\Psi^2 Z$. Terms in this form with structure constants have been canceled in the original ABJM action. Variation terms in the form of $\bar{\epsilon}\Psi\Psi^2 Z$ without structure constants are generated in three different ways: the δZ of $(\Psi\Psi)Z^2$, the $\delta B_\mu{}^A{}_B$ and the δA_μ of the Dirac term. Actually, terms obtained in the first ways gives

$$\begin{aligned}
&e\frac{1}{8}(\bar{\Psi}_{Ab}\gamma^\mu\Psi^{Bb})(\bar{\Psi}^{Aa}\gamma_\mu\epsilon_{BD})Z_a^D + c.c.\\
&e\frac{1}{8}(\bar{\Psi}_{Db}\gamma^\mu\Psi^{Bb})(\bar{\Psi}^{Aa}\gamma_\mu\epsilon_{BA})Z_a^D + c.c.\\
&e\frac{1}{16}(\bar{\Psi}_{Db}\gamma^\mu\Psi^{Db})(\bar{\Psi}^{Aa}\gamma_\mu\epsilon_{BA})Z_a^B + c.c.
\end{aligned} \tag{C.84}$$

in which the first and the second lines cancel with $\delta B_\mu{}^A{}_B$ of the Dirac term, and the third line cancels with δA_μ of the Dirac term[4]. That is, all such variation terms vanish.

Now, the variation terms left are of four kinds:

$$\bar{\epsilon}\chi\chi^2\Psi^2,\ \bar{\epsilon}\chi\chi^2 Z^4,\ \bar{\epsilon}\Psi\chi^2 Z^3,\ \bar{\epsilon}\chi\Psi^2 Z^2. \tag{C.85}$$

What we have got so far is that terms in the form of $(\bar{\Psi}^{Aa}\Psi^{Bb})(\bar{\epsilon}\chi)\cdot f$ vanish.

Though the rest of them are too difficult to check in full details, there are some strong hints the Lagrangian we gave is complete. For example, the fact that most of the coefficients appearing in the Lagrangian are determined uniquely by at least two independent calculations, convinces us to believe both the Lagrangian and its SUSY transformation rules, must be the component parts of the final gauged theory. Then all variation terms involving the derivative have vanished, providing a strong constraint on the new terms we can add to the Lagrangian. Actually, combining with both the analysis of indices and cancelations of those variation terms independent of derivative, it seems no new terms are allowed when one traverses all possible terms of dimension-three.

[4]It is another support for the existence of the additional U(1) group, as we mentioned in the last section of **Chapter** 5.

Bibliography

[1] Pär arvidsson. *Superconformal theories in six dimensions.* hep-th/0608014v1.

[2] Martin F.Sohnius. *Introducing supersymmetry. Phys. Rep.*, 128(1985) 39.

[3] P.C.West. *Introduction to Rigid Sueprsymmetric Theories.* hep-th/9805055v1.

[4] Antoine Van Proeyen. *Tools for supersymmetry.* hep-th/9910030v6.

[5] S.Coleman and J.Mandula. *All possible symmetries of the S matrix. Phys. Rev.*, 159(1967) 1251.

[6] P. Ramond. *Dual theory for free fermions. Phys.Rev.D*, **3**(1971), 2415.

[7] Y.A.Golfand and E.P.Likhtman. *Extension of the algebra of Poincare grou generators an voilation of P invariance. JETP Lett.*, 13(1971) 323.

[8] J.Wess and B.Zumino. *A lagrangian model invariant under supergauge transformation. Phys.Lett.*, **B**49(1974) 52.

[9] J.Wess and J.Bagger. *Supersymmtry and supergravity.* Princeton Unverisity Press, 1992.

[10] S.Weinberg. *The quantum theory of fields, vol.III.Supersymmetry.* Cambridge University Press, 2000.

[11] L.Frappat and P.Sorba. *Dictionary on Lie superalgebra.* hep-th/9607161v1.

[12] W. Nahm. *Supersymmetries and their representations. Nucl.Phys.*, **B**135 (1978) 149.

[13] P.van Nieuwenhuizen. *Supergravity. Phys.Rep.*, 68(1981) 189.

[14] E.S.Fradkin and A.A.Tseytlin. *Conformal Supergravity. Phys.Rep.*, 119(1985) 233.

