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Exercise Classes

Gravitation and Cosmology

Exercise Class 1 - Special Relativity

1. Show that the proper time $d\tau$ is invariant under a Lorentz transformation.

2. Write out $\Lambda^{\alpha}{}_{\beta}$ for a "standard" Lorentz boost ($c = 1$)

$$\begin{aligned} t' &= \gamma(v)(t - vx) \\ x' &= \gamma(v)(x - vt) \\ y' &= y \\ z' &= z. \end{aligned}$$

   What if $\mathbf{v}$ is not along the x-axis? Find also $\Lambda_{\alpha\beta}$ and $\Lambda^{\alpha\beta}$.

3. Suppose that a particle is moving with velocity u at an angle θ from the x-axis in the xy-plane of a system S. What is the corresponding angle θ' in a system S' moving with velocity v in the x-direction relative to S?

4. We have two rockets, A and B, moving in some system S with velocities u and v, respectively. Find the relative velocity of B in A's reference frame!

5. Show that

$$\begin{aligned} \frac{\partial}{\partial x^{\alpha}} F^{\alpha\beta} &= -J^{\beta} \\ \varepsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^{\beta}} F_{\gamma\delta} &= 0 \end{aligned}$$

   are the usual Maxwell equations.

6. Suppose that we have a scalar field $\phi(t, \mathbf{x})$. How fast is this field changing its value (per unit time) as measured by an observer moving with the four-velocity U^{α}?

Exercise Class 2 - Tensor Analysis

1. Find the metric for a flat two-dimensional surface expressed in polar coordinates.

2. Calculate the affine connection $\Gamma^{\lambda}_{\mu\nu}$ associated with the metric obtained in the previous exercise.

3. Starting from a flat two-dimensional space, described by Cartesian coordinates, perform the coordinate transformation

$$\begin{aligned} x' &= x + y \\ y' &= y. \end{aligned}$$

   Find the metric $g'_{\mu\nu}$!

4. Find the metric and affine connection on the surface of a two-sphere of radius a embedded in a Euclidean three-dimensional space.

5. Assume that $A^{\mu}B_{\mu\nu}$ is a covariant vector for all contravariant vectors A^{μ}. Show that $B_{\mu\nu}$ transforms as a covariant tensor.

Exercise Class 3 - Geodesics

1. Write down the equations of motion for a free particle on a flat two-dimensional surface expressed in polar coordinates.

2. Find all geodesics on the surface of a two-sphere of radius a embedded in a Euclidean three-dimensional space.

3. Determine all time-like and light-like geodesics for the two-dimensional metric

$$d\tau^2 = t^4 dt^2 - t^2 dx^2 \,.$$

4. Perform a parallel transport of a contravariant vector along a "latitude" on a two-sphere.

Exercise Class 4 - Curvature

1. Write out the covariant Laplacian $D_\mu D^\mu \equiv g^{\mu\nu} D_\mu D_\nu$ acting on a scalar for a two-dimensional flat space using polar coordinates.
2. Calculate the Riemann-Christoffel curvature tensor $R^\lambda{}_{\mu\nu\kappa}$ for a flat two-dimensional surface expressed in polar coordinates.
3. Calculate the curvature tensor, the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R on the surface of a two-sphere of radius a embedded in a Euclidean three-dimensional space.

Exercise Class 5 - The Schwarzschild Solution

1. Find the equations of motion for a massive particle in Schwarzschild geometry.

2. Using the Schwarzschild metric, find the proper length of the curves

   (a) $r = \theta = \text{const}$, $0 \leq \phi \leq 2\pi$

   (b) $\theta = \phi = \text{const}$, $r_1 \leq r \leq r_2$.

   Comment on the result!

3. Consider a massive particle in a space-time described by the Schwarzschild metric. For a given value of $j \equiv r^2\dot{\phi}$, is it possible to be in a circular orbit? If so, what is the value of r for this orbit?

Exercise Class 6 - Schwarzschild Cont.

1. Calculate the deflection of a light ray gracing the sun's surface.
2. Find the time for one revolution in the circular orbit at radius $r_c = 3a = 6MG$, as measured by
   (a) a comoving observer
   (b) an observer at rest at $r = r_c$
   (c) a distant ($r \to \infty$) observer at rest.
3. Calculate the coordinate time needed for a particle to fall into a black hole and pass the event horizon. Compare with the proper time measured by a comoving observer.

Exercise Class 7 - Symmetric Spaces and Cosmology

1. Find all Killing vectors on the two-sphere embedded in $\mathbf{R}^3$.

2. Consider the Robertson-Walker metric

$$d\tau^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right].$$

   Investigate the properties of the spatial part in the three cases $k = 1$, $k = 0$ and $k = -1$.

3. Starting from the conservation of energy and momentum

$$D_\mu T^{\mu\nu} \equiv T^{\mu\nu}{}_{;\mu} = 0$$

   in a space-time described by the Robertson-Walker metric, derive the relation

$$\frac{d}{dt}\left(\rho(t)R^3(t)\right) + p(t)\frac{d}{dt}\left(R^3(t)\right) = 0.$$