Home assignment 3 — Symmetry TIF310/FYM310, period II, 2020

Deadline Monday Jan. 8

Hand in solutions produced by TEX by mail to martin.cederwall@chalmers.se or printed in a box outside room Origo 6102. Good luck!

3.1. In 2-dimensional Minkowski space, let $u = \frac{1}{\sqrt{2}}(x^0 + x^1)$, $v = \frac{1}{\sqrt{2}}(x^0 - x^1)$. What is the metric in this coordinate system? Show that any transformation to new coordinates $u' = f(u)$, $v' = g(v)$, where f and g are strictly monotonous functions, is a conformal transformation.
Consider the infinitesimal form of such transformations. Which of them form the $\mathfrak{so}(2,2)$ subalgebra?

3.2. Let a Dynkin diagram be defined as a circle of $n+1$ nodes, where each node is connected to its two neighbours with single lines. Show that it can not be the Dynkin diagram of a finite-dimensional Lie algebra, *i.e.*, that a set of roots with the corresponding scalar products do not form a basis of Euclidean $(n+1)$-dimensional space. (The algebra A_n^+ defined by this Dynkin diagram is indeed infinite-dimensional, a so called affine Lie algebra.)

3.3. Consider the Maxwell field strength 2-form

$$\begin{aligned} F &= \frac{1}{4\pi r^3}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy) \\ &= \frac{1}{8\pi r^3}\epsilon_{ijk} x^i dx^j \wedge dx^k \\ &= \frac{\sin\theta}{4\pi} d\theta \wedge d\varphi\ , \end{aligned}$$

which is well defined outside the origin. Demonstrate the equality of the different expressions. What is the corresponding B-field? Show that F satisfies Maxwell's equations (without sources) for $r > 0$. Calculate the surface integral $\int_S F = \int_S \vec{B} \cdot d\vec{S}$, where S is a surface enclosing $r = 0$, and conclude that there is a magnetic monopole at $r = 0$. Find a 1-form A such that $dA = F$. Is it well defined everywhere outside the origin?

3.4. A construction of the 14-dimensional Lie algebra G_2. Inspect the root space of G_2, and verify that all roots of G_2 are weights of $\mathfrak{sl}(3)$, and that it consists of the weights for the adjoint (the roots) of $\mathfrak{sl}(3)$ together with the weight for the two 3-dimensional representations, which we can call $\mathbf{3}$ and $\bar{\mathbf{3}}$. There should be a formulation of G_2 with manifest $\mathfrak{sl}(3)$ and generators $J_m{}^n$, K_m and L^m. Construct the brackets by making some Ansatz and checking the Jacobi identities. (Hint: Only some of the Jacobi identities are "non-trivial", in the sense that they do not follow from the $\mathfrak{sl}(3)$ covariance.)

3.5. Symmetries of the Kepler problem. Consider the motion of a Newtonian particle with mass m in the central potential $V(\vec{r}) = -\frac{k}{r}$. Show that the components of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ fulfil $\{L_i, H\} = 0$, and are conserved charges. Which is the Lie algebra generated by these charges? Consider the Runge–Lenz vector

$$\vec{A} = \vec{p} \times \vec{L} - km\hat{r} \ .$$

The dimensionless vector $\frac{\vec{A}}{km}$ is the so called eccentricity vector. Show that $\vec{A}$ is conserved. It is convenient to rescale the Runge–Lenz vector to

$$\vec{B} = \frac{\vec{A}}{\sqrt{2m|E|}} \ ,$$

where E is the energy, for $E \neq 0$. Investigate the algebra of conserved charges under the Poisson bracket. It may be different in the cases $E < 0$, $E = 0$ and $E > 0$. Such "hidden symmetries" may be used to relate solutions to the equations of motion with the same energy to each other. Discuss (without explicit calculation) whether *e.g.* a circular orbit can be transformed into an elliptic (but non-circular) one using the corresponding hidden symmetry group.