Home assignment 1, 2021 — Symmetry TIF310/FYM310

Deadline Friday Nov. 19

Hand in solutions, preferrably produced by TEX, by mail to martin.cederwall@chalmers.se (or printed in a box outside room Origo 6102, which requires transversing locked doors). Good luck!

1.1. Consider the Lie algebra $\mathfrak{so}(4,\mathbb{R})$ of rotations in 4 euclidean dimensions.

a) Show that an antisymmetric tensor can be divided into a selfdual and an anti-selfdual part as $J_{ab} = J^{(+)}_{ab} + J^{(-)}_{ab}$, where

$$J^{(\pm)}_{ab} = \tfrac{1}{2}(J_{ab} \pm \tfrac{1}{2}\epsilon_{abcd}J_{cd}) \, ,$$

such that $J^{(+)}$ and $J^{(-)}$ do not mix under $\mathfrak{so}(4)$ transformations.

(*Hint: Given an antisymmetric tensor J, the dual tensor $J^\star$ can be defined as $J^\star_{ab} = \frac{1}{2}\epsilon_{abcd}J_{cd}$. Show that $(J^\star)^\star = J$, and that $(J^{(\pm)})^\star = \pm J^{(\pm)}$.*)

b) The algebra $\mathfrak{so}(4)$ is generated by antisymmetric matrices. Show that the two duality components each generate an $\mathfrak{su}(2)$, so that $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

c) Describe a 4-dimensional vector as a representation module of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

1.2. Consider the discrete group acting on vectors in $\mathbb{R}^2 = \mathbb{C}$ which is generated by reflections in the real line and in the line through the origin and the point $e^{2\pi i/3}$. Call these two group elements a and b. The statement that they "generate" a group means that one considers all elements obtained by applying them repeatedly, in some order. Each of the generators (being a reflection) squares to the identity element, $a^2 = I = b^2$. They generate cycles of length 2. What is the length of the cycle generated by ab? How many elements does the group contain? Make a complete table of the group composition, and show that it in fact is the "symmetric group" S_3 of permutations of three elements.

1.3. *a)* The Lie algebra $\mathfrak{so}(11)$ of rotations in 11 dimensions has an irreducible 55-dimensional representation. Describe the representation module in terms of tensors.

b) Same question, but for an irreducible 65-dimensional representation (this may require the use of some invariant tensor).

c) What is the tensor product of the 11-dimensional vector representation with itself?

1.4. Consider the Maxwell field strength 2-form on $\mathbb{R}^3\backslash\{0\}$:

$$F = \frac{1}{4\pi r^3}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy) \, ,$$

which is well defined outside the origin. What is the corresponding B-field? Show that F satisfies Maxwell's equations for $r > 0$. Calculate the surface integral $\int_S F = \int_S \vec{B} \cdot \vec{dS}$, where S is a surface enclosing $r = 0$, and conclude that there is a magnetic monopole at $r = 0$. Find a 1-form A such that $dA = F$. Is it well defined everywhere outside the origin?

(The "wedge" notation for forms is read so that a 2-form ω*, i.e., a tensor with 2 antisymmetric lower indices can is expressed as* $\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$*. For example,* $\omega = f(x, y, z) dy \wedge dz$ *means* $\omega_{23} = -\omega_{32} = f$*, and all other components* 0*. See also Section 4.3 in the lecture notes.)*