Home assignment 1 — Symmetry TIF310/FYM310

Deadline Monday Dec. 9

Hand in solutions produced by TEX by mail to martin.cederwall@chalmers.se or printed in a box outside room Origo 6102. Good luck!

1.1. Show that the real Lie algebra $\mathfrak{so}(4)$ of rotations in 4 euclidean dimensions is $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

1.2. In $2n$ dimensions with signature (n, n), consider the matrix $\gamma = \frac{1}{(2n)!\sqrt{|\det g|}} \epsilon^{M_1 \ldots M_{2n}} \gamma_{M_1} \ldots \gamma_{M_{2n}}$. In a basis where the metric is diagonal with entries ± 1, $\gamma = \gamma_1 \ldots \gamma_{2n}$. What is γ^2? How does it anti-commute with the γ matrices? Use γ to form projection operators on the two chiralities. If ψ is a spinor of definite chirality, what is the chirality of $v^M \gamma_M \psi$?
How are the properties of γ affected if the signature is changed? (Hint: one may think of multiplying some of the γ-matrices by i.)

1.3. Consider the Maxwell field strength 2-form

$$F = \frac{1}{4\pi r^3}(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy) \ ,$$

which is well defined outside the origin. What is the corresponding B-field? Show that F satisfies Maxwell's equations for $r > 0$. Calculate the surface integral $\int_S F = \int_S \vec{B} \cdot d\vec{S}$, where S is a surface enclosing $r = 0$, and conclude that there is a magnetic monopole at $r = 0$. Find a 1-form A such that $dA = F$. Is it well defined everywhere outside the origin?

1.4. Construct the two 3-dimensional $\mathfrak{sl}(3)$-modules by starting from the highest weights with Dynkin labels (10) and (01) and acting with lowering operators. If we denote a representation by the Dynkin labels of its highest weight, we can write $\mathbf{3} = (10)$, $\overline{\mathbf{3}} = (01)$. Determine, by some method, the tensor products $(10) \otimes (10)$ and $(10) \otimes (01)$ as direct sums of irreducible representations. Illustrate with sums of weights in a picture.

1.5. The *Weyl group* of a semi-simple Lie algebra is the discrete group generated by reflections in hyperplanes orthogonal to the simple roots. It is a symmetry of the root system. Reflection in the hyperplane orthogonal to α_i maps a vector β to $w_i(\beta) = \beta - \frac{2(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$. Describe the Weyl groups of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ (number of elements, multiplication table).

1.6. A construction of the 14-dimensional Lie algebra G_2. Inspecting the root space of G_2, one finds that all roots of G_2 are weights of $\mathfrak{sl}(3)$, and that it consists of the weights for the adjoint (the roots) of $\mathfrak{sl}(3)$ together with the weight for the two 3-dimensional representations, which we can call $\mathbf{3}$ and $\overline{\mathbf{3}}$. Therefore, the adjoint $\mathbf{14}$ of G_2 transforms as $\mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}$ under the subalgebra $\mathfrak{sl}(3) \subset G_2$. There must be a formulation of G_2 with manifest $\mathfrak{sl}(3)$ and generators $J_m{}^n$, K_m and L^m. Construct the brackets by making some Ansatz and checking the Jacobi identities.
(Hints: One is only allowed to use invariant tensors under $\mathfrak{sl}(3)$. Only some of the Jacobi identities are "non-trivial", in the sense that they do not follow from the $\mathfrak{sl}(3)$ covariance.)