Home assignment 2, 2021 — Symmetry TIF310/FYM310

Deadline Friday Dec. 10, 2021.

Hand in solutions in pdf format, produced by TEX, containing your name, by mail to martin.cederwall@chalmers.se (or printed in a box outside room Origo 6102 if you can get through the locked doors). Good luck!

2.1. "Tachyons" are particles travelling with speed higher than the speed of light. The presence of tachyons in a model is considered to be a sign of an instability, for example due to the choice of a false, unstable vacuum. In a classical field theory, we do not see the particles, but the analogous statement would be that there are wave solutions with space-like wave vectors. As a model for a tachyonic theory, consider the Klein–Gordon equation with $m^2 < 0$. Demonstrate that the "vacuum" $\phi = 0$ is unstable. Can the model be stabilised by adding a term proportional to ϕ^4? If so, what is the true (stable) vacuum?

2.2. Construct a vector v as the square of a $d = 4$ bosonic spinor λ as $v_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$. Show that $v^2 = 0$. Define and perform the corresponding construction in $d = 3$. (It works also in $d = 6$ and $d = 10$.)

2.3. Chiral spinor representations of (some real form of) $\mathfrak{so}(16)$ are 128-dimensional. Call the spinor modules $\mathbf{128}$ and $\mathbf{128}'$. They are both self-conjugate, meaning that $\mathbf{128}$ is its own dual module, and accordingly for $\mathbf{128}'$. Determine which antisymmetric tensors appear in $\mathbf{128}\otimes\mathbf{128}$, $\mathbf{128}\otimes\mathbf{128}'$ and $\mathbf{128}'\otimes\mathbf{128}'$, and which of these belong to the symmetric and antisymmetric parts of the tensor products. An argument with index structure and counting dimensions of modules is sufficient.

2.4. The simple Lie algebra $A_2 = \mathfrak{sl}(3)$ has two inequivalent 3-dimensional representations, $\mathbf{3}$ and $\overline{\mathbf{3}}$. Let us denote an irreducible representation by the Dynkin labels of its highest weight. Then, $\mathbf{3} = (10)$ and $\overline{\mathbf{3}} = (01)$. Show, using tensors, that $(10) \otimes (10)$ contains two irreducible representations, of which one is (01). Draw a picture of the weights in the other one, using the additivity of weights in a tensor product. What is its highest weight? Is it always true that the tensor product of a representation with highest weight λ with itself contains the representation with highest weight 2λ?

2.5. The energy-momentum (or stress-energy) tensor can be derived as the variation of an action with respect to the metric as

$$T^{mn} = \frac{2}{\sqrt{|g|}} \frac{\partial \mathscr{L}}{\partial g_{mn}} ,$$

where $\mathscr{L}$ is the Lagrangian density. This applies also for a theory defined in flat space, but then the metric has to be reinstated so that coordinate invariance of the action is manifest. Use this definition to derive the energy-momentum tensor for Maxwell theory, and identify the usual forms of the energy density as T^{00} and the Poynting vector as T^{0i}. Show that the Maxwell energy-momentum tensor is traceless precisely in $d = 4$, and relate this property to the invariance of the action under a rescaling of the metric. This is a sign of conformal invariance.
Hint: be careful with precisely how the metric enters $\mathscr{L}$.