<u>Home assignment 3, String theory, 2018-19</u>

0. Exercise problems from Zwiebach: 14.4, 17.5, 26.3.

1. Chiral spinors. Show that in any even dimension, $D = 2n$, the γ matrices can be combined into an equal number of creation and annihilation operators, and that this leads to the dimension 2^n of a Dirac spinor. Define $\gamma = \frac{1}{D!}\epsilon^{m_1 \ldots m_D}\gamma_{m_1} \ldots \gamma_{m_D}$. Show that $\gamma^2 = 1$ if the signature of space-time (*i.e.*, the numbers of time and space directions, respectively) is (n, n). Show that $\gamma^2 = 1$ for $D = 10$ with Minkowski signature and for $D = 8$ with Euclidean signature (in all these examples, spinors are real). Form projection operators on chiral spinors.

2. Superconformal algebras. The Virasoro algebra is given by the commutators

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \ ,$$

the second term being the central extension. Any algebra realised as commutators of (associative) operators must satisfy the *Jacobi identities*

$$[A, [B, C]] + \text{cyclic perm.} = 0 \ .$$

Explain why. You have already checked that the central term in the Virasoro algebra has this unique form (modulo redefinitions of L_0).

Consider fermionic generators G_r (here, the index r can belong to $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$) with conformal weight 3/2. The commutators $[L_m, G_r]$ are determined by the conformal weight of G. Given the value of c in the central term in the Virasoro subalgebra, what can be concluded about a possible central term in $\{G_r, G_s\} = 2L_{r+s} + \ldots$? (Be careful about signs and commutators/anti-commutators when deriving the relevant (graded) Jacobi identities!) Give the full list of (anti-)commutators between elements in the superconformal algebra.

3. BRST. Suppose one has a set of constraints T^A generating a gauge symmetry, and that $[T^A, T^B] = f^{AB}{}_C T^C$. Introduce fermionic ghosts c_A and their conjugates b^A, and show that there exists a "BRST operator" Q, formed as a linear combination of the terms $c_A T^A$ and $f^{AB}{}_C c_A c_B b^C$, that is nilpotent, $Q^2 = 0$.

What is the BRST operator for Maxwell theory?

(Physical states can be defined in terms of the *cohomology* of Q, *i.e.*, states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, modulo the equivalence relation $|\psi\rangle \approx |\psi\rangle + Q|\Lambda\rangle$.)

4. Gravitini. A gravitino is a massless Rarita–Schwinger field, *i.e.*, a "spin 3/2 field", which is a supersymmetric partner of the graviton. Consider a free graviton in Minkowski space. How many physical degrees of freedom does a (minimal) gravitino have in $D = 4$? In $D = 10$? Try to find a way to demonstrate how this counting arises. You may need to search for information.

5. $D = 10$ supersymmetry. The massless sector of an open superstring contains the fields of $D = 10$ super-Maxwell theory. These are a gauge potential A_μ and a chiral spinor ψ^α. The action is

$$S = -\frac{1}{g^2}\int d^{10}x \left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\psi\gamma^\mu\partial_\mu\psi\right) \ .$$

Find global supersymmetry transformations of A and ψ such that the action is invariant. (Hint: for dimensional reasons, the supersymmetry parameter must have dimension $-1/2$ (in powers of inverse length), when A has dimension 1 and ψ dimension 3/2.) Commute two such transformations, and show that the result is a translation, modulo gauge transformations, only if the fermion obeys its equation of motion. (The number of bosonic and fermionic degrees of freedom do not match until the equations of motion are imposed. For theories with a maximal number of supersymmetries, such as this one, and also the superstring, this posed problems with keeping manifest supersymmetry during quantisation for a long time.)

Show, *e.g.* using the gamma matrices below (a combination of the 0 and 9 directions is simplest), that if $p^2 = 0$, the equation $\gamma^\mu p_\mu\psi = 0$ leaves 8 local degrees of freedom ("polarisations") for the spinor field, corresponding to the GSO-projected Ramond ground states.

Some useful information about spinors and gamma matrices in ten dimensions:
A chiral spinor has 16 components (see Problem 1). Gamma matrices can be chosen as

$$(\Gamma^\mu)^A{}_B = \begin{pmatrix} 0 & (\gamma^\mu)^{\alpha\beta} \\ (\gamma^\mu)_{\alpha\beta} & 0 \end{pmatrix} \ ,$$

acting on spinors

$$\Psi^A = \begin{pmatrix} \psi^\alpha \\ \lambda_\alpha \end{pmatrix} \ .$$

The matrices γ^μ are symmetric. If ψ^α is chiral, then $(\gamma_\mu\psi)_\alpha$ is anti-chiral, so $(\gamma_{\mu\nu}\psi)^\alpha$ has the same chirality as ψ. Checking the commutator of supersymmetry transformations involves a so called Fierz identity $(\gamma_\mu)_{(\alpha\beta}(\gamma^\mu)_{\gamma\delta)} = 0$, *which holds in $D = 3, 4, 6, 10$.*

In terms of 8-dimensional spinors, the matrices can be split further, so that

$$\Gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} ,$$
$$\Gamma^I = \begin{pmatrix} 0 & \gamma^I \\ \gamma^I & 0 \end{pmatrix} , \quad I = 1, \ldots, 8 ,$$
$$\Gamma^9 = \begin{pmatrix} 0 & \sigma_9 \\ \sigma_9 & 0 \end{pmatrix} ,$$

where $\sigma_9 = \mathrm{diag}(1_8, -1_8)$, *and where*

$$\gamma^I = \begin{pmatrix} 0 & \tilde{\sigma}^I \\ \sigma^I & 0 \end{pmatrix}$$

are 8-dimensional γ *matrices. This method for constructing gamma matrices for Minkowski signature in* D *dimensions from the ones for Euclidean signature in* $D-2$ *dimensions is general.*