Home assignment 2 — Symmetry TIF310/FYM310

Deadline Wednesday Dec. 9, 2020

(Each problem in this set has approximately twice the weight/number of points compared to the problems in the first set.)

Hand in solutions in pdf format, produced by TEX, containing your name, by mail to martin.cederwall@chalmers.se (or printed in a box outside room Origo 6102 if you can get through the locked doors). Good luck!

2.1. a. Derive the explicit form of the Klein–Gordon equation in spherical coordinates (and time) in $d = 4$.

b. Find the static, spherically symmetric, solution to the Klein–Gordon equation with $m \neq 0$ in $d = 4$ corresponding to a point source at the origin.

2.2. Construct a vector v as the square of a $d = 4$ bosonic spinor λ as $v_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$. Show that $v^2 = 0$. Perform the corresponding construction in $d = 3$. (It works also in $d = 6$ and $d = 10$.)

2.3. Chiral spinors under (some real form of) $\mathfrak{so}(12)$ are 32-dimensional. Call the spinor modules $\mathbf{32}$ and $\mathbf{32}'$. They are both self-conjugate, meaning that $\mathbf{32}$ is its own dual module, and accordingly for $\mathbf{32}'$. Determine which antisymmetric tensors appear in $\mathbf{32} \otimes \mathbf{32}$, and which of these belong to the symmetric and antisymmetric part of the tensor product.

2.4. Consider the rank 4 simple Lie algebra $D_4 \simeq \mathfrak{so}(8)$. Draw its Dynkin diagram and write the corresponding Cartan matrix. The diagram has a lot of symmetry. Construct the highest weight representation corresponding for a representation with Dynkin label 1 for one of the simple roots at the outermost positions in the Dynkin diagram, and 0 for all other simple roots, by acting with lowering operators on the highest weight state. What is the dimension of such a representation? Interpret the three such representations.

2.5. The energy-momentum (or stress-energy) tensor can be derived as the variation of an action with respect to the metric as

$$T^{mn} = \frac{2}{\sqrt{|g|}} \frac{\partial \mathscr{L}}{\partial g_{mn}} ,$$

where $\mathscr{L}$ is the Lagrangian density. This applies also for a theory defined in flat space, but then the metric has to be reinstated so that coordinate invariance of the action is manifest.
Use this definition to derive the energy-momentum tensor for Maxwell theory, and identify the usual forms of the energy density as T^{00} and the Poynting vector as T^{0i}.
Show that the Maxwell energy-momentum tensor is traceless precisely in $d = 4$, and relate this property to the invariance of the action under a rescaling of the metric. This is a sign of conformal invariance.