Home assignment 2 — Symmetry TIF310/FYM310

Deadline Monday Jan. 6

Hand in solutions produced by TEX by mail to martin.cederwall@chalmers.se or printed in a box outside room Origo 6102. Good luck!

2.1. A non-linear realisation. Consider the quotient space $SL(2,\mathbb{R})/U(1)$, defined as equivalence classes of elements in $SL(2,\mathbb{R})$ modulo the right action of a $U(1)$. Two elements g and g' in $SL(2,\mathbb{R})$ (2×2 real matrices with unit determinant) are considered equivalent if they are related by a $U(1)$ transformation as $g' = gh$, where

$$h = e^{\theta j} , \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Show that almost all elements in $SL(2,\mathbb{R})$ are in the same equivalence class as an element of the form

$$g = \frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} .$$

Use this parametrisation to derive the transformation of the complex number $z = x + iy$ for such a representative of the equivalence class under the left action $g \mapsto Mg$ with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) , \quad ad - bc = 1 .$$

Show that the metric

$$ds^2 = \frac{dzd\bar{z}}{(\mathrm{Im}\, z)^2}$$

is invariant under $SL(2,\mathbb{R})$. (This is the so called Poincaré upper half plane, describing a 2-dimensional hyperbolic space with constant curvature.) Discuss what the $SL(2,\mathbb{R})$ isometry means. Is this a maximally symmetric space? Also, examine what the the word "almost" above means.

2.2. Consider the rank 2 simple Lie algebra $C_2 \simeq \mathfrak{sp}(4)$. This algebra has a 4-dimensional and a 5-dimensional representation. Describe these in a tensor language. Also, construct them as highest weight representations by acting with lowering operators on some highest weight states.

2.3. Consider an "inversion" of space (or space-time), defined by

$$x^m \mapsto x'^m = \frac{x^m}{x^2} .$$

Show that a special conformal transformation, on the infinitesimal form given in the lecture notes, is obtained by first performing an inversion, then a translation, and finally an inversion.

2.4. The Lie algebra F_4. There is a simple 52-dimensional Lie algebra F_4, with Dynkin diagram in the picture in the lecture notes. It has a subalgebra $\mathfrak{so}(9)$. The adjoint representation of F_4 becomes, seen as an $\mathfrak{so}(9)$ representation, the direct sum of the (36-dimensional) adjoint representation and a (16-dimensional) spinor representation. Construct the F_4 Lie bracket in an $\mathfrak{so}(9)$-covariant way, and check the Jacobi identities. (Information which can possibly be of help, and which may be used: The completely antisymmetric tensor product of 3 $\mathfrak{so}(9)$ spinors does not contain a spinor as one of its irreducible parts.)

2.5. A calculation with Fierz identities. The tensor product of two spinor representations contains antisymmetric tensors. This is due to the existence of invariant tensors $\gamma^{m_1\cdots m_p}$. As an example, take spinors in 9 dimensions. The dimension of the spinor representation S is 16. Count the number of matrices $\gamma^{m_1\cdots m_p}$ for different values of p, and determine, by demanding that the numbers sum up to the dimensions of the symmetric and antisymmetric parts of $S \otimes S$, which of these (with one index lowered to $(\gamma^{m_1\cdots m_p})_{\alpha\beta}$) are symmetric and which are antisymmetric in $\alpha\beta$.
Tensor products of more than two spinors are more difficult. There may be identities, so called Fierz identities, for products of more than one matrix $\gamma^{m_1\cdots m_p}$. Still in 9 dimensions, show that $\gamma_{mn[\alpha\beta}\gamma^{mn}{}_{\gamma\delta]} = 0$. One way of checking this is to contract the expression with all matrices $M^{[\gamma\delta]}$ (see above) and using the properties of the γ matrices. How does this Fierz identity relate to problem 2.4?

2.6. The energy-momentum (or stress-energy) tensor can be derived as the variation of an action with respect to the metric as

$$T^{mn} = 2\frac{\partial \mathscr{L}}{\partial g_{mn}} ,$$

where $\mathscr{L}$ is the Lagrangian density. This applies also for a theory defined in flat space, but then the metric has to be reinstated so that coordinate invariance of the action is manifest.
Use this definition to derive the energy-momentum tensor for Maxwell theory, and identify the usual forms of the energy density as T^{00} and the Poynting vector as T^{0i}.
Show that the Maxwell energy-momentum tensor is traceless precisely in $d = 4$, and relate this property to the invariance of the action under a rescaling of the metric. This is a sign of conformal invariance.

2.7. Symmetries of the Kepler problem. Consider the motion of a Newtonian particle with mass m in the central potential $V(\vec{r}) = -\frac{k}{r}$. Show that the components of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ fulfil $\{L_i, H\} = 0$, and are conserved charges. Which is the Lie algebra generated by these charges?
Consider the Runge–Lenz vector

$$\vec{A} = \vec{p} \times \vec{L} - km\hat{r} .$$

The dimensionless vector $\frac{\vec{A}}{km}$ is the so called eccentricity vector. Show that $\vec{A}$ is conserved. It is convenient to rescale the Runge–Lenz vector to

$$\vec{B} = \frac{\vec{A}}{\sqrt{2m|E|}} ,$$

where E is the energy, for $E \neq 0$. Investigate the algebra of conserved charges under the Poisson bracket. It may be different in the cases $E < 0$, $E = 0$ and $E > 0$. Such "hidden symmetries" may be used to relate solutions to the equations of motion with the same energy to each other.