[15] C.M.Hull and P.K. Townsend. *Unity of superstring dualities. Nucl.Phys.*, **B**438 (1995) 109, hep-th/9410167v2.

[16] E.Witten. *String theory dynamics in various dimensions. Nucl.Phys.*, **B**443 (1995) 85, hep-th/9503124v2.

[17] E.Cremmer, B.Julia and J.Scherk. *Supergravity theory in 11 dimensions. Phys.Lett*, **B**76(1978) 409.

[18] S.Deser, R.Jackiw and S.Templeton. *Topologically massive gauge theories. Annals.of Phy.*, **140** (1982) 372-411.

[19] S.Deser and J.H.Key. *Topologically massive supergravity. Phys.Lett*, **B**120 (1983) 97.

[20] P.van Nieuwenhuizen. *Three-dimensional conformal supergravity and Chern-Simons terms. Phys.Rev.D*, 32(1985) 872.

[21] E.Witten. *(2+1)-Dimensional gravity as an exactly soluble system. Nucl.Phys.*, **B**331 (1988/89) 46.

[22] J.H.Horne and E.Witten. *Conformal gravity in three dimensions as a gauge theory. Phys.Rev.Lett*, 62(1989) 501.

[23] U.Lindström and M.Roček. *Superconformal gravity in three dimensions as a gauge theory. Phys.Rev.Lett*, 62(1989) 2905.

[24] Ulf Gran and Bengt E.W. Nilsson. *Three-dimensional N=8 superconformal gravity and its couplings to BLG M2-branes.* hep-th/0809.4478v3.

[25] J.Bagger and N.Lambert. *Three-algebras and N=6 Chern-Simons gauge theories.* hep-th/0807.0163v2.

[26] Ofer Aharony,Oren Bergman,Daniel Louis Jafferis and Juan Maldacena. *N=6 superconformal Chern-Simons-matter theory, M2-branes and their gravity duals.* hep-th/0806.1218v3.

[27] Juan Maldacena. *The large N limit of superconformal field theories and supergravity.* hep-th/9711200v3.

[28] John H.Schwaz. *Superconformal Chern-Simons theories.* hep-th/0411077v2.

[29] J.Bagger and N.Lambert. *Modeling multiple M2's.* hep-th/0611108v3.

[30] J.Bagger and N.Lambert. *Gauge symmetry and supersymmetry of multiple M2-branes.* hep-th/07110955v2.

[31] A.Gustavsson. *Algebraic structures on paraller M2-branes.* hep-th/0709.1260v5.

[32] Mark Van Raamsdonk. *Comments on the Bagger-Lambert theory and multiple M2-branes.* hep-th/0803.3803v3.

[33] J.Bagger and N.Lambert. *Comments on multiple M2-branes.* hep-th/0712.3738v1.

[34] U.Gran,Bengt E.W.Nilsson and C.Petersson. *On relating multiple M2 and D2-branes.* hep-th/0804.1784v2.

[35] Bengt E.W.Nilsson and Jakob Palmkvist. *Superconformal M2-branes and generalized Jordan triple systems.* hep-th/0807.5134v2.

[36] M.A.Bandres, A.E.Lipstein and J.H.Schwarz. *Studies of the ABJM theory in a formulation with Manifest SU(4) R-symmetry.* hep-th/0807.0880v2.

[37] Jakob Palmkvist. *Three-algebras,triple systems and 3-graded Lie superalgebra.* hep-th/0905.2468v1.

[38] L.Brink, P.Di Vecchia and P.Howe. *A locally supersymmetric and reparameterization invariant action for the spinning string. Phys.Lett*, **B**65(1976) 471.

[39] A.M.Polyakov. *Quantum geometry of fermionic string. Phys.Lett*, **B**103(1981) 211.

[40] Xiaoyong Chu and Bengt E.W.Nilsson. *Three-dimensional topologically gauged N=6 ABJM type theories.* hep-th/0906.1655v1.

[41] Jeong-Hyuck Park. *Lectures on Gamma Matrix and supersymmetry. http://conf.kias.re.kr/ brane/2003/gamma.pdf.*

[42] S.Naito, K.Osada and T.Fukui. *Fierz identities and invariance of 11-dimensional supergravity action. Phys.Rev.D*, 34(1986) 536.