Home examination, Gravitation & Cosmology, 2007

To be handed in January 18, 2008

The maximum score for home assignments (10+10 points) and home examination (80 points) is 100 points. 50 points is minimum requirement (see the course web page for Chalmers, GU and ECTS grading).

1. Consider the d-dimensional geometries described by the metrics

$$(1) \qquad ds^2 = \frac{-dt^2 + \delta_{ij}dx^i dx^j}{t^2} \ ,$$

$$(2) \qquad ds^2 = \frac{dy^2 + \eta_{\alpha\beta}dx^\alpha dx^\beta}{y^2} \ ,$$

where δ and η are metric tensors for flat euclidean and minkowskian $(d-1)$-dimensional space, respectively. Calculate the curvature tensor in both cases, and describe the spaces in words.
(5 points)

2. The temperature of the background radiation of the universe is ≈ 2.7 K. The Hubble parameter is measured to be ≈ 70 (km/s)/Mpc. Suppose (somewhat contrary to the present observational status) that the universe is matter-dominated with exactly critical energy density, and has been so since atoms formed, matter became uncharged, electromagnetic radiation decoupled from matter and the universe became transparent. This transition can be assumed to have taken place at a temperature which is typical for atomic binding energies, $T \approx 5$ eV. Assume also that the universe before this decoupling was radiation-dominated. What is the age of the universe based on these measurements and assumptions, and what was its age at the time of decoupling?
(15 points)

3. A beacon radiating at a fixed frequency ν_0 is released at time $t = 0$ towards a black hole of mass M by an observer situated very far away from the black hole. The observer stays at a constant distance from the black hole while the probe is falling. Show that the frequency of the beacon (when it is close to the event horizon) as measured by the observer can be written as $\nu \sim e^{-\frac{t}{K}}$ for some constant K and relate the constant K to the mass of the black hole.
(10 points)

4. With the assumptions made in problem 2, show that the microwave background radiation we receive from different directions come from regions that at the time of decoupling were causally disconnected, *i.e.*, no information with a common source may have reached them during the time of existence of the universe.
(Still, the large-scale structure of the radiation is isotropic. This is the so-called "homogeneity problem" or "horizon problem": how can these regions then "know" that they should contain matter and radiation with the same density and temperature? It may be solved by "inflation": if the universe at some time underwent a very fast expansion, our visible part of the universe may come from a causally connected region. Incidentally, inflation can also solve the "flatness problem" and explain why the universe seems to be so close to being flat.)
(5 points)

5. A four-dimensional space-time is descibed by the metric

$$ds^2 = -2dudv + a^2(u)dx^2 + b^2(u)dy^2 \ .$$

Verify that the signature is $(-1, 1, 1, 1)$. Why is such a metric called a plane wave? Find all Killing vectors (some, but not necessarily all, are obvious in the sense that they don't require calculation). Calculate the energy-momentum tensor for the matter or radiation that acts as a source allowing this solution.
(15 points)

6. The metric of a rotating charged black hole can be written as:

$$d\tau^2 = (1-f)dt^2 - 2af\sin^2\theta dt d\phi - \Sigma(\frac{dr^2}{\Delta} + d\theta^2) - g\sin^2\theta d\phi^2 \ ,$$

where

$$\begin{aligned}
\Delta &= r^2 - (2mr - Q^2) + a^2 \ , \\
\Sigma &= r^2 + a^2\cos^2\theta \ , \\
f &= (2mr - Q^2)/\Sigma \ , \\
g &= r^2 + a^2 + fa^2\sin^2\theta \ , \\
m^2 &> Q^2 + a^2 \ ,
\end{aligned}$$

and m, Q, a, are mass, charge and angular momentum respectively and we set $G = 1$.

The metric is of course valid also outside a rotating charged object. Specialize on the case $Q = 0$. Verify that the solution solves Einstein's equations. For a circular orbit in the

equatorial plane: $\theta = \pi/2$ derive the law corresponding to Kepler's law $\Omega^2 = m/r^3$ for an object with angular velocity Ω at a radius r. Compute the time dilation factor for an observer moving in such an orbit, as judged by a distant stationary observer. Is the correction to the geometry around Earth due to its rotation a significant one?
(20 points)

7. Three-dimensional anti-deSitter space (AdS_3) is described by the metric

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \ ,$$

where the coordinates are confined to the hyperboloid

$$-u^2 - v^2 + x^2 + y^2 = -b^2 \ .$$

Change coordinates according to

$$\begin{aligned} u &= \sqrt{b^2 + r^2} \cos \frac{t}{b} \ , \\ v &= \sqrt{b^2 + r^2} \sin \frac{t}{b} \ , \\ x &= r \cos \phi \ , \\ y &= r \sin \phi \end{aligned}$$

(check that this defines a valid set of coordinates on the hyperboloid). Using *e.g.* the metric in the new coordinates, calculate the proper distance from the origin $r = 0$ to spatial infinity $r \to \infty$. Also find the coordinate time needed for a photon to travel this distance.
(10 points)